Conditional decoupling of random interlacements

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Abstract. We prove a conditional decoupling inequality for the model of random interlacements in dimension $d \geq 3$: the conditional law of random interlacements on a box (or a ball) A_1 given the (not very "bad") configuration on a "distant" set A_2 does not differ a lot from the unconditional law. The main method we use is a suitable modification of the *soft local time* method of Popov and Teixeira (2015), that allows dealing with conditional probabilities.

1. Introduction and results

Random interlacements were introduced by Sznitman (2010), to model the trace of the simple random walk on the discrete torus $\mathbb{Z}_n^d := \mathbb{Z}^d/n\mathbb{Z}^d$ or the discrete cylinder $\mathbb{Z} \times \mathbb{Z}^{d-1}$, in dimension $d \geq 3$. Detailed treatments and reviews of recent results can be found in the recent books Černý and Teixeira (2012); Drewitz et al. (2014); Sznitman (2012c). Loosely speaking, the model of random interlacements in $\mathbb{Z}^d$, $d \geq 3$, is a stationary Poissonian soup of bi-infinite simple random walk trajectories on the integer lattice. There is a parameter $u > 0$ entering the intensity measure of the Poisson process, the larger u is the more trajectories are thrown in. The sites of $\mathbb{Z}^d$ that are not touched by the trajectories constitute the *vacant set* $\mathcal{V}^u$, and the union of all trajectories constitutes the interlacement set $\mathcal{I}^u = \mathbb{Z}^d \setminus \mathcal{V}^u$. The random interlacements are constructed simultaneously for all $u > 0$ in such a way that $\mathcal{I}^{u_1} \subset \mathcal{I}^{u_2}$ if $u_1 < u_2$. In fact, the law of the vacant set at level u can be uniquely characterized by the following identity:

$$\mathbb{P}[A \subset \mathcal{V}^u] = \exp\left(-u\operatorname{cap}(A)\right), \tag{1.1}$$

where $\operatorname{cap}(A)$ is the *capacity* of a finite set $A \subset \mathbb{Z}^d$. Informally, the capacity measures how "big" is the set from the point of view of the walk, see Section 6.5 of Lawler and Limic (2010) for formal definitions, or Section 2 below.

The model of random interlacements naturally has more independence built in than just one random walk on the torus or the cylinder (because on a fixed set one observes traces of *independent* trajectories). Still, the analysis of random interlacements is difficult because of the long-range dependencies present there. For example, in (1.68) from Sznitman (2010) we can see that

$$\operatorname{Cov}(\mathbb{1}_{x \in \mathcal{I}^u}, \mathbb{1}_{y \in \mathcal{I}^u}) \sim \frac{c_d u}{\|x - y\|^{d-2}} \quad \text{as} \quad \|x - y\| \to \infty, \tag{1.2}$$

which means that the "degree of dependence" decreases polynomially in the distance.

Naturally, one is interested in "decoupling" the events supported on distant regions; that is, to argue that they are approximately independent to a certain degree. One possible approach to quantify that degree is the following: given finite sets $A_1, A_2 \subset \mathbb{Z}^d$ and functions $f_1 : \{0,1\}^{A_1} \to [0,1]$ and $f_2 : \{0,1\}^{A_2} \to [0,1]$ depending on the interlacements set intersected with A_1 and A_2 respectively, we have

$$\operatorname{Cov}_u(f_1, f_2) \leq c_d u \frac{\operatorname{cap}(A_1) \operatorname{cap}(A_2)}{\operatorname{dist}(A_1, A_2)^{d-2}}, \tag{1.3}$$

as proved in formula (2.15) of Sznitman (2010), see also (8.1.1) in Drewitz et al. (2014). However, the polynomial error term in (1.3) can complicate one's life in many applications (and, e.g. in the case when the diameters of these sets are of the same order as the distance between them, (1.3) is simply of no use); on the other hand, while (1.3) can be improved to some degree (de Bernardini et al., 2018+), the error term there should always be at least polynomial, as (1.2) shows. To circumvent this difficulty, one first may note that usually the "interesting" events/functions are *monotone* (i.e., increasing or decreasing). For e.g. increasing events, we know that their probabilities increase as the parameter u increases. Note also that the FKG inequality (see Teixeira, 2009, Theorem 3.1) gives us

$$\mathbb{E}^u[g_1 g_2] \geq \mathbb{E}^u[g_1] \mathbb{E}^u[g_2], \tag{1.4}$$

for *any* increasing functions g_1, g_2 with finite second moments. To complement the FKG inequality, we use *sprinkling*, i.e., we slightly change the intensity of random interlacements in order to decrease the error term; this approach was used in Sznitman (2010) and Sznitman (2012a). Then, in particular, in Popov and Teixeira (2015) it was proved that

$$\mathbb{E}^u[f_1 f_2] \leq \mathbb{E}^{(1+\varepsilon)u}[f_1] \mathbb{E}^{(1+\varepsilon)u}[f_2] + c_d(r+s)^d \exp(-c_d' \varepsilon^2 u s^{d-2}); \tag{1.5}$$

with $f_1 : \{0,1\}^{A_1} \to [0,1]$ and $f_2 : \{0,1\}^{A_2} \to [0,1]$ both increasing functions in the interlacements set, $r = \min(\operatorname{diam}(A_1), \operatorname{diam}(A_2))$, and $s = \operatorname{dist}(A_1, A_2)$. The same bound was also obtained for decreasing functions.

It is important to observe, however, that the decoupling in the above form may not always be useful for one's needs. Intuitively, one is tempted to understand inequalities like (1.3) as "what happens in one set does not influence a lot what happens in the other set". Now, consider the following situation. Suppose that on top of the random interlacements we have some additional stochastic process (e.g., a random walk) that "explores" the interlacement set in some way. Assume

that this process has already explored the interlacements in a given area, revealing a lot of information about it; think, for definiteness, that it simply revealed the interlacement set exactly. The probability of a particular configuration of the interlacement set is usually very small; so, (1.3) (even (1.5)!) will blow up when one divides by that probability, because of the error term. In fact, in the end of Section 2 we discuss a particular model of the random walk on the interlacement set, where our main results turn out to be useful.

This justifies the need for *conditional* decoupling, i.e., show that, given the configuration on some set, the law of the interlacement configuration on a distant set is still in some sense close to the unconditional law. This is what we are doing in this paper. To prove our results, the main method we use is a suitable modification (that allows dealing with conditional probabilities) of the *soft local time* method of Popov and Teixeira (2015). We hope that this modification will be useful in other contexts, for instance, for dealing with the decoupling properties of the *loop measures* Chang and Sapozhnikov (2016).

Another important observation is the following. There are strong connections between random interlacements and the Gaussian free field, see e.g. Sznitman (2012b,c). In particular, there are decoupling inequalities similar to (1.3) and (1.5) for the Gaussian free field as well, see Popov and Ráth (2015). Notice, however, that the decoupling-with-sprinkling result for the Gaussian free field (Theorem 1.2 of Popov and Ráth, 2015) is *already* conditional (the unconditional decoupling is obtained as a simple consequence, just by integration). On the other hand, note that the error terms in the conditional decoupling in the main result of this paper (Theorem 1.1) are much worse than that of (1.5); related to this is the fact that in the conditional setting the minimal distance between sets that permits the result to work is much bigger. A comparison with the situation for the Gaussian free field suggests that, hopefully, there is still much room for improvement for the conditional decoupling for random interlacements.

Let us briefly introduce a bit of notation in order to state our main result. Let $r > 0$ be sufficiently big, and let $s := s(r) > 0$, with $s = o(r)$. We define $A_1^\circ := A_1^\circ(r)$ to be the discrete Euclidean ball of radius r and center at the origin, that is

$$A_1^\circ := \{x \in \mathbb{Z}^d; \mathrm{dist}(x, 0) \le r\},$$

where $\mathrm{dist}(\cdot, \cdot)$ denotes de Euclidean distance. We also define $A_1^\square := A_1^\square(r)$ as a d-dimensional hypercube with side-length $2r + 1$ and center at the origin, that is

$$A_1^\square := \{x \in \mathbb{Z}^d; \|x\|_\infty \le r\},$$

where $\|\cdot\|_\infty$ denotes the ℓ_∞ norm. We then define the sets

$$A_2^\circ = A_2^\circ(r, s) := \{x \in \mathbb{Z}^d; \mathrm{dist}(x, 0) > r + 2s\},$$
$$A_2^\square = A_2^\square(r, s) := \{x \in \mathbb{Z}^d; \|x\|_\infty > r + 2s\}.$$

When there is no risk of confusion, or when the arguments presented work for both balls and smoothed hypercubes (which will be often so), we will omit the super-indexes $\circ, \square$.

We will now state our main result. Heuristically, it says the following: Let s be bounded from below by a power of r with an explicit given coefficient (strictly smaller than 1, depending only on the dimension d and whether A_1 is a ball or a hypercube). Let A_3 be a subset of A_2 with finite boundary, that is, A_3 is either
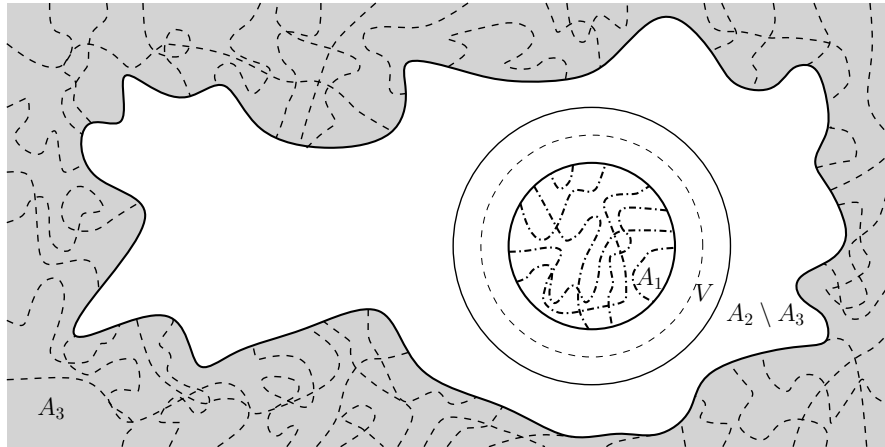
FIGURE 1.1. Our main result says that if the interlacements con-
figuration in a set $A_3 \subseteq A_2$ is not too weird, that is, it does not
belong to a set with stretched exponentially small probability (in s,
as $s \to \infty$), then with high probability (1 minus stretched expo-
nential in s) the distribution of the interlacements set intersected
with A_1 conditioned on the state of $\mathcal{I}^u \cap A_3$ can be well approxi-
mated by the unconditional distribution.

finite or has finite complement. If we pay a stretched exponentially small price
(in s) to guarantee that the interlacements configuration of $\mathcal{I}^u \cap A_3$ is not too
weird, then the distribution of $\mathcal{I}^u \cap A_1$ conditioned on this configuration is well
approximated by the unconditional distribution, with high probability (1 minus a
stretched exponential function of s).

Theorem 1.1. *Let the real numbers $b_\circ, b_\square$ be such that*

$$1 \leq b_\circ < \frac{2d-2}{d}, \tag{1.6}$$

$$1 \leq b_\square < \frac{4d-4}{3d-2}. \tag{1.7}$$

Then, define

$$a_\circ = 2d - 2 - db_\circ > 0, \tag{1.8}$$

$$a_\square = 4d - 4 - 3db_\square + 2b_\square > 0. \tag{1.9}$$

*From now on we will again omit the indexes $\circ, \square$. Recall that r is of the same order
as the diameter of A_1, and that s has the same order as the distance between A_1
and A_2. Assume $s = s(r) \asymp r^{\frac{1}{b}}$, let r be sufficiently big. Let $\varepsilon > 0$ be smaller
than $1/4$. Let A_3 be a subset of A_2 such that $|\partial A_3| < \infty$. Define $\mathcal{I}^u_{A_j} := \mathcal{I}^u \cap A_j$,
for $j = 1, 2, 3$.*

*Then there are positive constants c, c' depending only on the dimension d, and a
measurable (according to a suitably constructed σ-field) set $\mathcal{G} \in \{0,1\}^{A_3}$ such that*

$$\mathbb{P}^u\left[\mathcal{I}^u_{A_3} \in \mathcal{G}\right] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right),$$

and for any increasing function f on the interlacements set intersected with A_1, with $\|f\|_\infty \leq 1$, we have

$$\left(\mathbb{E}f(\mathcal{I}^{u(1-\varepsilon)}_{A_1}) - c\exp\left(-c'\varepsilon^2 us^a\right)\right)\mathbb{1}_{\mathcal{I}^u_{A_3}\in\mathcal{G}} \leq \mathbb{E}\left(f(\mathcal{I}^u_{A_1}) \mid \mathcal{I}^u_{A_3}\right)\mathbb{1}_{\mathcal{I}^u_{A_3}\in\mathcal{G}}$$

$$\leq \left(\mathbb{E}f(\mathcal{I}^{u(1+\varepsilon)}_{A_1}) + c\exp\left(-c'\varepsilon^2 us^a\right)\right)\mathbb{1}_{\mathcal{I}^u_{A_3}\in\mathcal{G}}.$$
$$(1.10)$$

We also obtain a result analogous to Theorem 1.1, but this time we allow the sprinkling factor to be arbitrarily big. This decreases the "precision" (in the result below, $\mathbb{E}f(\mathcal{I}^{u+u'}_{A_1})$ can be very different from $\mathbb{E}f(\mathcal{I}^u_{A_1})$), but, in compensation, the size of the complement of the "good" set as well as the "error term" become smaller.

Theorem 1.2. *Let $u' > u > 0$. We use the same definitions as in Theorem 1.1. There are positive constants c, c' depending only on the dimension d, and a measurable (according to the random interlacements σ-field) set $\mathcal{G}_{u'} \in \{0,1\}^{A_3}$ such that*

$$\mathbb{P}^u\left[\mathcal{I}^u_{A_3} \in \mathcal{G}_{u'}\right] \geq 1 - \exp\left(-c'u's^a\right),$$

and for any increasing function f on the interlacements set intersected with A_1, with $\|f\|_\infty \leq 1$, we have

$$\mathbb{E}\left(f(\mathcal{I}^u_{A_1}) \mid \mathcal{I}^u_{A_3}\right)\mathbb{1}_{\mathcal{I}^u_{A_3}\in\mathcal{G}_{u'}} \leq \left(\mathbb{E}f(\mathcal{I}^{u+u'}_{A_1}) + c\exp\left(-c'u's^a\right)\right)\mathbb{1}_{\mathcal{I}^u_{A_3}\in\mathcal{G}_{u'}}. \qquad (1.11)$$

Remark 1.3. We have to explain why we need to consider $A_3 \subset A_2$. Indeed, at first sight it seems that conditioning on a configuration on A_3 does not add generality to our results, since any fixed configuration on A_3 corresponds to a set of configurations on A_2. However, the problem with always setting $A_3 = A_2$ is the following: the "exceptional set" $\mathcal{G}^c$ will then be supported on the whole A_2, and this can be inconvenient for applications. For example, assume that we successively apply the conditional decoupling results to a process (such as the one of Section 1.1) that "explores" the interlacement environment. If that process has explored only a finite chunk of A_2, we would not be able to say if the configuration is "good" (i.e., belongs to $\mathcal{G}$) by only observing that finite chunk. This would force us to condition on the (configuration on the) whole A_2, which would mean that a subsequent application of a conditional decoupling may be difficult, since we already "revealed" some information about the configuration on a set which is "too big" (i.e., when we apply the decoupling result for the next time, the "new" A_1 may be inside the "previous" A_2)

Remark 1.4. In the course of the proof of the above theorems we actually prove a stronger result: the same conditional decoupling inequality holds true if we replace the sets $\mathcal{I}^u_{A_1} \subset A_1$ and $\mathcal{I}^u_{A_3} \subset A_3$ by sets of *random walk excursions* in A_1 and A_3 (we also have to replace the function f by an increasing function on the set of excursions). That is, the conditional decoupling continues to work when we replace the ranges of the excursions (which constitute the random interlacements set) by the actual excursions themselves. We chose to state the results in the above manner for the sake of clarity and brevity. Note that this remark also applies to the decoupling obtained by Popov and Teixeira (2015).

