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Does Eulerian percolation on $\mathbb{Z}^2$ percolate?

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Abstract. Eulerian percolation on $\mathbb{Z}^2$ with parameter p is the Bernoulli bond percolation with parameter p conditioned on the event that every site has an even degree. Eulerian percolation with parameter p coincides with the contours of the Ising model for a well-chosen parameter $\beta(p)$. This allows to study the percolation properties of Eulerian percolation in the ferromagnetic range $\beta(p) \geq 0$, corresponding to $p \leq 1/2$. To study the case $p > 1/2$, we provide a new coupling between Bernoulli percolations with parameters p and $1-p$, that increases connectivity properties and preserves the parity of the degrees of the sites. Some key ingredients of the proofs are couplings between Eulerian percolation, the Ising model and FK-percolation.

1. Introduction

In Bernoulli percolation with parameter p on a graph, each edge of the graph is independently open with probability p, and closed with probability $1-p$. Open edges then induce a random subgraph of the initial graph. Eulerian percolation with parameter p is the Bernoulli percolation with parameter p, but conditioned to be even, *i.e.* conditioned to the event that each vertex of the random subgraph has an even number of open edges touching it. In this paper, we aim to study the percolation properties of the Eulerian (or even) percolation on the edges of $\mathbb{Z}^2$. For independent Bernoulli bond percolation, there is a natural increasing coupling for parameters $p \in [0,1]$. This allows to define the critical parameter for the existence of an infinite connected component of open edges. When we condition by the Eulerian condition, we loose this monotonicity property, so that it is *a priori* not even clear that there is such a unique critical parameter for Eulerian percolation.

This paper has two parts.

1. On $\mathbb{Z}^2$, the event by which we want to condition has probability 0. The first step is thus to define properly the Eulerian percolation measures on the edges of $\mathbb{Z}^2$,

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by the mean of specifications in finite boxes and of Gibbs measures. Doing so, the Eulerian percolation measure with parameter p is given by the contours of the Ising model on the sites of the dual $\mathbb{Z}^2_* \sim \mathbb{Z}^2$ for a well-chosen parameter $\beta = \beta(p)$:

Theorem 1.1. *For every $p \in [0,1]$, there exists a unique Eulerian percolation measure on the edges of $\mathbb{Z}^2$ with opening parameter p, and we denote it by μ_p. It is the image by the contour application of any Gibbs measure for the Ising model on the dual graph $\mathbb{Z}^2_*$ of $\mathbb{Z}^2$, with parameter*

$$\beta = \beta(p) = \frac{1}{2}\log\frac{1-p}{p} \quad \Leftrightarrow p = \frac{1}{1+\exp(2\beta)}.$$

Moreover, μ_p is invariant and ergodic under the automorphism group of $\mathbb{Z}^2$.

The interpretation of the Ising model in terms of contours is known for a long time and was used to prove the phase transition of the Ising model. Note that Eulerian percolation with parameter $p < 1/2$, resp. $p > 1/2$, corresponds to the contours of the Ising model in the ferromagnetic range $\beta > 0$, resp. antiferromagnetic range $\beta < 0$. Theorem 1.1 is an extension of Theorem 5.2 of Grimmett and Janson (2009), that studies random even subgraphs on finite planar graphs. In the same paper, they mention the existence of a thermodynamic limit, but the question of uniqueness is not asked.

2. Connected components induced by open edges are called *open clusters*. We are interested in the probability, under the even percolation measure μ_p, of the percolation event:

$$\mathcal{C} = \text{“there exists an infinite open cluster”}.$$

Our first result consists in proving the almost-sure uniqueness of the infinite cluster when it exists:

Theorem 1.2. *For every $p \in [0,1]$, there exists μ_p-almost surely exactly one infinite cluster or μ_p-almost surely no infinite cluster.*

Note that the “even degree” condition induces dependencies between states of edges, that break the classical finite energy property. However, we can adapt the classical proof by using the interpretation in terms of contours of the Ising model. To study the percolation itself, we have at our disposal the results proved for the Ising model on $\mathbb{Z}^2$, especially in the ferromagnetic range. Remember that $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ is the critical value of the Ising model in $\mathbb{Z}^2$; we introduce the corresponding percolation parameter

$$p_{c,\text{even}} = \frac{1}{1+\exp(2\beta_c)} = 1 - \frac{1}{\sqrt{2}} < \frac{1}{2}.$$

We prove the following:

Theorem 1.3. *In terms of even percolation with parameter $p \in [0,1]$,*

- *for every $p \in [0, p_{c,\text{even}}]$, $\mu_p(\mathcal{C}) = 0$,*
- *for every $p \in (p_{c,\text{even}}, 1] \backslash \{1 - p_{c,\text{even}}\}$, $\mu_p(\mathcal{C}) = 1$.*

In terms of the Ising model with parameter $\beta \in \mathbb{R}$, these results correspond to:

- *for $\beta \geq \beta_c$, for every Gibbs measure with parameter β, contours a.s. do not percolate,*
- *for $\beta < \beta_c$ such that $\beta \neq -\beta_c$, for every Gibbs measure with parameter β, contours a.s. percolate.*

These results are summarized in the following table:

p	0		$p_{c,\text{even}}$ $= 1 - 1/\sqrt{2}$	$1/2$	$1 - p_{c,\text{even}}$ $= 1/\sqrt{2}$		1
$\beta(p)$	$+\infty$		β_c	0	$-\beta_c$		$-\infty$
μ_p		no perco.	]	perco.	?	perco.	

The percolation results for $p \le 1/2$ essentially follow from the results about percolation of spins in the Ising model in the ferromagnetic case $\beta > 0$. The Ising model in the antiferromagnetic case has been much less studied, so other kinds of arguments are needed for $p > 1/2$.

In order to settle the case $1/2 < p < 1 - p_{c,\text{even}}$, we provide a new coupling between Bernoulli percolations with parameters p and $1-p$, that increases connectivity properties and preserves the parity of the degrees of the sites. This induces a coupling between μ_p and μ_{1-p} that also increases connectivity (Lemma 4.2), so that percolation for $p \in (p_c, 1/2)$ implies also percolation for $p \in (1/2, 1 - p_{c,\text{even}})$.

For the case $p > 1 - p_{c,\text{even}}$, we use the coupling between the Ising model and FK-percolation. Our proof requires the identification of the critical parameter for percolation of closed edges in FK-percolation, using techniques from Duminil-Copin (2016).

