ALEA, Lat. Am. J. Probab. Math. Stat. 18, 129–165 (2021)
DOI: 10.30757/ALEA.v18-07

Non universality of fluctuations of outlier eigenvectors for block diagonal deformations of Wigner matrices

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Abstract. In this paper, we investigate the fluctuations of a unit eigenvector associated to an outlier in the spectrum of a spiked $N \times N$ complex Deformed Wigner matrix M_N. M_N is defined as follows: $M_N = W_N/\sqrt{N} + A_N$ where W_N is an $N \times N$ Hermitian Wigner matrix whose entries have a law μ satisfying a Poincaré inequality and the matrix A_N is a block diagonal matrix, with an eigenvalue θ of multiplicity one, generating an outlier in the spectrum of M_N. We prove that the fluctuations of the norm of the projection of a unit eigenvector corresponding to the outlier of M_N onto a unit eigenvector corresponding to θ are not universal. Indeed, we take away a fit approximation of its limit from this norm and prove the convergence to zero as N goes to ∞ of the Lévy–Prohorov distance between this rescaled quantity and the convolution of μ and a centered Gaussian distribution (whose variance may depend depend upon N and may not converge).

1. Introduction

To begin with, we introduce some notations.

- $M_N(\mathbb{C})$ is the set of $N \times N$ matrices with complex entries, $M_N^{sa}(\mathbb{C})$ the subset of self-adjoint elements of $M_N(\mathbb{C})$ and I_N the identity matrix.
- Tr_N denotes the trace and $\mathrm{tr}_N = \frac{1}{N}\mathrm{Tr}_N$ the normalized trace on $M_N(\mathbb{C})$.

Received by the editors October 11th, 2019; accepted October 27th, 2020.
2010 *Mathematics Subject Classification.* 15A18, 15A52, 60F05, 60B12, 46L54.
Key words and phrases. Random matrices; Outliers; eigenvectors; Fluctuations; Nonuniversality, Free probability.

- $\|.\|$ denotes the operator norm on $M_N(\mathbb{C})$.
- For any $X \in M_N^{sa}(\mathbb{C})$, $(\lambda_1(X), \ldots, \lambda_N(X))$ denote the eigenvalues of X ranked in decreasing order and the empirical spectral measure of X is defined by

$$\mu_X := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(X)}.$$

- For a probability measure τ on $\mathbb{R}$, $\mathrm{supp}(\tau)$ denotes the support of τ and $g_\tau : z \in \mathbb{C} \setminus \mathrm{supp}(\tau) \mapsto \int \frac{1}{z-x} d\tau(x)$ is the Stieltjes transform of τ.
- d_{LP} denotes the Lévy-Prohorov distance, which is a metric for the topology of the convergence in distribution.

1.1. *Wigner matrices.* Wigner matrices are complex Hermitian random matrices whose entries are independent (up to the symmetry condition). They were introduced by Wigner in the fifties, in connection with nuclear physics. Here, we will consider Hermitian Wigner matrices of the following form :

$$X_N = \frac{1}{\sqrt{N}} W_N$$

where W_N is an Hermitian matrix, $\{W_{ii}, \sqrt{2}\mathcal{R}W_{ij}, \sqrt{2}\mathcal{I}W_{ij}\}_{1\le i<j}$ are independent identically distributed random variables with law μ, with mean zero and variance σ^2. If the entries are independent Gaussian variables, $X_N =: X_N^G$ is a matrix from the Gaussian Unitary Ensemble (G.U.E.).

There is currently a quite precise knowledge of the asymptotic spectral properties (i.e. when the dimension of the matrix tends to infinity) of Wigner matrices. This understanding covers both the so-called global regime (asymptotic behavior of the spectral measure) and the local regime (asymptotic behavior of the extreme eigenvalues and eigenvectors, spacings...). Wigner proved that a precise description of the limiting spectrum of these matrices can be achieved.

Theorem 1.1. *Wigner (1955, 1958)*

$$\mu_{X_N} \xrightarrow{w} \mu_{sc} \ a.s. \ when \ N \to +\infty$$

where

$$\frac{d\mu_{sc}}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma,2\sigma]}(x) \tag{1.1}$$

is the so-called semi-circular distribution.

A priori, the convergence of the spectral measure does not prevent an asymptotically negligeable fraction of eigenvalues from going away from the limiting support (called *outliers* in the following). Actually, it turns out that Wigner matrices do not exhibit outliers.

Theorem 1.2. *Bai and Yin (1988) Assume that the entries of W_N has finite fourth moment, then almost surely,*

$$\lambda_1(X_N) \to 2\sigma \ and \ \lambda_N(X_N) \to -2\sigma \ when \ N \to +\infty.$$

In Tracy and Widom (1994), Tracy and Widom derived the limiting distribution (called the Tracy-Widom law) of the largest eigenvalue of a G.U.E. matrix.

Theorem 1.3. *Let $q : \mathbb{R} \to \mathbb{R}$ be the unique solution of the differential equation*

$$q''(x) = xq(x) + 2q(x)^3$$

such that $q(x) \sim_{x\to+\infty} Ai(x)$ where Ai is the Airy function, unique solution on $\mathbb{R}$ of the differential equation $f''(x) = xf(x)$ satisfying $f(x) \sim (4\pi\sqrt{x})^{1/2}\exp(-2/3x^{3/2})$ at $+\infty$. Then

$$\lim_{N\to+\infty} \mathbb{P}\left(\frac{N^{2/3}}{\sigma}\left(\frac{\lambda_1(X_N^G)}{\sqrt{N}} - 2\sigma\right) \leq s\right) = F_2(s),$$

where $F_2(s) = \exp\left(-\int_s^{+\infty}(x-s)q^2(x)dx\right)$.

The first main step to prove the universality conjecture for fluctuations of the largest eigenvalue of Wigner matrices has been achieved by Soshnikov (1999); in Lee and Yin (2014), a necessary and sufficient condition on off-diagonal entries of the Wigner matrix is established for the distribution of the largest eigenvalue to weakly converge to the Tracy-Widom distribution. We also refer to these papers for references on investigations on edge universality.

In regards to eigenvectors, it is well known that the matrix whose columns are the eigenvectors of a G.U.E. matrix can be chosen to be distributed according to the Haar measure on the unitary group. In the non-Gaussian case, the exact distribution of the eigenvectors cannot be computed. However, the eigenvectors of general Wigner matrices have been the object of a growing interest and in several papers, a delocalization and universality property were shown for the eigenvectors of these standard models (see among others Bloemendal et al., 2014; Erdős et al., 2009a,b; Knowles and Yin, 2013; Tao and Vu, 2012 and references therein). Heuristically, delocalization for a random matrix means that its normalized eigenvectors look like the vectors uniformly distributed over the unit sphere. Let us state for instance the following sample result.

Theorem 1.4. *(Isotropic delocalization, Theorem 2.16 from Bloemendal et al., 2014). Let X_N be a $N \times N$ Wigner matrix satisfying some technical assumptions. Let $v(1), \ldots, v(N)$ denote the normalized eigenvectors of X_N. Then, for any $C_1 > 0$ and $0 < \epsilon < 1/2$, there exists $C_2 > 0$ such that*

$$\sup_{1\leq i\leq N} |\langle v(i), u\rangle| \leq \frac{N^\epsilon}{\sqrt{N}},$$

for any fixed unit vector $u \in \mathbb{C}^N$, with probability at least $1 - C_2N^{-C_1}$.

1.2. *Deformed Wigner matrices.* Practical problems (in the theory of statistical learning, signal detection etc.) naturally lead to wonder about the spectrum reaction of a given random matrix after a deterministic perturbation. In those applications, the random matrix is the noise and the perturbed matrix is a noisy version of the information; the question is to know whether the observation of the spectral properties of the perturbed matrix can give access to significant parameters on the information. Theoretical results on these deformed random models may allow to establish statistical tests on these parameters. A typical illustration is the so-called BBP phenomenon (after Baik, Ben Arous, Péché) which put forward outliers (eigenvalues that move away from the rest of the spectrum) and their Gaussian fluctuations for spiked covariance matrices in Baik et al. (2005) and for low rank deformations of G.U.E. in Péché (2006).

In this paper, we consider additive perturbations of Wigner matrices. The pionner works on additive deformations go back to Pastur (1972) for the behavior of the limiting spectral distribution and to Füredi and Komlós (1981) for the behavior of the largest eigenvalue.

We refer to Capitaine and Donati-Martin (2017) and the references therein for a survey on spectral properties of deformed random matrices.

The model studied is as follows:

$$M_N := \frac{W_N}{\sqrt{N}} + A_N, \tag{1.2}$$

where

(W) W_N is a complex Wigner matrix, that is a $N \times N$ random Hermitian matrix such that $\{W_{ii}, \sqrt{2}\mathcal{R}W_{ij}, \sqrt{2}\mathcal{I}W_{ij}\}_{1\leq i<j}$ are independent identically distributed random variables with law μ. We assume that μ is a distribution with mean zero, variance σ^2, and satisfies a Poincaré inequality (see Appendix). Note that this condition implies that μ has moments of any order (see Corollary 3.2 and Proposition 1.10 in Ledoux, 2001).

(A) A_N is a $N \times N$ deterministic Hermitian matrix, whose spectral measure μ_{A_N} converges to a compactly supported probability measure ν. We assume that A_N has a fixed number q of eigenvalues, not depending on N, outside the support of ν called spikes, whereas the distance of the other eigenvalues to the support of ν goes to 0.

The empirical spectral distribution μ_{M_N} converges a.s. towards the probability measure $\lambda := \mu_{sc} \boxplus \nu$ where μ_{sc} is the semicircular distribution with variance σ^2 and $\boxplus$ denotes the free convolution, see Pastur (1972) (in this paper, the limiting distribution is given via a functional equation for its Stieltjes transform), Anderson et al. (2010, Theorem 5.4.5). We refer to Voiculescu et al. (1992); Mingo and Speicher (2017) for an introduction to free probability theory.

Concerning extremal eigenvalues, Capitaine et al. (2011) proved that the spikes of A_N can generate outliers for the limiting spectrum of M_N, i.e. eigenvalues outside the support of the limiting distribution λ. More precisely, Capitaine et al. (2011) proved the following (see Capitaine et al., 2011, Theorem 8.1 for a more general statement).

Proposition 1.5. *Capitaine et al. (2011) Assume that a spike θ with a fixed multiplicity k_0 in the spectrum of A_N satisfies :*

$$\theta \in \Theta_{\sigma,\nu} := \{u \in \mathbb{R}\backslash \mathrm{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\}. \tag{1.3}$$

Denote by $n_0+1, \dots, n_0+k_0$ the descending ranks of θ among the eigenvalues of A_N. Then the k_0 eigenvalues $(\lambda_{n_0+i}(M_N),\ 1 \leq i \leq k_0)$ converge almost surely outside the support of λ towards $\rho_\theta := \theta + \sigma^2 g_\nu(\theta)$. Moreover, these eigenvalues asymptotically separate from the rest of the spectrum since (with the conventions that $\lambda_0(M_N) = +\infty$ and $\lambda_{N+1}(M_N) = -\infty$) there exists $0 < \delta_0$ such that almost surely for all large N,

$$\lambda_{n_0}(M_N) > \rho_\theta + \delta_0 \ \ and \ \ \lambda_{n_0+k_0+1}(M_N) < \rho_\theta - \delta_0. \tag{1.4}$$

Note that Capitaine et al. (2011) assumes that the distribution μ is symmetric but this assumption can be removed. Indeed, this assumption is used for establishing Theorem 5.1 in Capitaine et al. (2011) that is now generalized by Theorem 1.1 in Belinschi and Capitaine (2017). Note also that Capitaine et al. (2011) assumes moreover that the support of ν has a finite number of connected components in order to prove Theorem 6.1 in Capitaine et al. (2011) but this assumption is removed in Theorem 2.3 in Capitaine and Péché (2016).

Remark 1.6. Note that

$$\{u \in \mathbb{R}\backslash \mathrm{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\} = \{u \in \mathbb{R}\backslash \mathrm{supp}(\nu), H'(u) > 0\}$$

and for any θ in this set, $\rho_\theta = \theta + \sigma^2 g_\nu(\theta) = H(\theta)$, where H is defined by (2.14) (see Section 2.4 below).

It turns out that we can also describe the angle between the eigenvector associated to the outlier of M_N and the corresponding eigenvector associated to the spike θ. Capitaine (2013) (see also Capitaine and Donati-Martin, 2017) proved

Proposition 1.7. *Capitaine (2013) We keep the notation and hypothesis of Proposition 1.5. Let ξ be a unit eigenvector associated to one of the eigenvalues $(\lambda_{n_0+i}(M_N),\ 1 \leq i \leq k_0)$. Then, a.s.*

$$\|P_{Ker(A_N-\theta I)}(\xi)\|^2 \longrightarrow_{N\to+\infty} \tau(\theta) := 1 - \sigma^2 \int \frac{1}{(\theta-x)^2} d\nu(x), \tag{1.5}$$

where P_E denotes the orthogonal projection onto any subspace E.

Note that fluctuations of outliers for deformed non-Gaussian Wigner matrices have been more extensively studied in the case of perturbations A_N of fixed rank r. We emphasize that the limiting distribution in the CLT for outliers depends on the localisation/delocalisation of the eigenvector of the spike. Roughly speaking, in the delocalized case, the limiting distribution of the fluctuations of the correponding outliers is Gaussian. In the localized case, the limiting distribution depends on the distribution μ of the entries and thus, this uncovers a non universality phenomenon. We refer to Capitaine et al. (2012) for these results.

We first recall the fluctuations of the largest eigenvalue $\lambda_1(M_N)$ when the matrix A_N is a diagonal matrix of rank 1 in the localized case.

Proposition 1.8. *Capitaine et al. (2009) Assume that $A_N = diag(\theta, 0, \ldots, 0)$ with $\theta > \sigma$. The fluctuations of $\lambda_1(M_N)$ around $\rho_\theta = \theta + \frac{\sigma^2}{\theta}$ are given by*

$$c_\theta \sqrt{N}(\lambda_1(M_N) - \rho_\theta) \overset{(law)}{\underset{N\to\infty}{\longrightarrow}} \mu \star N(0, v_\theta^2)$$

where $c_\theta = (1-\frac{\sigma^2}{\theta^2})^{-1}$, $v_\theta^2 = \frac{1}{2}\frac{m_4-3\sigma^4}{\theta^2} + \frac{\sigma^4}{\theta^2-\sigma^2}$ and m_4 denotes the fourth moment of μ.

Capitaine and Péché (2016) proved a fluctuation result for any outlier of a full rank deformation of a G.U.E. matrix. Their result yields the following

Proposition 1.9. *Capitaine and Péché (2016) Assume that W_N is a G.U.E. matrix (that is $\mu = \mathcal{N}(0, \sigma^2)$) and that A_N is a diagonal matrix with a spike $\lambda_{i_0}(A_N) =$*

$\theta \in \Theta_{\sigma,\nu}$ of multiplicity one and limiting spectral distribution ν. The fluctuations of $\lambda_{i_0}(M_N)$ around

$$\rho_\theta^{(N)} = \theta + \sigma^2 \frac{1}{N-1} \sum_{\lambda_j(A_N)\neq\theta} \frac{1}{\theta - \lambda_j(A_N)}$$

are given by[1]*:*

$$c_{\theta,\nu}\sqrt{N}(\lambda_1(M_N) - \rho_\theta^{(N)}) \xrightarrow[N\to\infty]{(law)} N(0, \sigma^2_{\theta,\nu})$$

where $c_{\theta,\nu} = \left(1 - \sigma^2 \int \frac{1}{(\theta-x)^2} d\nu(x)\right)^{-1}$ *and*

$$\sigma^2_{\theta,\nu} \quad = \quad \sigma^2 \left(1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\nu(x)\right)^{-1}.$$

1.3. *Main results.* In the following, we consider block diagonal perturbations. We focus on spikes of the perturbation with multiplicity one generating an outlier in the spectrum of the deformed Wigner model. Therefore, we shall consider the following assumption on A_N throughout the paper:

(A') *A_N satisfies* **(A)** *and*

$$A_N = \mathrm{diag}(A_p, A_{N-p}),$$

with A_{N-p} a $(N-p)\times(N-p)$ Hermitian matrix for some fixed integer p, $A_p = PDP^*$ is a fixed matrix (independent of N) where P a $p\times p$ unitary matrix and D is a diagonal matrix.