Remark 1.5. The above theorems can be proved in the same way if we replace the hypercube $A_1^\square$ by a box $[0, a_1] \times \cdots \times [0, a_d]$, with $c^{-1}r < a_i < cr$ for all $i = 1, \ldots, d$,

and some constant $c > 1$, and then replace the set $A_2^{\square}$ accordingly. We chose to prove the theorems for $A_1^{\square}$ only to simplify the notation. We also note that we prove the theorem for both balls and boxes because the error term obtained in the decoupling for balls is much smaller than the error obtained in the decoupling for boxes, but at the same time the decoupling between boxes tends to be more useful because boxes cover the space in a much more efficient manner.

Remark 1.6. For $d = 3$, the only way to obtain an exponentially small (instead of a *stretched exponentially* small) error term in equations (1.10) and (1.11) is to allow the distance $\sim s$ between the sets A_1 and A_2 to be of the same order of the minimal diameter $\sim r$.

Here is an overview of the paper. In Subsection 1.1, we discuss an application of some of our results. Section 2 is devoted to establish the notation and state further definitions. In Section 3 we show how we simulate the interlacements set $\mathcal{I}_{A_1}^u$ conditioned on the information given by $\mathcal{I}_{A_2}^u$ using a suitable version of the soft local times method. Finally, in Section 4, we prove the main theorem using a large deviations estimate for the soft local times associated with $\mathcal{I}_{A_1}^u$. The Appendix is then used to collect and derive the technical estimates we need.

Now, let us comment on how this paper relates to Popov and Teixeira (2015). As observed above, the present results neither imply nor follow from those of Popov and Teixeira (2015). The very main idea of the proof remains the same: construct the processes on the same probability space using the soft local times method, and then prove that one soft local time lies completely above the other with high probability. However, carrying this program out for the *conditional* decoupling is a much more nontrivial task, for the following reasons:

- There are some fundamental differences in the construction of the conditioned process itself. The unconditional excursion process was a Markov chain, with a lot of regenerations. On the other hand, here, given a specific configuration of the clothesline process defined in (3.7), the excursions of the conditioned process are independent, but not identically distributed (using a RWRE analogy, one may think of the conditioned excursion process as a process in a quenched random environment, where the realization of the random environment is a instance of the aforementioned clothesline process, whose law depends on the interlacements configuration on A_3).

- To control the soft local times, in Popov and Teixeira (2015) one basically only needed to control the entrance measures (given the starting point, the excursion was "free", i.e., just a simple random walk). Here, however, the excursion space has a more complicated base measure μ (cf. the definitions in Section 3): we have to take both the entrance and exit point into account, and the excursions' law is conditional. There is an additional problem that the estimates on the soft local time increment depend on the relative position of the two points, as well as on the domain's shape itself.

- Due to the above, the task of controlling the first two moments and the tails of soft local times becomes more complicated as well (as the reader may have inferred from the size of this paper). Also, it is probable that the final results are not sharp: we were not able to obtain the s^{d-2}-factor in the exponents (as in Popov and Teixeira, 2015), the distance between the sets needs to be polynomial with respect to the diameter, and also the results for balls and boxes are different (as contrasted to those of Popov and

Teixeira, 2015, where it was possible to obtain the result for fairly general
sets).

1.1. *An application: biased random walk on the interlacement set.* Let G be some
(possibly random) subset of $\mathbb{Z}^d$, $d \geq 2$. Fix a parameter $\beta > 0$, which accounts for
the bias; also, fix some non-zero vector $\ell \in \mathbb{Z}^d$. Let us define the *conductances* on
the edges of $\mathbb{Z}^d$ in the following way:

$$\mathcal{C}(x,y) = \begin{cases} e^{\beta(x+y)\cdot\ell}, & \text{if } x,y \text{ are neighbors and belong to } G, \\ 0, & \text{otherwise,} \end{cases}$$

and we call the collection of all conductances $\omega = \{\mathcal{C}(x,y), x,y \in \mathbb{Z}^d\}$ the ran-
dom environment. Consider a random walk $(X_n, n \geq 0)$ in this environment of
conductances; i.e., its transition probabilities are given by

$$P^\omega[X_{n+1} = y \mid X_n = x] = \frac{\mathcal{C}(x,y)}{\sum_z \mathcal{C}(x,z)}$$

(the superscript in P^ω indicates that we are dealing with the "quenched" probabil-
ities, i.e., when the underlying random graph / conductancies are already fixed).

There have been significant interest towards this model in recent years, mainly in
the case when G is the infinite cluster of supercritical Bernoulli percolation model,
see e.g. Berger et al. (2003); Sznitman (2003); Fribergh (2013). In particular, one
remarkable fact is the following: the walk is ballistic (transient and with positive
speed) in the direction of the drift if $\beta > 0$ is small enough; however, it moves only
sublinearly fast (its displacement is only of order t^a by time t with $a \in (0,1)$, as
proved in Fribergh and Hammond, 2014) for large values of β.

In the work Fribergh and Popov (2018) the case $G = \mathcal{I}^u$ was considered. It
turned out that in dimension $d = 3$, for *any* value of $\beta > 0$, although still transient
in the direction of the drift, the walk is not only sub-ballistic, but has also sub-
polynomial speed, in the sense that its distance to the origin grows slower than t^ε
for any $\varepsilon > 0$. This is also in contrast with the result that the walk on $\mathcal{I}^u$ without
any drift is diffusive (so, loosely speaking, its "speed" is $\sqrt{t}$), as shown in Procaccia
et al. (2016).

We will not describe all the details of Fribergh and Popov (2018) here, but
the main idea is the following. As in the case of the biased walk on the infinite
percolation cluster, to prove zero speed one needs to show that the walk frequently
gets caught in traps. These traps are "dead ends" of the environment looking in
the direction of the bias, see Figure 1.2. When the walk enters such a trap, the bias
prevents it from goint out, so there is a good chance that the walk will spend quite
a lot of time there, and this effectively leads to zero speed. Now, the crucial fact
is that, specifically in three dimensions, it is much cheaper to have a trap in the
interlacement set than in the (Bernoulli) percolation cluster. Indeed, it is possible
to show that the capacity of the dotted set on Figure 1.2 is of order $\frac{\ln t}{\ln \ln t}$ for any
fixed $\alpha < 1$. The formula (1.1) then shows that having a trap as above has only a
subpolynomial (in t) cost; also, it turns out that "forcing" a trajectory to create a
"dead end" as shown on the picture is not too costly as well.

So, when the walk advances in the direction of the bias, from time to time it will
encounter a trap and be trapped. However, to make such an argument rigorous, one
has to face the following difficulty. When the walk already explored some parts of
the environment and then came to an unexplored area, we can no longer use (1.1)
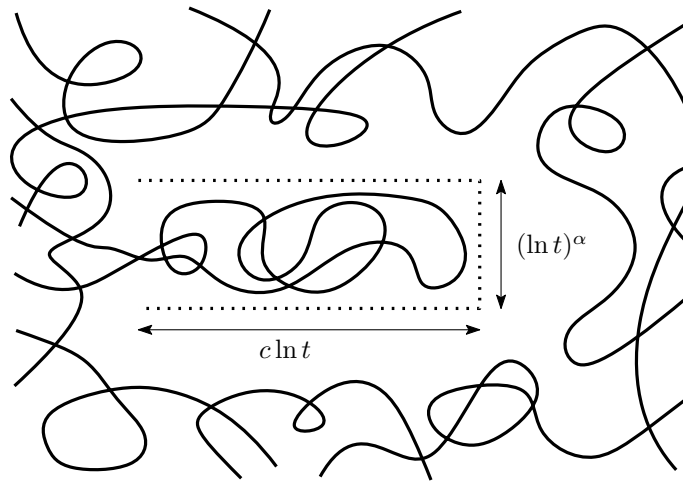
FIGURE 1.2. A trap for the random walk on the interlacement set (on this picture, the bias is directed along the first coordinate vector). Only the interlacements are shown; the trajectory of the RWRE X is not present on the picture.

to estimate the probability that there is a trap in front of it, due to the lack of independence. It is here that the conditional decoupling enters the scene: it is possible to use the main results of this paper to show that probability of having a trap in front of the particle (when it comes to an unexplored area) is not very small. As mentioned above, the detailed argument can be found in Fribergh and Popov (2018).

2. Definitions and notation

In this section we will introduce the basic definitions, conventions and notation used in this paper. We start by stating our convention regarding constants: c, c', c_1, c_2, c_3,... are always defined as strictly positive constants depending only on the dimension d. Constants can also change value from line to line, unless when the text explicitly states the contrary.

We let $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the Euclidean and ℓ_∞ norms in $\mathbb{Z}^d$ respectively. For $x, y \in \mathbb{Z}^d$, we also let $\mathrm{dist}(x, y) \equiv \|x - y\|$. We say that two vertices $x, y \in \mathbb{Z}^d$ are neighbors when $\|x - y\| = 1$, this notion introduces the usual nearest-neighbor graph structure in $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}_+$, we define

$$B(x, r) := \{y \in \mathbb{Z}^d; \|y - x\| \le r\},$$

the discrete ball in the Euclidean norm centered on x with radius r, and

$$B_\infty(x, r) := \{y \in \mathbb{Z}^d; \|y - x\|_\infty \le r\},$$

the discrete ball in the ℓ_∞-norm centered on x with radius r. Given a set $A \subseteq \mathbb{Z}^d$ we denote by

$$A^C := \{x \in \mathbb{Z}^d; x \notin A\}$$

its complement and by

$$\partial A := \left\{ x \in A; \text{ there exists } y \in A^C \text{ such that } \|x - y\| = 1 \right\}$$

its (internal) boundary.

For any set Z and any two functions $f, g : Z \to \mathbb{R}$, we write $f(z) \asymp g(z)$ to denote the fact that there exist two strictly positive constants, c_1 and c_2, such that $c_1 f(z) \leq g(z) \leq c_2 f(z)$ for all $z \in Z$. When Z is equal to $\mathbb{R}$ we say that $f(z) = o(g(z))$ when $\frac{f(z)}{g(z)}$ goes to 0 as $z \to \infty$.

Given $x \in \mathbb{Z}^d$, we let $\mathbb{P}_x$ denote the probability measure associated with the simple random walk in $\mathbb{Z}^d$ started at x. We will also let $(X_k, k \geq 0)$ denote the simple random walk process in $\mathbb{Z}^d$. Given a set $A \subset \mathbb{Z}^d$, we define the entrance time for the set A

$$H_A := \inf \left\{ k \geq 0; X_k \in A \right\}.$$

We also let the hitting time for A be defined as

$$\tilde{H}_A := \inf \left\{ k \geq 1; X_k \in A \right\}.$$

When A is finite we denote its harmonic measure by

$$e_A(x) = \mathbb{1}_{x \in A} \mathbb{P}_x \left[\tilde{H}_A = \infty \right] \text{ for } x \in \mathbb{Z}^d.$$

We are then able to define the capacity of the set A

$$\text{cap}(A) := \sum_{x \in A} e_A(x),$$

and the normalized harmonic measure

$$\overline{e}_A(x) := e_A(x) \, \text{cap}(A)^{-1}.$$

We now write down the definition of the Green function for the simple random walk in $\mathbb{Z}^d$: for $x, y \in \mathbb{Z}^d$, we let

$$G(x, y) := \sum_{k \geq 0} \mathbb{P}_x \left[X_k = y \right].$$

Theorem 1.5.4 of Lawler (2013) provides us with the following estimate on the Green function:

$$G(x, y) \asymp \frac{1}{1 + \|x - y\|^{d-2}}. \tag{2.1}$$

An application of the optional stopping theorem then gives us the following elementary result:

Lemma 2.1. *Let $0 < \rho_1 < \rho_2$ be sufficiently large real numbers, and let $x \in B(0, \rho_2) \setminus B(0, \rho_1)$. Then*

$$\mathbb{P}_x \left[H_{\partial B(0, \rho_1)} < H_{\partial B(0, \rho_2)} \right] = \frac{|x|^{-(d-2)} - (\rho_2)^{-(d-2)} + O((\rho_1)^{-(d-1)})}{(\rho_1)^{-(d-2)} - (\rho_2)^{-(d-2)}}. \tag{2.2}$$

Let us briefly discuss the definition of the measure associated with the random interlacements process intersected with a given finite set $A \subset \mathbb{Z}^d$. Assume we have constructed a probability space where, for every $i \geq 1$, there exists a simple random walk process $(X_k^{(i)}, k \geq 0)$ with starting distribution given by $\overline{e}_A(\cdot)$, and such that $(X_k^{(i)}, k \geq 0)$ is independent from $(X_k^{(j)}, k \geq 0)$ for $i \neq j$. We also assume that in this space we can construct an independent Poisson process $(J_u)_{u \geq 0}$ on the positive

real line with intensity $u \cdot \mathrm{cap}(A)$. The law of the random interlacements process $(\mathcal{I}^u)_{u \geq 0}$ intersected with the set A can then be characterized by

$$(\mathcal{I}^u \cap A)_{u \geq 0} \overset{d}{=} \Big(A \cap \bigcup_{i \leq J_u} \bigcup_{k \geq 0} X_k^{(i)} \Big)_{u \geq 0}, \tag{2.3}$$

as can be seen in Sznitman (2010), Proposition 1.3, or in the paragraph before (2.6) in Černý and Teixeira (2016). This definition gives rise to compatible measures in the following sense: Given two finite sets $K_1 \subset K_2 \subset \mathbb{Z}^d$, we have that $((\mathcal{I}^u \cap K_2)_{u \geq 0}) \cap K_1$ has the same law as $(\mathcal{I}^u \cap K_1)_{u \geq 0}$.

We now introduce the concept of the "smoothed hypercube". Throughout this paper we will need to use estimates on the hitting measure of a simple random walk on the sets A_1 and A_2. However, the sharp edges present on $A_1^\square$ and $A_2^\square$ greatly complicate such estimates. Instead we will prove our main results in the context of "smoothed" versions of these sets, that is, we let

$$A_1^\square := \{x \in \mathbb{Z}^d; B(x,s) \subset B_\infty(0,r)\}, \ \ A_2^\square := \big(\{x \in \mathbb{Z}^d; B(x,s) \subset B_\infty(0,r+2s)\} \big)^C.$$

We will later explain how to obtain our main results for the unsmoothed versions of these sets.

To finish the section, we define $V^\circ := V^\circ(r,s)$ to be the boundary set

$$V^\circ := \partial\{x_1 \in \mathbb{Z}^d, \mathrm{dist}(x_1, x_2) \leq s \text{ for some } x_2 \in A_1^\circ\},$$

and we define $V^\square := V^\square(r,s)$ in an analogous way. Since $s = o(r)$, we have

$$\mathrm{cap}(V) = \mathrm{cap}(A_2^C)(1 + o(1)) = \mathrm{cap}(A_1)(1 + o(1)),$$

and also, by Proposition 2.2.1 and equation (2.16) of Lawler (2013),

$$\mathrm{cap}(V) \asymp r^{d-2}. \tag{2.4}$$

3. Simulating excursions with soft local times

We begin the present section with a short recapitulation of the technique introduced in Popov and Teixeira (2015), the Soft Local Times method. This review will also be important in order to reformulate the results of Popov and Teixeira (2015) in a more general setting (the original paper was written with a Markovian setting in mind).

The method essentially allows us to simulate any number of random variables taking values in a state space Σ using a realization of a Poisson point process in $\Sigma \times \mathbb{R}_+$.

Let Σ be a locally compact Polish metric space, and let $\mathcal{B}(\Sigma)$ be its Borel σ-algebra. Let μ be a Radon measure over $\mathcal{B}(\Sigma)$, so that every compact set has finite μ-measure.