In independent Bernoulli bond percolation, $p \mapsto \mathbb{P}_p(\mathcal{C})$ is non-decreasing, and this follows from a natural coupling of percolation for all parameters $p \in [0,1]$. Here, conditioning by the Eulerian condition can break the positive association, even if the underlying graph is Eulerian (see Section 5). Thus, we can't deduce that percolation occurs for $p = 1 - p_{c,\text{even}}$ from the fact that we know that percolation occurs for smaller parameters. We naturally conjecture that $p_{c,\text{even}}$ is indeed the unique percolation threshold for Eulerian percolation on $\mathbb{Z}^2$:

Conjecture 1.4.
- *In terms of even percolation:* $\mu_{1-p_{c,\text{even}}}(\mathcal{C}) = 1$.
- *In terms of the Ising model: for every Gibbs measure with parameter $-\beta_c$, contours a.s. percolate.*

2. Eulerian percolation probability measures

On $\mathbb{Z}^2$, we consider the set of edges $\mathbb{E}^2$ between vertices at distance 1 for $\|.\|_1$. An edge configuration is an element $\omega \in \{0,1\}^{\mathbb{E}^2}$: if $\omega(e) = 1$, the edge e is present (or open) in the configuration ω, and if $\omega(e) = 0$, the edge is absent (or closed). For $x \in \mathbb{Z}^2$, we define the degree $d_\omega(x)$ of x in the configuration ω by setting

$$d_x(\omega) = \sum_{e \ni x} \omega(e).$$

An Eulerian edge configuration is then an element of

$$\Omega_{\text{EP}} = \{\omega \in \{0,1\}^{\mathbb{E}^2} : \forall x \in \mathbb{Z}^d,\ d_x(\omega) = 0\ [2]\}.$$

If $\Lambda \subset \mathbb{E}^2$, $\eta \in \{0,1\}^\Lambda$ and $\omega \in \{0,1\}^{\Lambda^c}$, we denote by $\eta\omega$ the concatenation of η inside Λ and of ω in Λ^c.

Gibbs measures for Eulerian percolation. A measure μ on $(\{0,1\}^{\mathbb{E}^2}, \mathcal{B}(\{0,1\}^{\mathbb{E}^2}))$ is a Gibbs measure for Eulerian percolation (or a Eulerian percolation measure) if it satisfies the two following conditions:

- $\mu(\Omega_{\mathrm{EP}}) = 1$;
- for each finite subset Λ of $\mathbb{E}^2$, and μ-almost every $\omega \in \Omega_{\mathrm{EP}}$, the conditional law of the configuration inside Λ conditionally on Λ^c is given by

$$\begin{aligned} \forall \eta \in \{0,1\}^\Lambda \quad \mu(\,\eta\,|\,\omega_{\Lambda^c}\,) &= \mathrm{Ber}(p)^{\otimes\{0,1\}^\Lambda}\,(\,\eta\,|\,\eta\omega_{\Lambda^c} \in \Omega_{\mathrm{EP}}\,) \stackrel{\text{def}}{=} \mu^p_{\Lambda,\omega}(\eta) \\ &= \frac{1}{Z^p_{\Lambda,\omega}} \left(\frac{p}{1-p}\right)^{\sum_{e\in\Lambda}\eta_e} \mathbf{1}_{\eta\omega_{\Lambda^c}\in\Omega_{\mathrm{EP}}} \end{aligned} \tag{2.1}$$

where $Z^p_{\Lambda,\omega}$ is a normalizing factor.

Thus $\mu^p_{\Lambda,\omega}$ is the law of independent Bernoulli percolation with parameter p on the edges of the finite set Λ, but conditioned to create a Eulerian configuration of $\mathbb{E}_2$ when concatenated with ω_{Λ^c} outside Λ. We denote by $\mathcal{G}_{\mathrm{EP}}(p)$ the set of Gibbs measures for Eulerian percolation with opening parameter p.

It can be proved that for any $\omega \in \Omega_{\mathrm{EP}}$, any accumulation point of the sequences $(\mu^p_{\Lambda_n,\omega})_{n\geq 0}$ with $\Lambda_n \uparrow \mathbb{E}^2$ is a Gibbs measure for Eulerian percolation. Conversely, each $\mu \in \mathcal{G}_{EP}(p)$ can be obtained as a limit of contour measures in finite boxes with random boundary conditions. We will use this property in our proof of Theorem 1.1.

Spin configurations and Eulerian percolation. A natural way to obtain a Eulerian configuration of the edges of a planar graph is to take the contours of a spin configuration of the sites of its dual, and this is what we describe now in the $\mathbb{Z}^2$ case.

Let $\mathbb{Z}^2_* = (1/2, 1/2) + \mathbb{Z}^2$ be the dual graph of $\mathbb{Z}^2$. The set $\mathbb{E}^2_*$ of edges of $\mathbb{Z}^2_*$ is the image of $\mathbb{E}^2$ by the translation with respect to the vector $(1/2, 1/2)$. If $e \in \mathbb{E}^2$, we denote by e_* its dual edge, *i.e.* the only edge in $\mathbb{E}^2_*$ that intersects e. We can map any spin configuration of $\{-1,+1\}^{\mathbb{Z}^2_*}$ to its contour in the following way:

$$\begin{array}{rccl} \Gamma: & \{-1,1\}^{\mathbb{Z}^2_*} & \longrightarrow & \Omega_{\mathrm{EP}} \\ & \sigma = (\sigma_{i_*})_{i_*\in\mathbb{Z}^2_*} & \longmapsto & (\eta_e)_{e\in\mathbb{E}^2}, \text{ with } \eta_e = \mathbb{1}_{\{\sigma_{i_*}\neq\sigma_{j_*}\}} \text{ if } e_* = \{i_*, j_*\} \end{array}.$$

Let us see that $\Gamma(\sigma) \in \Omega_{\mathrm{EP}}$. Indeed, set $\eta = \Gamma(\sigma)$, and fix $x \in \mathbb{Z}^2$. Let a_*, b_*, c_*, d_* be the four corners of the square with length side 1 in $\mathbb{Z}^2_*$ whose center is x: then the four edges issued from x are the dual edges of $\{a_*, b_*\}$, $\{b_*, c_*\}$, $\{c_*, d_*\}$ and $\{d_*, a_*\}$. Thus

$$(-1)^{d_x(\eta)} = (-\sigma_{a_*}\sigma_{b_*})(-\sigma_{b_*}\sigma_{c_*})(-\sigma_{c_*}\sigma_{d_*})(-\sigma_{d_*}\sigma_{a_*}) = 1.$$

So $\Gamma(\sigma) \in \Omega_{\mathrm{EP}}$.

Reciprocally, the dual of a planar Eulerian graph is bipartite (see for instance van Lint and Wilson, 2001, Theorem (34.4)), and there are exactly two spin configurations on the sites of a connected bipartite graph such that the extremities of every edge have different spins. In our $\mathbb{Z}^2$ case, fix a Eulerian edge configuration η. By setting $\sigma_\eta(0_*) = +1$, and for any $x_* \in \mathbb{Z}^2_*$, $\sigma_\eta(x_*)$ equals (-1) power the number of edges in η crossed by any path (in the dual) between 0_* and x_*, we properly define a spin configuration σ_η of $\mathbb{Z}^2_*$, and $\Gamma^{-1}(\eta) = \{\sigma_\eta, -\sigma_\eta\}$. Finally, the contour application Γ is surjective and two-to-one. As it is local, it is also continuous.

As we will see now, the Gibbs measures for Eulerian percolation can be obtained as the images by the contour application Γ of the Gibbs measures for the Ising model on $\mathbb{Z}^2_*$.

Gibbs measures for the Ising model on $\mathbb{Z}^2_*$. It is of course the same model as the Ising model on $\mathbb{Z}^2$, but to avoid confusion between the initial graph $\mathbb{Z}^2$ and its dual $\mathbb{Z}^2_*$ in the sequel, we present it directly in the dual $\mathbb{Z}^2_*$. Fix a parameter $\beta \in \mathbb{R}$.

