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On the percolative properties of the intersection of two independent interlacements

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Abstract. We prove the existence of non-trivial phase transitions for the intersection of two independent random interlacements and the complement of the intersection. Some asymptotic results about the phase curves are also obtained. Moreover, we show that at least one of these two sets percolates in high dimensions.

1. Introduction

The model of random interlacements was first introduced by Sznitman (2010) to clarify the local structure left by a simple random walk on a discrete torus running up to some time proportional to its volume. It has interesting percolative and geometric properties, and a lot of research has been done in this field, e.g., Procaccia et al. (2016); Sapozhnikov (2017); Sidoravicius and Sznitman (2009); Sznitman (2010).

More precisely, random interlacements are a Poisson point process whose "points" are doubly-infinite trajectories on $\mathbb{Z}^d$ ($d \geq 3$), with the intensity measure governed by a parameter $u > 0$. We let $\mathcal{I}^u$ denote the set of vertices visited by at least one of these trajectories and call it the *interlacement set at level* u. We let $\mathcal{V}^u$ denote the complement of $\mathcal{I}^u$ and call it the *vacant set at level* u. We refer to Section 2 for precise definitions.

In this article, we will consider two independent interlacements $\mathcal{I}_1^{u_1}$, $\mathcal{I}_2^{u_2}$ with intensity parameters u_1, u_2, and their vacant sets $\mathcal{V}_1^{u_1}$, $\mathcal{V}_2^{u_2}$. Let $\mathcal{K}^{u_1,u_2} = \mathcal{I}_1^{u_1} \cap \mathcal{I}_2^{u_2}$ be their intersection and $\mathcal{V}^{u_1,u_2} = \mathcal{V}_1^{u_1} \cup \mathcal{V}_2^{u_2}$ be the complement of the intersection. Superscripts will be omitted whenever no ambiguity arises.

We now present our main results on the percolative properties of the intersection and its complement. First, both $\mathcal{K}$ and $\mathcal{V}$ have at most one infinite connected component and undergo a non-trivial phase transition in u_1 and u_2. We also obtain some results about the asymptotic behavior of the phase curves. The phase curve of $\mathcal{V}$ will tend to the lines $x = u^+$ and $y = u^+$, where u^+ is a parameter between the two percolative thresholds of interlacements u_* and u_{**} defined in Section 2 (see Figure 1). The phase curve of $\mathcal{K}$ will tend to x-axis and also y-axis (see Figure 2).[1]

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[1]Currently, we do not know whether this curve hits the coordinate axes for $d = 3, 4$, see Theorem 1.2 b) and d).

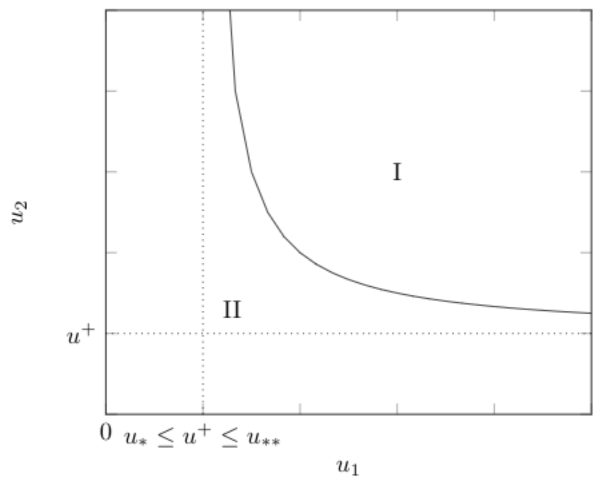

FIGURE 1.1. (Phase diagram of $\mathcal{V}$.) Region I: $\mathcal{V}$ does not percolate. Region II: $\mathcal{V}$ percolates.

