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On the affine recursion on $\mathbb{R}_+^d$ in the critical case

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Abstract. We fix $d \geq 2$ and denote $\mathcal{S}$ the semi-group of $d \times d$ matrices with non negative entries. We consider a sequence $(A_n, B_n)_{n\geq 1}$ of i.i.d. random variables with values in $\mathcal{S} \times \mathbb{R}_+^d$ and study the asymptotic behavior of the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}_+^d$ defined by:

$$\forall n \geq 0, \qquad X_{n+1} = A_{n+1}X_n + B_{n+1},$$

where X_0 is a fixed random variable. We assume that the Lyapunov exponent of the matrices A_n equals 0 and prove, under quite general hypotheses, that there exists up to a multiplicative constant a unique Radon measure m on $(\mathbb{R}^+)^d$ which is invariant for the chain $(X_n)_{n\geq 0}$; furthermore, this measure m is infinite. The existence of m relies on a recent work by T.D.C. Pham about fluctuations of the norm of product of random matrices (Pham, 2018). Its unicity is a consequence of a general property, called "local contractivity", highlighted about 20 years ago by M. Babillot, Ph. Bougerol et L. Elie in the case of the one dimensional affine recursion (Babillot et al., 1997).

1. Introduction

The Kesten's stochastic recurrence equation

$$X_{n+1} = a_{n+1}X_n + b_{n+1}$$

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on $\mathbb{R}$, where the $(a_n, b_n)_{n\geq 1}$ are independent and identically distributed (i.i.d.) random variables with values in $\mathbb{R}_{*+} \times \mathbb{R}$, has been extensively studied, with special attention given to the existence of a solution in law and its properties, especially the tails of the solution.

This process, called sometimes "random coefficients autoregressive models" occurs in different domains, in particular in economics; it has been studied intensively for several decades by many authors in various context. We refer to the book by D. Buraczewski, E. Damek & T. Mikosch (Buraczewski et al., 2016) for a general survey of the topic, a concentrate of recent results with comments and references.

Before the end of the 1990s, most of the authors studied the case when $\mathbb{E}[\ln a_1] < 0$; this condition ensures that this model has a unique stationary solution when $\mathbb{E}[\ln^+ |b_1|] < +\infty$.

Babillot et al. (1997), then Brofferio (2003), focus on the "critical case" $\mathbb{E}[\ln a_1] = 0$; they showed, under minimal assumptions on the distribution of the (a_n, b_n), that $(X_n)_n$ has a unique invariant Radon measure m, which is unbounded, and is recurrent on open sets of positive m-measure. The unicity is a consequence of a general property of stability of the trajectories at finite distance, called "local contractivity". This property is of interest for general iterated function systems (Peigné and Woess, 2011).

Simultaneously, the affine recursion $(X_n)_{n\geq 0}$ has been considered in dimension $d \geq 2$, the random variables a_n and b_n are replaced respectively by $d \times d$ random matrices A_n with real entries and random vectors B_n in $\mathbb{R}^d$. In this setting, the contractive case corresponds to the case when the Lyapunov exponent γ associated with the random matrices A_n is negative; various properties of the unique invariant probability have been obtained in this case, based on results of product of random matrices (see for instance Buraczewski et al., 2016, chap. 4 and references therein). As far as we know, the existence and unicity of an invariant Radon measure in the "critical case" $\gamma = 0$, is still an open question; the present paper proposes a partial answer to this problem, under some restrictive conditions on the matrices A_n and vectors B_n.

Let us introduce some notations. We fix $d \geq 2$ and endow $\mathbb{R}^d$ with the norm $|\cdot|$ defined by $|x| := \sum_{i=1}^{d} |x_i|$ for any column vector $x = (x_i)_{1\leq i\leq d}$. We denote $(e_i)_{1\leq i\leq d}$ the canonical basis of $\mathbb{R}^d$ and set $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}_{*+} =]0, +\infty[$.

Let $\mathcal{S}$ be the set of $d\times d$ matrices with nonnegative entries such that each column contains at least one positive entry. For any $A = (A(i,j))_{1\leq i,j\leq d} \in \mathcal{S}$, let

$$v(A) := \min_{1\leq j\leq d}\Big(\sum_{i=1}^{d} A(i,j)\Big) \quad \text{and} \quad \|A\| := \max_{1\leq j\leq d}\Big(\sum_{i=1}^{d} A(i,j)\Big).$$

The quantity $\|\cdot\|$ is a norm on $\mathcal{S}$ and $\|AB\| \leq \|A\| \times \|B\|$ for any $A, B \in \mathcal{S}$; furthermore, for any $A \in \mathcal{S}$ and $x \in \mathbb{R}_+^d$,

$$0 < v(A)\ |x| \leq |Ax| \leq \|A\|\ |x|. \tag{1.1}$$

Set $\mathfrak{n}(A) := \max\left(\frac{1}{v(A)}, \|A\|\right)$ and notice that $\mathfrak{n}(A) \geq 1$.

For any $0 < \delta \leq 1$, let $\mathcal{S}_\delta$ be the subset of matrices A in $\mathcal{S}$ such that, for any $1 \leq i, j, k \leq d$,

$$A(i,j) \geq \delta A(i,k). \tag{1.2}$$

Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space and $(A_n, B_n)_{n\geq 1}$ be a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{T}, \mathbb{P})$ with distribution μ on $\mathcal{S} \times \mathbb{R}_+^d$. We are interested in the recurrence properties of the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}_+^d$ defined inductively by $X_{n+1} = A_{n+1}X_n + B_{n+1}$ for any $n \geq 0$. By an easy induction, we may write, for any $n \geq 1$

$$X_n = A_{n,1}X_0 + B_{n,1}$$

with $A_{n,1} = A_n \cdots A_1$ and $B_{n,1} = B_n + \sum_{k=1}^{n-1} A_n \cdots A_{k+1} B_k$.

When $X_0 = x$ for some fixed $x \in \mathbb{R}_+^d$, we set $X_n = X_n^x$. The conditional probability with respect to the event $(X_0 = x)$ is denoted by $\mathbb{P}_x$; more generally, for any probability measure m on $\mathbb{R}_+^d$, we set $\mathbb{P}_m(\cdot) = \int_{\mathbb{R}_+^d} \mathbb{P}_x(\cdot) m(\mathrm{d}x)$.

For short, we introduce the following notations: $A_{n,m} := A_n \cdots A_m$ for any $n \geq m \geq 1$, with the convention $A_{n,m} =$I when $m > n$.

Firstly, we introduce some hypotheses on the distribution μ of (A_n, B_n); we denote $\bar{\mu}$ the distribution of the matrices A_n and fix $\delta \in]0, 1]$.

Hypotheses A(δ)

A1- $\mathbb{E}\left[(\ln \mathfrak{n}(A_1))^{2+\delta}\right] < +\infty$.

A2- *There exists no affine subspaces $\mathcal{A}$ of $\mathbb{R}^d$ such that $\mathcal{A} \cap \mathbb{R}_+^d$ is non-empty, bounded and invariant under the action of all elements of the support of $\bar{\mu}$.*

A3- $\bar{\mu}(\mathcal{S}_\delta) = 1$.

A4- *The upper Lyapunov exponent* $\gamma_{\bar{\mu}} = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[\ln \|A_1 \cdots A_n\|]$ *of* $\bar{\mu}$ *equals* 0.

A5- $\bar{\mu}\{A \in \mathcal{S}/v(A) \geq 1 + \delta\} > 0$.

Hypotheses B(δ) *The random variables B_k are $\mathbb{R}_+^d$-valued, $\mathbb{P}[|B_1| > 0] > 0$ and*

$$\mathbb{E}[(\ln^+ |B_1|)^{2+\delta}] < +\infty.$$

A Radon measure m on $\mathbb{R}_+^d$ is said to be invariant for the process $(X_n)_{n\geq 0}$ if and only if

$$\int_B \mathbb{P}[X_1^x \in B] m(\mathrm{d}x) = m(B)$$

for any Borel set $B \subset \mathbb{R}_+^d$ such that $m(B) < +\infty$.

Now, let us state the main result of this paper.

Theorem 1.1. *Assume hypotheses* $\mathbf{A}(\delta)$ *and* $\mathbf{B}(\delta)$ *hold. Then, the process $(X_n)_{n\geq 0}$ is conservative: for any $x \in \mathbb{R}_+^d$,*

$$\liminf_{n\to+\infty} |X_n^x| < +\infty \quad \mathbb{P}\text{-}a.s.$$

Furthermore,

1. *there exists on $\mathbb{R}_+^d$ a unique Radon measure m which is invariant for $(X_n)_{n\geq 0}$;*
2. *this measure has an infinite mass;*
3. *there exist a positive slowly varying function*[1] *L on $\mathbb{R}_+$ and positive constants a, b, c such that for any $t \geq 1$,*

$$L(t) \leq m\{x \in \mathbb{R}_+^d / ta \leq |x| \leq tb\} \leq cL(t).$$

[1] the function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying if $\lim_{t\to+\infty} \frac{L(tx)}{L(t)} = 1$ for any $x > 0$.

By Peigné and Woess (2011), this statement implies that the chain $(X_n)_{n\geq 0}$ is m-topologically null recurrent: in other words, for any open set $U \subset \mathbb{R}_+^d$ such that $0 < m(U) < +\infty$, the stopping time $\tau^U := \inf\{n \geq 1/X_n \in U\}$ is $\mathbb{P}_{m_U}$-a.s. finite and has infinite expectation with respect to $\mathbb{P}_{m_U}$, where m_U is the probability measure defined by $m_U(\cdot) = \dfrac{m(\cdot \cap U)}{m(U)}$.

Assertion 3 gives some general description on the tail of the mesure m. In dimension 1, a similar statement does exist in Babillot et al. (1997) and has been improved by Brofferio and Buraczewski (2015) (see also their previous work with E. Damek, Brofferio et al., 2012): when the distribution of the real random variables $\ln A_n$ is "aperiodic"[2], the measure m is in fact equivalent at infinity to the Lebesgue measure; in other words, the slowly varying function L which appears above is constant in this case. Such a result when $d \geq 2$ is out of the scope of the present paper and would require a detailed understanding of renewal theory for centered Markov walks.