A Gibbs measure γ for the Ising model with parameter β on $\mathbb{Z}^2_*$ is a probability measure on $(\{-1,+1\}^{\mathbb{Z}^2_*}, \mathcal{B}(\{-1,+1\}^{\mathbb{Z}^2_*}))$ such that: for any finite subset Λ of $\mathbb{Z}^2_*$, and for γ-almost every configuration $\tau \in \{-1,+1\}^{\mathbb{Z}^2_*}$, the conditional law of the configuration inside Λ conditionally on Λ^c is given by:

$$\forall \sigma \in \{-1,+1\}^{\Lambda} \quad \gamma(\,\sigma \,|\, \tau_{\Lambda^c}\,) \overset{\text{def}}{=} \Pi^{\beta}_{\Lambda,\tau}(\sigma) = \frac{1}{\mathcal{Z}^{\beta}_{\Lambda,\tau}} \exp\Big(\beta \sum_{\substack{e=\{x,y\}\in\mathbb{E}^2_* \\ e\cap\Lambda\neq\varnothing}} \sigma_x\sigma_y\Big),$$

with the convention $\sigma(y) = \tau(y)$ if $y \notin \Lambda$ and where $\mathcal{Z}^{\beta}_{\Lambda,\tau}$ is a normalizing factor. We denote by $\mathcal{G}(\beta)$ the set of Gibbs measures for the Ising model with parameter β.

Consider $\Pi^{\beta}_{\Lambda,\tau}$: when $\beta = 0$, the spins of sites inside Λ are i.i.d. and follow the uniform law in $\{-1,+1\}$; when $\beta > 0$, neighbour sites prefer to be in the same state (ferromagnetic case), while when $\beta < 0$, neighbour sites prefer to be in different states (anti-ferromagnetic case).

It is well known that any accumulation point of a sequence $(\Pi^{\beta}_{\Lambda_n,\tau_n})_{n\geq 0}$ with $\Lambda_n \uparrow \mathbb{Z}^2_*$ is a Gibbs measure for the Ising model with parameter β.

The Ising model has a phase transition: set $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ (see Onsager, 1944), then

- if $0 \leq \beta \leq \beta_c$, then there is a unique Gibbs measure;
- if $\beta > \beta_c$ then there are infinitely many Gibbs measures.

In the latter case, the set $\mathcal{G}(\beta)$ is the convex hull of two distinct extremal measures γ^+_β and γ^-_β, that can be deduced one from the other by exchanging the spin values. This result has been obtained independently by Aizenman (1980) and Higuchi (1981). See also Georgii and Higuchi (2000).

For $\beta < 0$, the Gibbs measures are obtained from $\mathcal{G}(-\beta)$ by changing the spins on the subset of even sites. In other words, if $S : \{-1,+1\}^{\mathbb{Z}^2_*} \to \{-1,+1\}^{\mathbb{Z}^2_*}$ is defined by:

$$\forall (i,j) \in \mathbb{Z}^2_*, \quad S(\sigma)_{i,j} = (-1)^{i+j}\sigma_{i,j}$$

then the measure μ_S defined by $\mu_S(A) = \mu(S^{-1}(A))$ belongs to $\mathcal{G}(-\beta)$ if and only if $\mu \in \mathcal{G}(\beta)$. For the details, see Chapter 6 in Georgii (1988).

To prove the uniqueness of the even percolation probability measure, we will need the following lemma:

Lemma 2.1. *Let $\sigma \in \{-1,1\}^{\mathbb{Z}^2_*}$, and consider the configuration $\eta \in \Omega_{EP}$ defined by $\eta = \Gamma(\sigma)$. Suppose that Λ is a finite simply connected subset of $\mathbb{Z}^2_*$, and denote by $E(\Lambda)$ the set of edges $e \in \mathbb{E}^2$ such that e_* has at least one end in Λ.*

Fix $p \in (0,1)$ and set $\beta = \beta(p) = \frac{1}{2}\log\frac{1-p}{p}$. Then, the probability $\mu^p_{E(\Lambda),\eta}$ is the image of $\Pi^{\beta(p)}_{\Lambda,\sigma}$ under the contour application Γ.

Proof: By construction, the image of $\Pi^{\beta}_{\Lambda,\sigma}$ under the map $\tau \mapsto \Gamma(\tau)$ is concentrated on configurations that coincide with η outside $E(\Lambda)$. Obviously it is the same for $\mu^p_{E(\Lambda),\eta}$, so we must focus on the behaviour of the edges in $E(\Lambda)$.

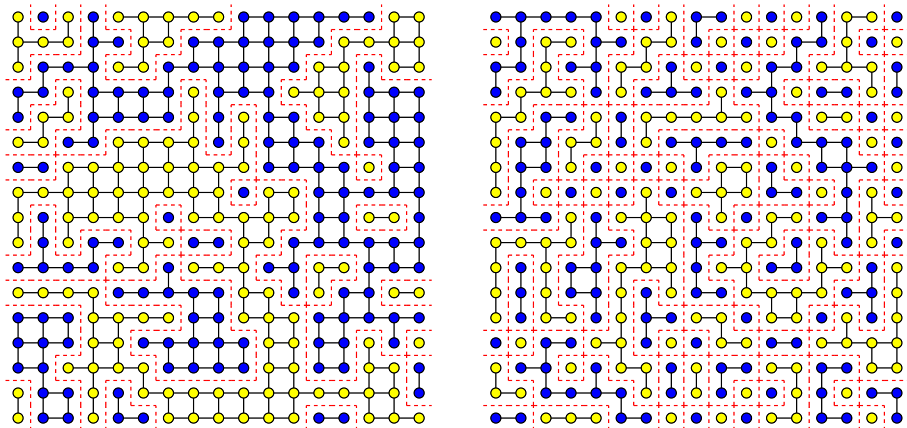


Figure 2.1. The mapping $\beta \longleftrightarrow -\beta$

Let $\eta' \in \Omega_{\mathrm{EP}}$ be such that η and η' coincide outside $E(\Lambda)$. There are exactly two spin configurations $\sigma', -\sigma'$ such that $\Gamma(\sigma') = \Gamma(-\sigma') = \eta'$. If x and y are two neighbours in Λ^c, then

$$\sigma_x\sigma_y = 1 - 2\eta_{(x,y)_*} = 1 - 2\eta'_{(x,y)_*} = \sigma'_x\sigma'_y,$$

so $\sigma_x\sigma'_x = \sigma_y\sigma'_y$. Since Λ^c is connected, it follows that one of the two spin configurations, say σ', coincides with σ on Λ^c (and $-\sigma'$ with $-\sigma$). Thus $\Pi^\beta_{\Lambda,\sigma}(-\sigma') = 0$ and $\Pi^\beta_{\Lambda,\sigma}(\sigma') > 0$, and:

$$\begin{aligned}
&\Pi^\beta_{\Lambda,\sigma}(\Gamma(.) = \eta') = \Pi^\beta_{\Lambda,\sigma}(\sigma') \\
&= \frac{1}{\mathcal{Z}^\beta_{\Lambda,\sigma}} \exp\left(\beta \sum_{\substack{e=\{x,y\}\in\mathbb{E}^2_* \\ e\cap\Lambda\neq\varnothing}} \sigma'_x\sigma'_y\right) = \frac{1}{\mathcal{Z}^\beta_{\Lambda,\sigma}} \exp\left(\beta \sum_{\substack{e=\{x,y\}\in\mathbb{E}^2_* \\ e\cap\Lambda\neq\varnothing}} (1-2\eta'_{(x,y)_*})\right) \\
&= \frac{1}{\mathcal{Z}^\beta_{\Lambda,\sigma}} \exp\left(\beta \sum_{e\in E(\Lambda)} (1-2\eta'_e)\right) = \frac{\exp(\beta|E(\Lambda)|)}{\mathcal{Z}^\beta_{\Lambda,\sigma}} \left(\frac{p}{1-p}\right)^{\sum_{e\in E(\Lambda)}\eta'_e} \\
&= \alpha_{\Lambda,\eta}\mu^p_{E(\Lambda),\eta}(\eta').
\end{aligned}$$