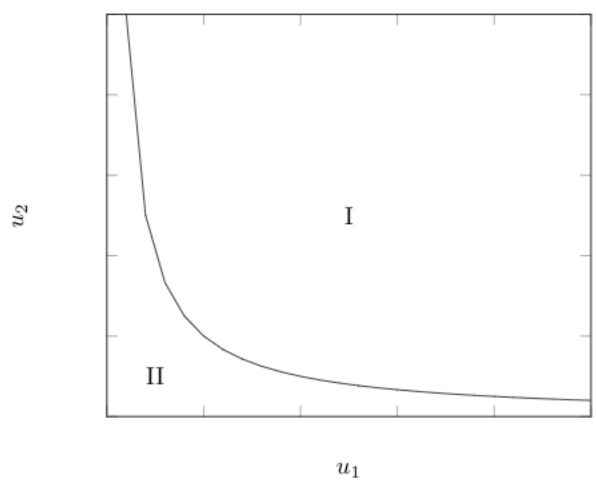

FIGURE 1.2. (Phase diagram of $\mathcal{K}$.) Region I: $\mathcal{K}$ percolates. Region II: $\mathcal{K}$ does not percolate.

We also research the phase graph of $\mathcal{K}$ and $\mathcal{V}$ put together and consider the questions whether there is a phase where two infinite components coexist and whether there is a phase where neither of them exists. It follows from the above asymptotic analysis that there exists a certain region such that both $\mathcal{K}$ and $\mathcal{V}$ percolate. The second question is hard and depends on the dimension, e.g., Bernoulli site percolation on $\mathbb{Z}^d$. In low dimensions, it might be the case that the occupied vertices and vacant vertices wrap each other. We claim that in high dimensions at least one of $\mathcal{K}$ and $\mathcal{V}$ percolates through showing that the phase curve of $\mathcal{K}$ lies below $\{(x, y) : x \geq 1, y \geq 1\}$. In this case, the phase graph of $\mathcal{K}$ and $\mathcal{V}$ put together is as follows (Figure 3).

The motivation of this article comes from the study of random walks. It is conjectured in Asselah and Schapira (2020); van den Berg et al. (2004) that two independent random walks conditioned to intersect many times will behave like two independent tilted interlacements. This article is also the starting point of giving large deviation bounds for the probability that the intersection of two independent random walks disconnects a large box (see Li, 2017; Sznitman, 2017) and the probability

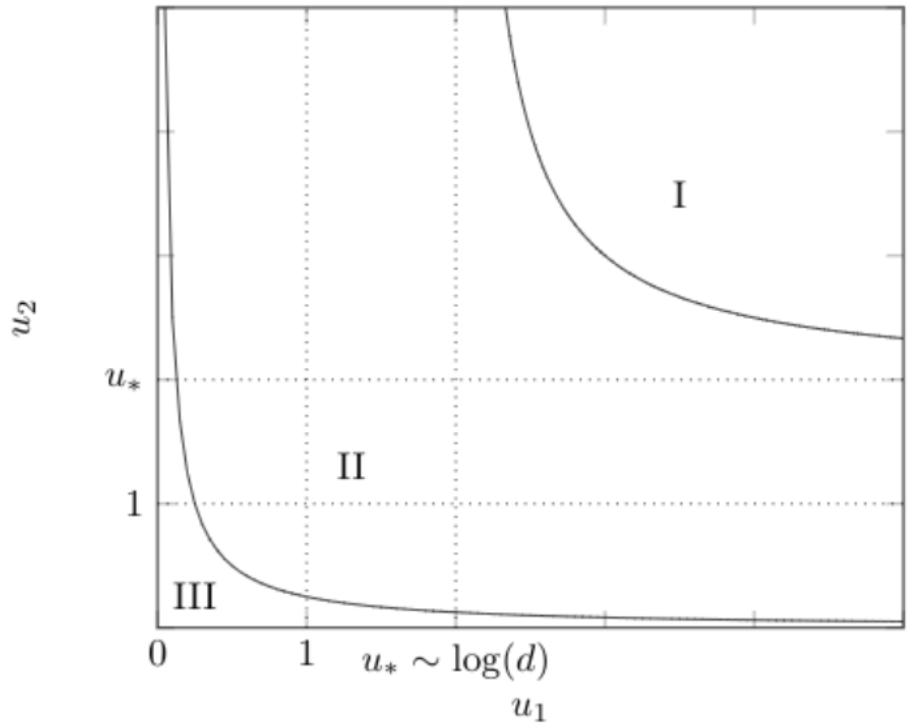

FIGURE 1.3. (Phase diagram of $\mathcal{K}$ and $\mathcal{V}$ in high dimensions.) Region I: $\mathcal{K}$ percolates but $\mathcal{V}$ does not percolate. Region II: both $\mathcal{K}$ and $\mathcal{V}$ percolate. Region III: $\mathcal{K}$ does not percolate but $\mathcal{V}$ percolates.

that the intersection of two independent random walks connects the original to the distance N (see Goswami et al., 2021).