Such measure space $(\Sigma, \mathcal{B}(\Sigma), \mu)$ is the usual setup for the construction of a Poisson point process on Σ. We consider the space of Radon point measures in $\Sigma \times \mathbb{R}_+$

$$L = \Big\{ \eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)}; z_\lambda \in \Sigma, v_\lambda \in \mathbb{R}_+ \text{ and } \eta(K) < \infty \text{ for all compact } K \Big\}, \tag{3.1}$$

endowed with the σ-algebra $\mathcal{D}$ generated by the evaluation maps

$$\eta \mapsto \eta(D), \ \ D \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Sigma).$$

We are then able to construct a Poisson point process η in the space $(L, \mathcal{D}, \mathbb{Q})$ with intensity measure given by $\mu \otimes dv$, where dv is the Lebesgue measure on $\mathbb{R}_+$, see Resnick (2008), Proposition 3.6 on p.130.

Let $X_1, X_2, \ldots, X_n$ be random variables on Σ such that X_1's distribution is absolutely continuous with respect to μ and, for all $i = 2, \ldots, n$, the law of X_i conditioned on the values taken by $X_1, \ldots, X_{i-1}$ is absolutely continuous with respect to μ. Using the process η constructed above, we define

$$g_1 : \Sigma \mapsto \mathbb{R}_+, \text{ the density function of } X_1 \text{ with respect to } \mu,$$

$$\xi_1 := \inf \big\{ t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg_1(z_\lambda) \geq v_\lambda \big\},$$

$$G_1(z) := \xi_1 \, g_1(z), \text{ for } z \in \Sigma, \tag{3.2}$$

$$(z_{\lambda_1}, v_{\lambda_1}), \text{ the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } G_1(z_{\lambda_1}) = v_{\lambda_1}.$$

Then, recursively, for $2 \leq k \leq n$ we define $g_k : \Sigma \mapsto \mathbb{R}_+$ to be the density function of X_k conditioned on the event $\{X_1 = z_{\lambda_1}, \ldots, X_{k-1} = z_{\lambda_{k-1}}\}$,

$$\xi_k := \inf \big\{ t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg_k(z_\lambda) + G_{k-1}(z_\lambda) \geq v_\lambda \big\},$$

$$G_k(z) := \xi_k \, g_k(z) + G_{k-1}(z), \text{ for } z \in \Sigma, \tag{3.3}$$

$$(z_{\lambda_k}, v_{\lambda_k}), \text{ the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } G_k(z_{\lambda_k}) = v_{\lambda_k}.$$

Using Proposition 4.1 of Popov and Teixeira (2015), we are able to state the following proposition:

Proposition 3.1. *The vector $(z_{\lambda_1}, \ldots, z_{\lambda_n})$ has the same law as $(X_1, \ldots, X_n)$.*

We call the function $G_n(z)$ the soft local time of the vector $(X_1, \ldots, X_n)$ up to time n with respect to the measure μ, or more usually simply the soft local time. If T is a stopping time with respect to the canonical filtration generated by the variables X_i, it is simple to define $G_T(z)$, the soft local time up to time T.

Note that by controlling the value of the soft local times function we will automatically control the values our random variables take, as the next corollary summarizes:

Corollary 3.2. *For any measurable function $h : \Sigma \to \mathbb{R}_+$ we have, using the same notation as above,*

$$\mathbb{Q}\Big[\{z_1, \ldots, z_T\} \subseteq \{z_\lambda; v_\lambda \leq h(z_\lambda)\}\Big] \geq \mathbb{Q}\Big[G_T(z) \leq h(z), \text{ for } \mu\text{-a.e. } z \in \Sigma\Big], \tag{3.4}$$

for any finite stopping time $T \geq 1$.

We will now use the above result in our random interlacements context. More specifically, we will show a way of simulating the intersection of the random interlacements set with a given subset of $\mathbb{Z}^d$ in such a way as to make explicit the dependence each random walk excursion has with its entrance and exit points on the subset. We refer the reader to Figure 3.3 for a brief overview of the arguments used in this section.

It is clear from (2.3) that in order to simulate the random interlacements set at level u in a bounded subset K of $\mathbb{Z}^d$ we need only to pick $N_K^u \overset{d}{=} \text{Poisson}(u \, \text{cap}(K))$ points in ∂K, each point chosen according to the measure $\overline{e}_K(\cdot)$, and from each point start a simple random walk.

We intend to study $\mathcal{I}_{A_1}^u = \mathcal{I}^u \cap A_1$, showing that this set is not that much influenced by the random interlacements set intersected with A_2, $\mathcal{I}_{A_2}^u = \mathcal{I}^u \cap A_2$.
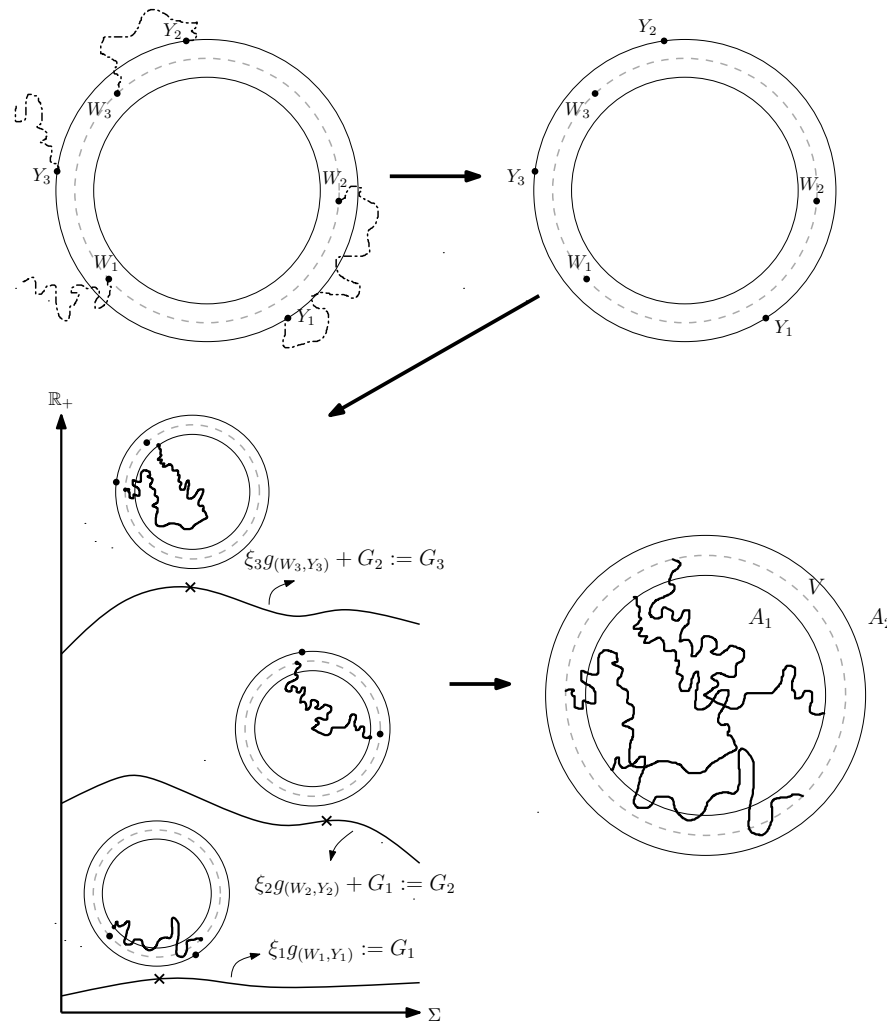
FIGURE 3.3. The figure shows how we use the soft local times technique to simulate the range of a simple random walk trajectory intersected with A_1. We first simulate a process of pairs of points $((W_k, Y_k), k \geq 0)$ denoting the entrance at V and exit at ∂A_2 of a simple random walk trajectory that starts at V. We then use the soft local times method to simulate the pieces of trajectory that lie between each of the pairs (W_k, Y_k).

We will later clarify what we mean by "influence". For now, we observe that the only "information" $\mathcal{I}_{A_1}^u$ receives from $\mathcal{I}_{A_2}^u$ is the location of the entrance and exit points of the excursions on ∂A_2 of the random walks that constitute $\mathcal{I}_{A_2}^u$.

Let us now describe how the process is constructed. We first generate the points of entrance at V and exit from A_2^C of each excursion on V of a random walk trajectory. These points will be the clothesline onto which we will hang the pieces of trajectory that meet A_1, we will do so using the soft local times method.

Let us define the successive return and departure times between V and A_2. Given a trajectory that starts at V, we define

$$
\begin{aligned}
D_0 &= 0, & R_1 &= H_{\partial A_2}, \\
D_1 &= H_V \circ \theta_{R_1} + R_1, & R_2 &= H_{\partial A_2} \circ \theta_{D_1} + D_1, \qquad (3.5) \\
D_2 &= H_V \circ \theta_{R_2} + R_2 & &\text{and so on.}
\end{aligned}
$$

We also define the random time

$$
T_\Delta = \inf\{k \geq 1; R_k = \infty\}, \qquad (3.6)
$$

which is almost surely finite, as the walk is transient.

Let $(X_n, n \geq 0)$ be the simple random walk with initial distribution given by $\overline{e}_V(\cdot)$. Let Δ be an artificial cemetery state. We construct a random sequence of elements of $(V \times \partial A_2) \cup \{\Delta\}$ in the following way: Conditioned on the event $\{T_\Delta = m\}$, we let

$$
\big((W_1, Y_1), \ldots, (W_{m-1}, Y_{m-1}), (W_m, Y_m), (W_{m+1}, Y_{m+1}), \ldots\big) \qquad (3.7)
$$
$$
= \big((X_{D_0}, X_{R_1}), \ldots, (X_{D_{m-2}}, X_{R_{m-1}}), \Delta, \Delta, \ldots\big).
$$

It is then elementary to prove that the process $((W_k, Y_k))_{k \geq 1}$ inherits the Markov property from the simple random walk. We call $((W_k, Y_k))_{k \geq 1}$ the clothesline process started at W_1. When there is no risk of confusion we will also denote by $\mathbb{P}_{w_0}$ the probability measure associated with the clothesline process started at a given point $w_0 \in V$.
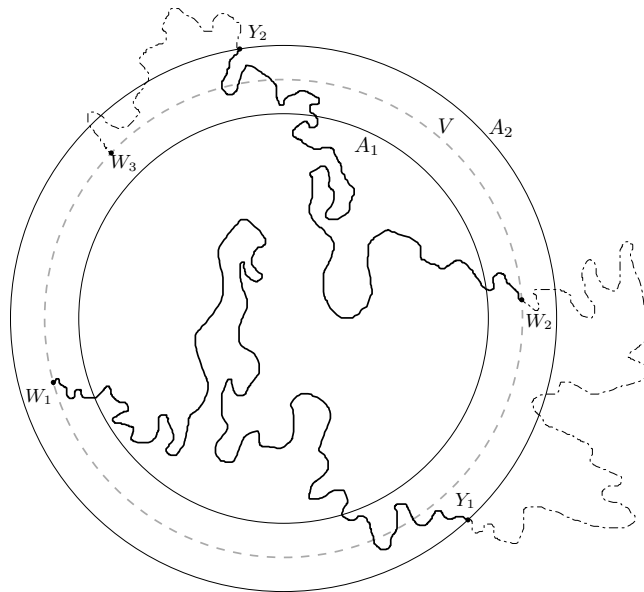

FIGURE 3.4. An example of the process $((W_k, Y_k))_{k \geq 1}$.

Let us now use the soft local times method to generate the trajectories inside A_1, given the entrance and exit points $((W_k, Y_k))_{k \geq 1}$. We first define the underlying

space Σ where our pieces of trajectories will live. We let $\mathcal{K}$ be the set of nearest-neighbor paths in A_2^C with one endpoint in ∂A_1 and the other in V,

$$\mathcal{K} := \big\{(x_0, x_1, \ldots, x_n); n \in \mathbb{N}, x_i \in A_2^C \text{ for } 1 \le i \le n, x_0 \in \partial A_1, x_n \in V \big\}. \quad (3.8)$$

We introduce yet another artificial state Θ for reasons that will be made clear in

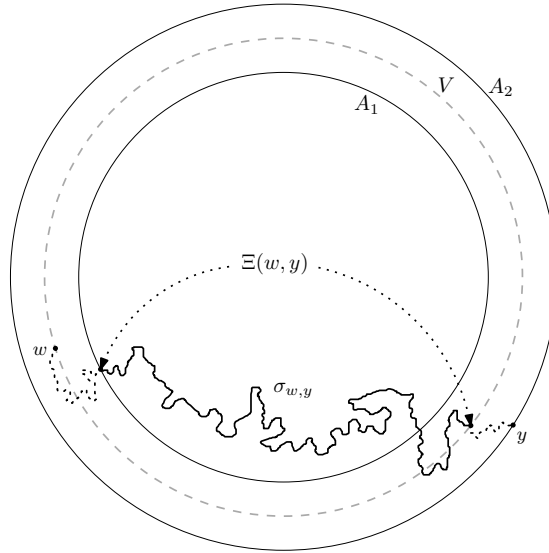

FIGURE 3.5. The definition of $\sigma(w, y)$ and $\Xi(w, y)$.

a few moments. We let $\Sigma := \mathcal{K} \cup \{\Theta\}$ and let μ be a measure on Σ defined in the following way: given $A \subseteq \Sigma$,

$$\mu(A) := \sum_{(x_0, \ldots, x_n) \in A} \tilde{\mathbb{P}}_{(x_0, x_n)}[X_0 = x_0, \ldots, X_n = x_n] + 1_{\{\Theta \in A\}}, \quad (3.9)$$

where $\tilde{\mathbb{P}}_{(x_0, x_n)}$ is the simple random walk measure conditioned on the event where x_0 is the walk's initial point and x_n is its last point on V before reaching ∂A_2. Notice that $\mu(\{\Theta\}) = 1$.

Given $(w, y) \in V \times \partial A_2$ we let $\hat{\mathbb{P}}_{w,y}$ be the measure associated with simple random walk starting at w conditioned on the event where y is the first point the walk hits in ∂A_2, that is:

$$\hat{\mathbb{P}}_{w,y}[\cdot] := \mathbb{P}_w[\,\cdot \mid X_{H_{\partial A_2}} = y] \quad (3.10)$$

We want to randomly select (according to the conditional simple random walk measure above) a piece of trajectory in A_1 given a starting point in V and an ending point in ∂A_2. Given $w \in V$ and $y \in \partial A_2$ we define the random element $\sigma_{w,y} \in \Sigma$ in the following way:

- Let $\mathcal{B}_{w,y}$ be a Bernoulli random variable with parameter $\hat{\mathbb{P}}_{w,y}[H_{\partial A_1} < H_{\partial A_2}]$.
- If $\mathcal{B}_{w,y} = 0$ we let $\sigma_{w,y} \equiv \Theta$.

- If $\mathcal{B}_{w,y} = 1$ we let, for $\mathfrak{A} \subseteq \mathcal{K}$:

$$\mathbb{P}[\sigma_{w,y} \in \mathfrak{A}] = \sum_{(a_0,\dots,a_n) \in \mathfrak{A}} \hat{\mathbb{P}}_{w,y} \left[\begin{array}{l} X_{H_{A_1}} = a_0, X_{H_{A_1}+1} = a_1, \dots, X_{H_{A_1}+n} = a_n, \\ X_k \notin V \text{ for every } k = H_{A_1} + n + 1, \dots, H_{A_2} \end{array} \right].$$

(3.11)

In other words, the random element $\sigma_{w,y} \in \Sigma$ will either be Θ, on the event where a random walk starting at w and exiting at y fails to reach A_1, or a simple random walk trajectory $(x_0, x_1, \dots, x_k) \in \mathcal{K}$ distributed so that x_0 is the first point in A_1 after the start at w and x_k is the last point in V before reaching $y \in \partial A_2$. We then define $g_{(w,y)} : \Sigma \mapsto \mathbb{R}_+$ to be the μ-density of $\sigma_{w,y}$. We refer to Figure 3.5.