Since we compare probability measures with the same support, $\alpha_{\Lambda,\eta} = 1$. □

Proof of Theorem 1.1. The existence of Gibbs measures for the even percolation follows from a compactness argument. It must be noticed that equation (2.1) is only defined for boundary conditions $\omega \in \Omega_{\mathrm{EP}}$, but the classical arguments can be adapted (see Georgii, 1988).

We now prove the uniqueness of the even percolation probability measure. First, let us see that all Gibbs measures for the Ising model with parameter β have the same image by the application Γ. Let $\gamma \in \mathcal{G}(\beta)$: there exists $\alpha \in [0,1]$ such that $\gamma = \alpha\gamma^+_\beta + (1-\alpha)\gamma^-_\beta$. Remember that γ^-_β is the image of γ^+_β by the exchange of spins, that leaves the contours unchanged. Thus, γ_β^+ and γ_β^+ induce the same measure on the contours; and so do their convex combinations.

Let now $\mu \in \mathcal{G}_{EP}(p)$. The measure μ can be obtained as a limit of contour measures in finite boxes with random boundary conditions. By Lemma (2.1), those measures can be obtained as the contours of Ising measures with random condition. By extracting a subsequence if necessary, we can see that μ is the contour measure associated to some Ising measures on the full lattice. This proves the uniqueness.

Finally, note that γ_β^+ is stationary and ergodic, and so does μ_p. □

3. Unicity of the infinite cluster in Eulerian percolation

Proof of Theorem 1.2. Since μ_p is ergodic and $\mathcal{C}$ is a translation-invariant event, we have $\mu_p(\mathcal{C}) \in \{0,1\}$. To prove the unicity of the infinite cluster, we now follow the famous proof by Burton and Keane (1989). The main point here is that the Eulerian percolation measure is not insertion tolerant: once a configuration is fixed outside a box, the even degree condition forbids some configurations inside the box. But the Ising model has the finite energy property, and we will thus use the representation of even percolation in terms of contours of the Ising model.

The number N of infinite clusters is translation-invariant, so the ergodicity of μ_p implies that it is μ_p-almost surely constant: there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N = k) = 1$. The first step consists in proving that $k \in \{0, 1, \infty\}$. So assume for contradiction that k is an integer larger than 2. Consider a finite box Λ, large enough to ensure that with positive probability (under μ_p), the box Λ intersects at least two infinite clusters. Using Theorem 1.1, this implies that with positive probability (under γ_β^+ for the parameter β corresponding to p), the contours of the Ising model present two infinite connected components that intersect Λ. But the Ising model has the finite energy property: by forcing the spins inside Λ to be a chessboard, we keep an event with positive probability, and we decrease the number of infinite clusters in the contours by at least one. Coming back to Eulerian percolation, this gives $\mu_p(N \le k-1) > 0$, which is a contradiction. See Newman and Schulman (1981a,b) for the first version of such an argument.

In the final step, we prove that $k = \infty$ is impossible. Assume by contradiction that $\mu_p(N = +\infty) = 1$. We work now with the spin configurations of the sites of $\mathbb{Z}^2_*$, under $\gamma^+_{\beta(p)}$.

By taking $L \in \mathbb{N}$ large enough, we can assume that the event E_L “the box $B_L = [-L, L]^2$ intersects at least 30 infinite clusters” has positive probability. Let $\partial\sigma_0$ be a spin configuration of the sites in $\partial_{int} B_L = B_L \backslash B_{L-1}$ such that

$$\gamma^+_{\beta(p)}(\sigma \in E_L,\, \sigma_{|\partial_{int} B_L} = \partial\sigma_0) > 0.$$

Take τ in this event. Each infinite cluster intersecting B_L crosses $\partial_{int} B_L$ via an open edge, and this edge sits between a $+1$ site and a -1 site.

Thus the 30 distinct infinite (edge) clusters intersecting B_L imply the existence of at least 15 clusters of $+1$ vertices in $\partial_{int} B_L$. To avoid geometric intricate details, we do not want to consider $+1$-clusters in $\partial_{int} B_L$ that are in the corners: we thus remove from our 15 clusters at most $12 = 3 \times 4$ clusters (the one containing the corner if it is a $+1$, and the nearest $+1$ cluster on each side). We are now left with at least 3 disjoint $+1$-clusters in $\partial_{int} B_L$, sitting near edges of distinct infinite clusters: they are far away enough so that we can draw, inside B_L, 3 paths of sites

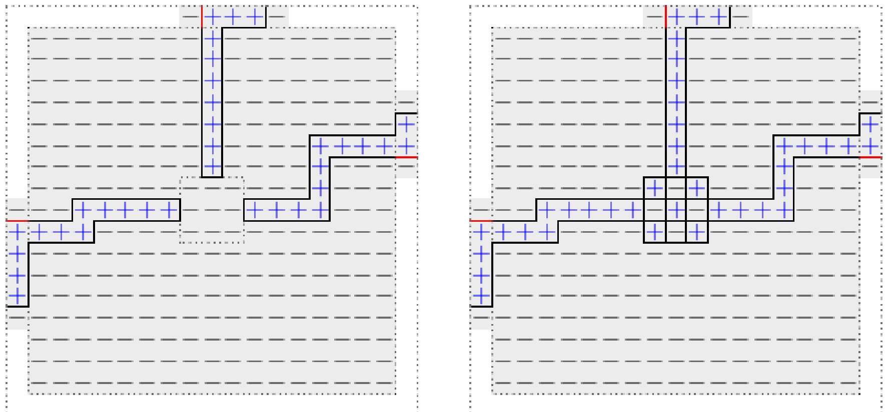


Figure 3.2. Construction of a trifurcation in B_{L_1}. Dotted squares are, from inside to outside, B_1, B_{L-1} and B_L. Red edges are, on the left, in three distinct infinite clusters of open edges.

linking these three clusters to three of the four centers of the sides of $\partial_{int} B_2$, in such a way that two distinct paths are not $*$-connected. See Figure 3.2.

Consider now the following spin configuration of B_{L-1}: all sites in the three paths are $+1$, all the other sites are -1. With this spin configuration, B_{L-1} intersects exactly three infinite clusters of open edges. If we change the spins of B_1 in a chessboard, B_{L-1} intersects exactly one infinite cluster of open edges. In this case, we say that 0 is a trifurcation. As $\gamma^+_{\beta(p)}$ has the finite energy property, we see that 0 has positive probability of being a trifurcation, and the end of the proof is as in the proof by Burton and Keane.