Next, we will state our results rigorously and briefly explain the main ideas after each theorem. Let $B(x, r)$ be the l^∞ ball centered at x and of radius r. For a finite subset K of $\mathbb{Z}^d$, let $\partial_i K$ be its inner boundary. First, we present the result about the percolative properties of $\mathcal{V}$.

Theorem 1.1 (Percolative properties of $\mathcal{V}$)**.**

a) The set $\mathcal{V}$ contains at most one infinite component a.s.

b) When $u_1 < u_*$ or $u_2 < u_*$, there is a.s. a unique infinite component in $\mathcal{V}$.

c) Given $u_1 > u_{**}$, there exists $C = C(u_1, d)$ such that for all $u_2 > C$, there are a.s. no infinite components in $\mathcal{V}$.

d) There exists a constant $u^+ \in [u_*, u_{**}]$ and a decreasing function $\Gamma : [u^+, +\infty) \to [u^+, +\infty]$ (only $\Gamma(u^+)$ can be $+\infty$) [2] such that $\mathcal{V}$ a.s. has a unique infinite component when

i. $u_1 < u^+$;

ii. $u_1 \geq u^+$ and $u_2 < \Gamma(u_1)$,

and $\mathcal{V}$ a.s. has no infinite components when $u_1 \geq u^+$ and $u_2 > \Gamma(u_1)$.

Claim a) is an elementary property of most percolation models on $\mathbb{Z}^d$ or more generally an amenable graph. The proof of it uses a variant of the Burton-Keane argument (Burton and Keane, 1989; Teixeira, 2009). Claim b) is immediate from the definition of u_*. For c), one can see $\mathcal{V}_2^{u_2}$ as a small perturbation when u_2 is large. Thus, Claim c) mainly says that the percolation of the vacant set of interlacements is stable under this fluctuation. The proof relies on the renormalization

[2] Currently, we do not know whether Γ is continuous.

argument introduced in Sidoravicius and Sznitman (2009) and local properties of random interlacements. The renormalization argument builds on an induction along the renormalization scheme and provides us with the decoupling inequalities (see Proposition 2.1). Thus, we only need to prove the "triggers", i.e., some local inequalities in a finite box $B(0,2L_0)$. Locally, with high probability $\mathcal{V}$ cannot have a large connected component, since with high probability $\mathcal{V}_1^{u_1}$ cannot connect $B(0,L_0)$ with $\partial_i B(0,2L_0)$ and $\mathcal{V}_2^{u_2}$ is empty in $B(0,2L_0)$ by enlarging u_2. Finally, combining b) and c), we can get d) instantaneously.

Next, we will present our result about the percolative properties of $\mathcal{K}$. These properties are different from those of the original model, i.e., random interlacements.

Theorem 1.2 (Percolative properties of $\mathcal{K}$)**.**

a) The set $\mathcal{K}$ has at most one infinite component a.s.

b) Given $u_1 > 0$, there exists a constant $C = C(u_1, d) < \infty$ such that $\mathcal{K}$ has a unique infinite component a.s. for all $u_2 > C$.

c) There exists a constant $c = c(d) > 0$ such that for all $u_1, u_2 < c$, there are a.s. no infinite components in $\mathcal{K}$.

d) Given $d \geq 5$ and $u_1 > 0$, there exists a constant $c = c(u_1, d) > 0$ such that for all $u_2 < c$, there are a.s. no infinite components in $\mathcal{K}$.