Assume that A_N has a spike of multiplicity one, which is an eigenvalue of A_p, $\theta = \lambda_{i_0}(A_N)$ for some i_0, satisfying (1.3). Without loss of generality, we can assume that $\theta = D_{11}$.

We set

$$W_N = \begin{pmatrix} W_p & Y^* \\ Y & W_{N-p} \end{pmatrix},$$

where $W_p \in M_p(\mathbb{C})$, $Y \in M_{(N-p)\times p}(\mathbb{C})$ and $W_{N-p} \in M_{N-p}(\mathbb{C})$.

The main results of this paper are the following Theorems 1.11 and 1.12 on non universality of fluctuations of an eigenvector associated with such an outlier. But, sticking to the approach of Capitaine (2020), we first establish Theorem 1.10 below, which is an extension in the non-Gaussian case of Proposition 1.8 and Proposition 1.9, in the block diagonal case.

Let M_N be defined by (1.2) with assumptions **(W)** and **(A')**. By Proposition 1.5,

$$\lambda_{i_0}(M_N) \to_{N\to+\infty} \rho_\theta \text{ a.s..} \tag{1.6}$$

Define

$$\rho_N = \theta + \sigma^2 g_{\mu_{A_{N-p}}}(\theta). \tag{1.7}$$

Note that

$$\rho_N \to_{N\to+\infty} \rho_\theta. \tag{1.8}$$

[1]They consider fluctuations around this point depending on N in order to not prescribe speed of convergence of μ_{A_N} to ν.

Theorem 1.10. *Let M_N be defined by (1.2) with assumptions* **(W)** *and* **(A')**. *Define*

$$\mathbf{C}_p = \begin{cases} {}^t com(\theta I_p - A_p), \text{ if } \theta I_p - A_p \neq 0 \\ I_p \text{ else} \end{cases}, \tag{1.9}$$

where $com(B)$ denotes the comatrix of a matrix B, and

$$c_\rho^{(1)} = \left(1 + \sigma^2 \int \frac{d\lambda(x)}{(\rho_\theta - x)^2}\right) \mathrm{Tr}_p(\mathbf{C}_p).$$

Then

$$d_{LP}(c_\rho^{(1)} \sqrt{N}(\lambda_{i_0}(M_N) - \rho_N), \Phi_N) \underset{N\to\infty}{\longrightarrow} 0 \tag{1.10}$$

where $\Phi_N = \mathrm{Tr}_p(\mathbf{C}_p W_p) + \mathcal{Z}_N$, W_p is the $p \times p$ upper left corner of the Wigner matrix W_N, $\mathcal{Z}_N$ is a Gaussian random variable, independent from W_p, with mean 0 and variance $v_\rho(N)$, with

$$v_\rho(N) = \mathrm{Tr}_p(\mathbf{C}_p^2)\sigma^4 \int \frac{d\lambda(x)}{(\rho_\theta - x)^2} + \frac{1}{2}(m_4 - 3\sigma^4)\kappa_N \sum_{i=1}^{p} ((\mathbf{C}_p)_{ii})^2,$$

$\lambda = \mu_{sc} \boxplus \nu$, $\kappa_N = \frac{1}{N-p}\sum_{i=1}^{N-p}(((\theta I_{N-p} - A_{N-p})^{-1})_{ii})^2$.
In particular, if A_{N-p} is diagonal, $c_\rho^{(1)}\sqrt{N}(\lambda_{i_0}(M_N) - \rho_N)$ converges in distribution to $\mathrm{Tr}_p(\mathbf{C}_p W_p) + \mathcal{Z}$ where $\mathcal{Z}$ is a Gaussian random variable, independent from W_p, with mean 0 and variance v_ρ, with

$$v_\rho = \mathrm{Tr}_p(\mathbf{C}_p^2)\sigma^4 \int \frac{d\lambda(x)}{(\rho_\theta - x)^2} + \frac{1}{2}(m_4 - 3\sigma^4) \int \frac{d\nu(x)}{(\theta - x)^2} \sum_{i=1}^{p} ((\mathbf{C}_p)_{ii})^2.$$

The aim of Theorems 1.11 and 1.12 below is to study the fluctuations associated to the a.s. convergence given in Proposition 1.7 above, for block diagonal perturbations. We first state an approximation result in distribution, in the spirit of Najim and Yao (2016) for perturbations A_N satisfying **(A')**.

Theorem 1.11. *Let M_N be defined by (1.2) with assumptions* **(W)** *and* **(A')**. *Let u_{i_0}, resp. v_{i_0} be a unit eigenvector associated to the spike θ of A_N, resp. the outlier $\lambda_{i_0}(M_N)$. Define $\tau_N(\theta)$ an approximation of $\tau(\theta)$ by*

$$\tau_N(\theta) = 1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\mu_{A_{N-p}}(x). \tag{1.11}$$

Then

$$d_{LP}(\sqrt{N}(|\langle u_{i_0}, v_{i_0}\rangle|^2 - \tau_N(\theta)), \Psi_N) \underset{N\to\infty}{\longrightarrow} 0$$

where the r.v. Ψ_N is given by

$$\Psi_N = (P^*(c_{\theta,\sigma} W_p + Z_{p,N})P)_{11},$$

where

$$c_{\theta,\sigma} = \sigma^2 g_\nu''(\theta), \tag{1.12}$$

W_p is the $p \times p$ upper left corner of the Wigner matrix W_N, $Z_{p,N}$ is a centered Gaussian Hermitian matrix of size p, independent from W_p, with independent entries (modulo the symmetry conditions). The diagonal coefficients are i.i.d. with variance

$$\sigma^4 B_{\theta,\nu} + \frac{1}{2}(m_4 - 3\sigma^4) A_{\theta,\nu,N} \tag{1.13}$$

where

$$B_{\theta,\nu} = -\frac{1}{6} g_\nu'''(\theta) - \frac{\sigma^2}{2} (g_\nu''(\theta))^2 \frac{1 + 2\sigma^2 g_\nu'(\theta)}{1 + \sigma^2 g_\nu'(\theta)}, \tag{1.14}$$

$A_{\theta,\nu,N} =$

$$\frac{1}{N-p} \sum_{i=1}^{N-p} \left(\sigma^2 g_\nu''(\theta)[(\theta I_{N-p} - A_{N-p})^{-1}]_{ii} - (1 + \sigma^2 g_\nu'(\theta))[(\theta I_{N-p} - A_{N-p})^{-2}]_{ii}\right)^2, \tag{1.15}$$

m_4 is the fourth moment of μ, g_ν is the Stieltjes transform of ν. The off diagonal elements $Z_{p,N}(i,j)$, $i < j$ are iid complex Gaussian with distribution Z such that $\mathbb{E}(Z^2) = 0$ and $\mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu}$.

In the case where the matrix A_{N-p} is a diagonal matrix, the sequence $(A_{\theta,\nu,N})_N$ defined in (1.15) converges as N tends to infinity. This leads to the following fluctuations result:

Theorem 1.12. *Let M_N be defined by (1.2) with assumptions* **(W)**, **(A')** *and A_{N-p} diagonal.*

Let u_{i_0}, resp. v_{i_0} be a unit eigenvector associated to the spike θ of A_N, resp. to the outlier $\lambda_{i_0}(M_N)$. Then,

$$\sqrt{N}(|\langle u_{i_0}, v_{i_0}\rangle|^2 - \tau_N(\theta)) \xrightarrow[N\to\infty]{(law)} (P^*(c_{\theta,\sigma} W_p + Z_p)P)_{11} \tag{1.16}$$

where $\tau_N(\theta)$ is defined by (1.11). W_p is the $p \times p$ upper left corner of the Wigner matrix W_N, Z_p is a centered Gaussian Hermitian matrix of size p, independent from W_p, with independent entries (modulo the symmetry condition). The diagonal coefficients are i.i.d. with variance

$$\frac{1}{2}(m_4 - 3\sigma^4) A_{\theta,\nu} + \sigma^4 B_{\theta,\nu} \tag{1.17}$$

and the off diagonal elements are iid complex Gaussian with distribution Z such that $\mathbb{E}(Z^2) = 0$ and $\mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu}$ where

$$\begin{cases} c_{\theta,\nu} = \sigma^2 g_\nu''(\theta), \\ A_{\theta,\nu} = -\frac{1}{6} g_\nu'''(\theta)(1 + \sigma^2 g_\nu'(\theta))^2 - 2\sigma^4 (g_\nu^{''}(\theta))^2 g_\nu'(\theta) - \sigma^2 (g_\nu^{''}(\theta))^2, \\ B_{\theta,\nu} = -\frac{1}{6} g_\nu'''(\theta) - \frac{\sigma^2}{2}(g_\nu''(\theta))^2 \frac{1 + 2\sigma^2 g_\nu'(\theta)}{1 + \sigma^2 g_\nu'(\theta)}, \end{cases} \tag{1.18}$$

m_4 is the fourth moment of μ and g_ν is the Stieltjes transform of ν.

In particular, we readily deduce the following corollary.

Corollary 1.13. *Let M_N be defined by (1.2) with assumptions* **(W)** *and* **(A')**. *Assume moreover that A_N is a diagonal matrix ($p = 1$). Let u_{i_0}, resp. v_{i_0} be a unit eigenvector associated to the spike θ of A_N, resp. the outlier $\lambda_{i_0}(M_N)$. Then,*

$$\sqrt{N}(|\langle u_{i_0}, v_{i_0}\rangle|^2 - \tau_N(\theta)) \xrightarrow[N\to\infty]{(law)} c_{\theta,\nu} W_{11} + Z \tag{1.19}$$

where Z is a centered Gaussian variable, independent from W_{11}, with variance :

$$\frac{1}{2}(m_4 - 3\sigma^4) A_{\theta,\nu} + \sigma^4 B_{\theta,\nu}. \tag{1.20}$$

See Eq.(1.18) for the definitions of $c_{\theta,\sigma}$, $A_{\theta,\nu}$, $B_{\theta,\nu}$.

Note that when the Wigner matrix is Gaussian, the choice of a diagonal matrix for A_N is not a restriction, due to the unitary invariance of the G.U.E..

The proof of Theorem 1.11 relies upon a representation, through Helffer-Sjöstrand formula, of the variable $|\langle u_{i_0}, v_{i_0}\rangle|^2$ in terms of the $p \times p$-matrix valued process $\{G_p(z), z \in \mathbb{C}\backslash\mathbb{R}\}$ where $G_p(z)$ denotes the principal submatrix of size p of the resolvant matrix $G(z) = (zI_N - M_N)^{-1}$. Then, the fluctuations of the process $\{G_p(z), z \in \mathbb{C}\backslash\mathbb{R}\}$ are analysed using Schur's formula which enables to express $\{G_p(z), z \in \mathbb{C}\backslash\mathbb{R}\}$ in terms of random sesquilinear forms. This approach is described in Section 4 where we also prove Theorem 1.12. Section 2 gathers preliminary results used in the proof of the main results. Therein, first we recall classical algebraic identities, Helffer-Sjöstrand's calculus, tightness criterion for random analytic process and some basic facts on free convolution with a semicircular distribution; later we establish some extension of central limit theorem for random quadratic forms and recall or deduce some results on deformed Wigner matrices from Belinschi and Capitaine (2017); Capitaine (2020). In Section 3, we first establish Theorem 1.10, sticking to the approach of Capitaine (2020). The last Section is an appendix reminding the reader about Poincaré inequality and concentration phenomenon.

For any integer number k, we will say that a matrix-valued function f_N on $\mathbb{C}\setminus\mathbb{R}$ is $O\left(\frac{1}{N^k}\right)$ if there exists a polynomial Q with nonnegative coefficients and an integer number d such that for all large N, for any z in $\mathbb{C}\setminus\mathbb{R}$,

$$\|f_N(z)\| \leq \frac{Q(|\Im z|^{-1})(|z|+1)^d}{N^k}.$$

For a family of functions $f_N^{(i)}$, $i \in \{1,\dots,N\}^2$, we will set $f_N^{(i)} = O^{(u)}\left(\frac{1}{N^k}\right)$ if for each i, $f_N^{(i)} = O\left(\frac{1}{N^k}\right)$ and moreover one can find a bound of the norm of each $f_N^{(i)}$ as above involving a common polynomial Q and a common d, that is not depending on i.

For two sequences $(X_N)_N$ and $(Y_N)_N$ of random variables, $X_N = Y_N + o_{\mathbb{P}}(1)$ means that $X_N - Y_N \longrightarrow_{N\to\infty} 0$ in probability.

2. Preliminaries

2.1. *Basic identities and inequalities.* For a matrix $A \in M_N(\mathbb{C})$ and I and J non empty subsets of $\{1,\dots,N\}$, we denote by $A_{I\times J}$ the submatrix of A obtained by keeping the rows with indices $i \in I$ and columns with indices $j \in J$. We set $A_I := A_{I\times I}$.

Proposition 2.1 (Schur inversion formula). *Let I be a non-empty subset of $\{1,\dots,N\}$ and $A \in M_N(\mathbb{C})$ such that A_I is invertible, then A is invertible if and only if $A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c}$ is invertible, in which case the following formulas hold:*

$$(A^{-1})_I = (A_I)^{-1} + (A_I)^{-1}A_{I\times I^c}(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}A_{I^c\times I}(A_I)^{-1},$$
$$(A^{-1})_{I\times I^c} = -(A_I)^{-1}A_{I\times I^c}(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1},$$
$$(A^{-1})_{I^c\times I} = -(A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}A_{I^c\times I}(A_I)^{-1},$$
$$(A^{-1})_{I^c} = (A_{I^c} - A_{I^c\times I}A_I^{-1}A_{I\times I^c})^{-1}.$$

Lemma 2.2. *For any matrix $B \in M_N(\mathbb{C})$ and for any fixed k, we have*

$$\sum_{l=1}^{N} |B_{lk}|^2 \leq \|B\|^2 \tag{2.1}$$

(or equivalently

$$\sum_{l=1}^{N} |B_{kl}|^2 \leq \|B\|^2.) \tag{2.2}$$

Therefore, we have

$$\frac{1}{N}\sum_{k,l=1}^{N} |B_{kl}|^2 \leq \|B\|^2. \tag{2.3}$$

Proof: Note that

$$\begin{aligned}\sum_{l=1}^{N} |B_{lk}|^2 &= \mathrm{Tr}_N(BE_{kk}B^*) \\ &= \mathrm{Tr}_N(B^*BE_{kk}) \\ &\leq \|B\|^2 \,\mathrm{Tr}_N(E_{kk}) = \|B\|^2.\end{aligned}$$

Now, since

$$\sum_{l=1}^{N} |B_{kl}|^2 = \sum_{l=1}^{N} |\overline{B_{kl}}|^2 = \sum_{l=1}^{N} |(B^*)_{lk}|^2 \text{ and } \|B^*\| = \|B\|,$$

(2.1) and (2.2) can be deduced from each other thanks to conjugate transposition. Finally (2.2) readily yields (2.3). □

Lemma 2.3 (Lemma A2 of Capitaine, 2020)**.** *Let A and H be $m \times m$ matrices such that, for some $K > 0$,*

$$\|A\| \leq K, \ \|H\| \leq K. \tag{2.4}$$

Then

$$\det(A+H) = \det(A) + \mathrm{Tr}_m\left({}^t com(A)H\right) + \epsilon,$$

where $com(A)$ denotes the comatrix of A, and there exists a constant $C_{m,K} > 0$, only depending on m and K, such that $|\epsilon| \leq C_{m,K}\|H\|^2$.