Given $z = (x_0, \dots, x_n) \in \mathcal{K}$ we denote by $\Xi(z)$ the pair (x_0, x_n), the path's starting and ending points. We also let $\Xi(\Theta) = \Theta$ so that $\Xi(z)$ is defined for all $z \in \Sigma$. For $(w, y) \in V \times \partial A_2$ we define $\Xi(w, y)$ to be the random element $\Xi(\sigma_{w,y})$.

Let us calculate $g_{(w,y)}$ using the above notation. For $\mathfrak{A} \subseteq \Sigma$ we want to express the probability $\mathbb{P}[\sigma_{w,y} \in \mathfrak{A}]$ as a μ-integral over $\mathfrak{A}$.

$$\begin{aligned} \mathbb{P}[\sigma_{w,y} \in \mathfrak{A}] &= \sum_{a \in \mathfrak{A}} \mathbb{P}[\sigma_{w,y} = a] \\ &= 1_{\{\Theta \in \mathfrak{A}\}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Theta] \\ &\quad + \sum_{\substack{a \in \mathfrak{A} \\ a \neq \Theta}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Xi(a)] \hat{\mathbb{P}}_{w,y}[a \mid \Xi(w, y) = \Xi(a)] \\ &= 1_{\{\Theta \in \mathfrak{A}\}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Theta] + \sum_{\substack{a \in \mathfrak{A} \\ a \neq \Theta}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Xi(a)] \tilde{\mathbb{P}}_{\Xi(a)}[a] \\ &= \sum_{a \in \mathfrak{A}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Xi(a)] \mu(a) \\ &= \int_{\mathfrak{A}} \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Xi(z)] \mu(\mathrm{d}z), \end{aligned}$$

(3.12)

so that $g_{(w,y)}(z) = \hat{\mathbb{P}}_{w,y}[\Xi(w, y) = \Xi(z)]$. Notice that the function $g_{(w,y)}(z)$ only depends on the pair $\Xi(z)$, the path's initial and ending points.

Let $(L, \mathcal{D}, \mathbb{Q})$ be the measure space of the Poisson point process on $\Sigma \times \mathbb{R}_+$ with intensity measure $\mu \otimes \mathrm{d}v$, where $\mathrm{d}v$ is the Lebesgue measure on $\mathbb{R}_+$. A weighted sum of functions $g_{(\cdot,\cdot)}$ indexed by clothesline processes $((W_k, Y_k))_{k \geq 1}$ will be the soft local time used to simulate the pieces of trajectory we need. This way we will be able to simulate the intersection of a simple random walk trajectory with A_1. As we have seen in the random interlacements process's definition, to simulate the interlacements set inside V we need a number $N_V^u \stackrel{d}{=} \mathrm{Poisson}(u\, \mathrm{cap}(V))$ of independent random walks. We will need the same number of independent clothesline processes. For such task we will need a much bigger probability space, easily definable as a product between the Poisson point process space and an infinite product of independent simple random walk spaces starting on V. We call this bigger space the global probability space, and denote by $\mathcal{P}$ its distribution. Abusing the notation slightly, we will also write $\mathbb{E}$ to denote the expectation associated with this probability space.

Given a clothesline process $((W_k, Y_k))_{k \geq 1}$, we define the trajectory's soft local time:

$$G(z) = \sum_{k=1}^{T_\Delta} \xi_k g_{(W_k, Y_k)}(z). \tag{3.13}$$

We will also need to consider the soft local time up to a random time $T \leq T_\Delta$:

$$G_T(z) = \sum_{k=1}^{T} \xi_k g_{(W_k, Y_k)}(z). \tag{3.14}$$

Analogously, we define for any deterministic time $n \geq 1$

$$G_n(z) = \sum_{k=1}^{n} \xi_k g_{(W_k, Y_k)}(z). \tag{3.15}$$

We denote by z_k the piece of trajectory randomly selected by the k-th soft local time, G_k.

As we have seen before, in order to simulate the random interlacements set at level u in A_1, we actually need a

$$N_V^u \stackrel{d}{=} \mathrm{Poisson}(u \operatorname{cap}(V))$$

number of random walk trajectories, each started at a point in V distributed as $\overline{e}_V(\cdot)$. For $j = 1, \ldots, N_V^u$ we let $((W_k^j, Y_k^j))_{k \geq 1}$ be a clothesline process started at W_1^j, so that $((W_k^j, Y_k^j))_{k \geq 1}$ is independent from $((W_k^i, Y_k^i))_{k \geq 1}$ for $i \neq j$, and so that W_1^j is distributed as $\overline{e}_V(\cdot)$. Let T_Δ^j be the killing time associated with $((W_k^j, Y_k^j))_{k \geq 1}$. We denote by

$$G^j(z) = \sum_{k=1}^{T_\Delta^j} \xi_k^j g_{(W_k^j, Y_k^j)}(z) \tag{3.16}$$

the soft local time associated with the j-th clothesline process. It should be clear from Proposition 3.1 that we can simulate all the random elements $(\sigma_{W_k^j, Y_k^j})_{j, k \geq 1}$ at the same time using only one realization of a Poisson point process in $\Sigma \times \mathbb{R}_+$. As the Corollary 3.2 shows, in order to control the values our random elements take we only need to control the function

$$G_u^\Sigma(z) = \sum_{j=1}^{N_V^u} G^j(z), \tag{3.17}$$

the soft local time associated with the whole process. With such objective in mind we for now set our goals at estimating the soft local time's moments. We first show an easier way to express the expectation of $G(z)$.

Proposition 3.3. *Using the same notation as above, we have*

$$\mathbb{E}(G(z)) = \mathbb{E}\Big(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(X_{D_{k-1}}, X_{R_k}) = \Xi(z)\}} \Big). \tag{3.18}$$

Proof: In fact,

$$
\begin{aligned}
\mathbb{E}(G(z)) &= \mathbb{E}\Big(\sum_{k=1}^{T_\Delta} g_{(W_k, Y_k)}(z) \Big) = \mathbb{E}\Big(\sum_{k=1}^{T_\Delta} \hat{\mathbb{P}}_{W_k, Y_k}[\Xi(W_k, Y_k) = \Xi(z)] \Big) \\
&= \mathbb{E}\Big(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(W_k, Y_k) = \Xi(z)\}} \Big) = \mathbb{E}\Big(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(X_{D_{k-1}}, X_{R_k}) = \Xi(z)\}} \Big).
\end{aligned}
\tag{3.19}
$$

$\square$

We have shown that the expectation of $G(z)$, for $z \neq \Theta$, is the same as the expectation of how many times a random walk started at W_1 will do an excursion on A_2^C with starting and ending points given by $\Xi(z)$.

The next proposition gives a bound on the second moment $\mathbb{E}(G(z)^2)$. It is a straightforward adaptation of Theorem 4.8 of Popov and Teixeira (2015) to our context and therefore we omit its proof. In what follows, we let $\mathbb{E}_w(G(z))$ denote the expectation of the soft local time $G(z)$ when the associated clothesline process is conditioned on starting at $w \in V$.

Proposition 3.4. *For any $w_0 \in V$,*

$$
\mathbb{E}_{w_0}\big(G(z)^2\big) \leq 2\mathbb{E}_{w_0}\big(G(z)\big)\big(\sup_{w' \in V} \mathbb{E}_{w'} G(z) + \sup_{w,y} g_{(w,y)}(z) \big).
\tag{3.20}
$$

For this paper's results, an estimate on the exponential moments of G will be essential. The next proposition, again adapted from Popov and Teixeira (2015) (Propositions 3.5 and 3.4 are proved in the context of Markov chains in the original paper), gives us such an estimate. Since the proof of the following result is essentially the same as the proof of Theorem 4.9 of Popov and Teixeira (2015), we will omit it.

Proposition 3.5. *Given $\hat{z} \in \Sigma$ and a measurable set $\Gamma \subset \Sigma$, let*

$$
\begin{aligned}
\alpha &= \inf \Big\{ \frac{g_{(w,y)}(z')}{g_{(w,y)}(\hat{z})}; (w,y) \in V \times \partial A_2, z' \in \Gamma \Big\}, \\
N(\Gamma) &= \#\{k \leq T_\Delta; z_k \in \Gamma\}, \quad \text{and} \\
\ell &\geq \sup_{(w,y) \in V \times \partial A_2} g_{(w,y)}(\hat{z}).
\end{aligned}
\tag{3.21}
$$

Then, for any $v \geq 2$,

$$
\begin{aligned}
&\mathcal{P}[G(\hat{z}) \geq v\ell] \\
&\leq \mathcal{P}[G(\hat{z}) \geq \ell]\Big(\exp\big\{ -\big(\tfrac{v}{2} - 1\big)\big\} + \sup_{w'} \mathcal{P}_{w'}\big[\eta(\Gamma \times [0, \tfrac{1}{2}v\ell\alpha]) \leq N(\Gamma) \big] \Big)
\end{aligned}
\tag{3.22}
$$

(note that $\eta(\Gamma \times [0, \tfrac{1}{2}v\ell\alpha])$ is a random variable with distribution Poisson$\big(\tfrac{1}{2}v\ell\alpha\mu(\Gamma)\big)$).

4. Conditional decoupling

We begin with an overview of the main argument presented in this section. We will sample the random interlacements set intersected with A_1 in two ways. In the first way we will sample $\mathcal{I}_{A_1}^u$ using G_u^Σ, that is, we will sample $\mathcal{I}_{A_1}^u$ using the soft local times indexed by a collection of independent clothesline processes. In the second way, we will construct a set made up from random walk trajectories in A_1 in a similar way to the construction of $\mathcal{I}_{A_1}^u$, the only difference will be that the soft

local times used in this second construction will be indexed by a given nonrandom sequence $\hat{\zeta}$ of pairs of points belonging $V \times \partial A_2$. We will denote this second random set by $\mathcal{I}^u_{A_1|\hat{\zeta}}$, and we will show using the soft local times method that $\mathcal{I}^u_{A_1|\hat{\zeta}}$ and $\mathcal{I}^u_{A_1}$ are usually very similar to each other. We then prove a similar result when the pairs of points that constitute the nonrandom sequence all belong to the boundary of a set contained in A_2.

Throughout this section we will again only differentiate between A_1° and $A_1^{\square}$ when the need arises.

For $\hat{u} > 0$, $\delta \in (0,1)$ and $z \in \mathcal{K}$, we define the interval

$$I^{\delta}_{\hat{u},z} := \Big[(1-\delta)\hat{u}\operatorname{cap}(V)\mathbb{E}[G(z)], (1+\delta)\hat{u}\operatorname{cap}(V)\mathbb{E}[G(z)] \Big]$$

and the event

$$D^{\delta}_{\hat{u}} := \{ G^{\Sigma}_{\hat{u}}(z) \in I^{\delta}_{\hat{u},z} \text{ for all } z \in \mathcal{K} \}.$$

The main technical goal of this section is to prove the following result:

Proposition 4.1. *Recall the definition of $a_{\circ}, a_{\square}$. Using the notation above defined, we have that, for $u > 0$ and $\varepsilon \in (0,1/4)$, there exist constants $c, c' > 0$ depending only on the dimension and on whether of the set A_1 is an Euclidean ball or a hypercube such that*

$$\mathcal{P}\Big[D^{\varepsilon/4}_u, D^{\varepsilon/4}_{u(1-\varepsilon)}, D^{\varepsilon/4}_{u(1+\varepsilon)} \Big] \geq 1 - c \exp\big(-c'\varepsilon^2 u s^a \big). \qquad (4.1)$$

We will prove the above proposition later, after establishing some preliminary results about large deviations for the soft local times we defined. We will first show how we obtain the Theorems 1.1 and 1.2 from (4.1).

Proposition 4.1 implies that each of the soft local times associated to the processes $\mathcal{I}^u_{A_1}$, $\mathcal{I}^{u(1-\varepsilon)}_{A_1}$ and $\mathcal{I}^{u(1+\varepsilon)}_{A_1}$ stay confined between the graphs of two explicit deterministic functions (as can partially be seen in Figure 4.6). This happens when we let the points of entrance at V and exit at ∂A_2 of the excursions on A_1 of the simple random walk trajectories of the interlacements process at level u be distributed according to the "right" law, that is, the law of the clothesline processes. That is, when we "average" these points according to this law we obtain a good concentration for the whole function G^{Σ}_u. Nevertheless, our goal is to obtain a similar concentration when these points are deterministic. The heuristic argument is that when something happens with high probability in the "annealed" law, then most of the times it will also happen with high probability in the "quenched" law. Therefore, for "many" of these deterministic entrance and exit points, we will also have a strong concentration inequality result similar to Proposition 4.1. We will introduce some new notation to make this argument rigorous and prove our main theorem.

We let

$$\Big(\mathcal{C}_u(V, \partial A_2), \mathcal{B}_u(V, \partial A_2), \mathbb{P}^u_{V, \partial A_2} \Big)$$

be a probability subspace in which the collection of clothesline processes $\{ (W^j_k, Y^j_k)^{T^j_\Delta}_{k=1} \}^{N^V_u}_{j=1}$ is defined, where each $\hat{\zeta} \in \mathcal{C}_u(V, \partial A_2)$ can be written as a finite collection of finite sequences of points belonging to V and ∂A_2:

$$\hat{\zeta} := \big\{ \hat{\zeta}_1, \ldots, \hat{\zeta}_K \big\},$$

where for each $j = 1, \dots, K$; $\hat{\zeta}_j$ is a finite sequence alternating between points of V and ∂A_2. We write

$$\hat{\zeta}_j = \left(\zeta_0^j, \dots, \zeta_{n(j)}^j \right),$$

where $n(j)$ is odd, every even entry belongs to V and every odd entry belongs to ∂A_2.

We then define the soft local times associated with $\hat{\zeta}$. Using the same realization of the Poisson point process on $\Sigma \times \mathbb{R}_+$ defined in Section 3, we construct the soft local times

$$G^{\hat{\zeta}_j}(z) := \sum_{k=0}^{\frac{n(j)-1}{2}} \tilde{\xi}_k^j g_{(\zeta_{2k}^j, \zeta_{2k+1}^j)}(z),$$

where $\tilde{\xi}_k^j$ is an exponential random variable defined in the manner of (3.13). We then define

$$G^{\hat{\zeta}}(z) := \sum_{j=1}^{K} G^{\hat{\zeta}_j}(z).$$

This function should be viewed as a quenched version of the soft local times G_u^Σ, when the collection of clothesline processes $\{(W_k^j, Y_k^j)_{k=1}^{T_\Delta^j}\}_{j=1}^{N_u^V}$ is given by the deterministic element $\hat{\zeta}$. We denote by $\mathcal{I}_{A_1|\hat{\zeta}}^u$ the interlacements process inside A_1 determined by the ranges of the excursions of Σ bellow $G^{\hat{\zeta}}$. $\mathcal{I}_{A_1|\hat{\zeta}}^u$ is distributed as the random interlacements process inside A_1 when its associated random walks excursions have entrance points at V and exit points at ∂A_2 given by $\hat{\zeta}$. The next proposition implies that $G^{\hat{\zeta}}$ is usually between $G_{u(1-\varepsilon)}^\Sigma$ and $G_{u(1+\varepsilon)}^\Sigma$ with high probability.