2. Random iterations and product of random matrices

2.1. *On stochastic dynamical systems.* The Markov chain $X_n, n \geq 0$, is a central example of the so-called "stochastic dynamical systems" $Z_n = Z_n^x$ on $\mathbb{R}^d$, or a closed subset C of $\mathbb{R}^d$, defined inductively by

$$Z_0^x = x \quad \text{and} \quad Z_{n+1}^x = f_{n+1}(Z_n^x) \quad \text{for all} \quad n \geq 0,$$

where x is a fixed point in C and $(f_n)_{n\geq 1}$ is a sequence of independent and identically distributed random variables with values in the set of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ (or from C to C).

The contraction properties of the maps f_n have a great influence on the recurrence/transience properties of the chain $(Z_n)_{n\geq 0}$. In Peigné and Woess (2011), one can find a quite general criteria which yields to the existence and uniqueness of an invariant Radon measure for the sequence $(Z_n)_{n\geq 0}$.

Firstly, we introduce the following "weak contraction property": a sequence $(F_n)_{n\geq 1}$ of continuous functions on $\mathbb{R}^d$ is said to be *locally contractive* when, for any $x, y \in E$ and any compact set $K \subset E$,

$$\lim_{n\to+\infty} |F_n(x) - F_n(y)| \; \mathbf{1}_K(F_n(x)) = 0.$$

This weak "contraction property" is of interest and yields to deep consequences in the context of stochastic dynamical systems. Let us recall the main result of Peigné and Woess (2011) and assume that, $\mathbb{P}$-a.s., the sequence $(F_n)_{n\geq 1} = (f_n \circ \cdots \circ f_1)_{n\geq 1}$ is locally contractive on $C \subset \mathbb{R}^d$. Then

(i) either $|Z_n^x| \to +\infty$ $\mathbb{P}$-a.s. (in this case we say that $(Z_n)_{n\geq 0}$ is *transient*);
(ii) or $\liminf_{n\to+\infty} |Z_n^x| < +\infty$ $\mathbb{P}$-a.s. (in this case we say that $(Z_n)_{n\geq 0}$ is *conservative*).

Furthermore, in the conservative case, there exists on C a unique invariant Radon measure m for $(Z_n)_{n\geq 0}$.

If m is infinite, for any open set $U \subset E$ such that $0 < m(U) < +\infty$, the stopping time $\tau^U := \inf\{n \geq 1/Z_n \in U\}$ is $\mathbb{P}_{m_U}$-a.s. finite and has infinite expectation with

[2] a probability distribution on $\mathbb{R}$ is aperiodic when its support is not contained in some $a\mathbb{Z}$, $a > 0$.

respect to $\mathbb{P}_{m_U}$, where m_U denotes the probability measure $m(\cdot \cap U)/m(U)$. This last property corresponds to the null recurrence behavior of the Markov chain in the context of denumerable state space.

Let us emphasize that we do not require here any hypothesis of irreducibility on $\mathbb{R}^d$, as for instance in Élie (1982) where it is assumed that the measure μ is spread out, which implies that the chain $(X_n)_{n\geq 0}$ is Harris recurrent.

Application to the affine recursion on $\mathbb{R}_+^d$

Recall that $(A_n, B_n)_{n\geq 1}$ is a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{T}, \mathbb{P})$ with distribution μ on $\mathcal{S} \times \mathbb{R}_+^d$. For any $n \geq 1$, we denote g_n the random map on $\mathbb{R}_+^d$ defined by:

$$\forall x \in \mathbb{R}_+^d \quad g_n(x) = A_n x + B_n.$$

Notice that, for any $x \in \mathbb{R}_+^d$ and $n \geq 1$,

$$X_n^x = g_n \circ \cdots \circ g_1(x).$$

We prove in section 3 that the stochastic dynamical system $(X_n)_{n\geq 0}$ is conservative and that, $\mathbb{P}$-a.s., the sequence $(g_n \circ \cdots \circ g_1)_{n\geq 1}$ is locally contractive on $\mathbb{R}_+^d$. By the general results stated above, this yields the first assertion of Theorem 1.1.

2.2. *On the semi-group of positive random matrices.* Let $\mathbb{X}$ be the standard simplex in $\mathbb{R}_+^d$ defined by

$$\mathbb{X} := \{x \in \mathbb{R}_+^d / |x| = 1\}$$

and let $\mathring{\mathbb{X}}$ be its interior: $\mathring{\mathbb{X}} = \{x = (x_i)_{1\leq i\leq d} / x_i > 0 \text{ and } |x| = 1\}$.

Endowed with the standard multiplication of matrices, the set $\mathcal{S}$ is a semigroup; we consider the two following actions of $\mathcal{S}$:

- the left linear action on $\mathbb{R}_+^d$ defined by $(A, x) \mapsto Ax$ for any $A \in \mathcal{S}$ and $x \in \mathbb{R}_+^d$,
- the left projective action on $\mathbb{X}$ defined by $(A, x) \mapsto A \cdot x := \dfrac{Ax}{|Ax|}$ for any $A \in \mathcal{S}$ and $x \in \mathbb{X}$.

Notice that, for any $A \in \mathcal{S}$ and $x \in \mathbb{X}$, it holds

$$Ax = |Ax| \ \frac{Ax}{|Ax|} = \exp(\rho(A, x)) \ A \cdot x,$$

with $\rho(A, x) = \ln |Ax|$. The function $\rho : \mathcal{S} \times \mathbb{X} \to \mathbb{R}$ satisfies the following "cocycle property":

$$\forall A, A' \in \mathcal{S}, \forall x \in \mathbb{X} \quad \rho(AA', x) = \rho(A, A' \cdot x) + \rho(A', x).$$

Hence, for any $n \geq 1$, any $A_1, \ldots, A_n \in \mathcal{S}$ and any $x \in X$,

$$A_{n,1}x \ = \exp(S_n(x)) \ \xi_n$$

with $\xi_k := A_k \cdots A_1 \cdot x, 1 \leq k \leq n$, and

$$S_n(x) = \rho(A_n, \xi_{n-1}) + \rho(A_{n-1}, \xi_{n-2}) + \cdots + \rho(A_1, x).$$

This decomposition is of interest in order to control the linear action of product of random matrices, the behavior of the process $(|A_{n,1}x|)_{n\geq 1}$ and in particular its fluctuations.

Now we focus on some important properties of the set $\mathcal{S}_\delta$.

Lemma 2.1. *The set $\mathcal{S}_\delta$ is a semi-group. Furthermore, for any $A, B \in \mathcal{S}_\delta$ and any $x \in \mathbb{R}_+^d$,*

$$\delta\|A\|\ |x| \le |Ax| \le \|A\|\ |x| \qquad \text{and} \qquad \delta\|A\|\ \|B\| \ \le\ \|AB\| \ \le\ \|A\|\ \|B\|. \tag{2.1}$$

This type of property was first introduced by Furstenberg and Kesten (1960). They consider another subset of $\mathcal{S}$, namely the set $\mathcal{S}'_\Delta$ of matrices A satisfying the stronger condition:

$$\forall 1 \le i,j,k,l \le p \qquad \frac{1}{\Delta}A(i,j) \le A(k,l) \le \Delta A(i,j).$$

The main difference between $\mathcal{S}_\delta$ and $\mathcal{S}'_\Delta$ is that, for $A \in \mathcal{S}_\delta$, inequality (1.2) holds only for entries in the same line. In particular, elements in $\mathcal{S}'_\Delta$ have only positive entries while a matrix $A \in \mathcal{S}_\delta$ can have null coefficients: more precisely, if one entry of A equals 0, the same holds for all entries in the same line.

The set $\mathcal{S}'_\Delta$ is a proper subset of S_δ for $\delta = 1/\Delta$ but is not a semi-group. Nevertheless the closed semi-group $T_{\mathcal{S}'_\Delta}$ it generates satisfies the following property: for any $A \in T_{\mathcal{S}'_\Delta}$ and $1 \le i,j,k,l \le p$,

$$\frac{1}{\Delta^2}A(i,j) \ \le\ A(k,l) \ \le\ \Delta^2 A(i,j).$$

In other words, $T_{\mathcal{S}'_\Delta} \subset \mathcal{S}'_{\Delta^2}$.

Proof of Lemma 2.1: Let $A, B \in \mathcal{S}_\delta$; for any $1 \le i,j,k \le d$,

$$(AB)(i,j) = \sum_{l=1}^d A(i,l)B(l,j) \ge \delta \sum_{l}^d A(i,l)B(l,k) = \delta(AB)(i,k),$$

hence $AB \in \mathcal{S}_\delta$.

Let us prove (2.1). Inequalities $|Ax| \le \|A\|\ |x|$ and $\|AB\| \le \|A\|\ \|B\|$ are obvious. Furthermore,

$$|Ax| = \sum_{i,j=1}^d A(i,j)x_j \ge \delta \sum_{j=1}^d x_j \left(\sum_{i=1}^d A(i,k)\right)$$

for any $1 \le k \le d$, which readily yields $|Ax| \ge \delta\|A\|\ |x|$. At last,

$$\begin{aligned}
\|AB\| = \max_{1\le k\le d}\sum_{i=1}^d AB(i,k) &= \max_{1\le k\le d}\sum_{i,j=1}^d A(i,j)B(j,k)\\
&\ge \delta \max_{1\le k,l\le d}\sum_{i,j=1}^d A(i,l)B(j,k)\\
&= \delta \max_{1\le l\le d}\sum_{i=1}^d A(i,l) \max_{1\le k\le d}\sum_{j=1}^d B(j,k) = \delta\|A\|\ \|B\|.
\end{aligned}$$

□

Let us highlight an interesting property of the action on the cone $\mathbb{R}_+^d$ of elements of the semi-group $\mathcal{S}_\delta$. For any $A \in \mathcal{S}$, denote ${}^tA$ its transpose matrix; if $A \in \mathcal{S}_\delta$, then, for $1 \le i,j \le d$,

$$\langle e_i, {}^tAe_j\rangle = A(j,i) \quad \text{while} \quad |{}^tAe_j| = \sum_{k=1}^d A(j,k) \le \frac{d}{\delta}A(j,i).$$

Hence, $\langle e_i, {}^tAe_j\rangle \geq \dfrac{\delta}{d}|{}^tAe_j|$. In other words,

$${}^tA(\mathbb{R}_+^d) \subset \mathcal{C}_{\frac{\delta}{d}},$$

where $\mathcal{C}_c, c > 0$, denotes the proper sub-cone of $\mathbb{R}_+^d$ defined by

$$\mathcal{C}_c = \left\{x \in \mathbb{R}_+^d / \langle e_i, x\rangle \geq c|x| \text{ for } i = 1, \ldots, d\right\}.$$

Following Hennion (1997), we endow $\mathbb{X}$ with a bounded distance $\mathfrak{d}$ such that any $A \in \mathcal{S}$ acts on $\mathbb{X}$ as a contraction with respect to $\mathfrak{d}$. In the following lemma, we just recall some fundamental properties of this distance.