Claim a) is elementary and its proof is the same as that of Theorem 1.1 a). For b), one can see $\mathcal{V}_2^{u_2}$ as a small fluctuation when u_2 is large. Thus, Claim b) mainly says that the percolation of the intersection $\mathcal{K} = \mathcal{I}_1^{u_1} \backslash \mathcal{V}_2^{u_2}$ is stable under this fluctuation. The proof uses local properties of random interlacements and the renormalization argument. Locally, random interlacements are strongly connected meaning that with high probability all the vertices of $\mathcal{I}^u$ in $B(0,N)$ are connected in $B(0,2N)$. Meanwhile, u_2 can be taken large such that with high probability $\mathcal{V}_2^{u_2}$ is empty in $B(0,2N)$. Thus, with high probability, $\mathcal{K}$ has a large connected component in $B(0,2N)$. Then, through the renormalization argument, these local properties can be pushed to the global ones. The rigorous proof is a little bit harder since the above mentioned event isn't monotone and we cannot use the decoupling inequalities directly to it. For the result of c), we consider the box of side length $2N$ and write $M = u_1 N^{d-2} = u_2 N^{d-2}$. First, pick a large M to offset the error terms in the decoupling inequalities. In $\mathcal{I}_1^{u_1} \cap B(x,2N)$ and $\mathcal{I}_2^{u_2} \cap B(x,2N)$, with high probability there are $O(M)$ independent random walks individually. Given M, one can take N large and simultaneously u_1 and u_2 small such that only with small probability $\partial_i B(0,N)$ is connected to $\partial_i B(0,2N)$ by the intersection. Then, we can use the renormalization argument to push these local properties to the global ones. The rigorous proof will need some concrete calculations on simple random walks. Claim d) is an improvement of Claim c) for $d \geq 5$. Its proof uses cut times of random walks Lawler (1996).

By now, two natural questions arise: is there a phase where two infinite components coexist? Similarly, is there a phase where neither of them exists? Our results above also shed some light to these questions. By Theorem 1.1 b) and Theorem 1.2 b), there exist choices of u_1 and u_2 such that both $\mathcal{K}$ and $\mathcal{V}$ have an infinite component. For the second question, we give an affirmative answer when the dimension is high. Together with Theorem 0.1 in Sznitman (2011), Theorem 1.1 b) and the following Theorem 1.4, we conclude that when the dimension is high, at least one of $\mathcal{K}$ and $\mathcal{V}$ has an infinite component. We summarize the discussion above into the following theorem.

Theorem 1.3.

a) There exists a phase such that $\mathcal{K}$ and $\mathcal{V}$ both have a unique infinite component a.s.

b) There exists $D_2 < \infty$ such that for all $d > D_2$ and $u_1, u_2 > 0$, at least one of $\mathcal{K}$ and $\mathcal{V}$ has a unique infinite component a.s.

A key ingredient of the proof of Theorem 1.3b) is the following Theorem 1.4.

Theorem 1.4. *There exists a constant D_1 such that for all $d > D_1$ and $u_1, u_2 > 1$, there is a.s. a unique infinite component in $\mathcal{K}$. In other words, the phase curve of $\mathcal{K}$ lies below the region $\{(x,y) : x \geq 1$ and $y \geq 1\}$.*

As a remark, we note that the lower bound 1 here is not optimal and can be improved to $\frac{1}{d^{1/2-\epsilon}}$. However, the argument cannot be extended to low dimensions as it involves some asymptotic analysis.

The proof of this theorem contains two parts: a local analysis and a renormalization argument similar to those in the proof of Theorem 1.1 and 1.2. Locally, in a hypercube $\{0,1\}^d$, random interlacements can stochastically dominate a Bernoulli site percolation. Thus, by the method in Alon et al. (2004), in high dimensions, in each hypercube with high probability there is a ubiquitous component (see Section 5 for the definition) connected to the neighboring ones (which also exist with high probability). Then, by the renormalization argument, there will be an infinite component.

Next, we will explain how this article is organized. In Section 2, we introduce some notation, make a brief introduction to random interlacements, explain the renormalization argument which we will use repeatedly in this article and give some estimates on simple random walks. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. The phase diagram of $\mathcal{K}$ and $\mathcal{V}$ put together is discussed in Section 5.

Finally, we explain the convention regarding constants in this work. All constants in this article are positive. Constants like c, C, ϵ, γ may change from place to place, while constants with subscripts like c_1, C_1, D_1 are kept fixed through the article. The constants in Section 3 and 4 may depend on d implicitly, while the constants in Section 5 will not.

2. Notation and useful results

In this section, we introduce notation, review some basic properties of random interlacements together with the renormalization argument introduced in Section 2 of Sidoravicius and Sznitman (2009) and collect some estimates related to simple random walks.