Proposition 4.2. *There exists a set $\mathcal{A} \in \mathcal{B}_u(V, \partial A_2)$ such that*

$$\mathbb{P}_{V,\partial A_2}^u[\mathcal{A}] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right),$$

and for all fixed $\hat{\zeta} \in \mathcal{A}$,

$$\mathcal{P}\left[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}\right]$$
$$\geq 1 - c\exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right).$$

Proof: Observe that (4.1) implies

$$\int \mathcal{P}\left[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}\right] \mathbb{P}_{V,\partial A_2}^u[\mathrm{d}\hat{\zeta}]$$
$$\geq 1 - c\exp\left(-c'\varepsilon^2 u s^a\right). \tag{4.2}$$

Let

$$\mathcal{A} := \left\{ \hat{\zeta} \in \mathcal{C}_u(V, \partial A_2) \text{ such that:} \right.$$

$$\left. \mathcal{P}\left[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}\right] \geq 1 - c\exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right) \right\}.$$

Then (4.2) implies

$$\mathbb{P}_{V,\partial A_2}^u[\mathcal{A}] + \left(1 - c\exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right)\right)\left(1 - \mathbb{P}_{V,\partial A_2}^u[\mathcal{A}]\right)$$

$$\geq 1 - c \exp\left(-c'\varepsilon^2 u s^a\right),$$

so that

$$\mathbb{P}^u_{V,\partial A_2}[\mathcal{A}] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right).$$

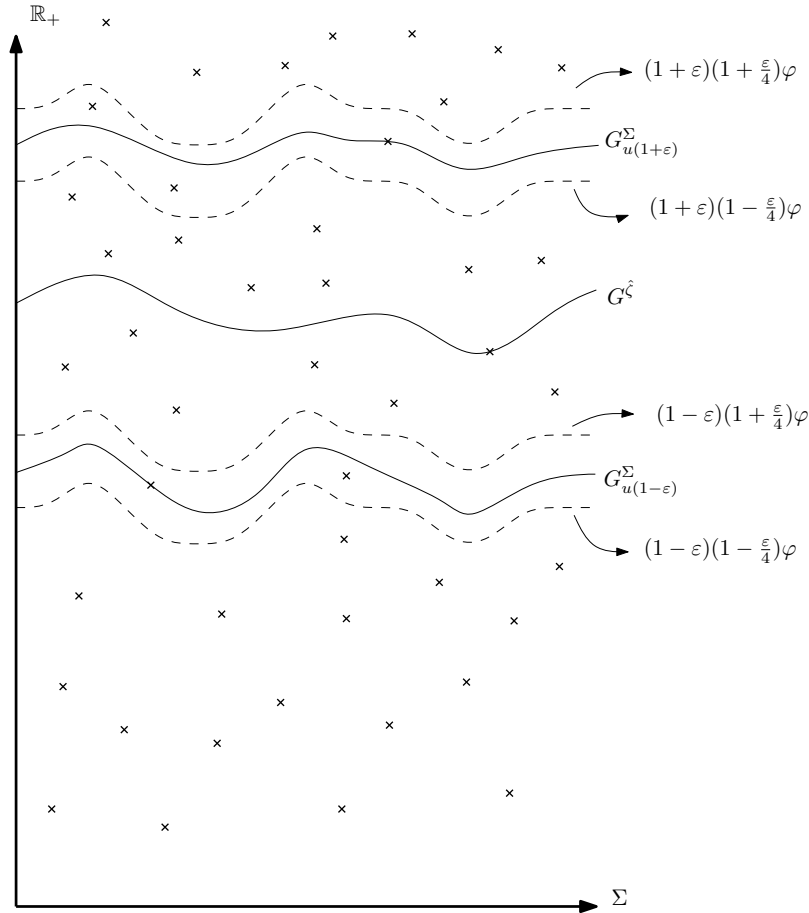This finishes the proof of the proposition.   □

FIGURE 4.6. When the sequence $\hat{\zeta}$ belongs to a well behaved set $\mathcal{A}$, the decoupling probability is greater than $1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^a\right)$. The symbol φ in the figure stands for the function $u \operatorname{cap}(V)\pi(\Xi(z))$. The figure shows the decoupling event, where $G^\Sigma_{u(1-\varepsilon)}(z) \leq G^{\hat{\zeta}}(z) \leq G^\Sigma_{u(1+\varepsilon)}(z)$ for all $z \in \mathcal{K}$.

Proposition 4.2 implies that, for $\hat{\zeta} \in \mathcal{A}$, there exists a process $(\hat{\mathcal{I}}^u_{A_1}, u \geq 0)$ distributed as the random interlacements set intersected with A_1, and a coupling

$\mathcal{P}$ such that, for all $\varepsilon > 0$ sufficiently small and $r > 0$ sufficiently big, we have

$$\mathcal{P}\big[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\zeta}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}\big] \geq 1 - c\exp\Big(-\frac{c'}{2}\varepsilon^2 us^a\Big). \tag{4.3}$$

To complete the proof of our main theorem we need to show that a result similar to Proposition 4.2 remains valid under a different conditioning.

Let $A_3 \subset A_2$ be such that $|\partial A_3| < \infty$. We enlarge our global probability space in order to make room for two new random elements. The first is $\Psi = \Psi(A_3)$, which is a random collection of points of ∂A_3 distributed as the intersection of ∂A_3 with the random walk trajectories that constitute the interlacements process $\mathcal{I}^u$. Given some possible realization $\hat{\psi}$ of Ψ, we define the second new random element to be $\mathcal{M}(\hat{\psi}) \in \mathcal{C}_u(V, \partial A_2)$, which is distributed as the clothesline process between V and ∂A_2 generated by the interlacements trajectories at level u conditioned on the fact that $\Psi = \hat{\psi}$. We can then construct a process $\mathcal{I}_{A_1|\hat{\psi}}^u$ which has the law of the the random interlacements process at level u intersected with A_1 conditioned on the event where the intersections of the interlacements' random walk trajectories with ∂A_3 are given by $\hat{\psi}$. We obtain

$$\mathcal{P}\big[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}\big] = \sum_{\hat{\zeta} \in \mathcal{C}_u(V, \partial A_2)} \mathcal{P}\big[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)} \mid \mathcal{M}(\hat{\psi}) = \hat{\zeta}\big]$$

$$\tag{4.4}$$

$$\times \mathcal{P}\big[\mathcal{M}(\hat{\psi}) = \hat{\zeta}\big]$$

$$= \sum_{\hat{\zeta} \in \mathcal{C}_u(V, \partial A_2)} \mathcal{P}\big[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\zeta}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}\big]\mathcal{P}\big[\mathcal{M}(\hat{\psi}) = \hat{\zeta}\big]$$

$$\geq \Big(1 - c\exp\Big(-\frac{c'}{2}\varepsilon^2 us^a\Big)\Big)\mathcal{P}\big[\mathcal{M}(\hat{\psi}) \in \mathcal{A}\big].$$

Let $\mathcal{E}$ be the set of all the $\hat{\psi}$'s such that

$$\mathcal{P}\big[\mathcal{M}(\hat{\psi}) \in \mathcal{A}^C\big] \geq \sqrt{\mathbb{P}_{V,\partial A_2}^u\big[\mathcal{A}^C\big]}.$$

Since

$$\mathbb{P}_{V,\partial A_2}^u\big[\mathcal{A}^C\big] = \int \mathcal{P}\big[\mathcal{M}(\hat{\psi}) \in \mathcal{A}^C\big]\mathcal{P}\big[\mathrm{d}\hat{\psi}\big] \geq \mathcal{P}\big[\mathcal{E}\big]\sqrt{\mathbb{P}_{V,\partial A_2}^u\big[\mathcal{A}^C\big]},$$

we have

$$\mathcal{P}\big[\mathcal{E}\big] \leq \sqrt{\mathbb{P}_{V,\partial A_2}^u\big[\mathcal{A}^C\big]}.$$

We have proved the following theorem, which implies Theorem 1.1:

Theorem 4.3. *Using the same notation as above, we have that, for constants $c, c' > 0$, there exists a set $\mathcal{E}$ made of possible values of Ψ such that*

$$\mathcal{P}\big[\Psi \in \mathcal{E}\big] \geq 1 - \exp\Big(-\frac{c'}{4}\varepsilon^2 us^a\Big),$$

and for all $\hat{\psi} \in \mathcal{E}$,

$$\mathcal{P}\big[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}\big] \geq 1 - c\exp\Big(-\frac{c'}{2}\varepsilon^2 us^a\Big). \tag{4.5}$$

We now present a brief proof of Theorem 1.2.

Proof of Theorem 1.2: Note that, on equation (4.15), δ can be any real number greater than 0, whereas in equation (4.16), we need to have $0 < \delta < 1$. Recall that $u' > u > 0$. We have, by substituting the appropriate δ in (4.17) and ignoring the union bound term cr^{2d-2},

$$\mathcal{P}\big[G_u^{\Sigma}(z) < G_{u+u'}^{\Sigma}(z)\big] \geq 1 - \mathcal{P}\big[G_u^{\Sigma}(z) > (u + u'4^{-1})\operatorname{cap}(V)\pi(\Xi(z))\big] \qquad (4.6)$$
$$- \mathcal{P}\big[G_{u+u'}^{\Sigma}(z) < 2^{-1}(u + u')\operatorname{cap}(V)\pi(\Xi(z))\big]$$
$$\geq 1 - \exp\left(-\frac{c}{4}(u + u')s^a\right) - \exp\left(-\frac{c}{16}\frac{(u')^2}{u^2}us^a\right)$$
$$\geq 1 - \exp\left(-c'u's^a\right).$$

Now, proceeding in the same manner as we did in the proof of Theorem 4.3, we are able to prove Theorem 1.2. $\qquad\square$

Remark 4.4. As we previously stated, the above theorems are also valid for the unsmoothed versions of the sets $A_1^{\square}$ and $A_2^{\square}$. This result follows after employing the same argument present in the above proofs, where one considers the law of the entrance and exit point of the interlacements' trajectories in the smaller set (the smoothed version of $A_2^{\square}$) generating a law of entrance and exit points in the larger set (the unsmoothed version of $A_2^{\square}$), and then performing a computation similar to (4.4) in order to obtain good concentration inequalities for the "quenched" law. One then obtains in the same way the result for $A_3 \subset A_2^{\square}$.

The remainder of the section is devoted to proving Proposition 4.1. We begin by stating the following bound

$$\sup_{\substack{w' \in V \\ y' \in \partial A_2}} \hat{\mathbb{P}}_{w',y'}\big[\Xi(w', y') = (w_0, y_0)\big] \leq cs^{-2(d-1)}, \qquad (4.7)$$

for which the proof is technical and we thus postpone it to subsection A.1 of the appendix.

Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0) \in \partial A_1 \times V$, and let $h := \operatorname{dist}(w_0, y_0)$. We let $F(w_0, y_0)$ stand for $G(z)$, making explicit the dependence of the soft local time on the endvertices $\Xi(z)$. We define

$$\pi(w_0, y_0) := \mathbb{E}(F(w_0, y_0)). \qquad (4.8)$$

We define $f_{A_1}(w_0, y_0)$ to be the probability that the simple random walk started at w_0 visits y_0 before hitting A_2. We will prove in the appendix (see Section A.1, propositions A.1 and A.2) the following bounds for these probabilities:

(i) Given $(w_0, y_0) \in A_1^{\circ} \times V^{\circ}$, there are constants $c_1, c_2 > 0$ such that

$$c_1 \frac{s^2}{h^d} \leq f_{A_1^{\circ}}(w_0, y_0) \leq c_2 \frac{s^2}{h^d}. \qquad (4.9)$$

(ii) Let $(w_0, y_0) \in A_1^{\square} \times V^{\square}$, and recall that $B_{\infty}(0, r + 2s)$ is the unsmoothed version of $A_2^{\square C}$. Let $\mathfrak{H}_i^{d-1}$; $i = 1, \ldots, 2d$; denote the $(d-1)$-dimensional hyperfaces of $B_{\infty}(0, r + 2s)$, and let $l_i^{w_0} := \min\{\operatorname{dist}(w_0, \mathfrak{H}_i^{d-1}), h\}$, and $l_i^{y_0} := \min\{\operatorname{dist}(y_0, \mathfrak{H}_i^{d-1}), h\}$. Then there are constants $c_1, c_2 > 0$ such that

$$c_1 \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{1}{h^{d-2}} \cdot \frac{l_1^{y_0} \ldots l_{2d}^{y_0}}{h^{2d}} \leq f_{A_1^{\square}}(w_0, y_0) \leq c_2 \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{1}{h^{d-2}} \cdot \frac{l_1^{y_0} \ldots l_{2d}^{y_0}}{h^{2d}}. \quad (4.10)$$

The following lemma, whose proof we also postpone to the appendix (Section $A.2$), gives us an estimate on $\pi(w_0, y_0)$.

Lemma 4.5. *Using the notation defined above we have, for constants* $c_1, c_2, c_3, c_4 > 0$:

(i) $c_1 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0) \leq \pi(w_0, y_0) \leq c_2 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0),$

(ii) $\mathbb{E}(F(w_0, y_0)^2) \leq c_3 \operatorname{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0).$

Moreover, since $\operatorname{dist}(w_0, y_0) \geq s$, *we have*

(iii) $\sup_{w_0, y_0} \pi(w_0, y_0) \leq c_4 \operatorname{cap}(V)^{-1} s^{-(d-1)}.$

We now provide a large deviation bound for $F(w_0, y_0)$.

Lemma 4.6. *There are constants* $c, c_1, c_2 > 0$ *such that for every* $(w_0, y_0) \in \partial A_1 \times V$, *we have*

$$\mathcal{P}\big[F(w_0, y_0) > vc s^{-2(d-1)}\big] \leq c_1 s^{2d-3} f_{A_1}(w_0, y_0) \operatorname{cap}(V)^{-1} e^{-c_2 v} \qquad (4.11)$$

for any $v \geq 2$ *(we can also assume* $c_2 \leq 1$ *without loss of generality).*

Proof: In the proof of this particular result it will be important for us to distinguish between the constants. We will use Proposition 3.5 for $F(w_0, y_0)$, with

$$\Gamma_{w_0, y_0} := \{(w_0', y_0') \in \partial A_1 \times V; \max\{\|w_0' - w_0\|, \|y_0' - y_0\|\} \leq c_4 s\},$$

with $0 < c_4 < 1$ defined in Section $A.3$ of the appendix.

Using the same notation as in Proposition 3.5, we note that (4.7) implies

$$\ell \leq c s^{-2(d-1)}$$

and observe that $\mu(\Gamma_{w_0, y_0}) \geq c_5 s^{2(d-1)}$ for some constant $c_5 > 0$. Also, as can be seen in Section $A.3$ of the appendix, we have

$$\alpha \geq c_3 > 0.$$

Chebyshev's inequality and Lemma 4.5 then imply

$$\mathcal{P}\big[T_l < \infty\big] \leq \mathcal{P}\big[F(w_0, y_0) > c s^{-2(d-1)}\big] \leq \frac{\pi(w_0, y_0)}{c s^{-2(d-1)}} \qquad (4.12)$$
$$\leq c_1 s^{2d-3} f_{A_1}(w_0, y_0) \operatorname{cap}(V)^{-1}.$$

We denote by $N(\Gamma_{w_0, y_0})$ the number of times the simple random walk trajectory associated with $F(w_0, y_0)$ makes an excursion of the form $z' \in \Sigma$ on A_2^C such that $\Xi(z') = (w', y') \in \Gamma_{w_0, y_0}$. We also let η_{w_0, y_0} stand for the number of points of the Poisson process associated with our soft local times that belong to $\Gamma_{w_0, y_0} \times \big[0, \frac{1}{2} v c c_3 s^{-2(d-1)}\big]$. We note that both definitions are consistent with Proposition 3.5 and write

$$\mathcal{P}\Big[\eta_{w_0, y_0} \leq N(\Gamma_{w_0, y_0})\Big] \leq \mathcal{P}\Big[\eta_{w_0, y_0} \leq \frac{c c_3 c_5 v}{4}\Big] + \mathcal{P}\Big[N(\Gamma_{w_0, y_0}) \geq \frac{c c_3 c_5 v}{4}\Big].$$

We claim that both terms in the right side of the above inequality are exponentially small in v. To see why this is true, observe that:

- η_{w_0, y_0} has Poisson distribution with parameter at least $\frac{c c_3 c_5 v}{2}$, and

- every time the simple random walk associated with $F(w_0, y_0)$ hits ∂A_2, with uniform positive probability the walk never reaches Γ_{w_0, y_0} again. This way $N(\Gamma_{w_0, y_0})$ is dominated by a Geometric(c_6) random variable, for some constant $c_6 < 1$.