4. Percolation properties of Eulerian percolation

This section is devoted to the proof of Theorem 1.3. We begin by recalling the coupling between the Ising model and the random cluster model (or FK-percolation). Concerning the random cluster model, we just recall the few results we need and we refer to Grimmett (2006) for a complete survey on this model.

The random cluster measure with parameters p and q on a finite graph $G = (V, E)$ is the probability measure on $\{0,1\}^E$ defined by:

$$\varphi^G_{p,q}(\eta) = \frac{1}{Z}\Big(\frac{p}{1-p}\Big)^{\sum_{i\in E}\eta_i} q^{k(\eta)},$$

where $k(\eta)$ is the number of connected components in the subgraph of G given by η, and Z is a normalizing constant.

On $\mathbb{Z}^2$, it is known that at least for $p \neq \frac{\sqrt{q}}{1+\sqrt{q}}$, there exists a unique infinite volume random cluster measure, that we denote by $\varphi_{p,q}$ (Theorem (6.17) in Grimmett, 2006). It is a probability measure on $\{0,1\}^{\mathbb{E}^2}$.

In our study of even percolation, we use two properties of the random cluster model: its link with the Ising model, and its duality property. For $\beta > 0$, $\beta \neq \beta_c$, let us set

$$f(\beta) = 1 - \exp(-2\beta).$$

(A1) From a spin configuration $\sigma \in \{-1, +1\}^{\mathbb{Z}^2}$ whose distribution is any Gibbs measure γ_β for the Ising model with parameter $\beta \geq 0$, one obtains a subgraph $\eta \in \{0,1\}^{\mathbb{E}^2}$ with distribution $\varphi_{f(\beta),2}$ by keeping independently each edge between identical spins with probability $f(\beta)$, and erasing all the edges between different spins. For finite graphs, this can be found in Theorem (1.13) in Grimmett (2006). For the $\mathbb{Z}^2$ case, Theorem (4.91) in Grimmett (2006) says that this erasing procedure allows to couple the wired boundary infinite volume random cluster measure $\varphi^1_{f(\beta),2}$ and the Ising measure γ^+_β.

For a subgraph $\eta \in \{0,1\}^{\mathbb{E}^2}$, we denote by $\eta^c \in \{0,1\}^{\mathbb{E}^2}$ the complementary subgraph of $\mathbb{Z}^2$, meaning that the open edges of η^c are exactly the closed edges of η. We denote by $\eta_* \in \{0,1\}^{\mathbb{E}^2_*}$ the dual graph of η: in η_*, the edge e_* is open if and only if e is closed. We naturally extend these notations to measures.

(A2) The random cluster model on $\mathbb{Z}^2$ has the following duality property (Theorem (6.13) in Grimmett, 2006): if η is distributed according to $\varphi^1_{p,2}$, then the distribution $(\varphi^1_{p,2})_*$ of η_* is equal to the free boundary infinite volume random cluster measure $\varphi^0_{p^*,2}$, where:

$$\frac{p^*}{1-p^*} = 2\,\frac{1-p}{p} \quad \Leftrightarrow \quad p^* = \frac{2-2p}{2-p}.$$

It was first derived by Onsager (1944) that the critical parameter $p_c(2)$ for percolation of open edges in the random cluster model is equal to the self-dual point, *i.e.* the only fixed point of the map $p \mapsto p^*$:

$$p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}.$$

Let us also recall that the Eulerian percolation measure μ_p on the edges of $\mathbb{Z}^2$ is obtained as the contours of any Ising measure with parameter $\beta(p)$ on $\mathbb{Z}^2_*$, and in particular as the contours of $\gamma^+_{\beta(p)}$ (Theorem 1.1). From the previous coupling between the Ising model and the random cluster model, we can deduce the following stochastic comparison (please note that this stochastic comparison has already been proved by Grimmett and Janson (2009), but we provide here a different and short proof):

Lemma 4.1. *For $p \leq 1/2$, we have the following stochastic ordering:*

$$\mu_p \preceq \varphi^0_{2p,2}, \qquad \textit{or equivalently,} \qquad (\varphi^0_{2p,2})^c \preceq \mu_{1-p}.$$

Proof: For $p \leq 1/2$, starting from an Ising configuration $\mathbb{Z}^2_*$ of distribution $\gamma^+_{\beta(p)}$, let us draw **all** the edges between identical spins. By Theorem 1.1, the configuration on the edges of $\mathbb{Z}^2_*$ that we obtain is distributed according to $(\mu_p)_*$.

By property (A1) above, this measure on the edges of $\mathbb{Z}^2_*$ stochastically dominates the distribution $\varphi^1_{f(\beta(p)),2}$:

$$\varphi^1_{f(\beta(p)),2} \preceq (\mu_p)_*,$$

see Figure 4.3 for an illustration. Taking the dual of graphs, we obtain:

$$\mu_p \preceq (\varphi^1_{f(\beta(p)),2})_* = \varphi^0_{q,2},$$

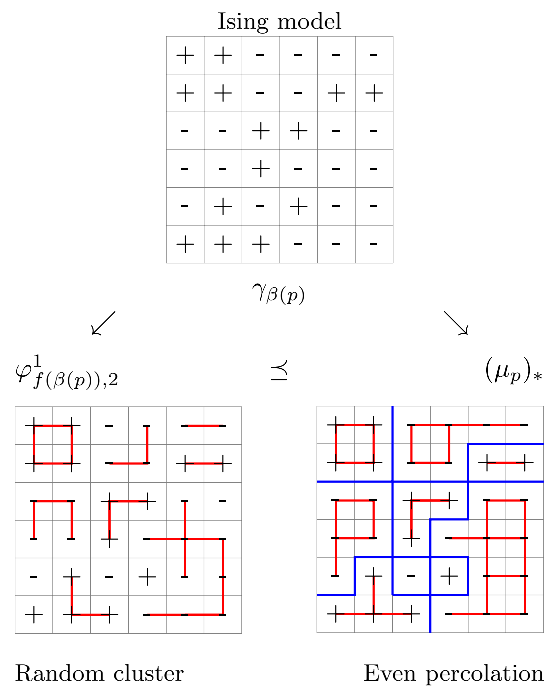


FIGURE 4.3. From a configuration distributed according to $\gamma_{\beta(p)}$, we construct a configuration distributed according to $\varphi^1_{f(\beta(p)),2}$ by keeping each edge between identical spins with probability $f(\beta)$ (red graph on the left), and a configuration distributed according to $(\mu_p)_*$ by keeping all edges between identical spins (red graph on the right: it is the dual graph of the blue contour graph, whose distribution is μ_p).

with, by property (A2),

$$q = f(\beta(p))_* = \frac{2-2f(\beta(p))}{2-f(\beta(p))} = \frac{2\exp(-2\beta(p))}{1+\exp(-2\beta(p))} = \frac{2\,\frac{p}{1-p}}{1+\frac{p}{1-p}} = 2p.$$

Thus, $\mu_p \preceq \varphi^0_{2p,2}$. Taking the complementary of configurations, we obtain the second stochastic comparison. $\square$

In the following lemma, we build, for $p<1/2$, a coupling between μ_p and μ_{1-p} that increases connectivity. For $x,y\in\mathbb{Z}^2$, let us denote $x\leftrightarrow y$ if x and y are connected by a path of open edges, meaning that they belong to the same open cluster.