For a real value a, we write $[a]$ for the largest integer $\leq a$. We consider the integer lattice $\mathbb{Z}^d$ with $d \geq 3$. Norms $|\cdot|_1$ and $|\cdot|_\infty$ represent the l^1-norms and l^∞-norms on $\mathbb{Z}^d$. We call two vertices x and y *-neighbors if $|x-y|_\infty = 1$ and nearest neighbors if $|x-y|_1 = 1$. We call a set $\pi = (y_1, y_2, ..., y_k) \subset \mathbb{Z}^d$ a *-neighbor path if y_i, y_{i+1} are *-neighbors for $1 \leq i \leq k-1$, and a nearest neighbor path if y_i, y_{i+1} are nearest neighbors for $1 \leq i \leq k-1$. A path is simple if y_is are all different from each other. Given K, L, U subsets of $\mathbb{Z}^d$, we say K and L are connected by U and write $K \overset{U}{\longleftrightarrow} L$, if there exists a nearest neighbor path with values in U which starts in K and ends in L. We denote by $B(x,N) = \{y \in \mathbb{Z}^d : |x-y|_\infty \leq N\}$ the closed l^∞ ball centered at x and of radius N. For a finite subset K of $\mathbb{Z}^d$, we write $\partial_i K = \{x \in K : x \text{ is a nearest neighbor of some point } y \notin K\}$ for its inner boundary.

Here is some notation about discrete-time simple random walks. P_x represents the discrete-time simple random walk X started at x on $\mathbb{Z}^d$. We write $P_{x,y}$ for two independent discrete-time simple random walks started at x and y respectively. Let K be a finite subset of $\mathbb{Z}^d$. We write τ_K for the first time that X hits K and τ_K^+ for the first positive time that X hits K. We denote the equilibrium measure of K by $e_K(x) = P_x\left[\tau_K^+ = \infty\right] \mathbb{1}_K(x)$ for $x \in \mathbb{Z}^d$, and the capacity of K by $\mathrm{cap}(K) = \sum_x e_K(x)$.

For functions $f, g : \mathbb{Z} \to R$, we write $f = O(g)$ if there is a constant C such that $f(x) \leq C \cdot g(x)$, for all $x \in \mathbb{Z}$. We write $f = \Omega(g)$ if there are constants c, C such that $c \cdot g(x) \leq f(x) \leq C \cdot g(x)$, for all $x \in \mathbb{Z}$.

2.1. *Random interlacements and the renormalization argument.* First, we briefly introduce the random interlacements.

Let W be the space of doubly-infinite nearest neighbor paths in $\mathbb{Z}^d$, and let W^* be the quotient space of W modulo time shift. π is the quotient map from W to W^*. By Chapter 5 of Drewitz et al. (2014a), we can define a Poisson point measure μ on W^* with the following local property. Given K a finite subset of $\mathbb{Z}^d$, we write W_K^* for the paths in W^* that pass through K and μ_K for μ restricted to W_K^*. Then

$$\mu_K = \sum_{i=1}^{N_K} \delta_{\pi(X_i)}, \tag{2.1}$$

where N_K is a random variable $\sim$ Poisson$(u \cdot \mathrm{cap}(K))$ and X_i is a doubly-infinite path in which $X_i(0)$ is a random point in K according to the equilibrium measure distribution. Conditional on $X_i(0)$, the positive side $\{X_i\}_{i\geq 0}$ is a simple random walk, and the opposite side $\{X_i\}_{i\leq 0}$ is a simple random walk conditional on $\{\tau_K^+ = \infty\}$ and independent of the positive one. Given N_K, all these N_K paths are conditionally independent. The set of points occupied by at least one path is called the interlacement set at level u, denoted by $\mathcal{I}^u$. The complement of it is called the vacant set, denoted by $\mathcal{V}^u$. The graph induced by the edges visited by random interlacements on $\mathbb{Z}^d$ is denoted by $\tilde{\mathcal{I}}^u$.

There is a more concise alternative definition of interlacements. The random interlacements $\mathcal{I}^u$ are a random subset of $\mathbb{Z}^d$ whose law is given by

$$P\left[\mathcal{I}^u \cap K = \emptyset\right] = e^{-u\cdot \mathrm{cap}(K)}, \text{ for all finite subset } K \text{ of } \mathbb{Z}^d. \tag{2.2}$$

Let $(\Omega_1, \mathcal{F}_1, P^{u_1})$ be the probability space on which $\mathcal{I}_1^{u_1}$ is defined (see (5.2.1) and (5.2.6) of Drewitz et al., 2014a for more details). Let $(\Omega_2, \mathcal{F}_2, P^{u_2})$ be the probability space on which $\mathcal{I}_2^{u_2}$ is defined. Finally, let $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \sigma(\mathcal{F}_1 \times \mathcal{F}_2), P^{u_1} \otimes P^{u_2})$ be the probability space on which $\mathcal{I}_1^{u_1}$ and $\mathcal{I}_2^{u_2}$ are jointly defined.