Lemma 2.2. *There exists a distance $\mathfrak{d}$ on $\mathbb{X}$ compatible with the standard topology of $\mathbb{X}$ satisfying the following properties:*

(1) $\sup\{\mathfrak{d}(x,y)/x, y \in \mathbb{X}\} = 1$.
(2) $|x-y| \leq 2\mathfrak{d}(x,y)$ *for any* $x, y \in \mathbb{X}$.
(3) *For any* $A \in \mathcal{S}$, *set* $[A] := \sup\{\mathfrak{d}(A\cdot x, A\cdot y)/x, y \in \mathbb{X}\}$*; then,*
   (a) $\mathfrak{d}(A\cdot x, A\cdot y) \leq [A]\mathfrak{d}(x,y)$ *for any* $x, y \in \mathbb{X}$*;*
   (b) $[AA'] \leq [A][A']$ *for any* $A, A' \in \mathcal{S}$*;*
(4) *There exists* $\rho_\delta \in]0,1[$ *such that* $[A] \leq \rho_\delta$ *for any* $A \in \mathcal{S}_\delta$.

Proof: The reader can find in Hennion (1997) a precise description of the properties of the distance $\mathfrak{d}$, that is defined as follows: for any $x, y \in \mathbb{R}_+^d \setminus \{0\}$, we write

$$\mathfrak{d}(x,y) := \frac{1 - m(x,y)m(y,x)}{1 + m(x,y)m(y,x)}$$

where $m(x,y) = \min\limits_{1\leq i\leq d}\left\{\dfrac{x_i}{y_i} | y_i > 0\right\}$. Notice that $\mathfrak{d}(x,y) = \mathfrak{d}(\lambda x, \mu y)$ for any $x, y \in \mathbb{R}_+^d \setminus \{0\}$ and $\lambda, \mu > 0$.

Properties 1 and 2 correspond to Lemma 10.2 and 10.4 in Hennion (1997). Property 3 is proved in Hennion (1997) Lemma 10.6 for matrices A with nonnegative entries such that each column **and** each line contains at least a positive entry. This property still holds for matrices in $\mathcal{S}$ that have some zero lines: heuristically, we can just restrict at the sub-simplex of $\mathbb{X}$ where it acts with positive entries. More formally, let $A \in \mathcal{S}$, fix i_0 such that $A(i_0, k) > 0$ for some $1 \leq k \leq d$ and denote B_A the element of $\mathcal{S}$ defined by:

$$B_A(i,j) = \begin{cases} A(i,j) & \text{if} \quad \sum_{k=1}^d A(i,k) > 0; \\ A(i_0,j) & \text{if} \quad \sum_{k=1}^d A(i,k) = 0. \end{cases}$$

Each column and each line of B_A contains a positive entry.

Notice that, for any x, y in $\mathbb{R}_+^d$ and $A \in \mathcal{S}$,

$$\mathfrak{d}(A\cdot x, A\cdot y) = \mathfrak{d}(Ax, Ay).$$

By a straightforward calculation,

$$m(Ax, Ay) = \min_{1\leq i\leq d}\left\{\frac{\sum_{k=1}^d x_k A(i,k)}{\sum_{k=1}^d y_k A(i,k)} | \sum_{k=1}^d y_k A(i,k) > 0\right\} = m(B_A x, B_A y),$$

thus $\mathfrak{d}(Ax, Ay) = \mathfrak{d}(B_A x, B_A y)$, $[A] = [B_A]$ and

$$\mathfrak{d}(A\cdot x, A\cdot y) = \mathfrak{d}(B_A\cdot x, B_A\cdot y) \leq [B_A]\mathfrak{d}(x,y) = [A]\mathfrak{d}(x,y).$$

This proves Property 3.a, then Property 3.b as in Hennion (1997). Let us now prove Property 4; for any $A \in \mathcal{S}_\delta$,

$$\frac{\sum_{k=1}^d x_k B_A(i,k)}{\sum_{k=1}^d y_k B_A(i,k)} \geq \delta^2 \frac{B_A(i,1)|x|}{B_A(i,1)|y|} = \delta^2 \frac{|x|}{|y|}.$$

Thus $m(Ax, Ay) \geq \delta^2 |x|/|y|$. The fact that the function $s \mapsto \frac{1-s}{1+s}$ is decreasing on $[0,1]$ yields $[A] \leq \frac{1-\delta^4}{1+\delta^4} < 1$. □

Property (4) of Lemma 2.2 readily implies that, for any $x, y \in \mathbb{X}$ and any $n \geq 0$,

$$\mathbb{E}\left[\mathfrak{d}\left(A_{n,1} \cdot x, A_{n,1} \cdot y\right)\right] \leq \rho_\delta^n. \tag{2.2}$$

As a direct consequence, the transition operator of the Markov chain $(A_{n,1} \cdot x)_{n \geq 0}$ on $\mathbb{X}$, restricted to the space of Lipschitz functions on $(\mathbb{X}, \mathfrak{d})$, is quasi-compact; we refer to Pham (2018) for a detailed proof.

2.3. *On fluctuations of the norm of product of random matrices.* In this subsection, we recall some recent result on fluctuations of the norm of product of random matrices. We consider a sequence of independent random matrices $(A_n)_{n \geq 0}$ with nonnegative coefficients, defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$ and with the same distribution $\bar{\mu}$ on $\mathcal{S}$. For any $n \geq 1$, denote $\mathcal{T}_n$ the σ-algebra generated by the random variables $A_1, \ \ldots \ , A_n$ and set $\mathcal{T}_0 = \{\emptyset, \Omega\}$.

We study here the left products of these random matrices defined as follows: $A_{n,m} = A_n A_{n-1} \cdots A_m$ for any $1 \leq m \leq n$; by convention $A_{n,m} = \mathrm{I}$ when $m > n$.

Fix $x \in \mathbb{X}$ and $a \geq 1$; the random variables

$$\tau^{x,a} := \min\{n \geq 1 : \, a|A_{n,1}x| \leq 1\} \quad \text{and} \quad \tau^a := \min\{n \geq 1 : \, a\|A_{n,1}\| \leq 1\}$$

are stopping times with respect to the canonical filtration $(\mathcal{T}_n)_{n \geq 0}$ associated with the sequence $(A_n)_{n \geq 1}$, with values in $\mathbb{N} \cup \{+\infty\}$. Furthermore $\tau^{x,a} \leq \tau^a$ $\mathbb{P}$-a.s.

Under hypotheses $\mathbf{A(\delta)}$, the sequence $(\ln \|A_{n,1}\|/\sqrt{n})_{n \geq 0}$ converges in distribution to a non degenerated and centered Gaussian distribution (with variance $\sigma^2 > 0$); by a standard argument in probability theory, it yields

$$\liminf_{n \to +\infty} \|A_{n,1}\| = 0 \quad \text{and} \quad \limsup_{n \to +\infty} \|A_{n,1}\| = +\infty \qquad \mathbb{P}\text{-a.s.} \tag{2.3}$$

Indeed, for any $c > 0$,

$$\begin{aligned} \mathbb{P}\left[\limsup_{n \to +\infty} \frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} > c\right] &\geq \mathbb{P}\left(\limsup_{n \to +\infty} \left[\frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} > c\right]\right) \\ &\geq \limsup_{n \to +\infty} \mathbb{P}\left[\frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} > c\right] = \frac{1}{\sqrt{2\pi}} \int_c^{+\infty} e^{-t^2/2} dt > 0. \end{aligned}$$

Now, observe that $\left[\limsup_{n \to +\infty} \frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} > c\right]$ is a tail event relative to the sequence $(A_n)_{n \geq 1}$; thus, by Kolmogorov's zero-one law, its probability equals 1. Hence

$$\mathbb{P}\left[\limsup_{n \to +\infty} \frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} = +\infty\right] = \mathbb{P}\left(\cap_{c \in \mathbb{N}} \left[\limsup_{n \to +\infty} \frac{\ln \|A_{n,1}\|}{\sigma\sqrt{n}} > c\right]\right) = 1.$$

Similarly,

$$\mathbb{P}\left[\liminf_{n\to+\infty}\frac{\ln\|A_{n,1}\|}{\sigma\sqrt{n}}<-c\right]\geq\mathbb{P}\left(\limsup_{n\to+\infty}\left[\frac{\ln\|A_{n,1}\|}{\sigma\sqrt{n}}<-c\right]\right)$$
$$\geq\frac{1}{\sqrt{2\pi}}\int_c^{+\infty}e^{-t^2/2}dt>0,$$

so that $\mathbb{P}\left[\liminf_{n\to+\infty}\frac{\ln\|A_{n,1}\|}{\sigma\sqrt{n}}=-\infty\right]=\mathbb{P}\left(\cap_{c\in\mathbb{N}}\left[\liminf_{n\to+\infty}\frac{\ln\|A_{n,1}\|}{\sigma\sqrt{n}}<-c\right]\right)=1.$ Property (2.3) follows immediately and implies that the stopping times $\tau^{x,a}$ and τ^a are $\mathbb{P}$-a.s. finite.

In Pham (2018), a precise estimate of the tail of the distribution of $\tau^{x,a}$ is obtained under a little bit different assumptions (Proposition 1.1 and Theorem 1.2); let us state the partial result we need in our context and explain briefly the amendments to the proofs given in Pham (2018).

Proposition 2.3. *Assume hypotheses* $\mathbf{A}(\delta)$. *Then, there exists a positive constant* κ *such that, for any* $x\in\mathbb{X}, a\geq 1$ *and* $n\geq 1$,

$$\mathbb{P}[\tau^{x,a}>n]=\mathbb{P}[a|A_1x|>1,\ \dots\ ,a|A_{n,1}x|>1]\leq\kappa\frac{1+\ln a}{\sqrt{n}}.$$

Our hypotheses **A2** and **A4** correspond exactly to **P2** and **P4** in Pham (2018); hypothesis **A5** is a little bit stronger than **P5**, it is more natural in our context.