We will often use the following obvious facts that for any $z \in \mathbb{C} \setminus \mathbb{R}$, for any $N \times N$ Hermitian matrix H,

$$\| (zI_N - H)^{-1} \| \leq |\Im z|^{-1} \tag{2.5}$$

and for any probability measure ν on $\mathbb{R}$, the Stieltjes transform g_ν satisfies for any $z \in \mathbb{C} \setminus \mathbb{R}$, $\Im z \Im g_\nu(z) < 0$, $|g_\nu(z)| \leq |\Im z|^{-1}$, and for any $\alpha \geq 0$, any $x \in \mathbb{R}$,

$$\frac{1}{|z - \alpha g_\nu(z) - x|} \leq |\Im z|^{-1}. \tag{2.6}$$

2.2. *Helffer-Sjöstrand's calculus.*

2.2.1. *Helffer-Sjöstrand's representation formula.* We recall Helffer-Sjöstrand's representation formula (see Benaych-Georges and Knowles, 2017, Proposition C.1): let $f \in C^{k+1}(\mathbb{R})$ with compact support and M be a Hermitian matrix; we have

$$f(M) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} F_k(f)(z)\ (M-z)^{-1} d^2 z \tag{2.7}$$

where $d^2 z$ denotes the Lebesgue measure on $\mathbb{C}$,

$$F_k(f)(x+iy) = \sum_{l=0}^{k} \frac{(iy)^l}{l!} f^{(l)}(x) \chi(y) \tag{2.8}$$

where $\chi : \mathbb{R} \to \mathbb{R}^+$ is a smooth compactly supported function such that $\chi \equiv 1$ in a neighborhood of 0, and $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.

The function $F_k(f)$ coincides with f on the real axis and is an extension to the complex plane.

Moreover

$$\bar{\partial} F_k(f)(x+iy) = \frac{1}{2} \frac{(iy)^k}{k!} f^{(k+1)}(x) \chi(y) + \frac{i}{2} \sum_{l=0}^{k} \frac{(iy)^l}{l!} f^{(l)}(x) \chi^{'}(y). \tag{2.9}$$

Thus, since $\chi \equiv 1$ in a neighborhood of 0, we have that, in a neighborhood of the real axis,

$$\bar{\partial} F_k(f)(x+iy) = \frac{1}{2} \frac{(iy)^k}{k!} f^{(k+1)}(x) = O(|y|^k) \text{ as } y \to 0. \tag{2.10}$$

2.2.2. *Computation of Helffer-Sjöstrand's integral.*

Proposition 2.4. *Let h be a smooth function with compact support in $(\rho_\theta - 2\delta, \rho_\theta + 2\delta)$ and satisfying $h \equiv 1$ on $[\rho_\theta - \delta, \rho_\theta + \delta]$. Let χ be a compactly supported function on $(-L, L)$, and $\chi = 1$ around 0. We denote by $D = (\rho_\theta - 2\delta, \rho_\theta + 2\delta) \times (-L, L)$. Let ϕ be a meromorphic function in D, with a pole in ρ_θ. Then,*

$$I(\phi) := \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} F_k(h)(z)\ \phi(z) d^2 z = -\mathrm{Res}(\phi, \rho_\theta) \tag{2.11}$$

where $F_k(h)$ is defined in (2.8). $\mathrm{Res}(\phi, \rho_\theta)$ *denotes the residue of the function ϕ at the point ρ_θ.*

Proof: Let ϵ small enough such that $F_k(h)(z) = 1$ for $z \in B(\rho_\theta, \epsilon)$. Set $D_\epsilon = D \backslash B(\rho_\theta, \epsilon)$. ϕ is holomorphic on D_ϵ. Since $F_k(h)$ has compact support in D, we have,

$$\begin{aligned} 0 = \int_{\partial D} F_k(h)(z)\ \phi(z) dz &= \int_{\partial D_\epsilon} F_k(h)(z) \phi(z) dz + \int_{\partial B(\rho_\theta, \epsilon)} F_k(h)(z)\ \phi(z) dz \\ &= 2i \int_{D_\epsilon} \bar{\partial} F_k(h)(z)\ \phi(z) d^2 z + \int_{\partial B(\rho_\theta, \epsilon)} \phi(z) dz \end{aligned}$$

where the first term is obtained by Green's formula using that $\bar{\partial}\phi(z) = 0$ on D_ϵ.

$$\int_{D_\epsilon} \bar{\partial} F_k(h)(z)\ \phi(z) d^2 z \underset{\epsilon \to 0}{\longrightarrow} \pi I(\phi)$$

and

$$\int_{\partial B(\rho_\theta,\epsilon)} F_k(h)(z)\ \phi(z)dz = 2i\pi \mathrm{Res}(\phi,\rho_\theta).$$

□

2.3. *Tightness criterion for a sequence of random analytic processes.* We recall here some results from Shirai (2012). Let $D \subset \mathbb{C}$ be an open set in the complex plane. Denote by $\mathcal{H}(D)$ the space of complex analytic functions in D, endowed with the uniform topology on compact set. For $f \in \mathcal{H}(D)$ and K a compact set of D, we denote $\|f\|_K = \sup_{z\in K}|f(z)|$. The space $\mathcal{H}(D)$ is equipped with the (topological) Borel σ-field $\mathcal{B}(\mathcal{H}(D))$ and the set of probability measures on $(\mathcal{H}(D);\mathcal{B}(\mathcal{H}(D)))$ is denoted by $\mathcal{P}(\mathcal{H}(D))$. By a random analytic function on D we mean an $\mathcal{H}(D)$-valued random variable on a probability space. The probability law of a random analytic function is uniquely determined by its finite dimensional distributions.

Proposition 2.5 (Proposition 2.5. in Shirai, 2012)**.** *Let f_n be a sequence of random analytic functions in D. If $\|f_n\|_K$ is tight for any compact set K, then $\mathcal{L}(f_n)$ is tight in $\mathcal{P}(\mathcal{H}(D))$.*

Using that, by Markov's inequality, for any $C > 0$ and any $r > 0$,

$$\mathbb{P}\left(\|f_n\|_K > C\right) \leq \frac{1}{C^r}\mathbb{E}\left(\|f_n\|_K^r\right), \tag{2.12}$$

the following lemma turns out to be useful to prove tightness results.

Lemma 2.6 (lemma 2.6 of Shirai, 2012)**.** *For any compact set K in D, there exists $\delta > 0$ such that*

$$\|f\|_K^r \leq (\pi\delta^2)^{-1}\int_{\overline{K_\delta}} |f(z)|^r\, m(dz),\ f \in \mathcal{H}(D),$$

for any $r > 0$, where $\overline{K_\delta} \subset D$ is the closure of the δ-neighborhood of K and m denotes the Lebesgue measure.

2.4. *Free convolution with a semicircular distribution.* Let τ be a probability measure on $\mathbb{R}$. Its Stieltjes transform $g_\tau : z \mapsto \int_{\mathbb{R}} \frac{1}{z-x} d\tau(x)$ is analytic on the complex upper half-plane $\mathbb{C}^+$. There exists a domain

$$D_{\alpha,\beta} = \{u + iv \in \mathbb{C}, |u| < \alpha v, v > \beta\}$$

on which g_τ is univalent. Let K_τ be its inverse function, defined on $g_\tau(D_{\alpha,\beta})$, and

$$R_\tau(z) = K_\tau(z) - \frac{1}{z}.$$

Given two probability measures τ and ν, there exists a unique probability measure λ such that

$$R_\lambda = R_\tau + R_\nu$$

on a domain where these functions are defined. The probability measure λ is called the free convolution of τ and ν and denoted by $\tau \boxplus \nu$.

The free convolution of probability measures has an important property, called subordination, which can be stated as follows: let τ and ν be two probability measures on $\mathbb{R}$; there exists an analytic map $\omega_{\tau,\nu} : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$\forall z \in \mathbb{C}^+,\ g_{\tau\boxplus\nu}(z) = g_\nu(\omega_{\tau,\nu}(z)).$$

This phenomenon was first observed by D. Voiculescu under a genericity assumption in Voiculescu (1993), and then proved in generality in Biane (1998) Theorem 3.1. Later, a new proof of this result was given in Belinschi and Bercovici (2007), using a fixed point theorem for analytic self-maps of the upper half-plane.

In Biane (1997), P. Biane provides a deep study of the free convolution by a semicircular distribution. We first recall here some of his results that will be useful in our approach. Let ν be a probability measure on $\mathbb{R}$ and μ_{sc} the semicircular distribution defined by (1.1). For any $z \in \mathbb{C} \setminus \operatorname{supp}(\mu_{sc} \boxplus \nu)$, the subordination function $\omega_{\mu_{sc},\nu}$ is given by

$$\omega_{\mu_{sc},\nu}(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z). \tag{2.13}$$

In the following, we will denote $\omega_{\mu_{sc},\nu}$ by ω. For any $x \in \mathbb{C} \setminus \operatorname{supp}(\nu)$, define

$$H(x) = x + \sigma^2 g_\nu(x). \tag{2.14}$$

We have for any $x \in \mathbb{C} \setminus \operatorname{supp}(\mu_{sc} \boxplus \nu)$,

$$H(\omega(x)) = x. \tag{2.15}$$

More precisely, the following one to one correspondance holds:

$$\mathbb{R} \setminus \operatorname{supp}(\mu_{sc} \boxplus \nu) \underset{H}{\overset{\omega}{\rightleftarrows}} \{u \in \mathbb{R} \setminus \operatorname{supp} \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2}\}. \tag{2.16}$$

Assume that A_N satisfies **(A)**. Let $\{\theta_j,\, 1 \le j \le q\}$ be the spiked eigenvalue of A_N outside $\operatorname{supp}(\nu)$. Furthermore, for all $\theta_j \in \Theta_{\sigma,\nu}$, where $\Theta_{\sigma,\nu}$ is defined by (1.3), we set

$$\rho_{\theta_j} := H(\theta_j) = \theta_j + \sigma^2 g_\nu(\theta_j) \tag{2.17}$$

which is outside the support of $\mu_{sc} \boxplus \nu$ according to (2.16), and we define

$$K_{\sigma,\nu}(\theta_1, \ldots, \theta_q) := \operatorname{supp}(\mu_{sc} \boxplus \nu) \bigcup \left\{\rho_{\theta_j},\, \theta_j \in \Theta_{\sigma,\nu}\right\}.$$

An important consequence of (2.16) is the following

Proposition 2.7 (Theorem 2.3 in Capitaine and Péché, 2016)**.** *For any $\delta > 0$,*

$$\operatorname{supp}(\mu_{sc} \boxplus \mu_{A_N}) \subset K_{\sigma,\nu}(\theta_1, \ldots, \theta_q) + (-\delta, \delta),$$

when N is large enough.

2.5. *Central limit theorem for processes of matrix valued random quadratic forms.*

Lemma 2.8 (Lemma 2.7 in Bai and Silverstein, 1998)**.** *There exists $C > 0$ such that for any $N \times N$ deterministic matrix $B = (b_{ij})_{1 \le i,j \le N}$, any random vectors $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$ in $\mathbb{C}^N$ with i.i.d. standardized entries ($\mathbb{E}(y_i) = 0$, $\mathbb{E}(|y_i|^2) = 1$, $\mathbb{E}(y_i^2) = 0$) such that $\mathbb{E}(|y_i|^4) < \kappa$ and any independent copy X of Y, one has*

$$\mathbb{E}(|Y^* BY - \operatorname{Tr}_N(B)|^2) \le C\kappa \operatorname{Tr}_N(B^* B),$$

$$\mathbb{E}(|Y^* BX|^2) \le C\kappa \operatorname{Tr}_N(B^* B).$$

In Najim and Yao (2016), the authors establish the following variation around the central limit theorem for martingales.

Lemma 2.9 (Lemma 5.6 in Najim and Yao, 2016). *Suppose that for each n $(Y_{nj}; 1 \leq j \leq r_n)$ is a $\mathbb{C}^d$-valued martingale difference sequence with respect to the increasing σ-field $\{\mathcal{G}_{n,j}; 1 \leq j \leq r_n\}$ having second moments. Write*

$$Y_{nj}^T = (Y_{nj}^1, \dots, Y_{nj}^d).$$

Assume moreover that $(\Theta_n(k,l))_n$ and $(\tilde{\Theta}_n(k,l))_n$ are uniformly bounded sequences of complex numbers, for $1 \leq k, l \leq d$. If

$$\sum_{j=1}^{r_n} \mathbb{E}\left(Y_{nj}^k \bar{Y}_{nj}^l \mid \mathcal{G}_{n,j-1}\right) - \Theta_n(k,l) \xrightarrow[n\to+\infty]{P} 0, \tag{2.18}$$

$$\sum_{j=1}^{r_n} \mathbb{E}\left(Y_{nj}^k Y_{nj}^l \mid \mathcal{G}_{n,j-1}\right) - \tilde{\Theta}_n(k,l) \xrightarrow[n\to+\infty]{P} 0, \tag{2.19}$$

and for each $\epsilon > 0$, $\sum_{j=1}^{r_n} \mathbb{E}\left(\|Y_{nj}\|^2 \mathbf{1}_{\|Y_{nj}\|>\epsilon}\right) \to_{n\to+\infty} 0$, then, for every bounded continuous function $f : \mathbb{C}^d \to \mathbb{R}$,

$$\mathbb{E}f(\sum_{j=1}^{r_n} Y_{nj}) - \mathbb{E}f(Z_n) \xrightarrow[n\to+\infty]{} 0, \tag{2.20}$$

where Z_n is a $\mathbb{C}^d$-valued centered Gaussian random vector with parameters

$$\mathbb{E}(Z_n Z_n^*) = (\Theta_n(k,l))_{k,l} \text{ and } \mathbb{E}(Z_n Z_n^T) = (\tilde{\Theta}_n(k,l))_{k,l}.$$

Following the lines of the proof of the central limit theorem for quadratic forms by Baik and Silverstein in the appendix of Capitaine et al. (2009) and using Lemma 2.9, we will establish the following extension.

Proposition 2.10. *Let $(z_1, \dots, z_q)$ be in I^q, where I is a subset of $\mathbb{C}$ such that $\forall z \in I, \bar{z} \in I$. For any z in $\{z_1, \dots, z_q\}$, let $B(z) = (b_{ij}(z))$ be a $N \times N$ matrix such that $(B(z))^* = B(\bar{z})$ and there exists a constant $a > 0$ (not depending on N) such that for any z in $\{z_1, \dots, z_q\}$, $\|B(z)\| \leq a$. Let p be a fixed integer number and $Y_N = (y_{ij})_{1\leq i\leq N, 1\leq j\leq p}$ be a $N \times p$ matrix which contains i.i.d. complex standardized entries with bounded fourth moment and such that $\mathbb{E}(y_{11}^2) = 0$. Set*

$$V_N = \Big((1/\sqrt{N})(Y_N^* B(z_1) Y_N - \mathrm{Tr_N}(B(z_1)) I_p), \dots, \\ (1/\sqrt{N})(Y_N^* B(z_q) Y_N - \mathrm{Tr_N}(B(z_q)) I_p)\Big).$$

If there exists uniformly bounded sequences $(f_N)_N$ and $(g_N)_N$ of functions on I^2 such that for any z_l, z_k,

$$\frac{1}{N}\sum_{i=1}^{N} (B(z_l))_{ii} (B(z_k))_{ii} = f_N(z_l, z_k) + o(1),$$

and

$$\mathrm{tr}_N(B(z_l) B(z_k)) = g_N(z_l, z_k) + o(1),$$

then $d_{LP}(V_N, \mathcal{V}_N) \to 0$ where $\mathcal{V}_N = (\mathcal{G}_N(z_1), \dots, \mathcal{G}_N(z_q))$, $\mathcal{G}_N$ is a centered Gaussian matrix valued process whose distribution is given as follows:

1) the processes $((\mathcal{G}_N(z))_{ij})_z$ for $1 \leq i \leq j \leq p$ are independent, and $(\mathcal{G}_N(z))_{ji} = \overline{(\mathcal{G}_N(\bar{z}))_{ij}}$.