Lemma 4.2. *Let $p\in(0,1/2)$. The law of the field $(\mathbb{1}_{\{x\leftrightarrow y\}})_{(x,y)\in\mathbb{Z}^2\times\mathbb{Z}^2}$ under μ_p is stochastically dominated by the law of the field $(\mathbb{1}_{\{x\leftrightarrow y\}})_{(x,y)\in\mathbb{Z}^2\times\mathbb{Z}^2}$ under μ_{1-p}.*

Proof: For every site $x\in\mathbb{Z}^2+(1/2,1/2)$, we consider the set $E_x\subset\mathbb{E}^2$ of its four surrounding edges, *i.e.* the four edges of the unit square with center x. Define

$G_2 = \{(x_1, x_2) \in \mathbb{Z}^2 + (1/2, 1/2) : \ x_1 + x_2 \in 2\mathbb{Z}\}$. Then $\mathbb{E}^2$ is the disjoint union of the E_x for $x \in G_2$.

We define $\Omega_x = (\{0,1\} \times \{0,1\})^{E_x}$. A point $(\omega_e, \tilde{\omega}_e)_{e \in E_x} \in (\{0,1\} \times \{0,1\})^{E_x}$ encodes two configurations of the four edges surrounding x: $(\omega_e)_{e \in E_x}$ and $(\tilde{\omega}_e)_{e \in E_x}$.

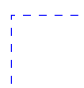
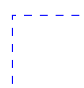
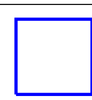
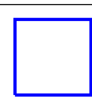
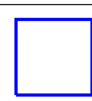
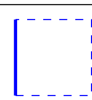
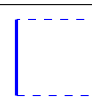
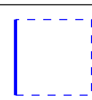
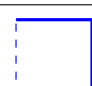
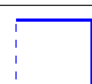
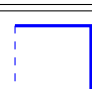
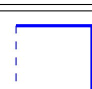
For $(\omega_e)_{e \in E_x}$ let us set $|\omega| = \sum_{e \in E_x} \omega_e$.

1. We first define a probability measure P on $\Omega_x = (\{0,1\} \times \{0,1\})^{E_x}$, whose first marginal is $\mathrm{Ber}(p)^{\otimes E_x}$, and whose second marginal is $\mathrm{Ber}(1-p)^{\otimes E_x}$. This probability P is defined by the table below, and has the property that P-almost surely, either $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$, or the configuration $(\tilde{\omega}_e)_{e \in E_x}$ is the complement of $(\omega_e)_{e \in E_x}$, which can be interpreted as the flip of the spin at x. This is possible since for any $(\alpha_e)_{e \in E_x} \in \{0,1\}^{E_x}$,

$$
\begin{aligned}
& \mathrm{Ber}(p)^{\otimes E_x}((\alpha_e)_{e \in E_x}) + \mathrm{Ber}(p)^{\otimes E_x}((1-\alpha_e)_{e \in E_x}) \\
= & \mathrm{Ber}(1-p)^{\otimes E_x}((1-\alpha_e)_{e \in E_x}) + \mathrm{Ber}(1-p)^{\otimes E_x}((\alpha_e)_{e \in E_x}) \\
= & p^{|\alpha|}(1-p)^{4-|\alpha|} + p^{4-|\alpha|}(1-p)^{|\alpha|}.
\end{aligned}
$$

In particular, P is such that there are the following possibilities for $(|\omega|, |\tilde{\omega}|)$:

- with probability $p^4 + (1-p)^4$, we have $(|\omega|, |\tilde{\omega}|) \in \{0,4\}^2$,
- with probability $4(p(1-p)^3 + (1-p)p^3)$, we have $(|\omega|, |\tilde{\omega}|) \in \{1,3\}^2$,
- with probability $6p^2(1-p)^2$, we have $|\omega| = |\tilde{\omega}| = 2$.

	$(\omega_e)_{e \in E_x}$	$(\tilde{\omega}_e)_{e \in E_x}$	probability under P	number of cases
$\lvert\omega\rvert = \lvert\tilde{\omega}\rvert = 0$ $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$			p^4	1
$\lvert\omega\rvert = 0, \lvert\tilde{\omega}\rvert = 4$ $(\omega_e)_{e \in E_x} = (1 - \tilde{\omega}_e)_{e \in E_x}$			$(1-p)^4 - p^4$	1
$\lvert\omega\rvert = \lvert\tilde{\omega}\rvert = 4$ $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$			p^4	1
$\lvert\omega\rvert = \lvert\tilde{\omega}\rvert = 1$ $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$			$p^3(1-p)$	$\binom{4}{1} = 4$
$\lvert\omega\rvert = 1, \lvert\tilde{\omega}\rvert = 3$ $(\omega_e)_{e \in E_x} = (1 - \tilde{\omega}_e)_{e \in E_x}$			$p(1-p)^3 -$ $p^3(1-p)$	$\binom{4}{1} = 4$
$\lvert\omega\rvert = \lvert\tilde{\omega}\rvert = 3$ $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$			$p^3(1-p)$	$\binom{4}{1} = 4$
$\lvert\omega\rvert = \lvert\tilde{\omega}\rvert = 2$ $(\omega_e)_{e \in E_x} = (\tilde{\omega}_e)_{e \in E_x}$			$p^2(1-p)^2$	$\binom{4}{2} = 6$

Because $p < 1/2$ and thus $p < 1 - p$, the probability measure P is well defined. One can easily check that P has the following properties.

(P1) The law of $(\omega_e)_{e\in E_x}$ under P is $\mathrm{Ber}(p)^{\otimes E_x}$, and the law of $(\tilde{\omega}_e)_{e\in E_x}$ under P is $\mathrm{Ber}(1-p)^{\otimes E_x}$.

(P2) $(\tilde{\omega}_e)_{e\in E_x}$ is more connected than $(\omega_e)_{e\in E_x}$: P-almost surely, if two corners of the squares are connected in $(\omega_e)_{e\in E_x}$, then they are connected in $(\tilde{\omega}_e)_{e\in E_x}$.

(P3) P-almost surely, the parity of the degree of each corner of the square is the same in the two configuration $(\omega_e)_{e\in E_x}$ and $(\tilde{\omega}_e)_{e\in E_x}$.

Note however that the coupling is not increasing: with probability $p(1-p)^3-p^3(1-p)>0$, $(\omega_e)_{e\in E_x}$ and $(\tilde{\omega}_e)_{e\in E_x}$ are not comparable.

2. We now extend the previous coupling to finite boxes of $\mathbb{E}^2$. Define, for $n\geq 1$, $\Lambda'_n=\{x\in G_2:\ \|x\|_\infty\leq n\}$ and denote by $E(\Lambda'_n)$ the subset of edges $e\in\mathbb{E}^2$ such that e_* has at least one end in Λ_n. Then $E(\Lambda'_n)$ is the disjoint union of the E_x for $x\in\Lambda'_n$. Set $\Delta_0=(\delta_0\otimes\delta_0)^{\otimes E_x}$.