Hypotheses **A3** and **P3** both imply the contraction property (2.2); this yields to the good spectral properties of the transition operator of the Markov chain $(A_{n,1}\cdot x)_{n\geq 0}$ on $\mathbb{X}$.

At last, existence of moments of order $2+\delta$ (our hypothesis **A1**) is sufficient instead of exponential moments **P1**. This ensures firstly that the function $t\mapsto P_t$ in Pham (2018), Proposition 2.3 is C^2, which is sufficient for this Proposition to hold. Secondly the martingale $(M_n)_{n\geq 0}$ which approximates the process $(S_n(x))_{n\geq 0}$ belongs to $\mathbb{L}^p$ for $p=2+\delta$ (and not for any $p>2$ as stated in Pham (2018) Proposition 2.6). This last property was useful in Pham (2018) to achieve the proof of Lemma 4.5, choosing p great enough in such a way $(p-1)\delta-\frac{1}{2}>2\varepsilon$ for some fixed constant $\varepsilon>0$. Recently, following the same strategy as C. Pham, M. Peigné and W. Woess have improved this part of the proof, by allowing various parameters (see Peigné and Woess, 2021+, Proof of Theorem 1.6 (d)).

As a direct consequence, a similar statement holds for the tail of the distribution of the stopping times τ^a; this is of interest in the sequel since the overestimations obtained do not depend on the starting point $x\in\mathbb{X}$ of the chain $(X_n)_{n\geq 0}$.

Corollary 2.4. *Assume hypotheses* $\mathbf{A}(\delta)$. *Then, for any* $a\geq 1$ *and* $n\geq 1$,

$$\mathbb{P}[\tau^a>n]=\mathbb{P}[a\|A_1\|>1,\ \dots\ ,a\|A_{n,1}\|>1]\leq\kappa(1+|\ln\delta|)\frac{1+\ln a}{\sqrt{n}}$$

where κ *is the constant given by Proposition 2.3.*

Proof: By Lemma 2.1, for any $k\geq 1$ and $x\in\mathbb{X}$,

$$\delta\|A_{k,1}\|\leq|A_{k,1}x|\leq\|A_{k,1}\|\qquad\mathbb{P}\text{-a.s.}$$

Proposition 2.3 yields

$$\mathbb{P}[\tau^a > n] \le \mathbb{P}[\tau^{x,a/\delta} > n] \le \kappa \frac{1+\ln a + |\ln \delta|}{\sqrt{n}} \le \kappa(1+|\ln\delta|)\frac{1+\ln a}{\sqrt{n}}.$$

□

3. Existence and uniqueness of an invariant Radon measure for $(X_n)_{n\geq 0}$

The Markov chain $(X_n)_{n\geq 1}$ is a stochastic dynamical system generated by the random maps $F_n : x \mapsto A_n x + B_n$ on $\mathbb{R}^d$. By section 2.1, in order to get the existence and the uniqueness of an invariant Radon measure for this process, it suffices to check that, under hypotheses $\mathbf{A}(\delta)$ and $\mathbf{B}(\delta)$, this process is conservative and the sequence $(F_n \circ \cdots \circ F_1)_{n\geq 1}$ is $\mathbb{P}$-a.s. locally contractive. This is the matter of the two following subsections.

3.1. *On the conservativity of the process* $(X_n)_{n\geq 0}$. Under hypotheses $\mathbf{A}(\delta)$, the sequences $(|A_{n,1}x|)_{n\geq 1}$ and $(\|A_{n,1}\|)_{n\geq 1}$ fluctuate $\mathbb{P}$-a.s. between 0 and $+\infty$; hence, the stopping times $\tau^{x,a}$ and τ^a are finite $\mathbb{P}$-a.s.

From now on, we fix $a > 1$ and set $\tau_0 = 0$, then for any $k \geq 1$, we denote

$$\tau_k := \inf\{n > \tau_{k-1}/a\|A_{n,\tau_{k-1}+1}\| \le 1\}.$$

Notice that $\tau_1 = \tau^a$ and for $k \geq 0$, the random variables τ_k are $\mathbb{P}$-a.s. finite stopping times with respect to the filtration $(\mathcal{T}_n)_{n\geq 0}$.

The process $(X_n)_{n\geq 0}$ is conservative if and only if for any $x \in \mathbb{R}^d_+$,

$$\mathbb{P}\left[\liminf_{n\to+\infty} |X_n^x| < +\infty\right] = 1.$$

This property holds in particular when

$$\mathbb{P}\left[\liminf_{k\to+\infty} |X_{\tau_k}^x| < +\infty\right] = 1. \tag{3.1}$$

Notice that $X_{\tau_k}^x = A_{\tau_k,1}x + B_{\tau_k,1}$ with

- $A_{\tau_k,1} = \displaystyle\prod_{\ell=1}^{k} A_{\tau_\ell,\tau_{\ell-1}+1} = \widetilde{A}_k \cdots \widetilde{A}_1,$
- $B_{\tau_k,1} = \displaystyle\sum_{\ell=1}^{k} A_{\tau_k}\cdots A_{\tau_\ell+1}\left(\sum_{j=\tau_{\ell-1}+1}^{\tau_\ell} A_{\tau_\ell,j+1}B_j\right) = \sum_{\ell=1}^{k} \widetilde{A}_k \cdots \widetilde{A}_{\ell+1}\widetilde{B}_\ell.$

The random variables $\widetilde{A}_\ell := A_{\tau_\ell,\tau_{\ell-1}+1}, \ell \geq 1$, are i.i.d. with the same distribution as $\widetilde{A}_1$; in other words, the sequence $(A_{\tau_n,1})_{n\geq 0}$ is a random walk on $\mathcal{S}$ with distribution $\mathcal{L}(\widetilde{A}_1)$ and for any $k \geq 1$,

$$\|A_{\tau_k,1}\| = \|\widetilde{A}_k \cdots \widetilde{A}_1\| \le \frac{1}{a^k} \qquad \mathbb{P}\text{-a.s.} \tag{3.2}$$

Similarly, the random variables $\widetilde{B}_\ell := \displaystyle\sum_{j=\tau_{\ell-1}+1}^{\tau_\ell} A_{\tau_\ell,j+1}B_j, \ell \geq 1$, are i.i.d. with the same distribution as $\widetilde{B}_1 = \displaystyle\sum_{j=1}^{\tau_1} A_{\tau_1,j+1}B_j$.

In order to prove (3.1), we first need to check that the $\widetilde{B}_\ell$ have logarithm moments. This is the aim of the following statement.

Lemma 3.1. *Under hypotheses* **A** *and* $\mathbf{B}(\delta)$,

$$\mathbb{E}\left[\ln(1+|\widetilde{B}_1|)\right] < +\infty. \tag{3.3}$$

The proof of (3.3) relies on the following classical result (see Élie, 1982 for a detailed argument):

Let $(U_\ell)_{\ell\geq 1}$ be a sequence of i.i.d. non negative random variables such that $\mathbb{P}[U_1 \neq 0] > 0$.

Then,

$$\limsup_{\ell\to+\infty} U_\ell^{1/\ell} < +\infty \quad \mathbb{P}\text{-}a.s. \qquad \Longrightarrow \qquad \mathbb{E}\Big[\ln(1+U_1)\Big] < +\infty \tag{3.4}$$

and

$$\mathbb{E}\Big[\ln(1+U_1)\Big] < +\infty \qquad \Longrightarrow \qquad \limsup_{\ell\to+\infty} U_\ell^{1/\ell} = 1 \quad \mathbb{P}\text{-a.s.} \tag{3.5}$$

Before to detail the proof of the lemma, let us explain how it yields (3.1). By combining (3.3) and (3.5), it holds $\limsup\limits_{\ell\to+\infty} |\widetilde{B}_\ell|^{1/\ell} = 1$, so that

$$\limsup_{\ell\to+\infty} |\widetilde{A}_1\cdots\widetilde{A}_{\ell-1}\widetilde{B}_\ell|^{1/\ell} \leq \limsup_{\ell\to+\infty} |\widetilde{A}_1\cdots\widetilde{A}_{\ell-1}|^{1/\ell} \times \limsup_{\ell\to+\infty} |\widetilde{B}_\ell|^{1/\ell} \leq \frac{1}{a} < 1 \ \mathbb{P}\text{-a.s.}$$

Hence, the series $\sum\limits_{\ell=1}^{+\infty} \widetilde{A}_1\cdots\widetilde{A}_{\ell-1}\widetilde{B}_\ell$ converges $\mathbb{P}$ a.s. to some random variable $\widetilde{B}_\infty$; this implies that $(B_{\tau_k,1})_{k\geq 1}$ converges in distribution towards $\widetilde{B}_\infty$, since $B_{\tau_k,1}$ has the same distribution as $\sum\limits_{\ell=1}^{k} \widetilde{A}_1\cdots\widetilde{A}_{\ell-1}\widetilde{B}_\ell$. By (3.2), the same property holds for the sequence $(X_{\tau_k})_{k\geq 0}$ for any $x\in\mathbb{R}_+^d$. Consequently,

$$\mathbb{P}[\liminf_{k\to+\infty} |X_{\tau_k}^x| < +\infty] = \mathbb{P}[|\widetilde{B}_\infty| < +\infty] = 1.$$

□

Proof of Lemma 3.1: By (3.4), it is sufficient to check that

$$\limsup_{\ell\to+\infty} |\widetilde{B_\ell}|^{1/\ell} < +\infty \quad \mathbb{P}\text{-a.s.}$$

For any $\ell \geq 1$, it holds

$$\begin{aligned}
|\widetilde{B}_\ell| &\leq \sum_{j=\tau_{\ell-1}+1}^{\tau_\ell} \|A_{\tau_\ell,j+1}\| \ |B_j| \\
&\leq \frac{1}{\delta}\|A_{\tau_\ell,\tau_{\ell-1}+1}\| \sum_{j=\tau_{\ell-1}+1}^{\tau_\ell} \frac{|B_j|}{\|A_{j,\tau_{\ell-1}+1}\|} \\
&\leq \frac{1}{\delta} \sum_{j=\tau_{\ell-1}+1}^{\tau_\ell} |B_j|
\end{aligned}$$

since $\|A_{\tau_\ell,\tau_{\ell-1}+1}\| \leq \frac{1}{a} < \|A_{j,\tau_{\ell-1}+1}\| \quad \mathbb{P}$-a.s. for $\tau_{\ell-1} < j < \tau_\ell$.