2) For $i \le p$,

$$\begin{aligned}\mathbb{E}((\mathcal{G}_N(z_l))_{ii}(\mathcal{G}_N(z_k))_{ii}) &= (\mathbb{E}(|y_{11}|^4 - 2)\,(f_N(z_l, z_k)) \\ &\quad + g_N(z_l, z_k)\end{aligned}$$

and

$$\mathbb{E}((\mathcal{G}_N(z_l))_{ii}\overline{(\mathcal{G}_N(z_k))_{ii}}) = \mathbb{E}((\mathcal{G}_N(z_1))_{ii}(\mathcal{G}_N(\bar{z}_2))_{ii}) \tag{2.21}$$

3) For $1 \le i \ne j \le p$,

$$\mathbb{E}((\mathcal{G}_N(z_l))_{ij}(\mathcal{G}_N(z_k))_{ij}) = 0$$

and

$$\mathbb{E}((\mathcal{G}_N(z_l))_{ij}\overline{(\mathcal{G}_N(z_k))_{ij}}) = g_N(z_l, \bar{z_k}). \tag{2.22}$$

Proof: First, for any B in $\{B(z), z = z_1, \ldots, z_q\}$ and any $1 \le s, t \le p$, one can write $(1/\sqrt{N})\Big(Y_N^* B Y_N - \mathrm{Tr}(B) I_p\Big)_{st}$ as a sum of martingale differences:

$$\begin{aligned}&(1/\sqrt{N})(Y_N^* B Y_N - \mathrm{Tr}(B) I_p)_{st} \\ &= (1/\sqrt{N}) \sum_{i=1}^{N} \Big((\bar{y}_{is} y_{it} - \delta_{st}) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt} b_{ij} + \bar{y}_{is} \sum_{j>i} y_{jt} b_{ij} \Big) \\ &= (1/\sqrt{N}) \sum_{i=1}^{N} \Big((\bar{y}_{is} y_{it} - \delta_{st}) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt} b_{ij} + y_{it} \sum_{j<i} \bar{y}_{js} b_{ji} \Big) \\ &= \sum_{i=1}^{N} (Z_i(B))_{st}\end{aligned}$$

where

$$(Z_i(B))_{st} = (1/\sqrt{N}) \Big((\bar{y}_{is} y_{it} - \delta_{st}) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt} b_{ij} + y_{it} \sum_{j<i} \bar{y}_{js} b_{ji} \Big).$$

Let $\mathcal{F}_{N,i}$ be the σ-field generated by $\{y_{1s}, \ldots, y_{is}, 1 \le s \le p\}$. Let also $\mathbb{E}_i(\cdot)$ denote conditional expectation with respect to $\mathcal{F}_{N,i}$. It is clear that for any B in $\{B(z), z \in I\}$, $Z_i(B) \in M_p(\mathbb{C}) \simeq \mathbb{C}^{p^2}$ is measurable with respect to $\mathcal{F}_{N,i}$ and satisfies $\mathbb{E}_{i-1}(Z_i(B)) = 0$.

We will show that the conditions of Lemma 2.9 are met for the $\mathbb{C}^{qp^2}$-valued martingale difference sequence $Y_{Ni}^T = (Z_i(B(z_1)), \ldots, Z_i(B(z_q)))$.
Write $(Z_i(B))_{st} = X_1^i + X_2^i$, with $X_1^i = (1/\sqrt{N})(\bar{y}_{is} y_{it} - \delta_{st}) b_{ii}$. Then for $\epsilon > 0$,

$$\sum_{i=1}^{N} \mathbb{E}(|X_1^i|^2\, 1_{(|X_1^i| \ge \epsilon)}) \le a^2 \mathbb{E}(|\bar{y}_{1s} y_{1t} - \delta_{st}|^2\, 1_{\{|\bar{y}_{1s} y_{1t} - \delta_{st}|^2 \ge \sqrt{N}\epsilon/a\}}) \to 0 \tag{2.23}$$

as $N \to \infty$, by dominated convergence theorem.
We have

$$\begin{aligned}&\mathbb{E}|\sum_{j<i} y_{jt} b_{ij}|^4 \\ &= \mathbb{E}(|y_{1t}|^4 \sum_{j<i} |b_{ij}|^4) + 2\mathbb{E}(\sum_{*} |b_{ij_1}|^2 |b_{ij_2}|^2) + \mathbb{E}(|y_{1t}^2|^2 \sum_{*} |b_{ij_1}|^2 |b_{ij_2}|^2) \\ &\le \mathbb{E}|y_{1t}|^4 \mathbb{E}[(\max_{i,j} |b_{ij}|)^2 \max_i (BB^*)_{ii}] + (2 + \mathbb{E}|y_{1t}^2|^2)\mathbb{E}[(BB^*)_{ii}^2] \\ &\le a^4 \big[\mathbb{E}|y_{1t}|^4 + 2 + \mathbb{E}|y_{1t}^2|^2\big]\end{aligned}$$

where the sum $\sum_*$ is over $\{j_1 < i,\, j_2 < i,\, j_1 \neq j_2\}$. Therefore $\mathbb{E}|X_2^i|^4 = o(N^{-1})$ so that for any $\epsilon > 0$,

$$\sum_{i=1}^N \mathbb{E}(|X_2^i|^2\, 1_{(|X_2^i| \geq \epsilon)}) \leq (1/\epsilon^2) \sum_{i=1}^N \mathbb{E}|X_2^i|^4 \to 0 \quad \text{as } N \to \infty. \tag{2.24}$$

Thus, by (2.23), (2.24) and (A.4) in Capitaine et al. (2009), $\{Z_i(B)\}$ satisfies the Lindeberg condition of Lemma 2.9.

Now, we shall verify condition (2.19) of Lemma 2.9. We have for any B and C in $\{B(z), z \in I\}$, for any $1 \leq s, t, s', t' \leq p$,

$$\sum_{i=1}^N \mathbb{E}_{i-1}(Z_i(B))_{st} Z_i(C)_{s't'}) \tag{2.25}$$

$$= \quad (1/N) \sum_{i=1}^N \Big\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} + \delta_{st'}\delta_{s't}(1-\delta_{ss'}))b_{ii}c_{ii} \tag{2.26}$$

$$+\mathbb{E}(|y_{11}|^2 \bar{y}_{11}) \left[\delta_{st}\delta_{ss'} \sum_{j<i} y_{jt'} c_{ij} b_{ii} + \delta_{s't'}\delta_{ss'} \sum_{j<i} y_{jt} b_{ij} c_{ii} \right]$$

$$+\mathbb{E}(|y_{11}|^2 y_{11}) \left[\delta_{st}\delta_{tt'} \sum_{j<i} \bar{y}_{js'} c_{ji} b_{ii} + \delta_{s't'}\delta_{tt'} \sum_{j<i} \bar{y}_{js} b_{ji} c_{ii} \right] \tag{2.27}$$

$$+\delta_{s't} \sum_{j<i} \bar{y}_{js} b_{ji} \sum_{j<i} y_{jt'} c_{ij} + \delta_{st'} \sum_{j<i} \bar{y}_{js'} c_{ji} \sum_{j<i} y_{jt} b_{ij} \Big\}.$$

Let B_L (resp. B_U denote the strictly lower (resp. upper) triangular part of B. We have, using Cauchy-Schwarz's inequality, that

$$\begin{aligned}
\mathbb{E}|(1/N)\sum_{i=1}^N c_{ii} \sum_{j<i} y_{jt} b_{ij}|^2 &= \mathbb{E}|(1/N)\sum_{j=1}^{N-1} y_{jt} \sum_{i>j} c_{ii} b_{ij}|^2 \\
&= (1/N^2)\mathbb{E}(\sum_{j=1}^{N-1} \sum_{i>j} c_{ii} b_{ij} \sum_{\underline{i}>j} \bar{c}_{\underline{i}\,\underline{i}} \bar{b}_{\underline{i}j}) \\
&= (1/N^2)\mathbb{E}(\sum_{i\underline{i}} c_{ii} \bar{c}_{\underline{i}\,\underline{i}} (B_L B_L^*)_{i\underline{i}}) \\
&\leq \mathbb{E}\big[(\max_i |c_{ii}|)^2 (1/N) (\sum_{i\underline{i}} |(B_L B_L^*)_{i\underline{i}}|^2)^{1/2}\big] \\
&= \mathbb{E}\big[(\max_i |c_{ii}|)^2 (1/N) \mathrm{Tr}((B_L B_L^*)^2)^{1/2}\big] \\
&\leq \mathbb{E}\big[(\max_i |c_{ii}|)^2 (1/\sqrt{N}) \|B_L\|^2\big].
\end{aligned}$$

We apply the following bound (due to R. Mathias, see Mathias, 1993): $\|B_L\| \leq \gamma_N \|B\|$ where $\gamma_N = O(\ln N)$, and the bounds $\|B\| \leq a$, $\|C\| \leq a$, to conclude that

$$(1/N) \sum_{i=1}^N c_{ii} \sum_{j<i} y_{jt} b_{ij} \xrightarrow[N \to +\infty]{P} 0.$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}_{i-1}(Z_i(B))_{st} Z_i(C)_{s't'}) \\
&= (1/N) \sum_{i=1}^{N} \Big\{ \left[(\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} + \delta_{st'}\delta_{s't}(1-\delta_{ss'})\right] b_{ii}c_{ii} \\
&\quad + \delta_{s't} \sum_{j<i} \bar{y}_{js} b_{ji} \sum_{j<i} y_{jt'} c_{ij} + \delta_{st'} \sum_{j<i} \bar{y}_{js'} c_{ji} \sum_{j<i} y_{jt} b_{ij} \Big\} + o_{\mathbb{P}}(1) \\
&= \left[(\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} + \delta_{st'}\delta_{s't}(1-\delta_{ss'})\right] \frac{1}{N} \sum_{i=1}^{N} b_{ii}c_{ii} \\
&\quad + \delta_{s't}(1/N)(Y_N)_s^* B_U C_L (Y_N)_{t'} + \delta_{st'}(1/N)(Y_N)_{s'}^* C_U B_L (Y_N)_t + o_{\mathbb{P}}(1)
\end{aligned}$$

Besides, from Lemma 2.8 we have

$$\begin{aligned}
\mathbb{E}\Big|(1/N)((Y_N)_s^* B_U C_L (Y_N)_{t'} - \delta_{st'} \mathrm{Tr}(B_U C_L))\Big|^2 &\leq (1/N^2)\mathbb{E}(\mathrm{Tr}(C_L^* B_U^* B_U C_L) \\
&\leq K \mathbb{E}\|B\|^2 \|C\|^2 \frac{\ln^4 N}{N} \rightarrow_{N\to\infty} 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}_{i-1}(Z_i(B))_{st} (Z_i(C))_{s't'} \\
&= \left[(\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} + \delta_{st'}\delta_{s't}(1-\delta_{ss'})\right] \frac{1}{N} \sum_{i=1}^{N} b_{ii}c_{ii} \\
&\quad + \delta_{st'}\delta_{s't} \frac{1}{N} \sum_{j<i} b_{ij}c_{ji} + \delta_{s't}\delta_{st'} \frac{1}{N} \sum_{j<i} b_{ji}c_{ij} + o_{\mathbb{P}}(1) \\
&= \left[(\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} + \delta_{st'}\delta_{s't}(1-\delta_{ss'})\right] \frac{1}{N} \sum_{i=1}^{N} b_{ii}c_{ii} \\
&\quad + \delta_{st'}\delta_{s't} \frac{1}{N} \mathrm{Tr}(BC) - \delta_{s't}\delta_{st'} \frac{1}{N} \sum_{i} b_{ii}c_{ii} + o_{\mathbb{P}}(1) \\
&= \left[(\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss'}\delta_{s't'} - \delta_{st'}\delta_{s't}\delta_{ss'})\right] \frac{1}{N} \sum_{i=1}^{N} b_{ii}c_{ii} \\
&\quad + \delta_{st'}\delta_{s't} \,\mathrm{tr}_N(BC) + o_{\mathbb{P}}(1).
\end{aligned}$$

Proposition 2.10 readily follows. □

2.6. *Preliminary results on deformed Wigner matrices.*

2.6.1. *Preliminary results from Belinschi and Capitaine (2017).* Note that, in Section 5 in Belinschi and Capitaine (2017), the authors consider for any fixed integer numbers m, r, t and any fixed $m \times m$ Hermitian matrices $\gamma, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_t$, the matrix model in $M_m(\mathbb{C}) \otimes M_N(\mathbb{C})$, $\gamma \otimes I_N + \sum_{v=1}^r \alpha_v \otimes \frac{W_N^{(v)}}{\sqrt{N}} + \sum_{u=1}^t \beta_u \otimes A_N^{(u)}$ where the $W_N^{(v)}$'s are independent more general Wigner matrices than ours and the $A_N^{(u)}$'s are deterministic matrices such that $\sup_N \|A_N^{(u)}\| < \infty$. Therefore, the results therein apply to our model by choosing $m = 1, r = t = 1$ and $\alpha_1 = 1 = \beta_1$.

For any $z \in \mathbb{C}\backslash\mathbb{R}$, $G(z) = [G_{ij}(z)]_{1\leq i,j\leq N} = (zI_N - M_N)^{-1}$ denotes the resolvant of M_N. Note that by (2.5),

$$\|G(z)\| \leq |\Im z|^{-1}. \tag{2.28}$$

For any $z \in \mathbb{C} \setminus \mathbb{R}$, define $g_N(z) = \mathbb{E}(\mathrm{tr}_N\, G(z))$ and denote by $\tilde{g}_N(z)$ the Stieltjes transform of $\mu_{sc} \boxplus \mu_{A_N}$.