Thus, $Q_n=P^{\otimes\Lambda'_n}\otimes\Delta_0^{\otimes G_2\backslash\Lambda'_n}$ is a probability measure on $(\{0,1\}\times\{0,1\})^{\mathbb{E}^2}$, where $(\omega_e,\tilde{\omega}_e)_{e\in\mathbb{E}^2}\in(\{0,1\}\times\{0,1\})^{\mathbb{E}^2}$ encodes two edges configurations on the whole plane: $\omega=(\omega_e)_{e\in\mathbb{E}^2}$ and $\tilde{\omega}=(\tilde{\omega}_e)_{e\in\mathbb{E}^2}$. From Properties (P1), (P2) and (P3), one gets:

(P1') The law of ω under Q_n is $\mathrm{Ber}(p)^{\otimes E(\Lambda'_n)}\otimes\delta_0^{\otimes\mathbb{E}^2\backslash E(\Lambda'_n)}$, and the law of $\tilde{\omega}$ under Q_n is $\mathrm{Ber}(1-p)^{\otimes E(\Lambda'_n)}\otimes\delta_0^{\otimes\mathbb{E}^2\backslash E(\Lambda'_n)}$.

(P2') Q_n-almost surely, if $x\overset{\omega}{\leftrightarrow}y$ then $x\overset{\tilde{\omega}}{\leftrightarrow}y$.

(P3') Q_n-almost surely, $\omega\in\Omega_{\mathrm{EP}}\iff\tilde{\omega}\in\Omega_{\mathrm{EP}}$.

3. Now we want to condition Q_n by the event that both configurations ω and $\tilde{\omega}$ are even. By Property (P3'), we have

$$\overline{Q}_n(.)\overset{def}{=}Q_n(.|\omega\in\Omega_{\mathrm{EP}},\ \tilde{\omega}\in\Omega_{\mathrm{EP}})=Q_n(.|\omega\in\Omega_{\mathrm{EP}})=Q_n(.|\tilde{\omega}\in\Omega_{\mathrm{EP}}).$$

Remember the definition of $\mu^p_{E(\Lambda'_n),0}$. With Property (P1'), one gets

$$\overline{Q}_n(\omega_{E(\Lambda'_n)}\in A)=\frac{Q_n(\omega_{E(\Lambda'_n)}\in A,\ \omega\in\Omega_{\mathrm{EP}})}{Q_n(\omega\in\Omega_{\mathrm{EP}})}=\mu^p_{E(\Lambda'_n),0}(A).$$

In the same manner, $\overline{Q}_n(\tilde{\omega}_{E(\Lambda'_n)}\in A)=\mu^{1-p}_{E(\Lambda'_n),0}(A)$. And we obtain

(P1") The law of ω under $\overline{Q}_n$ is $\mu^p_{E(\Lambda'_n),0}$, and the law of $\tilde{\omega}$ under $\overline{Q}_n$ is $\mu^{1-p}_{E(\Lambda'_n),0}$.

(P2") $\overline{Q}_n$-almost surely, if $x\overset{\omega}{\leftrightarrow}y$ then $x\overset{\tilde{\omega}}{\leftrightarrow}y$.

4. It remains to take limits when n goes to $+\infty$. We can extract a subsequence (n_k) such that $\overline{Q}_{n_k}$ converges to a probability measure $\overline{Q}$ when k tends to infinity. Thus both marginals $\mu^p_{E(\Lambda'_{n_k}),0}$ and $\mu^{1-p}_{E(\Lambda'_{n_k}),0}$ also converge when k tends to infinity to the marginals of $\overline{Q}$. As in the proof of Theorem 1.1, their limits are Gibbs measures for even percolation, so by uniqueness, they respectively converge to μ_p and μ_{1-p}. Thus,

(P1"') The law of ω under $\overline{Q}$ is μ_p, and the law of $\tilde{\omega}$ under $\overline{Q}$ is μ_{1-p}.

(P2"') $\overline{Q}$-almost surely, if $x\overset{\omega}{\leftrightarrow}y$ then $x\overset{\tilde{\omega}}{\leftrightarrow}y$.

So, the law of the field $(\mathbb{1}_{\{x\leftrightarrow y\}})_{(x,y)\in\mathbb{Z}^2\times\mathbb{Z}^2}$ under μ_p is stochastically dominated by the law of the field $(\mathbb{1}_{\{x\leftrightarrow y\}})_{(x,y)\in\mathbb{Z}^2\times\mathbb{Z}^2}$ under μ_{1-p}. $\square$

We can now proceed to the proof of Theorem 1.3: the proof is split into four cases, corresponding to ranges $(0, p_{c,\text{even}}]$, $(p_{c,\text{even}}, 1/2]$, $(1/2, 1-p_{c,\text{even}})$ and $(1-p_{c,\text{even}}, 1)$.

Proof of Theorem 1.3.

Case 1: if $p \in [0, p_{c,\text{even}}]$, then $\mu_p(\mathcal{C}) = 0$. Remember that $p_{c,\text{even}} < 1/2$, so Lemma 4.1 ensures that for every $p \le p_{c,\text{even}}$, $\mu_p \preceq \varphi^0_{2p,2}$, with $2p \le 2p_{c,\text{even}} = p_c(2)$. But in the random cluster model, there is by definition no percolation below the critical point, and no percolation at the critical point for the free boundary condition random cluster measure in dimension 2 (Theorem (6.17) in Grimmett (2006)).

Note that we can also prove that $\mu_p(\mathcal{C}) = 0$ for $p < p_{c,\text{even}}$ by a different argument that we only sketch here. We need the notion of $*$-neighbours: two sites $x_*, y_* \in \mathbb{Z}^2_*$ are $*$-neighbours if and only if $\|x_* - y_*\|_\infty = 1$. A $*$-chain is then a sequence of sites in $\mathbb{Z}^2_*$ such that two consecutive sites are $*$-neighbours. As stated in Theorem 1.1, Eulerian percolation with parameter $p < p_{c,\text{even}}$ corresponds to the contours of the Ising model under the Gibbs measure $\gamma^-_{\beta(p)}$, with parameter $\beta(p) > \beta_c$. Assume that there exists an infinite path in the contours of a spin configuration of the Ising model with parameter $\beta(p)$: the set of spins $+1$ along this infinite path constitutes an infinite $*$-chain of spins $+1$. But by Proposition 1 in Russo (1979), for $\beta > \beta_c$, under γ^-_β, the probability that there exists such an infinite $*$-chain of spins $+1$ is 0.