It remains to check that

$$\limsup_{\ell\to+\infty}\Big(\sum_{j=\tau_{\ell-1}+1}^{\tau_\ell}|B_j|\Big)^{1/\ell}<+\infty\quad \mathbb{P}\text{-a.s.}$$

Indeed, we prove the stronger convergence

$$\limsup_{\ell\to+\infty}\Big(\sum_{j=1}^{\tau_\ell}|B_j|\Big)^{1/\ell}<+\infty\quad \mathbb{P}\text{-a.s.}\tag{3.6}$$

Notice that, for any $\alpha>0$,

$$\begin{aligned}\ln\left(\sum_{j=1}^{\tau_\ell}|B_j|\right)^{1/\ell} &\le \frac{1}{\ell}\ln\Big(1+\sum_{j=1}^{\tau_\ell}|B_j|\Big)\\ &= \frac{\tau_\ell^\alpha}{\ell}\left(\frac{1}{\tau_\ell^\alpha}\ln\Big(1+\sum_{j=1}^{\tau_\ell}|B_j|\Big)\right)\end{aligned}$$

Recall that the random variables $\tau_{j+1}-\tau_j$ are i.i.d. with distribution $\mathcal{L}(\tau_1)$; furthermore, by Corollary 2.4, there exists $c(a)>0$ s.t.

$$\mathbb{P}[\tau_1>n]=\mathbb{P}[\tau_a>n]\sim\frac{c(a)}{\sqrt{n}}\quad\text{as}\quad n\to+\infty.$$

Hence,

- on the one hand, for any $\alpha<1/2$, it holds $\mathbb{E}(\tau_1^\alpha)<+\infty$, so that $\limsup_{\ell\to+\infty}\tau_\ell^\alpha/\ell<+\infty$ $\mathbb{P}$-a.s.;

- on the other hand, the inequality $\ln\Big(1+\sum_{j=1}^{\tau_\ell}|B_j|\Big)\le\ln\tau_\ell+\max_{1\le j\le\tau_\ell}\ln(1+|B_j|)$ yields

$$\begin{aligned}\limsup_{\ell\to+\infty}\frac{1}{\tau_\ell^\alpha}\ln\left(1+\sum_{j=1}^{\tau_\ell}|B_j|\right) &\le \limsup_{\ell\to+\infty}\frac{1}{\tau_\ell^\alpha}\max_{1\le j\le\tau_\ell}\ln(1+|B_j|)\\ &= \limsup_{\ell\to+\infty}\Big(\frac{1}{\tau_\ell}\max_{1\le j\le\tau_\ell}\Big(\ln(1+|B_j|)\Big)^{1/\alpha}\Big)^\alpha\\ &\le \limsup_{\ell\to+\infty}\Big(\frac{1}{\tau_\ell}\sum_{j=1}^{\tau_\ell}\Big(\ln(1+|B_j|)\Big)^{1/\alpha}\Big)^\alpha.\end{aligned}$$

By hypotheses **A1** and **B(**δ**)**, if $\alpha\ge\frac{1}{2+\delta}$, the random variable $\ln(1+|B_1|)^{1/\alpha}$ is integrable and the strong law of large numbers implies

$$\limsup_{\ell\to+\infty}\frac{1}{\tau_\ell^\alpha}\ln\Big(1+\sum_{j=1}^{\tau_\ell}|B_j|\Big)\le\left(\mathbb{E}\Big[\Big(\ln(1+|B_1|)\Big)^{1/\alpha}\Big]\right)^\alpha<+\infty.$$

The proof of (3.6) arrives choosing $\frac{1}{2+\delta}\le\alpha<\frac{1}{2}$, which achieves the proof of Lemma 3.1. □

3.2. *On the local contractivity of the process $(X_n)_{n\geq 0}$ on $\mathbb{R}_+^d$.* Local contractivity is a direct consequence of the following Lemma.

Lemma 3.2. *Assume that*

- $\displaystyle\sum_{n=0}^{+\infty} \mathbf{1}_{[\|A_{n,1}\|\leq 1]} = +\infty$ *$\mathbb{P}$-a.s.*
- *the B_k are $\mathbb{R}_+^d$-valued and $\mathbb{P}[B_1 \neq 0] > 0$.*

Then, $\mathbb{P}$-a.s., for any $x, y \in \mathbb{R}_+^d$ and any $K > 0$,

$$\lim_{n\to+\infty} |X_n^x - X_n^y| \mathbf{1}_{[|X_n^x|\leq K]} = 0.$$

Proof: We use here the argument developed in Brofferio and Buraczewski (2015), Theorem 1.2. Observe that

$$|X_n^x - X_n^y| \mathbf{1}_{[|X_n^x|\leq K]} \leq \|A_{n,1}\| |x-y| \mathbf{1}_{[|X_n^x|\leq K]} \leq \frac{K}{\frac{|X_n^x|}{\|A_{n,1}\|}} |x-y|$$

(with the convention $\frac{1}{0} = +\infty$). Fix $\epsilon > 0$ such that $p_\epsilon := \mathbb{P}\left[\frac{|B_1|}{\|A_1\|} \geq \epsilon\right] > 0$. We consider the sequences $(\varepsilon_k)_{k\geq 1}$ and $(\eta_k)_{k\geq 1}$ of Bernoulli random variables defined by: for any $k \geq 1$,

$$\varepsilon_k = \mathbf{1}_{[|B_k|/\|A_k\|\geq\epsilon]} \quad \text{and} \quad \eta_k = \mathbf{1}_{[\|A_{k-1,1}\|\leq 1]}.$$

For any $k \geq 1$, the random variable ε_k is independent on $(\eta_1, \ldots, \eta_k)$ and $\mathbb{P}[\varepsilon_k = 1] = p_\epsilon > 0$. Lemma 2.1 readily implies: for any $x \in \mathbb{R}_+^d$,

$$\frac{|X_n^x|}{\|A_{n,1}\|} \geq \frac{|B_{n,1}|}{\|A_{n,1}\|} = \sum_{k=1}^{n} \frac{|A_{n,k+1}B_k|}{\|A_{n,1}\|} \geq \delta \sum_{k=1}^{n} \frac{\|A_{n,k+1}\||B_k|}{\|A_{n,1}\|} \geq \delta \sum_{k=1}^{n} \frac{|B_k|}{\|A_{k,1}\|}$$

with

$$\sum_{k=1}^{n} \frac{|B_k|}{\|A_{k,1}\|} \geq \sum_{k=1}^{n} \frac{|B_k|}{\|A_k\|} \frac{1}{\|A_{k-1,1}\|} \geq \epsilon \sum_{k=1}^{n} \varepsilon_k \eta_k.$$

By hypothesis, it holds $\displaystyle\sum_{k=1}^{+\infty} \eta_k = +\infty$ $\mathbb{P}$-a.s.; consequently $\displaystyle\sum_{k=1}^{n} \varepsilon_k \eta_k \to +\infty$ $\mathbb{P}$-a.s., by the following statement.

Lemma 3.3. *Let $(\varepsilon_k)_{k\geq 1}$ and $(\eta_k)_{k\geq 1}$ be two sequences of Bernoulli random variables defined on $(\Omega, \mathcal{T}, \mathbb{P})$ such that*

1. $\displaystyle\sum_{k=1}^{+\infty} \eta_k = +\infty$ *$\mathbb{P}$-a.s.;*
2. *the ε_k are i.i.d. Bernoulli random variables with parameter $0 < p \leq 1$;*
3. *for any $k \geq 1$, the random variable ε_k is independent on $\eta_1, \ldots, \eta_k$.*

Then $\displaystyle\sum_{k=1}^{+\infty} \varepsilon_k \eta_k = +\infty$ *$\mathbb{P}$-a.s.*

□

Proof of Lemma 3.3: Let us introduce the sequence $(t_k)_{k\geq 1}$ of stopping times with respect to the filtration $(\sigma(\eta_1, \ldots, \eta_k))_{k\geq 1}$ defined by

$$t_0 = 1, \quad t_1 := \inf\{n \geq 1/\eta_n = 1\} \quad \text{and} \quad t_{k+1} := \inf\{n > t_k/\eta_n = 1\}.$$

By hypothesis 1. the stopping times t_k for $k \geq 1$ are $\mathbb{P}$-a.s. finite. Furthermore, by the strong Markov's property, hypotheses 2. and 3. yield: for any $i, j \geq 1$,

$$\begin{aligned}
&\mathbb{P}[\varepsilon_{t_i} = 0, \dots, \varepsilon_{t_{i+j}} = 0] \\
&= \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{[\varepsilon_{t_i}=0,\dots,\varepsilon_{t_{i+j}}=0]}/\eta_1, \dots, \eta_{t_{i+j}}, \varepsilon_1, \dots, \varepsilon_{t_{i+j}-1}]\right] \\
&= \mathbb{E}\left[\mathbf{1}_{[\varepsilon_{t_i}=0,\dots,\varepsilon_{t_{i+j-1}}=0]}\mathbb{P}[\varepsilon_{t_{i+j}} = 0/\eta_1, \dots, \eta_{t_{i+j}}, \varepsilon_1, \dots, \varepsilon_{t_{i+j}-1}]\right] \\
&= (1-p)\ \mathbb{P}[\varepsilon_{t_i} = 0, \dots, \varepsilon_{t_{i+j-1}} = 0] = \dots = (1-p)^j.
\end{aligned}$$

Hence $\mathbb{P}[\liminf_{i\to+\infty}[\varepsilon_{t_i} = 0]] = 0$ so that $\displaystyle\sum_{k=1}^{+\infty} \varepsilon_k \eta_k = \sum_{i=1}^{+\infty} \varepsilon_{t_i} \eta_{t_i} = +\infty$ $\mathbb{P}$-a.s. □

3.3. *Proof of Theorem 1.1 (1) and (2).*

1. We use the properties stated in subsection 2.1 about stochastic dynamical systems.

The existence of an invariant Radon measure m, follows from the conservativity of the process $(X_n)_{n\geq 0}$ proved in subsection 3.1.

The uniqueness of m is a consequence of the local contractivity of $(X_n)_{n\geq 0}$ established in subsection 3.2.

2. The fact that m is infinite is a direct consequence of Theorem 3.2-A in Bougerol and Picard (1992): indeed, if m was finite, then the Lyapunov exponent $\gamma_{\bar\mu}$ would be negative, contradiction.