Lemma 2.11. *For any* $(i,j) \in \{1,\dots,N\}^2$,

$$\mathbb{E}\left(G_{ij}(z)\right) = \left[\left((z - \sigma^2\tilde{g}_N(z))I_N - A_N\right)^{-1}\right]_{ij} + O^{(u)}\left(\frac{1}{\sqrt{N}}\right).$$

Proof: According to Corollary 5.5 in Belinschi and Capitaine (2017), for any $(i,j) \in \{1,\dots,N\}^2$,

$$\mathbb{E}\left(G_{ij}(z)\right) = \left(Y_N(z)\right)_{ij} + \frac{(1-\sqrt{-1})\kappa_3}{2\sqrt{2}N\sqrt{N}} \sum_{s,l=1}^{N} \left(Y_N(z)\right)_{il} \left(Y_N(z)\right)_{ss} \left(Y_N(z)\right)_{ll} \mathbb{E}\left(G_{sj}(z)\right)$$
$$+ O_{ij}^{(u)}(\frac{1}{N})$$

where

$$Y_N(z) = \left((z - g_N(z))I_N - A_N\right)^{-1}$$

and κ_3 is the third classical cumulant of μ. Note that $|\Im(z - g_N(z))| \geq |\Im z|$ so that, by (2.5)

$$\|Y_N(z)\| \leq \frac{1}{|\Im z|}. \tag{2.29}$$

Now,

$$\begin{aligned}
&\left|\sum_{s,l=1}^{N} \left(Y_N(z)\right)_{il} \left(Y_N(z)\right)_{ss} \left(Y_N(z)\right)_{ll} \mathbb{E}\left(G_{sj}(z)\right)\right| \\
&\quad\leq \sum_{s,l=1}^{N} \left|\left(Y_N(z)\right)_{il}\right| \left|\left(Y_N(z)\right)_{ss}\right| \left|\left(Y_N(z)\right)_{ll}\right| \left|\mathbb{E}\left(G_{sj}(z)\right)\right| \\
&\quad\leq |\Im z|^{-2} N \left(\sum_{s=1}^{N} \left|\mathbb{E}\left(G_{sj}(z)\right)\right|^2\right)^{1/2} \left(\sum_{l=1}^{N} \left|\left(Y_N(z)\right)_{il}\right|^2\right)^{1/2} \\
&\quad\leq N|\Im z|^{-4}
\end{aligned}$$

where we used Lemma 2.2, (2.29) and (2.28). Therefore

$$\mathbb{E}\left(G_{ij}(z)\right) = \left(Y_N(z)\right)_{ij} + O_{ij}^{(u)}(\frac{1}{\sqrt{N}}).$$

Now, according to (5.56) in Belinschi and Capitaine (2017),

$$\left\|Y_N(z) - \tilde{Y}_N(z)\right\| = O\left(\frac{1}{\sqrt{N}}\right)$$

where

$$\tilde{Y}_N(z) = \left((z - \tilde{g}_N(z))I_N - A_N\right)^{-1}. \tag{2.30}$$

Lemma 2.11 readily follows. $\square$

Lemma 2.12. *We have*

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ g_N(z) = \tilde{g}_N(z) + O(\frac{1}{N}).$$

Proof: According to Proposition 5.8 in Belinschi and Capitaine (2017), we have

$$g_N(z) - \tilde{g}_N(z) = \left(1 - \tilde{g}_N^{'}(z)\right) \tilde{L}_N(z) + O(\frac{1}{N\sqrt{N}})$$

where

$$\begin{aligned}
\tilde{L}_N(z) \quad &= \quad \frac{\kappa_4}{2N^3} \sum_{i,l=1}^{N} \left(\tilde{Y}_N(z)^2\right)_{ll} \left[\left(\tilde{Y}_N(z)\right)_{ii}\right]^2 \left(\tilde{Y}_N(z)\right)_{ll} \\
&\quad + \frac{\kappa_3(1+\sqrt{-1})}{2\sqrt{2}N^2\sqrt{N}} \sum_{i,l=1}^{N} \left(\tilde{Y}_N(z)^2\right)_{ll} \left(\tilde{Y}_N(z)\right)_{ii} \left(\tilde{Y}_N(z)\right)_{li} \\
&\quad + \frac{\kappa_3(1-\sqrt{-1})}{2\sqrt{2}N^2\sqrt{N}} \sum_{i,l=1}^{N} \left(\tilde{Y}_N(z)^2\right)_{il} \left(\tilde{Y}_N(z)\right)_{ii} \left(\tilde{Y}_N(z)\right)_{ll} \\
&\quad + \frac{\kappa_3(1-\sqrt{-1})}{2\sqrt{2}N^2\sqrt{N}} \sum_{i,l=1}^{N} \left(\tilde{Y}_N(z)^2\right)_{ll} \left(\tilde{Y}_N(z)\right)_{il} \left(\tilde{Y}_N(z)\right)_{ii},
\end{aligned}$$

and $\tilde{Y}_N$ is defined by (2.30). (Note that in Proposition 5.8 in Belinschi and Capitaine, 2017, in full generality the $\left(\tilde{Y}_N(z)\right)_{il}$'s are $m \times m$ matrices which a priori do not commute and $\tilde{G}_N(zI_m)$ is a $m \times m$ matrix too. But in the present paper, since $m = 1$, $\left(\tilde{Y}_N(z)\right)_{il}$'s are scalar and obviously commute and $\tilde{G}_N(zI_m) = \tilde{g}_N(z)$.) Note that, $|\Im(z - \tilde{g}_N(z))| \geq |\Im z|$ so that, by (2.5),

$$\|\tilde{Y}_N(z)\| \leq |\Im z|^{-1}.$$

Lemma 2.12 readily follows by using Cauchy-Schwarz's inequality and (2.3). □

Lemma 8.7 in Belinschi and Capitaine (2017) implies in particular the following variance estimates.

Lemma 2.13.

$$\mathrm{Var}(G_{ij}(z)) = O^{(u)}(1/N).$$

Lemma 2.14.

$$\mathrm{Var}(\mathrm{tr}_N\, G(z)) = O(1/N^2)$$

The following result is a corollary of Theorem 1.1 in Belinschi and Capitaine (2017).

Proposition 2.15 (Theorem 1.1 in Belinschi and Capitaine, 2017)**.** *Let $[b; c]$ be a real interval such that there exists $\delta > 0$ such that, for any large N, $[b + \delta; c + \delta]$ lies outside the support of $\mu_{sc} \boxplus \mu_{A_N}$. Then, almost surely, for all large N, there is no eigenvalue of M_N in $[b; c]$.*

2.6.2. *Quantitative asymptotic freeness.* Let $\epsilon_0 > 0$ be fixed such that $d(\rho_\theta, \mathrm{supp}(\mu_{sc} \boxplus \nu) \cup \{\rho_{\theta_j}, \theta_j \neq \theta\}) > \epsilon_0$ and $d(\theta, \mathrm{supp}(\nu) \cup \{\theta_j, \theta_j \neq \theta\}) > \epsilon_0$. Let Ω_N be the event on which there is no eigenvalue of $\frac{W_{N-p}}{\sqrt{N}} + A_{N-p}$ in $]\rho_\theta - \epsilon_0; \rho_\theta + \epsilon_0[$, $\lambda_{i_0}(M_N)$ is the unique eigenvalue of M_N in $]\rho_\theta - \epsilon_0/2; \rho_\theta + \epsilon_0/2[$ and $\left\| \frac{W_N}{\sqrt{N}} \right\| \leq 3$. Propositions 2.7, 2.15 applied to M_{N-p}, Proposition 1.5 and Bai-Yin's theorem lead that

$$\lim_{N \to +\infty} \mathbf{1}_{\Omega_N} = 1, \text{ a.s.} \tag{2.31}$$

Denoting by $\hat{G}_{N-p}$ the resolvent of the lower right submatrix of size $N-p$ of M_N and ρ_N being defined by (1.7), we have the following

Proposition 2.16.

$$\sqrt{N} \left\{ \mathrm{tr}_{N-p}\, \hat{G}_{N-p}(\rho_N) \mathbf{1}_{\Omega_N} - \int \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - x)} \right\}$$

goes to zero in probability.

Proof: We stick to the proof of Proposition 5.5 in Capitaine (2020). Using Proposition 2.7, for N large enough,

$$d(\rho_N, \mathrm{supp}(\mu_{sc} \boxplus \mu_{A_{N-p}})) > \epsilon_0/2 \tag{2.32}$$

and on Ω_N, $d\left(\{\rho_N, \lambda_{i_0}(M_N)\}, \mathrm{spect}\left(\frac{W_{N-p}}{\sqrt{N}} + A_{N-p}\right)\right) > \epsilon_0/2$, so that

$$\left\| \hat{G}_{N-p}(\rho_N) \right\| \leq \frac{2}{\epsilon_0}, \ \left\| \hat{G}_{N-p}(\lambda_{i_0}(M_N)) \right\| \leq \frac{2}{\epsilon_0}. \tag{2.33}$$

Moreover, there exists $K > 0$ such that for any $x \in \mathrm{supp}(\mu_{sc} \boxplus \mu_{A_{N-p}})$,

$$|\rho_N - x| \leq K \text{ and on } \ \Omega_N, \ \left\| (\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right\| \leq K.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function with support in $\{\epsilon_0/4 \leq |x| \leq 2K\}$ and such that $g \equiv 1$ on $\{\epsilon_0/2 \leq |x| \leq K\}$. $f : x \mapsto \frac{g(x)}{x}$ is a $\mathcal{C}^\infty$ function with compact support. Note that

$$\int \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - x)} = \int f\,(\rho_N - x)\, d\mu_{sc} \boxplus \mu_{A_{N-p}}(x) \tag{2.34}$$

$$\text{and on } \ \Omega_N, \ \hat{G}_{N-p}(\rho_N) = f\left(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}\right). \tag{2.35}$$

According to Lemma 2.12, for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\sqrt{N}\, \mathrm{tr}_{N-p}\, \mathbb{E}\left[\hat{G}_{N-p}(\rho_N - z)\right] = \sqrt{N} \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - z - x)} + o^{(z)}(1), \tag{2.36}$$

where there exist polynomials Q_1 and Q_2 with non negative coefficients and $(d,k) \in \mathbb{N}^2$ such that

$$\|o^{(z)}(1)\| \leq \frac{Q_1(|\Im z|^{-1})(|z|+1)^d}{\sqrt{N}} \leq \frac{1}{\sqrt{N}} \frac{Q_2(|\Im z|)(|z|+1)^d}{|\Im z|^k}. \tag{2.37}$$

Therefore, by Helffer-Sjöstrand functional calculus (see Section 2.2),

$$\sqrt{N}\, \mathrm{tr}_{N-p}\, \mathbb{E}\left(f\left(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right)\right)$$

$$= \frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{R}} \bar\partial F_k(f)(z)\sqrt{N}\,\mathrm{tr}_{N-p}\,\mathbb{E}\left[\hat G_{N-p}(\rho_N - z)\right] d^2z$$

and

$$\sqrt{N}\int f\left((\rho_N - x\right) d\mu_{sc}\boxplus\mu_{A_{N-p}}(x) = \frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{R}} \bar\partial F_k(f)(z)\sqrt{N}\frac{d\mu_{sc}\boxplus\mu_{A_{N-p}}(x)}{(\rho_N - z - x)} d^2z.$$

Hence, using (2.36) and (2.34), we can deduce that

$$\begin{aligned}&\sqrt{N}\,\mathrm{tr}_{N-p}\,\mathbb{E}\left(f\left(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}\right)\right)\\ =\ & \sqrt{N}\int \frac{d\mu_{sc}\boxplus\mu_{A_{N-p}}(x)}{(\rho_N - x)} + \frac{1}{\pi}\int_{z\in\mathbb{C}\setminus\mathbb{R}} \partial F_k(f)(z) o^{(z)}(1) d^2z.\end{aligned}$$

Note that since f and χ are compactly supported, the last integral is an integral on a bounded set of $\mathbb{C}$ and according to (2.37) and (2.10),

$$\left\|\frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{R}} \partial F_k(f)(z) o^{(z)}(1) d^2z\right\| \leq \frac{C}{\sqrt{N}}.$$

Thus,

$$\sqrt{N}\Big\{\mathbb{E}\,\mathrm{tr}_{N-p}\left(f\Big(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}\Big)\right) - \int \frac{d\mu_{sc}\boxplus\mu_{A_{N-p}}(x)}{\Big(\rho_N - x\Big)}\Big\} \to_{N\to+\infty} 0. \tag{2.38}$$

Define $k : M_{N-p}^{sa}(\mathbb{C}) \to \mathbb{C}$ by

$$k(X) = \mathrm{tr}_{N-p}\left[f\left(\rho_N I_{N-p} - X - A_{N-p}\right)\right].$$

Applying Poincaré inequality, we get that

$$\mathbb{E}\left(\left|k(\frac{W_{N-p}}{\sqrt{N}}) - \mathbb{E}(k(\frac{W_{N-p}}{\sqrt{N}}))\right|^2\right) \leq \frac{C}{N}\mathbb{E}\left(\left\|\mathrm{grad}k\left(\frac{W_{N-p}}{\sqrt{N}}\right)\right\|_e^2\right),$$

with

$$\|\mathrm{grad}k(X)\|_e^2 = \sup_{w\in S_1(M_{N-p}^{sa}(\mathbb{C}))}\left|\frac{d}{dt}k(X+tw)_{|t=0}\right|^2.$$

Since f is a Lipschitz function on $\mathbb{R}$ with Lipschitz constant C_L, its extension on Hermitian matrices is C_L-Lipschitz with respect to the norm $\|M\|_e = (\mathrm{Tr}_{N-p} M^2)^{1/2}$. Therefore,

$$\sup_{w\in S_1(M_{N-p}^{sa}(\mathbb{C}))}\left|\frac{d}{dt}k(X+tw)_{|t=0}\right|^2 \leq \frac{C}{N},$$

and then

$$\mathbb{E}\left(\left|\sqrt{N}\left\{k\left(\frac{W_{N-p}}{\sqrt{N}}\right) - \mathbb{E}\left(k\left(\frac{W_{N-p}}{\sqrt{N}}\right)\right)\right\}\right|^2\right) \leq \frac{C}{N}.$$

It readily follows that

$$\begin{aligned}&\sqrt{N}\,\mathrm{tr}_{N-p}\left\{f(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}) - \mathbb{E}\left(f(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p})\right)\right\}\\ &= o_{\mathbb{P}}(1)\end{aligned} \tag{2.39}$$

Proposition 2.16 follows from (2.35), (2.38), (2.39) and (2.31). □

3. Proof of Theorem 1.10

The approach to prove (1.10) is the one of Capitaine (2020). On Ω_N, defined at the beginning of Section 2.6.2, we have by Proposition 2.1,

$$\det\left(X_p(N)\right) = 0, \tag{3.1}$$

where

$$X_p(N) = \lambda_{i_0}(M_N) I_p - \frac{W_p}{\sqrt{N}} - A_p - \frac{1}{N} Y^* \hat{G}_{N-p}(\lambda_{i_0}(M_N)) Y,$$

and $\hat{G}_{N-p}$ is the resolvent of $\frac{W_{N-p}}{\sqrt{N}} + A_{N-p}$. Let ρ_N be as defined by (1.7). Using the identity

$$\hat{G}_{N-p}(\rho_N) - \hat{G}_{N-p}(\lambda_{i_0}(M_N)) = (\lambda_{i_0}(M_N) - \rho_N) \hat{G}_{N-p}(\rho_N) \hat{G}_{N-p}(\lambda_{i_0}(M_N)),$$

we have

$$X_p(N) = H_p(N) + X_p^{(0)},$$

where

$$X_p^{(0)} = \theta I_p - A_p,$$

$$\begin{aligned} H_p(N) \;&=\; (\lambda_{i_0}(M_N) - \rho_N) I_p - \Delta_1(N) - \Delta_2(N) \\ &\quad + (\lambda_{i_0}(M_N) - \rho_N) r_1(N) - \frac{W_p}{\sqrt{N}} - (\lambda_{i_0}(M_N) - \rho_N)^2 r_2(N) \end{aligned}$$

with

$$r_1(N) = \frac{1}{N} Y^* \hat{G}_{N-p}(\rho_N)^2 \mathbf{1}_{\Omega_N} Y,$$

$$r_2(N) \;=\; \frac{1}{N} Y^* \hat{G}_{N-p}(\rho_N)^2 \hat{G}_{N-p}(\lambda_{i_0}(M_N)) \mathbf{1}_{\Omega_N} Y,$$

$$\Delta_1(N) = \frac{1}{N} Y^* \hat{G}_{N-p}(\rho_N) \mathbf{1}_{\Omega_N} Y - \frac{\sigma^2}{N} \operatorname{Tr}_{N-p}\left(\hat{G}_{N-p}(\rho_N) \mathbf{1}_{\Omega_N}\right),$$

$$\Delta_2(N) = \frac{1}{N} \operatorname{Tr}_{N-p}\left(\hat{G}_{N-p}(\rho_N) \mathbf{1}_{\Omega_N}\right) - \int \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - x)}.$$

First, by Lemma 2.8 we have that,

$$r_1(N) - \sigma^2 \frac{1}{N} \operatorname{Tr}_{N-p}(\hat{G}_{N-p}(\rho_N)^2) \mathbf{1}_{\Omega_N} I_p = o_{\mathbb{P}}(1).$$

By (2.31), (2.35), (1.8), (2.32) and asymptotic freeness of $\frac{W_{N-p}}{\sqrt{N}}$ and A_{N-p} (see Theorem 5.4.5 Anderson et al., 2010),

$$\frac{1}{N} \operatorname{Tr}_{N-p}(\hat{G}_{N-p}(\rho_N)^2) \mathbf{1}_{\Omega_N} \underset{N\to\infty}{\longrightarrow} \int \frac{d\lambda(x)}{(\rho_\theta - x)^2} \text{ almost surely.} \tag{3.2}$$