Together with (4.12) and Proposition 3.5, this finishes the proof of the lemma. $\quad\square$

We are now able to complete the proof of Proposition 4.1, that in turn implies our main results:

Proof of Proposition 4.1: Let $\Psi_{w_0, y_0}(\lambda) = \mathbb{E}(e^{\lambda F(w_0, y_0)})$ be the moment generating function of $F(w_0, y_0)$. We are going to use the bounds above to estimate Ψ_{w_0, y_0}. It is elementary to obtain that $e^t - 1 \leq t + t^2$ for $t \in [0, 1]$. With this observation in mind, we write for $0 \leq \lambda \leq \frac{c_2 s^{2(d-1)}}{2c}$, where c and c_2 are the same as in the theorem above:

$$\Psi_{w_0, y_0}(\lambda) - 1 =$$

$$= \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)\mathbb{1}_{\lambda F(w_0, y_0) \leq 1} + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)\mathbb{1}_{\lambda F(w_0, y_0) > 1}$$

$$\leq \mathbb{E}(\lambda F(w_0, y_0) + \lambda^2 F(w_0, y_0)^2) + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)\mathbb{1}_{\lambda F(w_0, y_0) > 1}$$

$$\leq \lambda \pi(w_0, y_0) + c_1 \lambda^2 \operatorname{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0) + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)\mathbb{1}_{\lambda F(w_0, y_0) > 1}$$

$$\leq \lambda \pi(w_0, y_0) + c' \lambda^2 \operatorname{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0) + \lambda \int_{\lambda^{-1}}^{\infty} e^{\lambda y} \mathcal{P}\big[F(w_0, y_0) > y\big] dy$$

$$\leq \lambda \pi(w_0, y_0)$$

$$\quad + f_{A_1}(w_0, y_0) \operatorname{cap}(V)^{-1} \left(c' \lambda^2 s^{-2d+2} + \lambda c' s^{2d-3} \int_{\lambda^{-1}}^{\infty} \exp\left(\frac{-c_2 s^{2(d-1)} y}{2c}\right) dy\right)$$

$$\leq \lambda \pi(w_0, y_0)$$

$$\quad + f_{A_1}(w_0, y_0) \operatorname{cap}(V)^{-1} \left(c' \lambda^2 s^{-2d+2} + c' \lambda s^{-1} \exp\left(\frac{-c_2 s^{2(d-1)} \lambda^{-1}}{2c}\right)\right)$$

$$\leq \lambda \pi(w_0, y_0) + c' \lambda^2 \operatorname{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0), \tag{4.13}$$

where we used Lemma 4.5 and Lemma 4.6. Now since $e^{-t} - 1 \leq -t + t^2$ for all $t \geq 0$, we obtain for $\lambda \geq 0$

$$\Psi_{w_0, y_0}(-\lambda) - 1 \leq -\lambda \pi(w_0, y_0) + c\lambda^2 \operatorname{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0), \tag{4.14}$$

(the large deviation bound of Lemma 4.6 is not necessary is this case).

Observe that if $(\chi_j, j \geq 1)$ are i.i.d. random variables with common moment generating function Ψ and N is an independent Poisson random variable with parameter θ, then

$$\mathbb{E} \exp\big(\lambda \sum_{j=1}^{N} \chi_j\big) = e^{(\theta(\Psi(\lambda) - 1))}.$$

We let $F_j(w_0, y_0)$ denote the expectation $\mathbb{E}(G^j(z))$ defined in (3.16), when $z \in \Sigma$ is such that $\Xi(z) = (w_0, y_0)$. Using Lemma 4.5 and (4.13), we have, for $N_{\check{u}}^V \stackrel{d}{=}$

Poisson($\hat{u}\operatorname{cap}(V)$) and any $\delta > 0$

$$\mathcal{P}\big[G_{\hat{u}}^{\Sigma}(z) \geq (1+\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0)\big] =$$

$$= \mathcal{P}\Big[\sum_{j=1}^{N_{\hat{u}}^V} F_j(w_0, y_0) \geq (1+\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0)\Big]$$

$$\leq \frac{\mathbb{E}(\exp\big(\lambda \sum_{j=1}^{N_{\hat{u}}^V} F_j(w_0, y_0)\big))}{\exp\big(\lambda(1+\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0)\big)}$$

$$\leq \exp\big(-\lambda(1+\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0) + \hat{u}\operatorname{cap}(V)(\Psi_{w_0, y_0}(\lambda) - 1)\big)$$

$$\leq \exp\big(-(\lambda\delta\hat{u}\operatorname{cap}(V)\pi(w_0, y_0) - c'\lambda^2\hat{u}s^{-2d+2}f_{A_1}(w_0, y_0))\big)$$

$$\leq \exp\big(-(\lambda\delta\hat{u}cs^{-1}f_{A_1}(w_0, y_0) - c'\lambda^2\hat{u}s^{-2d+2}f_{A_1}(w_0, y_0))\big). \tag{4.15}$$

Analogously, with (4.14) instead of (4.13), we obtain

$$\mathcal{P}\big[G_{\hat{u}}^{\Sigma}(z) \leq (1-\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0)\big] \tag{4.16}$$
$$\leq \exp\big(-(\lambda\delta\hat{u}cs^{-1} - c'\lambda^2\hat{u}s^{-2d+2})f_{A_1}(w_0, y_0)\big).$$

We choose $\lambda = c_7\delta s^{2d-3}$ with c_7 small enough so that $\lambda \leq \frac{c_2 s^{2(d-1)}}{2c}$, and observe that the bounds for $f_{A_1}(w_0, y_0)$ given in (4.9) and (4.10) imply

$$\inf_{w_0, y_0} f_{A_1^\square}(w_0, y_0) \geq cs^{2d}r^{-3d+2},$$

$$\inf_{w_0, y_0} f_{A_1^\circ}(w_0, y_0) \geq cs^2 r^{-d}.$$

Recall the definition of $b_\circ$, a number such that

$$1 \leq b_\circ < \frac{2d-2}{d},$$

and the definition of $b_\square$, a number such that

$$1 \leq b_\square < \frac{4d-4}{3d-2}.$$

Recall that $s = s(r) \asymp r^{\frac{1}{b}}$. Then, since $a_\circ := 2d - 2 - db_\circ > 0$ and $a_\square := 4d - 4 - 3db_\square + 2b_\square > 0$, we have

$$\mathcal{P}\big[G_{\hat{u}}^{\Sigma}(z) \geq (1+\delta)\hat{u}\operatorname{cap}(V)\pi(w_0, y_0)\big] \leq \exp\big(-c\delta^2\hat{u}s^a\big).$$

Using the union bound (note that $\partial A_1 \times V$ has $O(r^{2(d-1)})$ elements),

$$\mathcal{P}\big[(1-\delta)\hat{u}\operatorname{cap}(V)\pi(\Xi(z)) \leq G_{\hat{u}}^{\Sigma}(z) \leq (1+\delta)\hat{u}\operatorname{cap}(V)\pi(\Xi(z)), \text{ for all } z \in \mathcal{K}\big]$$
$$\geq 1 - cr^{2(d-1)}\exp\big(-c'\delta^2\hat{u}s^a\big). \tag{4.17}$$

Observe that we can suppose $c' \leq 1$ without loss of generality. Using (4.17) and the union bound we obtain, for $\varepsilon \in (0, 1/4)$,

$$\mathcal{P}\Big[D_u^{\varepsilon/4}, D_{u(1-\varepsilon)}^{\varepsilon/4}, D_{u(1+\varepsilon)}^{\varepsilon/4}\Big] \geq 1 - cr^{2(d-1)}\exp\big(-c'\varepsilon^2 us^a\big).$$

Since $s = s(r) \asymp r^{\frac{1}{b}}$, by replacing the constants c and c' in the above equation we obtain (4.1) and finish the proof of Proposition 4.1. $\qquad\square$

Appendix A. Technical estimates

A.1. *Bounding the relevant probabilities.* For $w_0 \in \partial A_1$ and $y_0 \in V$ we want to study the supremum

$$\sup_{\substack{w' \in V \\ y' \in \partial A_2}} \hat{\mathbb{P}}_{w',y'}\big[\Xi(w',y') = (w_0,y_0)\big]. \tag{A.1}$$

Furthermore, we need to bound the "hanging" probability $\hat{\mathbb{P}}_{w,y}\big[\Xi(w,y) = (w_0,y_0)\big]$ for arbitrary $w \in V$ and $y \in \partial A_2$ from above and from bellow.

Given a finite nearest neighbor path γ, we denote by $|\gamma|$ its length. We will say that a path γ belongs to an event E if E occurs every time the simple random walk $(X_k, k \geq 0)$ first $|\gamma|$ steps coincide with γ. We also let $\mathbb{P}_x\big[\gamma\big]$ denote the probability that the first $|\gamma|$ steps of the simple random walk started at x coincide with γ.

In order to avoid a cumbersome notation we now introduce what, hopefully, will be a simpler way to denote our events of interest. For $w, y_0 \in V$, $w_0 \in \partial A_1$ and $y \in \partial A_2$ we define:

- $w \xrightarrow{1} w_0$: The collection of all finite nearest-neighbor trajectories starting at w that do not reach neither ∂A_1 nor ∂A_2, except at its ending point $w_0 \in \partial A_1$. Note that this collection can be thought of as the event where the simple random walk started at w hits ∂A_1 for the first time at w_0 before reaching ∂A_2.

- $w_0 \xrightarrow{2} y_0$: The collection of all finite nearest-neighbor trajectories starting at w_0 and ending at y_0 without reaching ∂A_2.

- $y_0 \xrightarrow{3} y$: The collection of all finite nearest-neighbor trajectories starting at y_0 that hit ∂A_2 for the first time at y before returning to V. Note that this collection can be thought of as the event where the simple random walk started at y_0 hits ∂A_2 before returning to V and its entrance point in ∂A_2 is y.

- $w \xrightarrow{4} y$: The event where the entrance point in ∂A_2 of the simple random walk started at w is y. This event can also be regarded as a collection of simple random walk trajectories starting at w and hitting ∂A_2 for the first time at y.

We also let $w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y$ be the "concatenation" of the first three collections, where the first trajectory's ending point becomes the second trajectory's starting point and so on. That is, if $\gamma \in w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y$ then γ is the concatenation of three distinct paths: $\gamma_1 \in w \xrightarrow{1} w_0$, $\gamma_2 \in w_0 \xrightarrow{2} y_0$, $\gamma_3 \in y_0 \xrightarrow{3} y$. With our new notation the hanging probability becomes

$$\hat{\mathbb{P}}_{w,y}\big[\Xi(w,y) = (w_0,y_0)\big] = \mathbb{P}_w\big[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y\big]$$
$$= \frac{\mathbb{P}_w\big[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y\big]}{\mathbb{P}_w\big[w \xrightarrow{4} y\big]}. \tag{A.2}$$
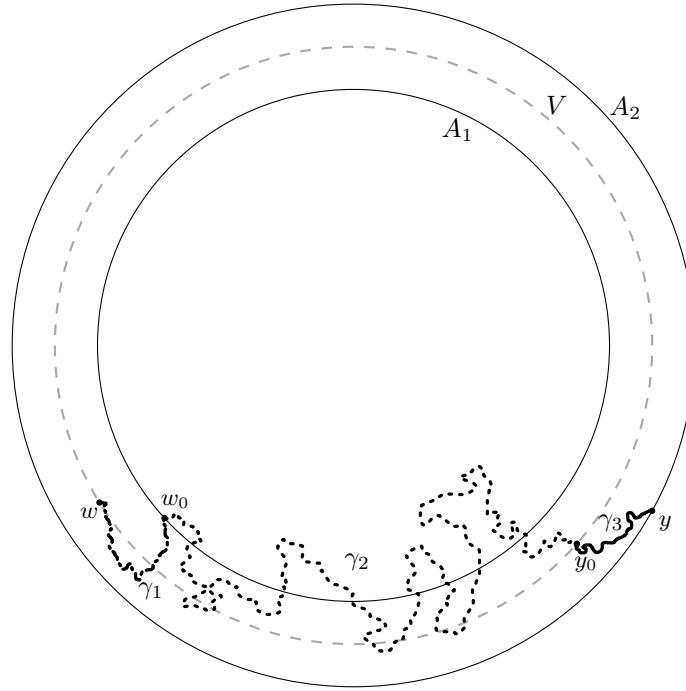
FIGURE A.7. γ as the concatenation of the three paths γ_1, γ_2 and γ_3.

We have

$$
\mathbb{P}_w\left[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y\right] = \sum_{\gamma \in w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y} \frac{1}{(2d)^{|\gamma|}}
$$
$$
= \sum_{\gamma_1 \in w \xrightarrow{1} w_0} \frac{1}{(2d)^{|\gamma_1|}} \sum_{\gamma_2 \in w_0 \xrightarrow{2} y_0} \frac{1}{(2d)^{|\gamma_2|}} \sum_{\gamma_3 \in y_0 \xrightarrow{3} y} \frac{1}{(2d)^{|\gamma_3|}}.
$$

(A.3)

Let us focus on the second sum, $\sum_{\gamma_2 \in w_0 \xrightarrow{2} y_0} \frac{1}{(2d)^{|\gamma_2|}}$, for a moment. Each $\gamma_2 \in w_0 \xrightarrow{2} y_0$ can be seen as the concatenation of one path γ_2^0 responsible for the walk's first visit to y_0 and a sequence of paths $\gamma_2^1, \ldots, \gamma_2^k$ associated with the returns the walk makes to y_0 before hitting ∂A_2, see Figure A.8. So

$$
\sum_{\gamma_2} \mathbb{P}_{w_0}\left[\gamma_2\right] = \sum_{\gamma_2^0} \mathbb{P}_{w_0}\left[\gamma_2^0\right] \sum_{k \geq 1} \sum_{\gamma_2^1, \ldots, \gamma_2^k} \mathbb{P}_{y_0}\left[\gamma_2^1\right] \ldots \mathbb{P}_{y_0}\left[\gamma_2^k\right].
$$

(A.4)

But for a fixed $k_0 > 0$, the last sum $\sum_{k \geq k_0} \sum_{\gamma_2^1, \ldots, \gamma_2^{k_0}} \mathbb{P}_{y_0}\left[\gamma_2^1\right] \ldots \mathbb{P}_{y_0}\left[\gamma_2^{k_0}\right]$ equals the probability that the simple random walk started at y_0 returns to y_0 at least k_0 times before hitting ∂A_2. Since the walk is transient, we can use the strong Markov property to show that there exists a constant $0 < c < 1$ such that

$$
\sum_{k \geq k_0} \sum_{\gamma_2^1, \ldots, \gamma_2^{k_0}} \mathbb{P}_{y_0}\left[\gamma_2^1\right] \ldots \mathbb{P}_{y_0}\left[\gamma_2^{k_0}\right] < c^{k_0}.
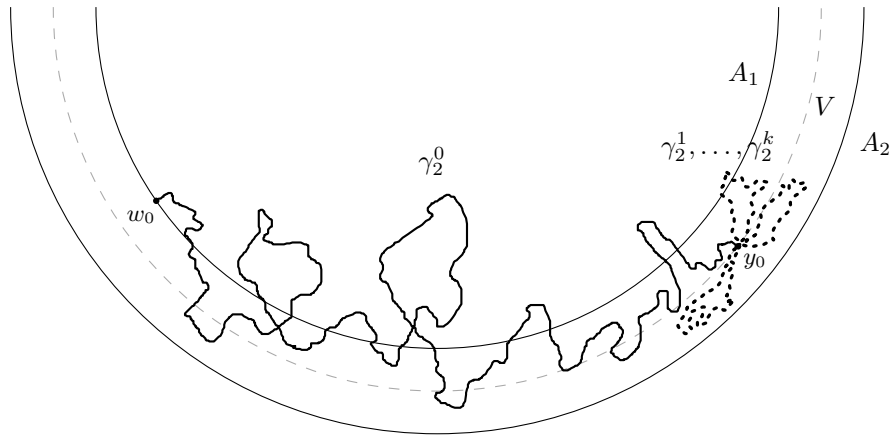$$

(A.5)

FIGURE A.8. γ_2 as the concatenation of the paths $\gamma_2^0, \gamma_2^1, \ldots, \gamma_2^k$.