4. Estimation on the tail of the invariant measure m

In this section, we prove the second assertion of Theorem 1.1; this is a direct consequence of the following statement, where the slowly varying function L is explicit. Firstly, we introduce some notation: for any $t > 0$ and any compact set $K \subset \mathbb{R}_+^d$;

$$tK = \{tx \in \mathbb{R}_+^d / x \in K\}.$$

Proposition 4.1. *Assume hypotheses* $\mathbf{A}(\delta)$ *and* $\mathbf{B}(\delta)$ *hold and let* m *be the unique (up to a multiplicative constant) invariant Radon measure for the process* $(X_n)_{n\geq 0}$*. Then, there exists a compact set* $K_\circ \subset \mathbb{R}_+^d \setminus \{0\}$ *such that*

(1) *the function* $L : t \mapsto m(tK_\circ)$ *is positive and slowly varying on* $\mathbb{R}_+$,
(2) *the family* $(m_t)_{t\geq 1}$ *of normalized measures on* $\mathbb{R}_+^d \setminus \{0\}$ *defined by*

$$m_t(K) := \frac{m(tK)}{L(t)} \tag{4.1}$$

is vaguely relatively compact. In particular, there exist $0 < a < b$ *and* $c > 1$ *such that for any* $t \geq 1$,

$$L(t) \leq m\Big\{x \in \mathbb{R}_+^d / ta \leq |x| \leq tb\Big\} \leq cL(t).$$

4.1. *Preliminary results.* First, we prove the following statement.

Lemma 4.2. *Under hypotheses* $\mathbf{A}(\delta)$ *and* $\mathbf{B}(\delta)$, *there exists a compact set* $K_\circ \subset \mathbb{R}_+^d \setminus \{0\}$ *such that*

(1) *the quantity* $m(tK_\circ)$ *is positive for any* $t \geq 1$;

(2) *for every compact set* $K \subset \mathbb{R}_+^d \setminus \{0\}$*, there exists a positive constant* κ_K *such that*

$$\forall t > 1, \qquad m(tK) \leq \kappa_K m(tK_\circ). \tag{4.2}$$

In other words, setting $L(t) := m(tK_\circ)$, inequality (4.2) states that the family $(m_t)_{t\geq 1}$ is vaguely relatively compact.

Proof of Lemma 4.2: We consider the family $(\mathcal{A}_R)_{R>0}$ of closed "annuli" (in the sense of the norm $|\cdot|$) defined by: for any $R > 0$,

$$\mathcal{A}_R = \left\{ x \in \mathbb{R}_+^d \setminus \{0\} / \frac{1}{R} \leq |x| \leq R \right\}.$$

For $a, b > 0$, we denote by $\mathcal{V}(a, b)$ the subset of elements $g = (A, B) \in S_\delta \times \mathbb{R}_+^d \setminus \{0\}$ such that

$$\|A\| + |B| < ab \text{ and } \|A\| > \frac{1}{\delta}\frac{b}{a}.$$

The set $\mathcal{V}(a, b)$ is trivially empty when $ab \leq \frac{1}{\delta}\frac{b}{a}$ i.e $a \leq \frac{1}{\sqrt{\delta}}$; hence, since $0 < \delta \leq 1$, we assume from now on $a > 1$, so that $\mathcal{V}(a, b)$ is not empty.

From now on, we fix two radii $r < R$ in $(1, +\infty)$.

Recall also that, to simplify the notations, we denote by g both the couple $(A, B) \in \mathcal{S} \times \mathbb{R}_+^d \setminus \{0\}$ and the map $x \mapsto Ax + B$ on $\mathbb{R}_+^d$; the "linear" component of g is $A = A(g)$ and its "translation component" is $B = B(g)$.

The proof of the Lemma is decomposed in 4 steps.

Step 1. For any $t > 0, s > 1/r$ and $g \in \mathcal{V}\left(\frac{R}{r}, \frac{t}{s}\right)$, it holds $g(s\mathcal{A}_r) \subset t\mathcal{A}_R$.

Indeed, $g = (A, B)$, we get the following inequalities for $x \in s\mathcal{A}_r$:

$$|g(x)| \leq \|A\| \times |x| + |B| \leq (\|A\| + |B|)sr < tR$$

and

$$|g(x)| \geq |Ax| \geq \delta\|A\| \times |x| \geq \delta\|A\|\frac{s}{r} > \frac{t}{R}.$$

Step 2. $m(t\mathcal{A}_R) > 0$ for $R > 0$ great enough and any $t > 1$.

By hypotheses **A2** and **A4**, there exists $N \geq 1$ and an element $g = (A, B)$ in $\operatorname{supp}(\mu^{*N})$ such that the spectral radius $\rho(A)$ of A is greater than 1.

Notice that, for any $n \geq 1$ and $x \in \mathbb{R}_+^d$,

$$g^n(x) = A(g^n)x + B(g^n)$$

with $A(g^n) = A^n$ and $B(g^n) := \sum_{k=0}^{n-1} A^k B$.

First, there exists a constant $\beta > 0$ such that $|B(g^n)| \leq \beta\|A^n\|$. Indeed, by Lemma 2.1,

$$\frac{|B(g^n)|}{\|A(g^n)\|} = \frac{\left|\sum_{k=0}^{n-1} A^k B\right|}{\|A^n\|} \leq \frac{1}{\delta}|B| \sum_{k=0}^{n-1} \frac{1}{\|A^{n-k}\|} \leq \frac{1}{\delta}|B| \sum_{i=1}^{+\infty} \frac{1}{\|A^i\|} =: \beta,$$

with $\beta < +\infty$ since $\rho(A) > 1$.

Second, for any $t > 1$, set $n_t := \inf\{n \geq 1/t \leq \|A^n\|\}$. Notice that $n_t < +\infty$ since $\|A^n\| \to +\infty$. By the inequality $\|A^{n_t-1}\| < t \leq \|A^{n_t}\|$, for $k > \max\{(1 + \beta)\|A\|, \frac{1}{\delta}\}$,

$$\|A(g^{n_t})\| + |B(g^{n_t})| \leq (1 + \beta)\|A^{n_t}\| \leq (1 + \beta)\|A\| \times \|A^{n_t-1}\| \leq (1 + \beta)\|A\|t < kt$$

and

$$\|A(g^{n_t})\| = \|A^{n_t}\| \geq \frac{1}{\delta}\delta t > \frac{1}{\delta}\frac{t}{k}.$$

Hence, for $T > \max\{1+\beta, \frac{1}{\delta}\}$,

$$g^{n_t} \in \mathcal{V}(T,t) \quad \forall t > 1. \tag{4.3}$$

Last, we fix $r_0 > 1$ such that $m(\mathcal{A}_{r_0}) > 0$. For $R > \max\{1+\beta, \frac{1}{\delta}\}r_0$ and any $t > 1$, it holds

$$\begin{aligned} m(t\mathcal{A}_R) &= (\mu^{Nn_t} * m)(t\mathcal{A}_R) \\ &\geq \int \mathbf{1}_{\mathcal{V}(\frac{R}{r_0},t)}(g)\mathbf{1}_{t\mathcal{A}_R}(g(x))\mathrm{d}\mu^{Nn_t}(g)\mathrm{d}m(x) \\ &\geq \int \mathbf{1}_{\mathcal{V}(\frac{R}{r_0},t)}(g)\mathbf{1}_{g(\mathcal{A}_{r_0})}(g(x))\mathrm{d}\mu^{Nn_t}(g)\mathrm{d}m(x) \quad \text{(by Step 1, with } s=1) \\ &\geq \mu^{Nn_t}\left(\mathcal{V}\Big(\frac{R}{r_0},t\Big)\right)m(\mathcal{A}_{r_0}) \end{aligned}$$

with $\mu^{Nn_t}\left(\mathcal{V}\Big(\frac{R}{r_0},t\Big)\right) > 0$ since $g^{n_t} \in \mathrm{supp}(\mu^{Nn_t}) \cap \mathcal{V}(\frac{R}{r_0},t)$ and $\mathcal{V}(\frac{R}{r_0},t)$ is open. The proof of Step 2 is complete.

Step 3. For any $r > 1$, there exists $R_r > 0$ such that, for $R \geq R_r$ and $s > 0$,

$$\forall t > 1 \quad m(ts\mathcal{A}_r) \leq \kappa_s m(t\mathcal{A}_R), \tag{4.4}$$

for some constant $\kappa_s = \kappa_s(r,R) > 0$.

Case $s < 1$.

Assume $R > \max\{1+\beta, \frac{1}{\delta}\}r$, so that $g^{n_{1/s}} \in \mathcal{V}(\frac{R}{r}, \frac{1}{s})$ by (4.3). Consequently, as above,

$$\begin{aligned} m(t\mathcal{A}_R) &= (\mu^{Nn_{1/s}} * m)(t\mathcal{A}_R) \\ &\geq \int \mathbf{1}_{\mathcal{V}(\frac{R}{r},\frac{1}{s})}(g)\mathbf{1}_{t\mathcal{A}_R}(g(x))\mathrm{d}\mu^{Nn_{1/s}}(g)\mathrm{d}m(x) \\ &\geq \int \mathbf{1}_{\mathcal{V}(\frac{R}{r},\frac{1}{s})}(g)\mathbf{1}_{g(ts\mathcal{A}_r)}(g(x))\mathrm{d}\mu^{Nn_{1/s}}(g)\mathrm{d}m(x) \quad \text{(by Step 1)} \\ &\geq \mu^{Nn_{1/s}}\left(\mathcal{V}\Big(\frac{R}{r},\frac{1}{s}\Big)\right)m(ts\mathcal{A}_r). \end{aligned}$$

Inequality (4.4) holds with $\kappa_s = \frac{1}{\mu^{Nn_{1/s}}(\mathcal{V}(\frac{R}{r},\frac{1}{s}))}$.

Case $s \geq 1$.