Therefore,

$$r_1(N) \underset{N\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \sigma^2 \int \frac{d\lambda(x)}{(\rho_\theta - x)^2} I_p. \tag{3.3}$$

Now on Ω_N, using (2.33),

$$\|r_2(N)\| \leq \left\|\hat{G}_{N-p}(\rho_N) \mathbf{1}_{\Omega_N}\right\|^2 \left\|\hat{G}_{N-p}(\lambda_{i_0}(M_N)) \mathbf{1}_{\Omega_N}\right\| \frac{\|Y\|^2}{N} \leq 9 \left(\frac{2}{\epsilon_0}\right)^2. \tag{3.4}$$

By Lemma 2.8,

$$\Delta_1(N) = o_{\mathbb{P}}(1). \tag{3.5}$$

Lemma 3.1.
$$\frac{1}{N-p}\sum_{j=1}^{N-p}\left[\hat{G}_{N-p}(\rho_N)_{jj}\right]^2\mathbf{1}_{\Omega_N}=\frac{1}{N-p}\sum_{j=1}^{N-p}\left[[(\theta-A_{N-p})^{-1}]_{jj}\right]^2+o_{\mathbb{P}}(1).$$

Proof: By (4.29) which will be proved below, for any $r>0$,

$$\begin{aligned}&\frac{1}{N-p}\sum_{j=1}^{N-p}\left[\hat{G}_{N-p}(\rho_\theta+\frac{i}{r})_{jj}\right]^2\mathbf{1}_{\Omega_N}\\&=\frac{1}{N-p}\sum_{j=1}^{N-p}\left[[(\omega(\rho_\theta+\frac{i}{r})-A_{N-p})^{-1}]_{jj}\right]^2+o_{\mathbb{P}}(1),\end{aligned}$$

with ω defined by (2.13). Now, using resolvent identity, one can easily obtain that there exists some constant $C(\epsilon_0)$ such that for any $r>0$ and any N,

$$\begin{aligned}&\left|\frac{1}{N-p}\sum_{j=1}^{N-p}\left[\hat{G}_{N-p}(\rho_N)_{jj}\right]^2\mathbf{1}_{\Omega_N}-\frac{1}{N-p}\sum_{j=1}^{N-p}\left[\hat{G}_{N-p}(\rho_\theta+\frac{i}{r})_{jj}\right]^2\mathbf{1}_{\Omega_N}\right|\\&\leq C(\epsilon_0)\left(\frac{1}{r}+\rho_N-\rho\right).\end{aligned}$$

Moreover, $\omega(\rho_\theta)=\theta$ so that for all large N, $d(\omega(\rho_\theta)\,,\mathrm{supp}(\mu_{A_{N-p}})>\epsilon_0/2$. Now, choose r_0 large enough such that for all large N, $\forall r\geq r_0$, $d(\omega(\rho_\theta+\frac{i}{r}),\mathrm{supp}(\mu_{A_{N-p}})>\epsilon_0/4$. Using resolvent identity, one can easily obtain that there exists some constant $C(\epsilon_0)$ such that for all large N, $\forall r\geq r_0$,

$$\begin{aligned}&\left|\frac{1}{N-p}\sum_{j=1}^{N-p}\left[[(\omega(\rho_\theta+\frac{i}{r})-A_{N-p})^{-1}]_{jj}\right]^2-\frac{1}{N-p}\sum_{j=1}^{N-p}\left[[(\omega(\rho_\theta)-A_{N-p})^{-1}]_{jj}\right]^2\right|\\&\leq C(\epsilon_0)/r.\end{aligned}$$

Lemma 3.1 follows by letting N go to infinity and then r go to infinity. □

(3.2), Lemma 3.1 and Proposition 2.10 yield that
$$\sqrt{N}\Delta_1(N)^2=o_{\mathbb{P}}(1),\tag{3.6}$$
Now, one can prove that by Proposition 2.16, we have
$$\sqrt{N}\Delta_2(N)=o_{\mathbb{P}}(1).\tag{3.7}$$
Thus (1.6), (1.8), (3.3), (3.4), (3.5) and (3.7) yield that
$$H_p(N)=o_{\mathbb{P}}(1).\tag{3.8}$$
Therefore, according to Lemma 2.3 (using (3.8)) and (3.1), with a probability going to one as N goes to infinity,

$$\begin{aligned}0&=\det X_p(N)\\&=\det(X_p^{(0)}+H_p(N))\\&=\det(X_p^{(0)})+\mathrm{Tr}_p\left[B_{X_p^{(0)}}H_p(N)\right]+\epsilon_N\\&=\mathrm{Tr}_p\left[B_{X_p^{(0)}}H_p(N)\right]+\epsilon_N,\end{aligned}$$

where

$$B_{X_p^{(0)}} = {}^t com(X_p^{(0)}),$$

$$\epsilon_N = O(\|H_p(N)\|^2).$$

Thus, using (1.6), (1.8), (3.3), (3.4), (3.7) and (3.6),

$$\sqrt{N}\epsilon_N = o_{\mathbb{P}}(\sqrt{N}(\lambda_{i_0}(M_N) - \rho_N)) + o_{\mathbb{P}}(1).$$

Hence, with a probability going to one as N goes to infinity,

$$\sqrt{N}(\lambda_{i_0}(M_N) - \rho_N)\left[Tr_p B_{X_p^{(0)}}(I_p + r_1(N)) + o_{\mathbb{P}}(1)\right]$$

$$= Tr_p\left[B_{X_p^{(0)}}\left(\sqrt{N}\Delta_1(N) + W_p\right)\right] + o_{\mathbb{P}}(1).$$

(1.10) readily follows from (3.2), Lemma 3.1, Proposition 2.10, the independence of $\Delta_1(N)$ and W_p. When A_{N-p} is diagonal, the result follows using (4.29).

4. Proofs of Theorems 1.11 and 1.12

Let M_N be defined by (1.2) with assumptions **(W)** and **(A')**. We denote by $\lambda_i(A_N)$, resp. $\lambda_i(M_N)$, the eigenvalues of A_N, resp. M_N and u_i, resp. v_i the normalized associated eigenvectors. According to assumption **(A)**, there exists $\delta > 0$ such that for all large N, the distance from θ to the rest of the spectrum (that is the other eigenvalues of A_N except θ) is greater than δ. Moreover, from Proposition 1.5, we know that a.s.

$$\lambda_{i_0}(M_N) \longrightarrow_{N\to+\infty} \rho_\theta = \theta + \sigma^2 g_\nu(\theta). \tag{4.1}$$

and there exists $\delta_0 > 0$ such that almost surely for all large N, the distance from ρ_θ to the rest of the spectrum (that is the other eigenvalues of M_N except $\lambda_{i_0}(M_N)$) is greater than δ_0.

Throughout this section, h is a smooth function with support in $]\rho_\theta - \delta_0/2; \rho_\theta + \delta_0/2[$ which is equal to 1 near ρ_θ.

4.1. *Representation in terms of resolvent.* The aim of this section is to be brought back to the study of the fluctuations of the $p \times p$-matrix valued process $\{G_p(z), z \in \mathbb{C}\backslash\mathbb{R}\}$ where $G_p(z)$ denotes the principal submatrix of size p of the resolvant matrix $G(z) = (zI_N - M_N)^{-1}$.

Proposition 4.1. *Almost surely, for all large N,*

$$\sqrt{N}(|\langle u_{i_0}, v_{i_0}\rangle|^2 - \tau_N(\theta))$$
$$= -\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial F_k(h)(z)\left(P^*\sqrt{N}\left(G_p(z) - \Lambda_p(z)\right)P\right)_{11} d^2z,$$

where $\tau_N(\theta)$ is defined by (1.11),

$$\Lambda_p(z) = (zI_p - A_p - \sigma^2 \tilde g_{N-p}(z) I_p)^{-1} \tag{4.2}$$

and $\tilde g_{N-p}(z)$ is the Stieltjes transform of $\mu_{sc} \boxplus \mu_{A_{N-p}}$.

Proposition 4.1 readily follows from the two preliminaries Lemmas 4.2 and 4.3.

Lemma 4.2. *Almost surely, for all large N, for any integer number k,*

$$|\langle u_{i_0}, v_{i_0}\rangle|^2 = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\,(P^*G_p(z)P)_{11}\,d^2z, \tag{4.3}$$

where $F_k(h)$ is defined by (2.8).

Proof: Let f be any smooth function with support in $]\theta-\delta/2;\theta+\delta/2[$ which is equal to 1 near θ. From the formula

$$\mathrm{Tr}_N(h(M_N)f(A_N)) = \sum_{i,j=1}^{N} h(\lambda_i(M_N))f(\lambda_j(A_N))|\langle u_j, v_i\rangle|^2, \tag{4.4}$$

we easily deduce that, almost surely, for all large N,

$$|\langle u_{i_0}, v_{i_0}\rangle|^2 = \sum_{i,j=1}^{p}(P^*)_{1i}h(M_N)_{ij}P_{j1}. \tag{4.5}$$

By Helffer-Sjöstrand's representation formula (2.7), we can write $h(M_N)_{ij}$ as

$$h(M_N)_{ij} = -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}F_k(h)(z)\;G_{ij}(z)d^2z,$$

so that

$$\sum_{i,j=p}^{p}(P^*)_{1i}h(M_N)_{ij}P_{j1} = -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}F_k(h)(z)\,(P^*G_p(z)P)_{11}\,d^2z. \tag{4.6}$$

Lemma 4.2 follows from (4.5) and (4.6). □

Lemma 4.3. *For N large enough,*

$$\begin{aligned}\tau_N(\theta) &= 1-\sigma^2\int\frac{1}{(\theta-x)^2}d\mu_{A_{N-p}}(x)\\ &= -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}F_k(h)(z)\left(P^*(zI_p-A_p-\sigma^2\tilde{g}_{N-p}(z)I_p)^{-1}P\right)_{11}d^2z\end{aligned}$$

where $\tilde{g}_{N-p}(z)$ is the Stieltjes transform of $\mu_{sc}\boxplus\mu_{A_{N-p}}$.

Proof: Note that

$$\left(P^*(zI_p-A_p-\sigma^2\tilde{g}_{N-p}(z)I_p)^{-1}P\right)_{11} = \frac{1}{z-\sigma^2\tilde{g}_{N-p}(z)-\theta}.$$

Let us define for any $z\in\mathbb{C}\setminus\mathrm{supp}(\mu_{A_{N-p}})$,

$$H_{N-p}(z) = z+\sigma^2 g_{\mu_{A_{N-p}}}(z) \tag{4.7}$$

and for any $z\in\mathbb{C}\setminus\mathrm{supp}(\mu_{sc}\boxplus\mu_{A_{N-p}})$,

$$\omega_{N-p}(z) = z-\sigma^2 g_{\mu_{sc}\boxplus\mu_{A_{N-p}}}(z) = z-\sigma^2\tilde{g}_{N-p}(z). \tag{4.8}$$

Note that, for any $z\in\mathbb{C}\backslash\mathbb{R}$, $|\Im\omega_{N-p}(z)|\geq|\Im z|>0$. Moreover, according to (2.16), the following one to one correspondance holds:

$$\mathbb{R}\backslash\mathrm{supp}(\mu_{sc}\boxplus\mu_{A_{N-p}})\ \underset{H_{N-p}}{\overset{\omega_{N-p}}{\rightleftarrows}}\ \{u\in\mathbb{R}\backslash\mathrm{supp}\ (\mu_{A_{N-p}}),\int\frac{1}{(u-x)^2}d\mu_{A_{N-p}}(x)<\frac{1}{\sigma^2}\}$$

and for any $x \in \mathbb{C} \setminus \operatorname{supp}(\mu_{sc} \boxplus \mu_{A_{N-p}})$, $H_{N-p}(\omega_{N-p}(x)) = x$. Hence $\rho_N = H_{N-p}(\theta) = \theta + \sigma^2 g_{\mu_{A_{N-p}}}(\theta)$ is the single pole of $\frac{1}{z-\sigma^2\tilde{g}_{N-p}(z)-\theta}$ in $\mathbb{C}$. Therefore, (1.8) and Proposition 2.4 (used with $\phi(z) = \frac{1}{\omega_{N-p}(z)-\theta)}$) imply that

$$-\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\,\frac{1}{z-\sigma^2\tilde{g}_{N-p}(z)-\theta}d^2z = \frac{1}{\omega'_{N-p}(\rho_N)} = H'_{N-p}(\theta) = \tau_N(\theta). \quad (4.9)$$

□

We now consider the process

$$\xi_N(z) = \left(P^*\sqrt{N}\left(G_p(z) - \Lambda_p(z)\right)P\right)_{11}. \quad (4.10)$$

where Λ_p is defined by (4.2).

4.2. *Tightness of the sequence of processes* $\{\xi_N\}_N$.

Proposition 4.4. *ξ_N is tight on $\mathcal{H}(\mathbb{C}\setminus\mathbb{R})$.*

Proof: $\xi_N : z \mapsto \left(P^*\sqrt{N}\left(G_p(z) - \Lambda_p(z)\right)P\right)_{11}$ is analytic on $\mathbb{C}\setminus\mathbb{R}$. Let K be a compact set in $\mathbb{C}\setminus\mathbb{R}$. According to Lemma 2.6, there exists $\delta > 0$ such that $\overline{K_\delta} \subset \mathbb{C}\setminus\mathbb{R}$ and for any $r > 0$,

$$\|\xi_N\|_K^r \le (\pi\delta^2)^{-1}\int_{\overline{K_\delta}} |\xi_N(z)|^r\, m(dz).$$

Therefore

$$\begin{aligned}\mathbb{E}\left(\|\xi_N\|_K^r\right) &\le (\pi\delta^2)^{-1}\int_{\overline{K_\delta}} \mathbb{E}\left(|\xi_N(z)|^r\right) m(dz) && (4.11)\\ &\le (\pi\delta^2)^{-1}\sup_{z\in\overline{K_\delta}} \mathbb{E}\left(|\xi_N(z)|^r\right) m(\overline{K_\delta}). && (4.12)\end{aligned}$$

In order to prove the tightness of ξ_N, using (2.12) and (4.12), we are going to show that, for any compact set $K \subset \mathbb{C}\setminus\mathbb{R}$, there exists a constant $C' > 0$ such that for all large N,

$$\sup_{z\in K}\mathbb{E}\left(|\xi_N(z)|^2\right) < C'. \quad (4.13)$$

We have for any z_1 and z_2 in $\mathbb{C}\setminus\mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\left(\xi_N(z_1)\xi_N(z_2)\right)\\ &= N\sum_{i,j,u,v=1}^{p} P^*_{1i}P_{j1}P^*_{1u}P_{v1}\mathbb{E}\left(\left(G_{ij}(z_1) - (\Lambda_p(z_1))_{ij}\right)\left(G_{uv}(z_2) - (\Lambda_p(z_2))_{uv}\right)\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\left(G_{ij}(z_1) - (\Lambda_p(z_1))_{ij}\right)\left(G_{uv}(z_2) - (\Lambda_p(z_2))_{uv}\right)\right)\\ &\quad= \mathbb{E}\left(\left(G_{ij}(z_1) - \mathbb{E}\left(G_{ij}(z_1)\right)\right)\left(G_{uv}(z_2) - \mathbb{E}\left(G_{uv}(z_2)\right)\right)\right)\\ &\qquad+\left[\mathbb{E}\left(G_{ij}(z_1)\right) - (\Lambda_p(z_1))_{ij}\right]\left[\mathbb{E}\left(G_{uv}(z_2)\right) - (\Lambda_p(z_2))_{uv}\right].\end{aligned}$$