Random interlacements are a typical model of percolation with long-range correlation. It has been known that the interlacement set itself almost surely has a unique infinite component as shown in (2.21) of Sznitman (2010), while the vacant set undergoes a non-trivial phase transition in u (see Theorem 4.3 of Sznitman, 2010, Theorem 3.4 of Sidoravicius and Sznitman, 2009 and Theorem 3.1 of Popov and Teixeira, 2015). There are two percolative thresholds of interlacements u_* and u_{**} that we will use in this paper:

$$u_* = \inf\left\{u : P\left[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\right] = 0\right\}, \tag{2.3}$$

$$u_{**} = \inf\left\{u : \liminf_{L\to\infty} P\left[B(0,L) \overset{\mathcal{V}^u}{\longleftrightarrow} \partial_i B(0,2L)\right] = 0\right\}. \tag{2.4}$$

When u is above u_*, there are a.s. no infinite clusters in the vacant set. When u is above u_{**}, each component of the vacant set is exponentially small. There is another critical parameter $\overline{u}$ introduced in Theorem 1.1 of Drewitz et al. (2014b). It is plausible, but unproven at the moment, that actually $\overline{u} = u_* = u_{**}$, which is one of the most important open problems in this field. In the context of the level-set percolation of Gaussian free field, a model which bears similar properties to random interlacements, the parallel problem has been solved recently in Theorem 1.1 of Duminil-Copin et al. (2020).

Next, we will state the renormalization argument first introduced in Chapter 2 of Sidoravicius and Sznitman (2009). The idea is to zoom in on a large box layer upon layer along the renormalization scheme. In each layer, we can decouple the configurations in two boxes that are far apart with small error. The version we present here is from Chapter 8 of Drewitz et al. (2014a). Let L_0 and l_0 be two positive integers chosen according to the context, and $L_n = L_0 \cdot l_0^n$. We define the renormalized lattice graph $\mathbb{G}_n$ as

$$\mathbb{G}_n = L_n\mathbb{Z}^d = \left\{L_n x : x \in \mathbb{Z}^d\right\}.$$

For $x \in \mathbb{Z}^d$ and $n \geq 0$, let

$$\Lambda_{x,n} = \mathbb{G}_{n-1} \cap B(x, L_n)$$

be a renormalized box with side length L_n. We call an event $G_{x,0}$, or for short G_x, seed event if it is measurable with respect to the configuration in $B(x, 2L_0)$ and shift-invariant, i.e., $\psi \in G_x$ if and only if $\psi(\cdot - x) \in G_0$, where x is a vertex of $\mathbb{Z}^d$. We also hope G_x monotone. For $x \in \mathbb{Z}^d$ and $n \geq 1$, we write

$$G_{x,n} = \bigcup_{x_1,x_2 \in \Lambda_{x,n}; |x_1 - x_2|_\infty > \frac{L_n}{100}} G_{x_1,n-1} \cap G_{x_2,n-1}. \tag{2.5}$$

$G_{x,n}$ means that there exists a dyadic tree whose leaves are separated apart and satisfy G_x.

A typical scenario where we will use this event is in the following claim. When there is a *-neighbor or nearest neighbor path connecting 0 to $\partial_i B(0, L_n)$ such that every vertex of this path satisfies G_x, then $G_{x,n}$ happens. This claim can be proved by induction and it is used in the proof of Theorem 1.1c) and Theorem 1.2 b), c).

The following decoupling inequalities are a variant of Theorem 8.5 of Drewitz et al. (2014a) and they are used repeatedly in our proofs. They follow from the idea of renormalization and the sprinkling technique in Proposition 3.1 of Sznitman (2010).