As in Step 2, by hypotheses **A2** and **A4**, there exist $N \geq 1$ and $g_- = (A_-, B_-)$ in $\mathrm{supp}(\mu^{*N})$ such that the spectral radius $\rho(A_-)$ of A_- is less than 1. First, as above, for any $n \geq 1$, the norm $|B(g_-^n)|$ is smaller than $\beta_- := \sum_{k=0}^{+\infty}\|A_-^k\| \times |B_-|$. Second, for any $s \geq 1$, set $m_s := \inf\{m \geq 1/\frac{1}{s} \geq \|A_-^m\|\}$. Notice that $m_s < +\infty$ (since $\|A_-^m\| \to 0$) and

$$\|A_-^{m_s-1}\| > \frac{1}{s} \geq \|A_-^{m_s}\| \geq \delta\|A_-\| \times \|A_-^{m_s-1}\| > \delta\|A_-\|\frac{1}{s}$$

so that $g_-^{m_s}$ belongs to the set

$$\mathcal{U}(s) := \{g = (A,B)/\delta\|A_-\|\frac{1}{s} < \|A\| \leq \frac{1}{s} \text{ and } |B| \leq \beta_-\},$$

and $\mu^{Nm_s}(\mathcal{U}(s)) > 0$.

Let us choose $R > \max\{\frac{r}{\delta^2\|A_-\|}, r + \beta_-\}$. For $g \in \mathcal{U}(s)$ and $x \in ts\mathcal{A}_r$,

$$|g(x)| \leq \|A\|tsr + |B| \leq tr + \beta_- < t(r+\beta_-) < tR \qquad \text{(since } t > 1)$$

and

$$|g(x)| \geq \delta\|A\|\frac{ts}{r} \geq \delta^2\|A_-\|\frac{t}{r} > \frac{t}{R}$$

that is $g(ts\mathcal{A}_r) \subset t\mathcal{A}_R$. This yields, reasoning as in step 2,

$$\forall t > 1, \forall s \geq 1, \qquad \frac{m(ts\mathcal{A}_r)}{m(t\mathcal{A}_R)} \leq \frac{1}{\mu^{Nm_s}(\mathcal{U}(s))} < +\infty.$$

Step 4. By (4.4), Lemma 4.2 holds for $K_\circ := \mathcal{A}_R$ and any compact set $K \subset \mathbb{R}_+^d \setminus \{0\}$ of the form $s\mathcal{A}_r$ with $s > 0$. To extend this result to a generic compact set $K \subset \mathbb{R}_+^d \setminus \{0\}$, we just observe that such a compact set satisfies $K \subset \bigcup_{\ell=1}^k s_\ell\mathcal{A}_r$, for some nonnegative reals $s_1, \ldots, s_k$ (depending on K); we take $\kappa_K = \sum_{n=1}^k \kappa_{s_n}$. $\square$

Before concluding this section, we state some general result about harmonic functions for random walks on topological semigroups; it will be useful to achieve the proof of Theorem 1.1 (iii). It relies on standard arguments in potential theory but we did not find any precise reference in the literature; for the sake of completeness, we detail the proof in the Appendix.

Lemma 4.3. *Let T be a locally compact Hausdorff topological semi-group (with identity e) and $\mu_\circ$ be a Borel probability on $\mathcal{S}$. Let*

$$T_{\mu_\circ} = \overline{\bigcup_{n=0} \operatorname{supp}(\mu_\circ^n)}$$

be the closed sub-semigroup of T generated by the support of $\mu_\circ$. The"conservative part" $R_{\mu_\circ}$ of $T_{\mu_\circ}$ is defined by

$$R_{\mu_\circ} := \left\{ s \in T_{\mu_\circ} / \sum_{n=1}^{+\infty} \mu_\circ^n(V_s) = +\infty \quad \textit{for all} \quad \textit{open neighborhood } V_s \textit{ of } s \right\}.$$

Then

(1) *$R_{\mu_\circ}$ is a closed ideal of $T_{\mu_\circ}$, i.e. $R_{\mu_\circ}T_{\mu_\circ} \subseteq R_{\mu_\circ}$*

(2) *Let h be a continuous superharmonic function for the right random walk with law $\mu_\circ$ on T, that is a function $h : T_{\mu_\circ} \to [0, +\infty)$ such that $Ph(s_0) := \int h(s_0 s)\mathrm{d}\mu_\circ(s) \leq h(s_0)$ for all $s_0 \in T$.*

Then $h(rs) = h(r)$ for all $r \in R_{\mu_\circ}$ and $s \in T_{\mu_\circ}$.

Let us emphasize that, in this general setting, the ideal $R_{\mu_\circ}$ may be empty; furthermore, when $R_{\mu_\circ} \neq \emptyset$, it may not coincide with the semigroup $T_{\mu_\circ}$. For instance, in the context of product of elements in $\mathcal{S}_\delta$, the conservative part $R_{\mu_\circ}$ is included in the set of rank 1 matrices, which is a proper subset of $T_{\mu_\circ}$.

4.2. *Proof of Theorem 1.1 (iii).* We follow the strategy developed in Babillot et al. (1997) and Brofferio et al. (2012). The proof is decomposed into 3 steps. Recall that $\overline{\mu}$ denotes the law of the random variable A_1 and that its support is included in $\mathcal{S}_\delta$. In the sequel, we apply Lemma 4.3 with $T = \mathcal{S}_\delta$.

Step 1. There exists $A_0 \in R_{\overline{\mu}}$ such that

- $\mathrm{rank} A_0 = 1$;
- $\mathrm{Im} A_0 = \mathbb{R}v_0$ and $A_0 v_0 = \lambda_0 v_0$, for some $v_0 \in \mathbb{X}$ and $\lambda_0 > 1$.

The Markov chain $(A_{n,1} \cdot x, |A_{n,1}x|)_{n\geq 0}$ being recurrent on $\mathbb{X} \times \mathbb{R}^+$, it holds, for $M \geq 1$ great enough,

$$\sum_{n=0}^{+\infty} \mathbb{P}\left(\frac{1}{\delta^2} \leq \|A_{n,1}\| \leq M\right) = +\infty. \tag{4.5}$$

Since $D := \left\{A \in \mathcal{S}_\delta / \frac{1}{\delta^2} \leq \|A\| \leq M\right\}$ is compact in $\mathcal{S}_\delta \setminus \{0\}$, the set $R_{\overline{\mu}} \cap D$ is non empty. Otherwise, D is included in $\mathcal{S}_\delta \setminus R_{\overline{\mu}}$ and there exists a finite cover $V_1, \ldots, V_k$ of D with open sets V_i such that $\displaystyle\sum_{i=1}^{k} \mu^n(V_i) < +\infty$ for $i = 1, \ldots, k$. Contradiction with (4.5).

From now on, we fix some element $A_0 \in R_{\overline{\mu}} \cap D$. First, let us check that rank(A_0)=1.

By definition of $R_{\overline{\mu}}$, for any open set $\mathcal{O} \subset \mathcal{S}_\delta$ which contains A_0, it holds

$$\sum_{n=0}^{+\infty} \mathbb{P}(A_{n,1} \in \mathcal{O}) = +\infty.$$

For any $x, y \in \mathbb{X}$ and $\varepsilon > 0$, the open set $\mathcal{O}_{x,y,\varepsilon} := \{A \in \mathcal{S}_\delta / \mathfrak{d}(A \cdot x, A \cdot y) > \varepsilon\}$ does not contain A_0; indeed, by (2.2),

$$\sum_{n=0}^{+\infty} \mathbb{P}(\mathfrak{d}(A_{n,1} \cdot x, A_{n,1} \cdot y) > \varepsilon) \leq \frac{1}{\varepsilon} \sum_{n=0}^{+\infty} \mathbb{E}(\mathfrak{d}(A_{n,1} \cdot x, A_{n,1} \cdot y)) < +\infty.$$

Hence, for any $x, y \in \mathbb{X}$ and $\varepsilon > 0$, it holds $A_0 \notin \mathcal{O}_{x,y,\varepsilon}$, thus $\mathfrak{d}(A_0 \cdot x, A_0 \cdot y) \leq \varepsilon$. Letting $\epsilon \to 0$ yields $\mathfrak{d}(A_0 \cdot x, A_0 \cdot y) = 0$ for any $x, y \in \mathbb{X}$; in other words, rank $A_0 = 1$.

Let $v_0 \in \mathbb{R}^d, v_0 \neq 0$, such that Im $A_0 = \mathbb{R}v_0$. By the Perron-Frobenius's theorem, the matrix A_0 has a dominant and simple eigenvalue λ_0 with eigenvector $v_0 \in \mathbb{X}$; furthermore, since $A_0 \in D$,

$$\lambda_0 = |A_0 v_0| \geq \delta \|A_0\| \geq \frac{1}{\delta} > 1.$$

$\square$

Now, we introduce the function L. For any compact set $J \subseteq \mathbb{R}_+$, set $K_J := Jv_0 + \mathrm{Ker} A_0$; the set $K_J \cap \mathbb{R}^d_+$ is compact in $\mathbb{R}^d_+ \setminus \{0\}$.

We consider the intervals $I_N := [\lambda_0^{-N}, \lambda_0^N]$ for $N \geq 1$; by Lemma 4.2, for N great enough, the family of measure

$$K \mapsto m_t(K) := \frac{m(tK)}{m(tK_{I_N})}$$

is vaguely relatively compact. We fix such an integer N and set $L(t) := m(tK_{I_N})$.

We claim that the function L is slowly varying. First, we need to state some properties of cluster points of the family $(m_t)_{t>0}$, this is the purpose of the following step.

Step 2. Any weak cluster point $\eta = \lim_{i \to +\infty} m_{t_i}$ of the family $(m_t)_{t>0}$ satisfies

$$\int_{\mathbb{R}^d_+} \phi(AA'x) \mathrm{d}\eta(x) = \int_{\mathbb{R}^d_+} \phi(Ax) \mathrm{d}\eta(x) \tag{4.6}$$

for any $A \in R_{\overline{\mu}}, A' \in S_{\overline{\mu}}$ and any Lipschitz, compactly supported function ϕ on $\mathbb{R}_+^d \setminus \{0\}$.

We fix a Lipschitz, compactly supported function $\phi : \mathbb{R}_+^d \setminus \{0\} \to \mathbb{R}$; we denote by $[\phi]$ its Lipschitz coefficient and chose $M > 0$ such that the support of ϕ is included in $\mathcal{A}_M \cap \mathbb{R}_+^d$.

Set $h_\phi(A) := \int \phi(Ax)\mathrm{d}\eta(x)$ for any $A \in R_{\overline{\mu}}$. The fact that $A \in \mathcal{S}$ ensures that

$$A^{-1}\mathcal{A}_M \cap \mathbb{R}_+^d \subset \mathcal{A}_{NM} \cap \mathbb{R}_+^d$$

with $N = \mathfrak{n}(A) \geq 1$; hence $|h_\phi(A)| \leq |\phi|_\infty \eta(\mathcal{A}_{NM}) < +\infty$, which proves that h_ϕ is bounded on $R_{\overline{\mu}}$.