According to Lemma 2.11 and the block diagonal structure of A_N, we have for any $l = 1, 2$,

$$\mathbb{E}\left(G_p(z_l)\right) = \left((z_l - \sigma^2\tilde{g}_N(z_l))I_p - A_p\right)^{-1} + O(1/\sqrt{N}), \quad (4.14)$$

where $\tilde{g}_N$ is the Stieltjes transform of $\mu_{A_N} \boxplus \mu_{sc}$. First set

$$\check{G}_{N-p}(w) = \left(zI_{N-p} - \frac{W_{N-p}}{\sqrt{N-p}} - A_{N-p} \right)^{-1}.$$

By (2.5), we have

$$\|\check{G}_{N-p}(z)\| \leq |\Im z|^{-1}. \tag{4.15}$$

Note that

$$\begin{aligned} \hat{G}_{N-p}(z) &= \check{G}_{N-p}(z) \\ &+ \frac{p}{\sqrt{N-p}(\sqrt{N}+\sqrt{N-p})} \times \Big(I_{N-p} - \hat{G}_{N-p}(z)\,(zI_{N-p} - A_{N-p}) \Big)\, \check{G}_{N-p}(z). \end{aligned} \tag{4.16}$$

Now, Lemma 2.12 yields that

$$\tilde{g}_N(z) = \mathbb{E}\left(\operatorname{tr}_N G(z)\right) + O(1/N)$$

and

$$\tilde{g}_{N-p}(z) = \mathbb{E}\left(\operatorname{tr}_{N-p} \check{G}_{N-p}(z)\right) + O(1/N),$$

and then, using (4.16), that

$$\tilde{g}_{N-p}(z) = \mathbb{E}\left(\operatorname{tr}_{N-p} \hat{G}_{N-p}(z)\right) + O(1/N). \tag{4.17}$$

Since by (A.1.12) in Bai and Silverstein (2010), we have:

$$\mathbb{E}\left(\operatorname{tr}_N G(z)\right) = \mathbb{E}\left(\operatorname{tr}_{N-p} \hat{G}_{N-p}(z)\right) + O(1/N),$$

we can deduce that $\tilde{g}_N(z) = \tilde{g}_{N-p}(z) + O(1/N)$, and thus that, for any $1 \leq i, j \leq p$,

$$\mathbb{E}\left(G_{ij}(z_l)\right) = (\Lambda_p(z_l))_{ij} + O(1/\sqrt{N}).$$

Moreover, by Lemma 2.13,

$$\operatorname{Var}(G_{ij}(z_l)) = O(1/N).$$

It readily follows that there exist polynomials P_1 and P_2 with nonnegative coefficients such that

$$\mathbb{E}\left(\xi_N(z_1)\xi_N(z_2)\right) \leq P_1\left(|\Im z_1|^{-1}\right) P_2\left(|\Im z_2|^{-1}\right)$$

and then there exists some polynomial P_3 with nonnegative coefficients such that for all large N and all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathbb{E}\left(|\xi_N(z)|^2\right) \leq P_3\left(|\Im z|^{-1}\right). \tag{4.18}$$

This implies (4.13). Therefore, for any compact subset K in $\mathbb{C} \setminus \mathbb{R}$, there exists a constant $C > 0$ such that for all large N, $\mathbb{E}\left(\|\xi_N(z)\|_K^2\right) < C$ and the tightness of ξ_N in $\mathcal{H}(\mathbb{C} \setminus \mathbb{R})$ follows from (2.12) and Proposition 2.5. $\square$

4.3. *Finite dimensional distributions of ξ_N.* Set

$$\nabla_N(z) = \sqrt{N}\left(G_p(z) - \Lambda_p(z)\right).$$

We use Proposition 2.1 for the inversion of

$$zI_N - M_N = \begin{pmatrix} zI_p - \frac{1}{\sqrt{N}}W_p - A_p & -\frac{1}{\sqrt{N}}Y^* \\ -\frac{1}{\sqrt{N}}Y & zI_{N-p} - M_{N-p} \end{pmatrix}$$

where M_{N-p} is the lower right submatrix of size $N-p$ of M_N, leading to:

$$G_p(z) = (zI_p - \frac{1}{\sqrt{N}}W_p - A_p - \frac{1}{N}Y^*\hat{G}_{N-p}(z)Y)^{-1} \tag{4.19}$$

where $\hat{G}_{N-p}$ is the resolvent of M_{N-p}. Thus,

$$\nabla_N(z) = G_p(z)(W_p + \sqrt{N}(\frac{1}{N}Y^*\hat{G}_{N-p}(z)Y - \sigma^2\tilde{g}_{N-p}(z)I_p))\Lambda_p(z). \tag{4.20}$$

Lemma 4.5. *Define for $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\tilde{\nabla}_N(z) = (zI_p - A_p - \sigma^2 g(z)I_p)^{-1}(W_p + Q_N(z))(zI_p - A_p - \sigma^2 g(z)I_p)^{-1}, \tag{4.21}$$

where $Q_N(z)$ is the following matrix of size p :

$$Q_N(z) = \frac{1}{\sqrt{N}}(Y^*\hat{G}_{N-p}(z)Y - \sigma^2\operatorname{Tr}_{N-p}(\hat{G}_{N-p}(z))I_p), z \in \mathbb{C}\setminus\mathbb{R} \tag{4.22}$$

For any $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\nabla_N(z) - \tilde{\nabla}_N(z) \xrightarrow[N\to\infty]{\mathbb{P}} 0.$$

Proof: Obviously, we have

$$\Lambda_p(z) \xrightarrow[N\to\infty]{} (zI_p - A_p - \sigma^2 g(z)I_p)^{-1}, \tag{4.23}$$

and (4.14) and Lemma 2.13 yield that

$$G_p(z) \xrightarrow[N\to\infty]{\mathbb{P}} (zI_p - A_p - \sigma^2 g(z)I_p)^{-1}. \tag{4.24}$$

We write

$$\begin{aligned} &\sqrt{N}(\tfrac{1}{N}Y^*\hat{G}_{N-p}(z)Y - \sigma^2\tilde{g}_{N-p}(z)I_p) \\ &\qquad = \frac{1}{\sqrt{N}}(Y^*\hat{G}_{N-p}(z)Y - \sigma^2\operatorname{Tr}_{N-p}(\hat{G}_{N-p}(z))I_p) \\ &\qquad\quad + \sqrt{N}\sigma^2(\operatorname{tr}_{N-p}(\hat{G}_{N-p}(z))I_p - \tilde{g}_{N-p}(z)I_p). \end{aligned} \tag{4.25}$$

(4.17) and Lemma 2.14 yield that

$$\sqrt{N}\sigma^2(\operatorname{tr}_{N-p}(\hat{G}_{N-p}(z))I_p - \tilde{g}_{N-p}(z)I_p) \xrightarrow[N\to\infty]{\mathbb{P}} 0. \tag{4.26}$$

Lemma 4.5 readily follows from (4.20), (4.25), (4.23), (4.24), (4.26) and the tightness of $Q_N(z)$ (which readily follows from Lemma 2.8), by using Slutsky's theorem and classical operations on convergence in probability. □

We now state an approximation result in distribution for the finite dimensional distributions of the process $\{Q_N(z), z \in \mathbb{C}\backslash\mathbb{R}\}$.

Lemma 4.6. *Let $(z_1, \ldots, z_q)$ be in $(\mathbb{C} \setminus \mathbb{R})^q$. Set $V_N = (Q_N(z_1), \ldots, Q_N(z_q))$. Then, under the assumptions of Theorem 1.11*

$$d_{LP}(V_N, (\mathcal{G}_N(z_1), \ldots, \mathcal{G}_N(z_q))) \to 0$$

where $\mathcal{G}_N$ is a a centered matrix valued Gaussian process whose distribution is given as follows :
1) the processes $((\mathcal{G}_N)_{ij}(z))_z$ for $1 \leq i \leq j \leq p$ are independent, and $(\mathcal{G}_N)_{ji}(z) = \overline{(\mathcal{G}_N)_{ij}(\bar{z})}$.
2) For $i \leq p$,

$$\mathbb{E}((\mathcal{G}_N)_{ii}(z_k)(\mathcal{G}_N)_{ii}(z_l)) \tag{4.27}$$
$$= \frac{(m_4 - 3\sigma^4)}{2(N-p)} \sum_{i=1}^{N-p} ((z_k - \sigma^2 g(z_k) - A_{N-p})^{-1})_{ii}((z_l - \sigma^2 g(z_l) - A_{N-p})^{-1})_{ii}$$
$$+\sigma^4 \int \frac{1}{(z_k - x)(z_l - x)} d\lambda(x) \tag{4.28}$$

3) For $1 \leq i \neq j \leq p$,

$$\mathbb{E}((\mathcal{G}_N)_{ij}(z_k)(\mathcal{G}_N)_{ij}(z_l)) = 0$$

and

$$\mathbb{E}((\mathcal{G}_N)_{ij}(z_k)\overline{(\mathcal{G}_N)_{ij}(z_l)}) = \sigma^4 \int \frac{1}{(z_k - x)(\bar{z}_l - x)} d\lambda(x).$$

Proof: We apply Proposition 2.10 to the matrices $B(z) = \hat{G}_{N-p}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Note that these matrices are random but they are independent of the $N \times p$ matrix Y.
In order to conclude, we need to show that

$$I_N := \frac{1}{N-p} \sum_{i=1}^{N-p} ((\hat{G}_{N-p}(z_1))_{ii}(\hat{G}_{N-p}(z_2))_{ii}) = f_N(z_1, z_2) + o(1) \tag{4.29}$$

with

$$f_N(z_1, z_2) = \frac{1}{N-p} \sum_{i=1}^{N-p} ((z_k - \sigma^2 g(z_k) - A_{N-p})^{-1})_{ii}((z_l - \sigma^2 g(z_l) - A_{N-p})^{-1})_{ii}$$

and

$$J_N := \mathrm{tr}_{N-p}(\hat{G}_{N-p}(z_1)\hat{G}_{N-p}(z_2)) \xrightarrow[N\to\infty]{\mathbb{P}} \int \frac{1}{(z_1 - x)(z_2 - x)} d\lambda(x).$$

The second convergence follows from the convergence of $\mu_{M_{N-p}}$ towards λ.
For the first one, Lemma 2.11 and (4.16) yield that, for $k \leq N - p$,

$$\begin{aligned}\mathbb{E}\left((\hat{G}_{N-p}(z))_{kk}\right) &= \left[\left((z - \sigma^2 \tilde{g}_{N-p}(z))I_N - A_{N-p}\right)^{-1}\right]_{kk} + O\left(\frac{1}{\sqrt{N}}\right) \\ &= \left[\left((z - \sigma^2 g(z))I_N - A_{N-p}\right)^{-1}\right]_{kk} + o^{(u)}(1).\end{aligned}$$

Thus, using Lemma 2.13, we can deduce that

$$\mathbb{E}((\hat{G}_{N-p}(z_1))_{kk}(\hat{G}_{N-p}(z_2))_{kk})$$
$$= \left[(z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1}\right]_{kk} \left[(z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1}\right]_{kk} + o^{(u)}(1) \tag{4.30}$$

From Lemma 5.1 in the Appendix, using that $f_N(W) = \frac{1}{N-p}\sum_{i=1}^{N-p}[(z_1 - W - A_{N-p})^{-1}]_{ii}[(z_2 - W - A_{N-p})^{-1}]_{ii}$ is Lipschitz with constant $|\Im(z_1)|^{-2}|\Im(z_2)|^{-1} + |\Im(z_1)|^{-1}|\Im(z_2)|^{-2}$, we can deduce that

$$\frac{1}{N-p}\sum_{i=1}^{N-p}(\hat{G}_{N-p}(z_1))_{ii}(\hat{G}_{N-p}(z_2))_{ii}$$
$$= \frac{1}{N-p}\sum_{i=1}^{N-p}\mathbb{E}\left((\hat{G}_{N-p}(z_1))_{ii}(\hat{G}_{N-p}(z_2))_{ii}\right) + o_{\mathbb{P}}(1).$$

We also use to obtain (4.28) from Proposition 2.10 that for $\sigma^2 = 1$,

$$\mathbb{E}(|y_{11}|^4) - 2 = \frac{1}{2}(m_4 - 3),$$

where we recall that μ is the distribution of $\sqrt{2}\Re y_{11}$ and $\sqrt{2}\Im y_{11}$. □

Corollary 4.7. *Under the assumptions of Theorem 1.11, for any* $(z_1,\dots,z_q)$ *be in* $(\mathbb{C}\setminus\mathbb{R})^q$,

$$d_{LP}((\xi_N(z_1),\dots,\xi_N(z_q)),(\mathcal{T}_N(z_1),\dots,\mathcal{T}_N(z_q)))\to 0$$

where ξ_N *is the process defined by* (4.10) *and*

$$\mathcal{T}_N(z) = (z-\theta-\sigma^2 g(z))^{-2}\left(P^*(W_p + \mathcal{G}_N(z))P\right)_{11} \tag{4.31}$$

$\mathcal{G}_N$ *being a centered Gaussian matrix valued process independent from* W_p *whose distribution is described in Lemma 4.6.*

4.4. *Fluctuations of the eigenvector.* Recall from Proposition 4.1 that, for any $k \in \mathbb{N}^*$, Φ_N defined as

$$\sqrt{N}(|\langle u_{i_0}, v_{i_0}\rangle|^2 - \tau_N(\theta))$$

has the representation

$$\Phi_N = -\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}F_k(h)(z)\xi_N(z)d^2z. \tag{4.32}$$

We follow the proof of Lemma 6.3 in Najim and Yao (2016) based upon the following estimates, from (4.18) :

$$\sup_N \mathbb{E}(|\xi_N(z)|) \le P_3(|\Im(z)|^{-1}), \tag{4.33}$$

and from Lemma 4.6 and (2.6), ($\mathcal{T}_N$ being defined by (4.31))

$$\sup_N \mathbb{E}(|\mathcal{T}_N(z)|) \le P_5(|\Im(z)|^{-1}), \tag{4.34}$$

where P_3, P_4 and P_5 are some polynomial with nonnegative coefficients. Hence in the following, in (4.32), we choose k greater than the degrees of P_3, P_4 and P_5.

Proposition 4.8. *Under the assumptions of Theorem 1.11,* $d_{LP}(\Phi_N,\tilde{\Phi}_N)\to 0$ *where* $\tilde{\Phi}_N$ *is given by*

$$\tilde{\Phi}_N = (P^*(c_{\theta,\nu}W_p + Z_{p,N})P)_{11},$$

where W_p *is a Wigner matrix of size* p, $Z_{p,N}$ *is a centered Gaussian Hermitian matrix of size* p *with independent entries (modulo the symmetry condition); the diagonal coefficients are iid with variance*

$$\frac{1}{2}(m_4 - 3\sigma^4)A_{\theta,\nu,N} + \sigma^4 B_{\theta,\nu} \tag{4.35}$$

and the off diagonal elements are i.i.d. complex Gaussian with distribution Z such that $\mathbb{E}(Z^2) = 0$ and $\mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu}$.

See Eq.(1.18) for the definitions of $c_{\theta,\sigma}$, $B_{\theta,\nu}$ and (1.15) for the definition of $A_{\theta,\nu,N}$.

Proof: For the reader's convenience, we repeat here the strategy of Lemma 6.3 in Najim and Yao (2016). For any $0 < \epsilon < 1$ small enough such that $\chi \equiv 1$ on $]-\epsilon;\epsilon[$, define

$$D_\epsilon = \{z \in \mathbb{C}, |\Im(z)| \geq \epsilon\}.$$

Set

$$U_N = \int_{\mathbb{C}} \bar{\partial} F_k(h)(z)\xi_N(z)d^2z,\ U_N^\epsilon = \int_{D_\epsilon} \bar{\partial} F_k(h)(z)\xi_N(z)d^2z$$

and

$$V_N = \int_{\mathbb{C}} \bar{\partial} F_k(h)(z)\mathcal{T}_N(z)d^2z,\ V_N^\epsilon = \int_{D_\epsilon} \bar{\partial} F_k(h)(z)\mathcal{T}_N(z)d^2z,$$

where $\mathcal{T}_N$ is defined by (4.31). Let f be a bounded continuous complex function on $\mathbb{C}$.