Proposition 2.1 (Decoupling inequalities for two interlacements)**.** *For $d \geq 3$ and $\epsilon > 0$, there exists an integer $A = A(d, \epsilon)$ such that for all $n \geq 0$, $L_0 \geq 1$ and $l_0 \geq A$, the following two statements hold:*

1. *if G_x is an increasing seed event, then for all ${u_1}^- \leq (1-\epsilon)u_1$ and ${u_2}^- \leq (1-\epsilon)u_2$*

$$\begin{aligned} &\mathbb{P}\left[\mathcal{K}^{{u_1}^-, {u_2}^-} \in G_{0,n}\right] \\ \leq &(2l_0+1)^{d\cdot 2^{n+1}} \left[\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in G_0\right] + \epsilon({u_1}^-, L_0, l_0) + \epsilon({u_2}^-, L_0, l_0)\right]^{2^n}; \end{aligned} \tag{2.6}$$

2. *if G_x is a decreasing seed event, then for all ${u_1}^+ \geq (1+\epsilon)u_1$ and ${u_2}^+ \geq (1+\epsilon)u_2$*

$$\begin{aligned} &\mathbb{P}\left[\mathcal{K}^{{u_1}^+, {u_2}^+} \in G_{0,n}\right] \\ \leq &(2l_0+1)^{d\cdot 2^{n+1}} \left[\mathbb{P}[\mathcal{K}^{u_1,u_2} \in G_0] + \epsilon(u_1, L_0, l_0) + \epsilon(u_2, L_0, l_0)\right]^{2^n}, \end{aligned} \tag{2.7}$$

where

$$\epsilon(u, L_0, l_0) = \frac{2e^{-uL_0^{d-2} l_0^{\frac{d-2}{2}}}}{1 - e^{-uL_0^{d-2} l_0^{\frac{d-2}{2}}}}. \tag{2.8}$$

Proof: The proof is similar to Theorem 8.5 in Drewitz et al. (2014a) despite that the claim here involves two independent interlacements. We need to change the coupling in Theorem 7.9 of Drewitz et al. (2014a) into a coupling of two independent copies of the point measures there and the term ϵ there should be changed into $\epsilon_1 + \epsilon_2$, where $\epsilon_1 = \epsilon(u_{1-}, u_{1+}, S_1, S_2, U_1, U_2)$ and $\epsilon_2 = \epsilon(u_{2-}, u_{2+}, S_1, S_2, U_1, U_2)$. The term $\epsilon(u_-, n)$ in (8.1.9) and (8.1.10) of Drewitz et al. (2014a) should be replaced by $\epsilon(u_{1-}, n) + \epsilon(u_{2-}, n)$. Equation (8.3.4) of Drewitz et al. (2014a) is still true since $a^m + b^m + c^m \leq (a+b+c)^m$. □

2.2. *Estimates about SRW.* Here, we present some results about simple random walks on $\mathbb{Z}^d (d \geq 3)$ that will be used. For $x, y \in \mathbb{Z}^d$, the Green function is denoted by $G(x, y) = \sum_{n=0}^{\infty} P_x(X_n = y)$. The following lemma is very simple and we give a proof just for completeness. It is used in Section 5 to prove that interlacements can dominate Bernoulli site percolation in a hypercube $\{0, 1\}^d$.

Lemma 2.2. *There exists a constant $c_1 > 0$ such that for all $d \geq 3$,*

$$P_0\left[\tau^+_{\{0,1\}^d} = \infty\right] > c_1. \tag{2.9}$$

Proof: Thanks to symmetry and the strong Markov property,

$$\sum_{x\in\{0,1\}^d} G(0,x) = E_0\left[\sum_{i=0}^{\infty} \mathbb{1}_{\{X_i\in\{0,1\}^d\}}\right] = \frac{1}{P_0\left[\tau^+_{\{0,1\}^d} = \infty\right]}.$$

Thus, it is sufficient to prove that

$$\sum_{x\in\{0,1\}^d} G(0,x) < \frac{1}{c_1}.$$

By (2.10), p.243 in Montroll (1956), we have

$$G(0,x) = \int_0^\infty e^{-u}\prod_{i=1}^d I_{x_i}\left(\frac{u}{d}\right)du, \text{ for } x = (x_1, ..., x_d) \in \mathbb{Z}^d,$$

where

$$I_n(u) = \frac{1}{\pi}\int_0^\pi e^{u\cos\theta}\cos n\theta d\theta, \ u \in \mathbb{C}.$$

We get that

$$\begin{aligned}\sum_{x\in\{0,1\}^d} G(0,x) &= \int_0^\infty e^{-u}\sum_{x\in\{0,1\}^d}\prod_{i=1}^d I_{x_i}\left(\frac{u}{d}\right)du\\ &= d\int_0^\infty\left(\frac{I_0(u)+I_1(u)}{e^u}\right)^d du.\end{aligned}$$