We have thus shown the existence of a constant $c > 0$ such that

$$\sum_{\gamma_2^0} \mathbb{P}_{w_0}\left[\gamma_2^0\right] \leq \sum_{\gamma_2} \mathbb{P}_{w_0}\left[\gamma_2\right] \leq c \sum_{\gamma_2^0} \mathbb{P}_{w_0}\left[\gamma_2^0\right] \tag{A.6}$$

where γ_2^0 represents any nearest neighbor path that starts at w_0 and ends at its only visit to y_0, without ever reaching ∂A_2. Let us update our collection's definition in view of this last computation. We denote by

- $w_0 \xrightarrow{2'} y_0$: The collection of all finite nearest-neighbor paths starting at w_0 and ending at their first visit to y_0, without hitting ∂A_2. This collection now can be thought of as the event where the simple random walk started at w_0 makes a visit to y_0 before hitting ∂A_2.

Combining $(A.3)$ with $(A.6)$ we get

$$\mathbb{P}_w\left[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y\right] \leq c\mathbb{P}_w\left[w \xrightarrow{1} w_0\right]\mathbb{P}_{w_0}\left[w_0 \xrightarrow{2'} y_0\right]\mathbb{P}_{y_0}\left[y_0 \xrightarrow{3} y\right]. \tag{A.7}$$

Our work will now reside in giving upper bounds for these probabilities.

Using standard techniques from simple random walk theory, such as exit and entrance measure estimates, the Green function's estimate (2.1) and Lemma 2.1, one can prove the following result:

Proposition A.1. *Recall that $h := \mathrm{dist}(w_0, y_0)$ and define*

$$h_1 := \mathrm{dist}(w, w_0) \tag{A.8}$$
$$h_3 := \mathrm{dist}(y, y_0)$$
$$h_4 := \mathrm{dist}(w, y)$$

Regarding the sets A_1°, V° and A_2°, we have that there exist constants $c_k > 0$, $k = 1, \ldots, 9$, such that

$$c_1 e^{\frac{-c_2 h_1}{s}} s^{-(d-1)} \leq \mathbb{P}_w\left[w \xrightarrow{1} w_0\right] \leq c_2 e^{\frac{-c_2 h_1}{s}} s^{-(d-1)}, \tag{A.9}$$

$$c_3 e^{\frac{-c_4 h_3}{s}} s^{-(d-1)} s^{-1} \leq \mathbb{P}_w\left[y \xrightarrow{3} y_0\right] \leq c_4 e^{\frac{-c_4 h_3}{s}} s^{-(d-1)} s^{-1}, \tag{A.10}$$

$$c_5 \frac{s^2}{h^d} \le \mathbb{P}_{w_0}\left[w_0 \xrightarrow{2'} y_0\right] \le c_6 \frac{s^2}{h^d}, \tag{A.11}$$

$$c_7 \frac{s}{h_4^d} \le \mathbb{P}_{w_0}\left[w \xrightarrow{4} y\right] \le c_8 \frac{s}{h_4^d}. \tag{A.12}$$

$$\sup_{\substack{w \in V \\ y \in \partial A_2}} \mathbb{P}_w\left[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y\right] \le c_9 s^{-2(d-1)}. \tag{A.13}$$

We will briefly describe how the random walk trajectories should behave in order to satisfy the conditions in each of the above events, which should give a general idea about how to prove the above result rigorously. We will omit the details for the sake of brevity.

Sketch of the proof: In (A.9) the walk started at w has to cross $\sim h_1 s^{-1}$ subsets of $A_2^C \setminus A_1$ with diameter $\sim s$ without reaching either ∂A_2 or ∂A_1, which implies the exponential factor; and then after coming within distance $\sim s$ of w_0 it has to enter ∂A_1 though w_0, from which the $\sim s^{-(d-1)}$ term comes from.

The bound (A.10) follows from similar reasoning as in (A.9), but now the walk started at y_0 pays a price $\sim s^{-1}$ in order to reach y without returning to V.

In (A.11), the walk started at w_0 has to escape the proximity with ∂A_2, which it does with probability $\sim s h^{-1}$, then reach a ball with radius $\sim s$ centered at y_0 without ever reaching ∂A_2, which costs $\sim s^{d-1} h^{-(d-1)}$, before finally reaching y_0 from this ball, which has a cost of $s^{-(d-2)}$.

In (A.12), the random walk started at w escapes the proximity with ∂A_2 with probability $\sim s h_4^{-1}$ and then hits ∂A_2 for the first time at y with probability $h_4^{-(d-1)}$.

Finally, (A.13) is implied by the above bounds in an elementary way. $\square$

We will now focus on sets $A_1^\square, V^\square$ and $A_2^\square$, and the related simple random walk probabilities. We will continue to use the notation introduced in (A.8) in this new context.

$w \xrightarrow{1} w_0$, $y_0 \xrightarrow{3} y$: The bounds for the probabilities of these events have the same order as the bounds for the corresponding events when the underlying sets are Euclidean balls, and their proofs are essentially the same.

$w_0 \xrightarrow{2'} y_0$: If $h < 100s$, an application of a Green's function estimate gives the desired result. We then assume $h > 100s$. Define $\tilde{B}_x$ to be the discrete ball in the ℓ_∞-norm centered in x with radius $\frac{h}{4\sqrt{d}}$.

We will break up the path $\gamma_2^0 \in w_0 \xrightarrow{2'} y_0$ into pieces that are easier to work with. Let $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$, $y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C$. We define the collection of finite paths:

- $w_0 \xrightarrow{5} w_4$: The collection of all finite nearest-neighbor paths starting at w_0 whose only intersection with $\partial \tilde{B}_{w_0} \cup \partial A_2$ is at its ending point $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$.

- $w_4 \xrightarrow{6} y_4$: The collection of finite nearest-neighbor paths starting at w_4 and ending at y_4, without intersecting ∂A_2.

- $y_4 \xrightarrow{7} y_0$: The collection of all finite nearest-neighbor paths that start at y_4, never return to $\partial \tilde{B}_{y_0} \cap A_2^C$, and end at y_0 without ever reaching ∂A_2.
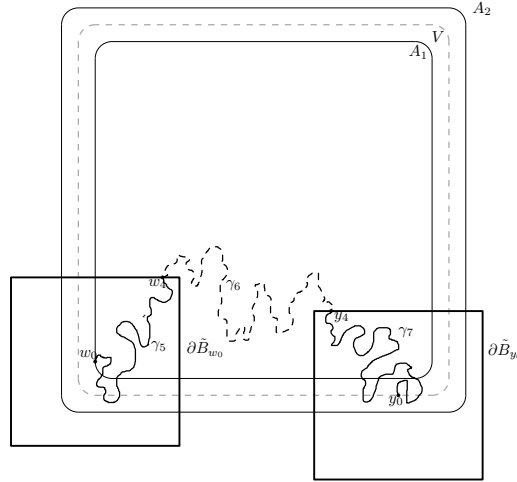
FIGURE A.9. Definition of the paths γ_5, γ_6 and γ_7.

As before, we denote by $w_0 \xrightarrow{5} w_4 \xrightarrow{6} y_4 \xrightarrow{7} y_0$ the concatenation of these three collections.

Analogously to what we noted at the start of this section, we observe that $\gamma_2^0 \in w_0 \xrightarrow{2'} y_0$ if and only if there exists $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$ and $y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C$ such that γ_2^0 is the concatenation of three paths: $\gamma_5 \in w_0 \xrightarrow{5} w_4$, $\gamma_6 \in w_4 \xrightarrow{6} y_4$, and $\gamma_7 \in y_4 \xrightarrow{7} y_0$.

We also define

- $w_4 \xrightarrow{6'} y_4$ The collection of finite simple random walk trajectories starting at w_4 and ending at its first visit to y_4 without intersecting ∂A_2. This collection can also be seen as the event where the simple random walk started at w_4 visits y_4 before it hits ∂A_2.

Using the same argument used to obtain the bound ($A.6$), we can find a constant $c > 1$ such that

$$\mathbb{P}_{w_4}\left[w_4 \xrightarrow{6} y_4\right] \leq \sum_{\gamma_6 \in w_4 \xrightarrow{6} y_4} \frac{1}{(2d)^{|\gamma_6|}} \leq c\mathbb{P}_{w_4}\left[w_4 \xrightarrow{6'} y_4\right]. \qquad (A.14)$$

We then have

$$\mathbb{P}_{w_0}\left[w_0 \xrightarrow{2'} y_0\right] = \sum_{\substack{w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C \\ \gamma_5 \in w_0 \xrightarrow{5} w_4}} \frac{1}{(2d)^{|\gamma_5|}} \sum_{\substack{y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C \\ \gamma_6 \in w_4 \xrightarrow{6} y_4}} \frac{1}{(2d)^{|\gamma_6|}} \sum_{\gamma_7 \in y_4 \xrightarrow{6} y_0} \frac{1}{(2d)^{|\gamma_7|}}$$

$$\leq c \sum_{w_4} \mathbb{P}_{w_0}\left[w_0 \xrightarrow{5} w_4\right] \sum_{y_4} \mathbb{P}_{w_4}\left[w_4 \xrightarrow{6'} y_4\right] \mathbb{P}_{y_4}\left[y_4 \xrightarrow{7} y_0\right]. \quad (A.15)$$

We then use the Green's function estimate (2.1) to bound $\mathbb{P}_{w_4}\left[w_4 \xrightarrow{6'} y_4\right]$ by ch^{d-2} (note that $\operatorname{dist}(w_4, y_4) = O(h)$). Using this bound on the above inequality, we

obtain

$$\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \le ch^{d-2} \sum_{w_4} \mathbb{P}_{w_0}\big[w_0 \xrightarrow{5} w_4\big] \sum_{y_4} \mathbb{P}_{y_4}\big[y_4 \xrightarrow{7} y_0\big]. \tag{A.16}$$

Note that

$$\sum_{w_4} \mathbb{P}_{w_0}\big[w_0 \xrightarrow{5} w_4\big] = \mathbb{P}_{w_0}\big[H_{\partial \tilde{B}_{w_0}} \le H_{\partial A_2}\big], \tag{A.17}$$

and using the simple random walk's reversibility, we also have

$$\sum_{y_4} \mathbb{P}_{y_4}\big[y_4 \xrightarrow{5} y_0\big] = \mathbb{P}_{y_0}\big[H_{\partial \tilde{B}_{y_0}} \le H_{\partial A_2}\big], \tag{A.18}$$

so that we obtain the following bound

$$\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \le \frac{c}{h^{d-2}} \mathbb{P}_{w_0}\big[H_{\partial \tilde{B}_{w_0}} \le H_{\partial A_2}\big]\mathbb{P}_{y_0}\big[H_{\partial \tilde{B}_{y_0}} \le H_{\partial A_2}\big]. \tag{A.19}$$

We still have to obtain bounds for these last two probabilities. Since they are similarly defined, the bound for both of them follows from the same arguments, and thus we will only provide a bound for $\mathbb{P}_{w_0}\big[H_{\partial \tilde{B}_{w_0}} \le H_{\partial A_2}\big]$.

We will do so by looking at the projections of the random walk trajectory in each of the d orthogonal axes. Since we will need to look at these projections independently, we will change our object of study from the simple random walk on $\mathbb{Z}^d$ to the continuous time simple random walk on $\mathbb{Z}^d$ with waiting times between steps distributed as $\mathrm{Exp}(1)$ random variables. Since we will be studying properties of the random walk's trajectories, this change of framework will in no way impact the probabilities of interest.

We recall that $B_\infty(0, r+2s)$ is the unsmoothed version of A_2^C. To simplify our notation we will work with the following translate of $B_\infty(0, r+2s)$:

$$\mathfrak{H}_{r+2s} := \big\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : 0 \le x_i \le 2r+4s+1, \text{ for all } i = 1, \ldots, d\big\}.$$

Without loss of generality we assume that $0 \in \mathbb{Z}^d$ is the point belonging to

$$\{0, 2r+4s+1\}^d$$

which is closest to w_0. We denote $w_0 \equiv (w_0^1, \ldots, w_0^d)$. The projection of the simple random walk in the jth coordinate axis is itself a continuous time random walk started at w_0^j with waiting time between jumps given by a $\mathrm{Exp}(d)$ random variable, and, as we already noted, these projected random walks are independent from each other.

Given w_0, we divide the d directions of $\mathbb{Z}^d$ in two kinds. The first kind will be such that $w_0^j \le h(4\sqrt{d})^{-1}$, the second will be such that $w_0^j > h(4\sqrt{d})^{-1}$. We assume without loss of generality the first d_0 directions to be of the first kind and the remaining directions to be of the second kind. Our goal now is establishing the following bound:

$$\mathbb{P}\big[H_{\partial \tilde{B}_{w_0}} \le H_{\partial A_2}\big] \le c \prod_{1 \le i \le d_0} \frac{w_0^i}{h}. \tag{A.20}$$

This inequality can be proved using the independence of the random walk projections on the coordinate axes and rather standard techniques, and therefore we will omit its proof. It can be interpreted as a version of the gambler's ruin estimate (see Section 5.1 of Lawler and Limic, 2010) where each of the walk's projections has to reach distance $\sim h$ from ∂A_2 without ever hitting the associated projections of ∂A_2 in the process.

We will change the notation so that we are able to express (A.20) in a way that does not uses the fact that $\{0\}^d$ is the vertex of $\{0, r + 2s\}^d$ which is closest to w_0. Recall the definition of $\mathfrak{H}_i^{d-1}$, $i = 1, \ldots, 2d$; the $(d - 1)$- dimensional hyperfaces of $\mathfrak{H}_{r+2s}$, of $l_i^{w_0} := \min\{\text{dist}(w_0, \mathfrak{H}_i^{d-1}), h\}$, and of $l_i^{y_0} := \min\{\text{dist}(y_0, \mathfrak{H}_i^{d-1}), h\}$. We have that (A.20) implies

$$\mathbb{P}_{w_0}\big[H_{\partial \tilde{B}_{w_0}} \leq H_{\partial A_2}\big] \leq c \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}}.$$

Together with ($A.19$), this shows

$$\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \leq ch^{-(d-2)} \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \ldots l_{2d}^{y_0}}{h^{2d}}. \tag{A.21}$$

We need a matching lower bound. Again we assume $h > 100s$, since otherwise the lower bound follows immediately from using (2.1). We define

$$w_5 := \Big(w_0^1 + \frac{h}{4\sqrt{d}}, \ldots, w_0^{d_0} + \frac{h}{4\sqrt{d}}, w_0^{d_0+1}, \ldots, w_0^d\Big), \tag{A.22}$$

We analogously define y_5: Let $e_{i_{d_1}}, \ldots, e_{i_{d_k}}$ be the vectors in the orthonormal basis of $\mathbb{R}^d$ corresponding to the directions in which the ball $B_\infty\big(y_0, \frac{h}{4\sqrt{d}}\big)$ passes the limits of the hypercube $\mathfrak{H}_{r+2s}$. y_5 is defined to be the point in A_2^C such that

$$l = d_1, \ldots, d_k \implies |\langle y_5 - y_0, e_{i_l}\rangle| = \frac{h}{4\sqrt{d}},$$

$$n \neq d_1, \ldots, d_k \implies |\langle y_5 - y_0, e_{i_n}\rangle| = 0$$

and

$$B_\infty\Big(y_5, \frac{h}{4\sqrt{d}}\Big) \subseteq \mathfrak{H}_{r+2s}.$$

Our plan is to describe an event contained in $w_0 \xrightarrow{2'} y_0$ with probability matching that of the right side of ($A.21$). We let

$$\hat{B}_{w_5} := \hat{B}_\infty\Big(w_5, \frac{h}{16\sqrt{d}}\Big),$$

and

$$\hat{B}_{y_5} := \hat{B}_\infty\Big(y_5, \frac{h}{16\sqrt{d}}\Big),$$

For $w_6 \in \partial \hat{B}_{w_5}$ and $y_6 \in \partial \hat{B}_{y_5}$, we define the events

- $w_0 \xrightarrow{8} w_6$: The event where the random walk started at w_0 hits $\partial \hat{B}_{w_5}$ before hitting ∂A_2 and its entrance point in $\partial \hat{B}_{w_5}$ is w_6.