We have

$$\begin{aligned} |\mathbb{E}(f(U_N)) - \mathbb{E}(f(V_N))| &\leq |\mathbb{E}(f(U_N)) - \mathbb{E}(f(U_N^\epsilon))| \\ &\quad + |\mathbb{E}(f(U_N^\epsilon)) - \mathbb{E}(f(V_N^\epsilon))| \\ &\quad + |\mathbb{E}(f(V_N^\epsilon)) - \mathbb{E}(f(V_N))| \end{aligned}$$

Let $\delta > 0$.

i) For any $\eta > 0$ and $K > 0$, we have

$$\begin{aligned} &|\mathbb{E}(f(U_N)) - \mathbb{E}(f(U_N^\epsilon))| \\ &\leq \left|\mathbb{E}(f(U_N) - f(U_N^\epsilon))\,\mathbf{1}_{|U_N - U_N^\epsilon| > \eta}\right| && (4.36) \\ &\quad + \left|\mathbb{E}(f(U_N) - f(U_N^\epsilon))\,\mathbf{1}_{|U_N - U_N^\epsilon| \leq \eta, |U_N| \vee |U_N^\epsilon| > K}\right| && (4.37) \\ &\quad + \left|\mathbb{E}(f(U_N) - f(U_N^\epsilon))\,\mathbf{1}_{|U_N - U_N^\epsilon| \leq \eta, |U_N| \vee |U_N^\epsilon| \leq K}\right| && (4.38) \end{aligned}$$

In the following, the constant $C > 0$ may vary from line to line. By (2.10), for any $z = x + iy$ in a neighborhood of the real axis,

$$|\bar{\partial} F_k(h)(z)| \leq C|y|^k. \tag{4.39}$$

(2.9), (4.33) and (4.39) readily yield that for any $\epsilon > 0$, for any N,

$$\mathbb{E}(|U_N|) \vee \mathbb{E}(|U_N^\epsilon|) \leq \int_{\mathbb{C}} |\bar{\partial} F_k(h)(z)| \mathbb{E}|\xi_N(z)| d^2z \leq C,$$

and therefore

$$\begin{aligned} \mathbb{P}(|U_N| \vee |U_N^\epsilon| > K) &\leq \mathbb{P}(|U_N| > K) + \mathbb{P}(|U_N^\epsilon| > K) \\ &\leq \frac{2}{K}\mathbb{E}(|U_N|) \vee \mathbb{E}(|U_N^\epsilon|) \leq \frac{C}{K}. \end{aligned}$$

Thus, we can choose K such that, for any $\epsilon > 0$, any $\eta > 0$ and any N, the RHS in (4.37) is smaller than δ. Now, since f is uniformly continuous on $\{z \in \mathbb{C} \setminus \mathbb{R}, |z| \leq K\}$, one can choose η small enough such that, for any $\epsilon > 0$ and any N, (4.38) are smaller that δ. Finally, by using (4.33) and (4.39), for any ϵ small enough, for any N,

$$\mathbb{E}\left|U_N - U_N^\epsilon\right| = \mathbb{E}\left|\int_{\{z,|\Im(z)|<\epsilon\}} \bar{\partial}F_k(h)(z)\xi_N(z)d^2z\right| \leq C\epsilon, \tag{4.40}$$

and then

$$\mathbb{P}\left(|U_N - U_N^\epsilon| > \eta\right) \leq \frac{C}{\eta}\epsilon.$$

The term $|\mathbb{E}\left(f(V_N^\epsilon)\right) - \mathbb{E}\left(f(V_N)\right)|$ is treated in the same way, using (4.34).
ii) Note that since h and χ are compactly supported, $\int_{D_\epsilon} \bar{\partial}F_k(h)(z)\xi_N(z)d^2z$ may be seen as an integral on a fixed compact set $K_\epsilon \subset D_\epsilon$. Proposition 4.4 readily yields that ξ_N is tight on the set $\mathcal{C}(K_\epsilon)$ of complex continuous functions on K_ϵ. Thus, using Corollary 4.7 , since $f \mapsto \int_{D_\epsilon} \bar{\partial}F_k(h)(z)f(z)d^2z$ is continuous on $\mathcal{C}(K_\epsilon)$, it remains to prove the tightness of the process $\{\mathcal{T}_N(.)\}$ on any compact set K in $\{z, |\Im z| \geq \epsilon\}$ to deduce from Lemma 5.7 in Najim and Yao (2016) that:

$$|\mathbb{E}\left(f(U_N^\epsilon)\right) - \mathbb{E}\left(f(V_N^\epsilon)\right)| \to_{N\to+\infty} 0.$$

We postpone the proof of the tightness of the process $\{\mathcal{T}_N(.)\}$ (see Lemma 4.9 below) Thus, up to the proof of Lemma 4.9, the convergence to 0 of $d_{LP}(\Phi_n, -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\mathcal{T}_N(z)d^2z)$ follows.

It remains to compute $c_{\theta,\nu}, B_{\theta,\nu}, A_{\theta,\nu,N}$. The computation of $c_{\theta,\nu}$ follows from Proposition 2.4:

$$c_{\theta,\nu} = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\frac{1}{(z-\sigma^2 g(z)-\theta)^2}dz = \mathrm{Res}(\frac{1}{(z-\sigma^2 g(z)-\theta)^2}, \rho_\theta).$$

A straightforward computation gives

$$\mathrm{Res}(\frac{1}{(z-\sigma^2 g(z)-\theta)^2}, \rho_\theta) = -\frac{\omega''(\rho_\theta)}{(\omega'(\rho_\theta))^3} = H''(\theta)$$

where $\omega(z) = z - \sigma^2 g(z)$ $(g := g_\lambda)$ and $H(z) = z + \sigma^2 g_\nu(z)$.

From Lemma 4.6 and using Fubini theorem, the diagonal entries of Z_N have variance equal to

$$\frac{1}{2}(m_4 - 3\sigma^4)A_{\theta,\nu,N} + \sigma^4 B_{\theta,\nu}$$

and the off diagonal entries of Z_N have variance equal to $\sigma^4 B_{\theta,\nu}$ where

$$A_{\theta,\nu,N} = \frac{1}{N-p}\sum_{i=1}^{N-p}\left(\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\frac{[(z-\sigma^2 g(z)-A_{N-p})^{-1}]_{ii}}{(z-\sigma^2 g(z)-\theta)^2}d^2z\right)^2 d\nu(x)$$

$$B_{\theta,\nu} = \int_{\mathbb{R}}\left(\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}F_k(h)(z)\frac{1}{(z-\sigma^2 g(z)-\theta)^2(z-x)}d^2z\right)^2 d\lambda(x).$$

The functions $\phi_i(z) = \frac{[(z-\sigma^2 g(z)-A_{N-p})^{-1}]_{ii}}{(z-\sigma^2 g(z)-\theta)^2}$ for $i \leq N-p$ and $\phi_x(z) = \frac{1}{(z-\sigma^2 g(z)-\theta)^2(z-x)}$ for $x \in \mathrm{supp}(\lambda)$ satisfy the hypothesis of Proposition 2.4.

Straightforward computations lead to:

$$A_{\theta,\nu,N} = \frac{1}{N-p}\sum_{i=1}^{N-p}(\mathrm{Res}(\frac{[(z-\sigma^2 g(z)-A_{N-p})^{-1}]_{ii}}{(z-\sigma^2 g(z)-\theta)^2},\rho_\theta))^2$$
$$= \frac{1}{N-p}\sum_{i=1}^{N-p}\left(H''(\theta)[(\theta I_{N-p}-A_{N-p})^{-1}]_{ii} - H'(\theta)[(\theta I_{N-p}-A_{N-p})^{-2}]_{ii}\right)^2$$
$$= \frac{1}{N-p}\sum_{i=1}^{N-p}\left(\sigma^2 g_\nu''(\theta)[(\theta I_{N-p}-A_{N-p})^{-1}]_{ii}-(1+\sigma^2 g_\nu'(\theta))[(\theta I_{N-p}-A_{N-p})^{-2}]_{ii}\right)^2$$

and

$$\begin{aligned} B_{\theta,\nu} &= \int_{\mathbb{R}}(\mathrm{Res}(\frac{1}{(z-\sigma^2 g(z)-\theta)^2(z-x)},\rho_\theta))^2 d\lambda(x) \\ &= -\frac{1}{6}g_\nu'''(\theta) - \frac{\sigma^2}{2}(g_\nu''(\theta))^2\frac{1+2\sigma^2 g_\nu'(\theta)}{1+\sigma^2 g_\nu'(\theta)}. \end{aligned}$$

□

We now prove the following Lemma, used in the proof of the above Proposition.

Lemma 4.9. *Let K be a compact subset in $\{z, |\Im z| \geq \epsilon\}$, for some $\epsilon > 0$. The process $\{\mathcal{T}_N(.)\}$ defined in (4.31) is a tight sequence on K, more precisely,*

$$\sup_{z_1,z_2\in K, n\in\mathbb{N}} \frac{\mathbb{E}(|\mathcal{T}_N(z_1)-\mathcal{T}_N(z_2)|^2)}{|z_1-z_2|^2} < \infty. \tag{4.41}$$

Proof: From Lemma 4.6,

$$\mathbb{E}(|(\mathcal{G}_N)_{ij}(z_1)-(\mathcal{G}_N)_{ij}(z_2)|^2) = \delta_{ij}\frac{1}{2}(m_4-3\sigma^4)\times$$
$$\frac{1}{N-p}\sum_{i=1}^{N-p}(|((z_1-\sigma^2 g(z_1)-A_{N-p})^{-1})_{ii}-((z_2-\sigma^2 g(z_2)-A_{N-p})^{-1})_{ii}|^2)$$
$$+\sigma^4\int|\frac{1}{z_1-x}-\frac{1}{z_2-x}|^2 d\lambda(x).$$

From the resolvent identity,

$$(z_1-\sigma^2 g(z_1)-A_{N-p})^{-1}-(z_2-\sigma^2 g(z_2)-A_{N-p})^{-1}$$
$$=(z_2-z_1-\sigma^2(g(z_2)-g(z_1)))(z_1-\sigma^2 g(z_1)-A_{N-p})^{-1}(z_2-\sigma^2 g(z_2)-A_{N-p})^{-1},$$

thus,

$$\begin{aligned} &|((z_1-\sigma^2 g(z_1)-A_{N-p})^{-1})_{ii}-((z_2-\sigma^2 g(z_2)-A_{N-p})^{-1})_{ii}| \\ &\qquad\leq \frac{1}{\varepsilon^2}(1+\frac{\sigma^2}{\epsilon^2})|z_1-z_2| \end{aligned}$$

and thus,

$$\begin{aligned} &\frac{1}{N-p}\sum_{i=1}^{N-p}(|((z_1-\sigma^2 g(z_1)-A_{N-p})^{-1})_{ii}-((z_2-\sigma^2 g(z_2)-A_{N-p})^{-1})_{ii}|^2) \\ &\qquad\leq \frac{1}{\varepsilon^4}(1+\frac{\sigma^2}{\epsilon^2})^2|z_1-z_2|^2. \end{aligned}$$

Since moreover

$$\int |\frac{1}{z_1 - x} - \frac{1}{z_2 - x}|^2 d\lambda(x) \leq |\Im z_1|^2 |z_2|^2 |z_1 - z_2|^2 \leq \frac{1}{\varepsilon^4} |z_1 - z_2|^2$$

(4.41) readily follows. The tightness follows from Kolmogorov's criterion (see Billingsley, 1999). □

Proposition 4.8 and Proposition 4.1 readily yield Theorem 1.11.

4.5. *Proof of Theorem 1.12.* Theorem 1.12 follows from Theorem 1.11 once we proved that $A_{\theta,\nu,N}$ converge to $A_{\theta,\nu}$.

Lemma 4.10. *Assume that the matrix A_N satisfies* **(A')** *with A_{N-p} diagonal. Then, the sequence $(A_{\theta,\nu,N})_N$ defined by (1.15) converges to $A_{\theta,\nu}$ defined by (1.18).*

Proof: Denote by d_i the eigenvalues of A_{N-p}.

$$\begin{aligned} A_{\theta,\nu,N} &= \frac{1}{N-p} \sum_{i=1}^{N-p} \left(H''(\theta)(\theta - d_i)^{-1} - H'(\theta)(\theta - d_i)^{-2} \right)^2 \\ &\longrightarrow_{N\to\infty} (H''(\theta))^2 \int \frac{1}{(\theta - x)^2} d\nu(x) - 2H'(\theta)H''(\theta) \int \frac{1}{(\theta - x)^3} d\nu(x) \\ &\qquad + (H'(\theta))^2 \int \frac{1}{(\theta - x)^4} d\nu(x) \\ &= -(H''(\theta))^2 g'_\nu(\theta) - H'(\theta)H''(\theta) g''_\nu(\theta) - \frac{1}{6} (H'(\theta))^2 g'''_\nu(\theta) \end{aligned}$$

Using, $H'(\theta) = 1 + \sigma^2 g'_\nu(\theta)$ and $H''(\theta) = \sigma^2 g''_\nu(\theta)$, we obtain the formula for $A_{\theta,\nu}$ given in (1.18). □

5. Appendix: Poincaré inequality and concentration phenomenon

A probabilty μ satisfies a Poincaré inequality if for any $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{C}$ such that f and f' are in $L^2(\mu)$,

$$\mathbf{V}(f) \leq C_{PI} \int |f'|^2 d\mu,$$

with $\mathbf{V}(f) = \int |f - \int f d\mu|^2 d\mu$.
If the law of a random variable X satisfies the Poincaré inequality with constant C_{PI} then, for any fixed $\alpha \neq 0$, the law of αX satisfies the Poincaré inequality with constant $\alpha^2 C_{PI}$.
Assume that probability measures $\mu_1, \ldots, \mu_M$ on $\mathbb{R}$ satisfy the Poincaré inequality with constant $C_{PI}(1), \ldots, C_{PI}(M)$ respectively. Then the product measure $\mu_1 \otimes \cdots \otimes \mu_M$ on $\mathbb{R}^M$ satisfies the Poincaré inequality with constant $C^*_{PI} = \max_{i \in \{1,\ldots,M\}} C_{PI}(i)$ in the sense that for any differentiable function f such that f and its gradient grad$f$ are in $L^2(\mu_1 \otimes \cdots \otimes \mu_M)$,

$$\mathbf{V}(f) \leq C^*_{PI} \int \|\mathrm{grad} f\|_2^2 d\mu_1 \otimes \cdots \otimes \mu_M$$

with $\mathbf{V}(f) = \int |f - \int f d\mu_1 \otimes \cdots \otimes \mu_M|^2 d\mu_1 \otimes \cdots \otimes \mu_M$.

Lemma 5.1 (Lemma 4.4.3 and Exercise 4.4.5 in Anderson et al., 2010 or Chapter 3 in Ledoux, 2001)**.** *Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^{\mathbb{M}}$ which satisfies a Poincaré inequality with constant C_{PI}. Then there exists $K_1 > 0$ and $K_2 > 0$ such that, for any Lipschitz function F on $\mathbb{R}^M$ with Lipschitz constant $|F|_{Lip}$,*

$$\forall \epsilon > 0,\, \mathbb{P}\left(|F - \mathbb{E}_{\mathbb{P}}(F)| > \epsilon\right) \leq K_1 \exp\left(-\frac{\epsilon}{K_2\sqrt{C_{PI}}|F|_{Lip}}\right).$$

Acknowledgements

The authors want to thank an anonymous referee who encouraged them to establish more general results and to provide a better readability, which led to an overall improvement of the paper.

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