A similar argument shows that h_ϕ is continuous on $R_{\overline{\mu}}$. Indeed, if $A_n \to A$, then $A_n^{-1}\mathcal{A}_M \cap \mathbb{R}_+^d \subset \mathcal{A}_{N'M} \cap \mathbb{R}_+^d$ for some $N' \geq 1$ and all $n \geq 1$. Thus, $|\phi(A_n x)| \leq |\phi|_\infty \mathbf{1}_{\mathcal{A}_{N'M}}(x)$ and $\phi(A_n x) \to \phi(Ax)$ as $n \to +\infty$, for all $x \in \mathbb{R}_+^d$. One concludes using the dominated convergence theorem.

Now, observe that for all $(A, B) \in \mathcal{S} \times \mathbb{R}_+^d$ and any $t > 0$,

$$\Big| |t^{-1}(Ax + B)| - |t^{-1}(Ax)| \Big| \leq t^{-1}|B|.$$

Then, for all $t > 2M|B|$ and $x \in \mathbb{R}_+^d$,

$$|\phi(t^{-1}(Ax + B)) - \phi(t^{-1}(Ax))| \leq [\phi] t^{-1}|B| \mathbf{1}_{\mathcal{A}_{2M}}(t^{-1}(Ax)).$$

This yields

$$\begin{aligned}
&\limsup_{i\to+\infty} \frac{\left|\int \phi(t_i^{-1}(Ax + B))\mathrm{d}m(x) - \int \phi(t_i^{-1}(Ax))\mathrm{d}m(x)\right|}{L(t_i)} \\
&\leq [\phi]\eta(\mathcal{A}_{2N(A)M}) \limsup_{i\to+\infty} t_i^{-1}|B| = 0.
\end{aligned}$$

Consequently, the function h_ϕ is superharmonic: indeed,

$$\begin{aligned}
&\int_{\mathcal{S}} h_\phi(AA')\mathrm{d}\overline{\mu}(A') \\
&= \int_{\mathcal{S}} \lim_{i\to+\infty} \frac{\int_{\mathbb{R}_+^d} \phi(t_i^{-1}AA'x)\mathrm{d}m(x)}{L(t_i)} \mathrm{d}\overline{\mu}(A') \\
&= \int_{\mathcal{S}\times\mathbb{R}_+^d} \lim_{i\to+\infty} \frac{\int_{\mathbb{R}_+^d} \phi\left(t_i^{-1}A(A'x + B')\right)\mathrm{d}m(x)}{L(t_i)} \mathrm{d}\mu(A', B') \\
&\leq \liminf_{i\to+\infty} \int_{\mathcal{S}} \int_{\mathbb{R}_+^d} \frac{\phi\left(t_i^{-1}A(A'x + B')\right)}{L(t_i)} \mathrm{d}m(x)\mathrm{d}\mu(A', B') \quad \text{by Fatou's Lemma} \\
&\leq \liminf_{i\to+\infty} \frac{\int_{\mathbb{R}_+^d} \phi\left(t_i^{-1}Ax\right)\mathrm{d}m(x)}{L(t_i)} = h_\phi(A) \quad \text{since } m \text{ is } \mu\text{-invariant.}
\end{aligned}$$

Thus, by Lemma 4.3, equality $h_\phi(AA') = h_\phi(A)$ holds for all $A \in R_{\overline{\mu}}$ and $A' \in \S_{\overline{\mu}}$. □

Step 3. The function $L : t \mapsto L(t) = m(tK_{I_N})$ is slowly varying

We must demonstrate that, for all $s > 0$,

$$\lim_{t\to+\infty} \frac{L(ts)}{L(t)} = \lim_{t\to+\infty} \frac{m(tsK_{I_N})}{m(tK_{I_N})} = 1.$$

Let $(t_i)_i$ be a sequence in $\mathbb{R}$ which tends to $+\infty$; by Lemma 4.2, there exists a subsequence $(t_{i_j})_j$ such that $(m_{t_{i_j}})_j$ converges weakly to some limit measure η [3]. It is sufficient to check that

$$\frac{\eta(sK_{I_N})}{\eta(K_{I_N})} = \lim_{j\to+\infty} \frac{m(t_{i_j} sK_{I_N})}{m(t_{i_j} K_{I_N})} = 1.$$

First, since $A_0 v_0 = \lambda_0 v_0$, for any $J \subseteq \mathbb{R}_{*+}$ it holds

$$A_0(K_J) = A_0(Jv_0 + \mathrm{Ker}A_0) = \lambda_0 J v_0 \quad \text{and} \quad A_0^{-1}(K_J) = \frac{1}{\lambda_0} K_J = K_{\frac{1}{\lambda_0}J}.$$

Since $A_0 \in R_{\overline{\mu}}$, Lemma 4.3 yields $A_0^k \in R_{\overline{\mu}}$ for any $k \geq 1$. Hence

$$\eta(\lambda_0^{-k} K_J) = \eta(A_0^{-k} A_0^{-1}(K_{g_0 J})) = \eta(A_0^{-1}(K_J)) = \eta(K_J).$$

The same relation holds also for negative $k \in \mathbb{Z}$, noticing

$$\eta(\lambda_0^{-k} K_J) = \eta(K_{\lambda_0^{-k} J}) = \eta(\lambda_0^{-(-k)} K_{\lambda_0^{-k} J}) = \eta(K_J). \tag{4.7}$$

In other words $\eta(sK_J) = \eta(K_J)$ for any interval J and $s \in \{\lambda_0^\ell / \ell \in \mathbb{Z}\}$. Now, if we specify the interval J, this property holds for generic s in $\mathbb{R}_+^*$; namely, set $J = I_N = [\lambda_0^{-N}, \lambda_0^N[$, choose some integer k_s such that $\lambda_0^{k_s}$ belongs to $[s\lambda_0^{-N}, s\lambda_0^N[$ and write

$$\begin{aligned}
\eta(sK_J) &= \eta(K_{[s\lambda_0^{-N}, s\lambda_0^N[}) = \eta\left(K_{[s\lambda_0^{-N}, \lambda_0^{k_s}[}\right) + \eta\left(K_{[\lambda_0^{k_s}, s\lambda_0^N[}\right) \\
&= \eta\left(K_{[s\lambda_0^{-N+2N}, \lambda_0^{k_s+2N}[}\right) + \eta\left(K_{[\lambda_0^{k_s}, s\lambda_0^N[}\right) \qquad \text{by} \quad (4.7) \\
&= \eta\left(K_{[\lambda_0^{k_s}, s\lambda_0^{k_s+2N}[}\right) \\
&= \eta\left(K_{[\lambda_0^{k_s-(k_s+2N)}, s\lambda_0^{k_s+2N-(k_s+2N)}[}\right) \quad \text{again by} \quad (4.7) \\
&= \eta(K_J).
\end{aligned}$$

This achieves the proof of Step 3. Proposition 4.1 follows.

□

4.3. *Appendix: proof of Lemma 4.3.* 1. Obviously, $R_{\mu_\circ}$ is closed and $R_{\mu_\circ} \subseteq T_{\mu_\circ}$. To check it is an ideal of $T_{\mu_\circ}$, let us fix $r \in R_{\mu_\circ}, s \in T_{\mu_\circ}$ and let V_{rs} be an open neighborhood of $rs \in T$. By continuity of the map $p : (s_1, s_2) \mapsto s_1 s_2$ on $T \times T$, there exist open neighborhoods V_r of r and V_s of s such that $V_r \times V_s \subset p^{-1}(V_{rs})$ (in other words $V_r V_s \subseteq V_{rs}$.) Fix $N \geq 1$ such that $\mu_\circ^N(V_s) > 0$. Then

$$\sum_{n=1}^{+\infty} \mu_\circ^n(V_{rs}) \geq \sum_{n=1}^{+\infty} \mu_\circ^{n+N}(V_{rs}) \geq \sum_{n=1}^{+\infty} \mu_\circ^n(V_r)\mu_\circ^N(V_s) = +\infty$$

which proves that $rs \in R_{\mu_\circ}$.

2. First, notice that the restriction to $R_{\mu_\circ}$ of any positive superharmonic function on T is harmonic on $R_{\mu_\circ}$; in other words, if $Ph(s_0) := \int h(s_0 s) \mathrm{d}\mu_\circ(s) \leq h(s_0)$ for any $s_0 \in T$ then $Ph(r) = h(r)$ for any $r \in R_{\mu_\circ}$.

[3]We do not know if the whole sequence does converge to η, the argument developed here does not reach to this property.

We fix $r \in R_{\mu_\circ}$, set $a_r := h(r) - Ph(r) \geq 0$ and suppose that $a_r > 0$. Then, since h and Ph are continuous, there exists an open neighborhood V of r such that $h - Ph \geq \frac{a}{2}\mathbf{1}_V$. Hence, for every $N \in \mathbb{Z}_+$,

$$\begin{aligned}\frac{a}{2}\sum_{n=0}^{N}\mu_\circ^n(V) &= \frac{a}{2}\sum_{n=0}^{N}P^n\mathbf{1}_V(e)\\ &\leq \sum_{n=0}^{N}P^n(h-Ph)(e) = h(e) - P^{N+1}h(e) \leq h(e) < +\infty.\end{aligned}$$

This yields $a = 0$ since $\sum_{n=0}^{N}\mu_\circ^n(V) \to +\infty$ as $N \to +\infty$.

Second, let us consider the function h' defined by $h'(s_0) = \min\{h(s_0), h(r)\}$ for any $s_0 \in T$. We claim that h' is superharmonic. Indeed, for any $s_0 \in T$,

$$Ph'(s_0) \leq \min\{Ph(s_0), h(r)\} \leq \min\{h(s_0), h(r)\} = h'(s_0).$$

Thus, for every $n \in \mathbb{Z}_+$, it holds

$$h(r) = h'(r) = P^n h'(r) = \int \min\{h(rs), h(r)\}\mathrm{d}\mu_\circ^n(s)$$

and $h(rs) \geq h(r)$ for $\mu_\circ^n$ almost all s and the equality $h(r) = P^n h(r) = \int h(rs)\mathrm{d}\mu_\circ^n(s)$ readily implies $h(rs) = h(s)$ for $\mu_\circ^n$-almost all s. By continuity of h, the equality holds for all $s \in T_{\mu_\circ}$.

□

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