- $w_6 \xrightarrow{9} y_6$: The event where the random walk started at w_6 visits $y_6 \in \partial \hat{B}_{y_5}$ before reaching ∂A_2.

- $y_6 \xrightarrow{10} y_0$: The event where the simple random walk started at y_6 hits y_0 before returning to $\partial \hat{B}_{y_5}$.
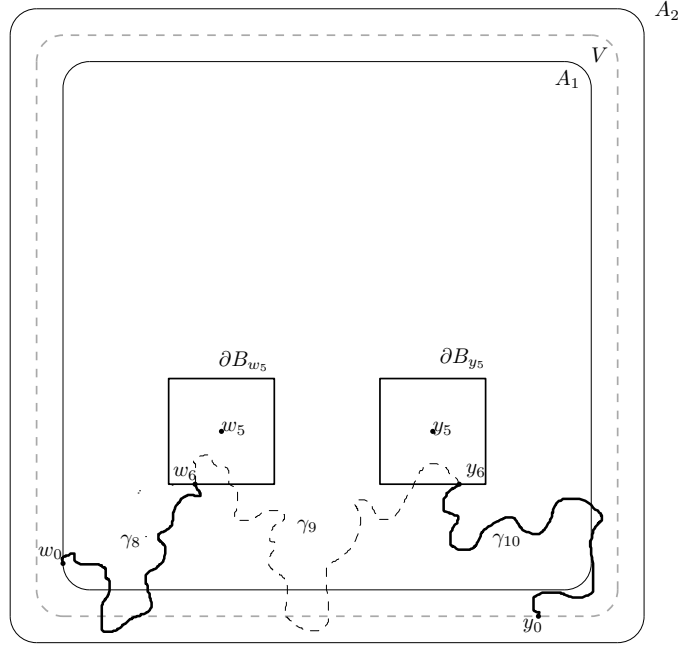
FIGURE A.10. Definition of the paths γ_8, γ_9 and γ_{10}.

And we denote by $w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0$ the "concatenation" of these three events, that is, the path γ belongs to the event $w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0$ if and only if γ is the concatenation of three paths: $\gamma_8 \in w_0 \xrightarrow{8} w_6$, $\gamma_9 \in w_6 \xrightarrow{9} y_6$ and $\gamma_{10} \in y_6 \xrightarrow{10} y_0$. It is then clear that

$$\bigcup_{w_6}\bigcup_{y_6} w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0 \subset w_0 \xrightarrow{2'} y_0,$$

so that; summing over $\gamma_8 \in w_0 \xrightarrow{8} w_6$, $\gamma_9 \in w_6 \xrightarrow{9} y_6$ and $\gamma_{10} \in y_6 \xrightarrow{10} y_0$; we have

$$\begin{aligned}
\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] &\geq \sum_{w_6}\sum_{\gamma_8} \frac{1}{(2d)^{|\gamma_8|}} \sum_{y_6}\sum_{\gamma_9} \frac{1}{(2d)^{|\gamma_9|}} \sum_{\gamma_{10}} \frac{1}{(2d)^{|\gamma_{10}|}} \\
&= \sum_{w_6} \mathbb{P}_{w_0}\big[w_0 \xrightarrow{8} w_6\big] \sum_{y_6} \mathbb{P}_{w_6}\big[w_6 \xrightarrow{9} y_6\big] \mathbb{P}_{y_6}\big[y_6 \xrightarrow{10} y_0\big] \qquad \text{(A.23)} \\
&\geq \frac{c}{h^{d-2}} \sum_{w_6} \mathbb{P}_{w_0}\big[w_0 \xrightarrow{8} w_6\big] \sum_{y_6} \mathbb{P}_{y_6}\big[y_6 \xrightarrow{10} y_0\big],
\end{aligned}$$

where we bounded $\mathbb{P}_{w_6}\big[w_6 \xrightarrow{9} y_6\big]$ from below by ch^{d-2} using a Green's function estimate and the fact that the distance of both w_6 and y_6 from ∂A_2 has order h.

Due to the simple random walk's reversibility, we obtain

$$\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \geq \frac{c}{h^{d-2}} \mathbb{P}_{w_0}\big[H_{\partial \hat{B}_{w_5}} < H_{\partial A_2}\big] \mathbb{P}_{y_0}\big[H_{\partial \hat{B}_{y_5}} < H_{\partial A_2}\big]. \qquad \text{(A.24)}$$

The bound for $\mathbb{P}_{w_0}\big[H_{\partial\hat{B}_{w_5}} < H_{\partial A_2}\big]$ and the bound for $\mathbb{P}_{y_0}\big[H_{\partial\hat{B}_{y_5}} < H_{\partial A_2}\big]$ are obtained in the same way. All we need is the following inequality:

$$\mathbb{P}_{w_0}\big[H_{\partial\hat{B}_{w_5}} < H_{\partial A_2}\big] \geq c \prod_{1\leq i\leq d_0} \frac{w_0^i}{h}, \qquad (A.25)$$

but again this bound follows from standard arguments from random walk theory, where we essentially use the independence of the random walk projections, the gambler's ruin estimate (Section 5.1 of Lawler and Limic, 2010) and the central limit theorem in order to construct an event where all random walk projections hit the respective projections of $\partial\hat{B}_{w_5}$ at around the same time without ever hitting the projections of ∂A_2. We will therefore omit the details.

Together with (A.24), (A.25) implies, using the general notation,

$$ch^{-(d-2)}\frac{l_1^{w_0}\ldots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{l_1^{y_0}\ldots l_{2d}^{y_0}}{h^{2d}} \leq \mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \leq c'h^{-(d-2)}\frac{l_1^{w_0}\ldots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{l_1^{y_0}\ldots l_{2d}^{y_0}}{h^{2d}}.$$
$$(A.26)$$

$\boldsymbol{w \xrightarrow{4} y}$: We suppose $h_4 > 100s$, since (2.1) implies the case when $h_4 < 100s$. We will start with the lower bound. Let B_1 be a discrete Euclidean ball of radius s contained in A_2^C such that $\partial A_2 \cap B_1 = \{y\}$. Let B_2 be a discrete Euclidean ball of radius $\frac{s}{4}$ concentric with B_1. Then, the probability that the walk started at y hits B_2 before returning to ∂A_2 is bigger than the probability that it hits B_2 before returning to B_1, and has order s^{-1}. Now, for every point $\tilde{y} \in \partial B_2$, we bound $\mathbb{P}_w\big[w \xrightarrow{2'} \tilde{y}\big]$ from below in exactly the same way as we bounded $\mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big]$. So that, using the walk's reversibility, the fact that $h_4 > 100s$, and the same notation introduced above, we have

$$\mathbb{P}_w\big[w \xrightarrow{4} y\big] \geq \sum_{\tilde{y}\in\partial B_2} \mathbb{P}_w\big[w \xrightarrow{2'} \tilde{y}\big]\mathbb{P}_y\big[H_{\partial B_2} < H_{\partial A_2}, X_{H_{\partial B_2}} = \tilde{y}\big]$$

$$\geq cs^{-1}\inf_{\tilde{y}\in\partial B_2}\mathbb{P}_w\big[w \xrightarrow{2'} \tilde{y}\big]$$

$$\geq cs^{-1}h_4^{-(d-2)}\inf_{\tilde{y}\in\partial B_2}\frac{l_1^{w_0}\ldots l_{2d}^{w_0}}{h_4^{2d}} \cdot \frac{l_1^{\tilde{y}}\ldots l_{2d}^{\tilde{y}}}{h_4^{2d}}.$$

For the upper bound, let B_1' be a discrete Euclidean ball of radius s contained in $A_2 \cup \partial A_2$ such that $\partial A_2 \cap B_1' = \{y\}$. Let B_2' be a discrete Euclidean ball of radius $2s$ concentric with B_1'. Then

$$\mathbb{P}_w\big[w \xrightarrow{4} y\big] \leq \sum_{\hat{y}\in\partial B_2'} \mathbb{P}_w\big[w \xrightarrow{2'} \hat{y}\big]\mathbb{P}_y\big[H_{\partial B_2'} < H_{\partial A_2}, X_{H_{\partial B_2'}} = \hat{y}\big]$$

$$\leq cs^{-1}\inf_{\hat{y}\in\partial B_2'}\mathbb{P}_w\big[w \xrightarrow{2'} \hat{y}\big]$$

$$\leq cs^{-1}h_4^{-(d-2)}\sup_{\hat{y}\in\partial B_2'}\frac{l_1^{w_0}\ldots l_{2d}^{w_0}}{h_4^{2d}}\frac{l_1^{\hat{y}}\ldots l_{2d}^{\hat{y}}}{h_4^{2d}}.$$

We have proved the following proposition:

Proposition A.2. *Regarding the sets $A_1^\square, V^\square$ and $A_2^\square$, we have that, using the notation defined above, for some constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 > 0$, the*

following bounds are valid:

$$c_1 e^{\frac{-c_2 h_1}{s}} s^{-(d-1)} \le \mathbb{P}_w\big[w \xrightarrow{1} w_0\big] \le c_2 e^{\frac{-c_2 h_1}{s}} s^{-(d-1)},$$

$$c_3 e^{\frac{-c_4 h_3}{s}} s^{-(d-1)} s^{-1} \le \mathbb{P}_w\big[y \xrightarrow{3} y_0\big] \le c_4 e^{\frac{-c_4 h_3}{s}} s^{-(d-1)} s^{-1},$$

$$c_5 h^{-(d-2)} \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \ldots l_{2d}^{y_0}}{h^{2d}} \le \mathbb{P}_{w_0}\big[w_0 \xrightarrow{2'} y_0\big] \le c_6 h^{-(d-2)} \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \ldots l_{2d}^{y_0}}{h^{2d}},$$

$$c_7 s^{-1} h_4^{-(d-2)} \inf_{\tilde{y} \in \partial B_2} \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h_4^{2d}} \frac{l_1^{\tilde{y}} \ldots l_{2d}^{\tilde{y}}}{h_4^{2d}} \le \mathbb{P}_{w_0}\big[w \xrightarrow{4} y\big]$$

$$\le c_8 s^{-1} h_4^{-(d-2)} \sup_{\hat{y} \in \partial B_2'} \frac{l_1^{w_0} \ldots l_{2d}^{w_0}}{h_4^{2d}} \frac{l_1^{\hat{y}} \ldots l_{2d}^{\hat{y}}}{h_4^{2d}}.$$

$$\sup_{\substack{w \in V \\ y \in \partial A_2}} \mathbb{P}_w\big[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y\big] \le c_9 s^{-2(d-1)}.$$

A.2. *Proof of Lemma 4.5.* Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0)$, and again let h stand for the Euclidean distance between w_0 and y_0. We let $\pi(w_0, y_0)$ be defined in the same way as in (4.5). Given a simple random walk trajectory ϱ started in a set B containing V, we define $\mathcal{C}_{w_0, y_0}^B(\varrho)$ to be the function that counts how many times the random walk trajectory ϱ makes an excursion on A_2^C that enters A_1 at w_0, and y_0 is the last point such excursion visits on V before reaching ∂A_2. We let $\mathcal{C}_{w_0, y_0}^B$ denote the random variable $\mathcal{C}_{w_0, y_0}^B(\bar{\varrho})$ when $\bar{\varrho}$'s first point is chosen according to $\bar{e}_B$. Proposition 3.3 then implies

$$\pi(w_0, y_0) = \mathbb{E}(\mathcal{C}_{w_0, y_0}^V).$$

Define $\tilde{V} := \partial B(0, 3(r + s))$, the discrete sphere of radius $3(r + s)$. We define

$$\tilde{\pi}(w_0, y_0) := \mathbb{E}(\mathcal{C}_{w_0, y_0}^{\tilde{V}}).$$

From the compatibility of the laws defined in (2.3), one can see that (see also the proof of Lemma 6.2 of Popov and Teixeira, 2015):

$$u \operatorname{cap}(\tilde{V}) \mathbb{E}(\mathcal{C}_{w_0, y_0}^{\tilde{V}}) = u \operatorname{cap}(V) \mathbb{E}(\mathcal{C}_{w_0, y_0}^V).$$

Since $\operatorname{cap}(\tilde{V}) \asymp \operatorname{cap}(V)$, if we successfully estimate $\tilde{\pi}(w_0, y_0)$ we will automatically be provided with an estimate for $\pi(w_0, y_0)$. We changed the problem from estimating $\pi(w_0, y_0)$ to estimating $\tilde{\pi}(w_0, y_0)$ so that the distance between the simple random walk's starting point and w_0 does not affect our calculations.

First we note that $\mathcal{C}_{w_0, y_0}^{\tilde{V}}$ is dominated by a Geometric (c_1) random variable, for some $0 < c_1 < 1$. This follows from the fact that every time the simple random walk exits A_2^C, with probability uniformly greater than some constant $1 - c_1 > 0$, the walk never returns to w_0. This way, it will be sufficient to estimate the probability $\mathcal{P}[\mathcal{C}_{w_0, y_0}^{\tilde{V}} \ge 1]$ for our purposes.

So, for a walk started at $\tilde{V}$ to reach w_0, it first has to hit a discrete sphere ∂B_1 of radius $\frac{s}{2}$ centered on w_0. The probability of such event is of order $\frac{s^{d-2}}{r^{d-2}}$, by Proposition 6.4.2 of Lawler and Limic (2010).

Let B_2 be a discrete Euclidean ball of radius s contained in A_1 such that $B_2 \cap A_1 = \{w_0\}$. We also let B_3 be a discrete Euclidean ball of radius $2s$ lying outside

A_1 such that $B_3 \cap A_1 = \{w_0\}$. Using Proposition 6.5.4 of Lawler and Limic (2010) we have, for any $x' \in \partial B_1 \cap A_!^C$ and some constant $c_2 > 0$:

$$\mathbb{P}_{x'}\left[X_{H_{A_1}} = w_0\right] \leq \mathbb{P}_{x'}\left[X_{H_{B_2}} = w_0\right] \leq c_2 s^{-(d-1)}.$$

Then, recalling the notation $f_{A_1}(w_0, y_0) := \mathbb{P}_{w_0}\left[w_0 \xrightarrow{2'} y_0\right]$ and the fact that $\operatorname{cap}(V) \asymp r^{(d-2)}$, and using the strong Markov property, we get, for constants $c, c_1 > 0$:

$$\pi(w_0, y_0) \leq c\mathcal{P}[\mathcal{C}_{w_0,y_0}^{\tilde{V}} \geq 1] \leq c_1 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0). \qquad (A.27)$$

For the lower bound, we let B_4 be a discrete Euclidean ball of radius $\frac{s}{4}$ contained in $A_2^C \setminus B(0, r+s)$ such that for every $x \in B_4$, $\operatorname{dist}(x, w_0) \leq 2s$. Using the strong Markov property, we get

$$\mathcal{P}\left[\mathcal{C}_{w_0,y_0}^{\tilde{V}} \geq 1\right] \geq \inf_{x \in \tilde{V}} \mathbb{P}_x\left[H_{B_4} < \infty\right] \inf_{x'' \in B_4} \mathbb{P}_x\left[X_{B_3} = w_0\right] f_{A_1}(w_0, y_0),$$

so that, using Proposition 6.4.2 of Lawler and Limic (2010) we have, for some constant $c_3 > 0$,

$$\pi(w_0, y_0) \geq c_3 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0).$$

The part (ii) then follows from (i) and Proposition 3.4.

A.3. *A lower bound for α.* Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0)$, let $c_4 > 0$ be some positive real number. For

$$\Gamma_{w_0,y_0} := \{(w_0', y_0') \in V \times \partial A_2; \max\{||w_0' - w_0||, ||y_0' - y_0||\} \leq c_4 s\}$$

and

$$\alpha := \inf\left\{\frac{g_{(w,y)}(z')}{g_{(w,y)}(\hat{z})}; (w,y) \in V \times \partial A_2, z' \in \Gamma_{w_0,y_0}, \hat{z} \in \mathcal{K}\right\},$$

we need to find a constant lower bound for α. But this is just a consequence of the bounds obtained in Propositions A.1 and A.2.

Acknowledgments

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