Case 2: if $p \in (p_{c,\text{even}}, 1/2]$, then $\mu_p(\mathcal{C}) = 1$. Let us set

$$\mathcal{C}^+ = \{\sigma \in \{-1,+1\}^{\mathbb{Z}^2_*} : \text{ there is an infinite chain of spins } +1 \text{ in } \sigma\},$$
$$\mathcal{C}^+_* = \{\sigma \in \{-1,+1\}^{\mathbb{Z}^2_*} : \text{ there is an infinite } *\text{-chain of spins } +1 \text{ in } \sigma\}.$$

Let $\sigma \in \mathcal{C}^+_* \cap (\mathcal{C}^+)^c$, and let δ be an infinite $*$-chain of spins $+1$ in σ. For each spin $+1$ along δ, let us consider the cluster of spins $+1$ to which it belongs. Since $\omega \notin \mathcal{C}^+$, these clusters are finite. The union of the contours of these clusters is an infinite connected subgraph of $\mathbb{Z}^2$. Indeed, let $x_1, x_2 \in \mathbb{Z}^2_*$ be the coordinates of two consecutive spins $+1$ of the $*$-chain δ. If $\sigma(x_1)$ and $\sigma(x_2)$ are not in the same cluster of spins $+1$, it means that the step from x_1 to x_2 in δ is diagonal (with spins -1 in the opposite diagonal), and that the contours of the clusters of $\sigma(x_1)$ and $\sigma(x_2)$ meet at point $(x_1+x_2)/2$. Thus, any two consecutive points of δ are such that the contours of their clusters are connected (or possibly the same). By induction, one can then prove that the union of the contours of all the clusters of spins $+1$ of δ is a connected subgraph of $\mathbb{Z}^2$.

It follows from Theorem 1.1 that for any $p \in (0,1)$, $\mu_p(\mathcal{C}) \ge \gamma^+_\beta(\mathcal{C}^+_* \cap (\mathcal{C}^+)^c)$. For $\beta \in [0, \beta_c)$, we have $\gamma^+_\beta = \gamma^-_\beta = \gamma_\beta$, and

- $\gamma^+_\beta(\mathcal{C}^+) = 0$, by Proposition 1 in Coniglio et al. (1976),
- $\gamma^+_\beta(\mathcal{C}^+_*) = 1$, by Theorem 1 in Higuchi (1993).

Thus, $\gamma^+_\beta(\mathcal{C}^+_* \cap (\mathcal{C}^+)^c) = 1$. It follows that for $p \in (p_c, 1/2]$, $\mu_p(\mathcal{C}) = 1$.

Case 3: if $p \in (1/2, 1-p_{c,\text{even}})$, then $\mu_p(\mathcal{C}) = 1$. As seen in Case 2, there is percolation under μ_p for any $p \in (p_{c,\text{even}}, 1/2]$, which by Lemma 4.2, implies that there is percolation under μ_{1-p} for any $(1-p) \in (1/2, 1-p_{c,\text{even}})$.

Case 4: if $p \in (p_{c,\text{even}}, 1)$, then $\mu_p(\mathcal{C}) = 1$. Let us set

$$\mathcal{D} = \{\eta \in \{0,1\}^{\mathbb{E}^2} : \text{ there is an infinite cluster in } \eta^c\}.$$

The event $\mathcal{D}$ is non-increasing, and $p \mapsto \varphi_{p,2}$ is stochastically increasing (Theorem (3.21) in Grimmett (2006)), so the map $p \mapsto \varphi_{p,2}(\mathcal{D})$ is non-increasing: there exists a critical value $\overline{p}_c(2) \in [0,1]$ such that $\varphi_{p,2}(\mathcal{D}) > 0$ for $p < \overline{p}_c(2)$ and $\varphi_{p,2}(\mathcal{D}) = 0$ for $p > \overline{p}_c(2)$. In words, $\overline{p}_c(2)$ is the critical parameter for percolation of closed edges in the random cluster model.

By Lemma 4.1, when $p \geq 1/2$, we have $(\varphi^0_{2(1-p),2})^c \preceq \mu_p$; so

$$\varphi^0_{2(1-p),2}(\mathcal{D}) = (\varphi^0_{2(1-p),2})^c(\mathcal{C}) \leq \mu_p(\mathcal{C}).$$

Thus if $2(1-p) < \overline{p}_c(2)$ and $p \geq 1/2$, which is equivalent to $p > \max(1/2, 1 - \overline{p}_c(2)/2)$, $\mu_p(\mathcal{C}) > 0$ and thus, by the 0–1 law, $\mu_p(\mathcal{C}) = 1$. So, to end the proof of Theorem 1.3, it only remains to identify the critical parameter $\overline{p}_c(2)$ for percolation of closed edges in the random cluster model:

Lemma 4.3. $\overline{p}_c(2) = \dfrac{\sqrt{2}}{1+\sqrt{2}}$.

Sketch of proof: We can adapt either the proof by Beffara and Duminil-Copin (2012) or by Duminil-Copin (2016). We check here that the key ingredients for the proof of Theorem 1.2 in Duminil-Copin (2016) are still valid:

- Monotonicity of FK measures with respect to the boundary conditions: the FK measure in a finite box is stochastically increasing with respect to the boundary condition, so the law of closed edges in FK-percolation is stochastically decreasing with respect to the boundary condition.
- Self-duality property of FK measures: the self-duality property works as well for closed bonds as for open bonds, and the self-dual point is the same:
$$\overline{p}_c(2) = p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}.$$
- The study of the variation of $\varphi_{p,q}(A)$ with respect to p (Theorem 3.12 in Grimmett, 2006): it does not depend on the reference measure and can be applied as well with $\mu(\omega) = q^{N(\omega)}$ (FK-percolation) as with $\mu(\omega) = q^{N(\overline{\omega})}$ (percolation of the closed bonds of FK-percolation).

□

5. Questions and discussions

The study of Bernoulli bond percolation on a graph $G = (V, E)$ intensively uses the following properties of the product measure $\text{Ber}(p)^{\otimes E}$:

- monotonicity: for every increasing event A, the map $p \mapsto \text{Ber}(p)^{\otimes E}(A)$ is non-decreasing.
- positive association: for every pair of increasing events A, B,
$$\text{Ber}(p)^{\otimes E}(A \cap B) \geq \text{Ber}(p)^{\otimes E}(A)\,\text{Ber}(p)^{\otimes E}(B),$$

It is natural to ask if these properties could be preserved for the Eulerian percolation measure with opening parameter p on G:

$$\mu_p(.) = \text{Ber}(p)^{\otimes E}(\cdot|\text{the subgraph of open edges is Eulerian}).$$

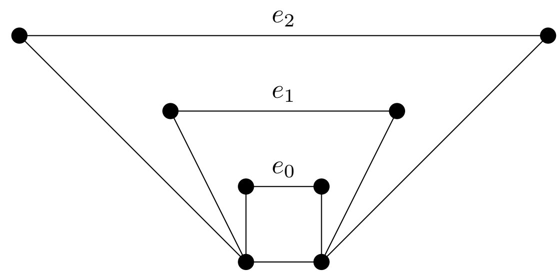


FIGURE 5.4. The finite Eulerian graph G.

Figure 5.4 presents a finite Eulerian graph G where the monotonicity property is preserved whereas the positive association property is lost.

For $i = 1, 2$, let us introduce the event $C_i =$"the edges e_0 and e_i are both open'. We can check by brute-force that $(\mu_p)_{p\in[0,1]}$ is non-decreasing for the stochastic order (the computer program can be found in our preprint Garet et al., 2016). However, computations show that the non-decreasing events C_1 and C_2 are not positively correlated for $p < 0,42$.

We conjecture that the monotonicity result should be more general:

Conjecture 5.1. *Let $G = (V, E)$ be a Eulerian graph. Then, the sequence of Eulerian percolation measures $(\mu_p)_{p\in[0,1]}$ on $\{0,1\}^E$ is stochastically non-decreasing.*

Note that Cammarota and Russo (1991) proved related results supporting this conjecture.

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