Denote $(I_0(u) + I_1(u))/e^u$ by $Z(u)$. Then, $Z(0) = 1$. By some easy calculations, we can take a constant $A \in (1,\infty)$ and $B = 1/(4e^A) + 1/(4e^{2A})$ such that

$$Z(u) \leq \frac{1}{\sqrt{u}}, \text{ for all } u \geq A,$$

and

$$Z(u) \leq 1 - Bu, \text{ for all } u \leq A.$$

Therefore,

$$\begin{aligned}\sum_{x\in\{0,1\}^d} G(0,x) &= d\int_0^A (Z(u))^d du + d\int_A^\infty (Z(u))^d du\\ &\leq d\int_0^A (1-Bu)^d du + d\int_A^\infty\left(\frac{1}{\sqrt{u}}\right)^d du\\ &\leq \frac{1}{B}\frac{d}{d+1} + A^{-\frac{d}{2}+1}\frac{d}{\frac{d}{2}-1}.\end{aligned}$$

Note that $d \geq 3$, $A > 1$ and A, B are independent of d. Hence, there exists a constant $c_1 > 0$ independent of d such that $\sum_{x\in\{0,1\}^d} G(0,x) < 1/c_1$. This completes the proof of (2.9). □

Remark 2.3. With more careful calculations, we can get that the left-hand side in (2.9) tends to $1/2$ as d tends to ∞. This coincides with the heuristic that whenever X_i leaves $\{0,1\}^d$ in high dimensions, it will not come back any more.

The following notation is defined when $d = 3$. Let $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$. The disc centered at x of radius M is denoted by

$$D(x, M) = \{(y_1, y_2, y_3) : |y_1 - x_1|, |y_2 - x_2| \leq M, y_3 = x_3\}.$$

We denote one quarter of the disc centered at x of radius M by

$$D^+(x, M) = \{(y_1, y_2, y_3) : x_1 \leq y_1 \leq x_1 + M, x_2 \leq y_2 \leq x_2 + M, y_3 = x_3\}.$$

The first exit time of a simple random walk is written by $\xi_m = \inf\{i \geq 0 : X_i \notin B(0, m)\}$. The following proposition is used to prove Theorem 1.2c).

Proposition 2.4. *For $d = 3$, there exists $C_1 > 0$ such that for all $M \geq 1$,*

$$\max_{x,y \in \partial_i B(0,2M)} P_{x,y}\left[X^1[0,\infty) \cap X^2[0,\infty) \cap \partial_i B(0, M) \neq \emptyset\right] < \frac{C_1}{\log(M)}, \tag{2.10}$$

where X^1 and X^2 are two independent simple random walks starting from x and y.

The order $\Omega(1/\log(M))$ here is right. The proof is similar to Section 3.4 in Lawler (1991). The expectation of the intersection in $\partial_i B(0, M)$ is $O(1)$. Intuitively, when X^1 and X^2 intersect in $\partial_i B(0, M)$, then with high probability they will have $\Omega(\log(M))$ intersection points in $\partial_i B(0, M)$. Thus, the probability that X^1 and X^2 intersect in $\partial_i B(0, M)$ is $\Omega(1/\log(M))$. However, there are no natural stopping times, which makes the rigorous proof difficult. To prove this proposition, we will need two lemmas.

Lemma 2.5. *For $d = 3$, there exists $\epsilon > 0$ and a positive integer N such that for all $M \geq N$,*

$$\min_{x \in B(0,M+1)} P_x\left[\sum_{i=0}^{\xi_{2M}-1} \mathbb{1}_{\{X_i \in D^+(0,2M)\}} > \epsilon M\right] > \epsilon.$$

Proof: Write y for $(3M/2, 3M/2, 0)$. By comparing the simple random walk with Brownian motion, there exists $\mu > 0$ such that for M large

$$\min_{x \in B(0,M+1)} P_x\left[\tau_{D(y,\frac{1}{4}M)} < \xi_{2M}\right] > \mu. \tag{2.11}$$

Furthermore, we can prove the following inequality with some $\gamma > 0$ and large M:

$$P_0\left[\sum_{i=0}^{\xi_{M/4}} \mathbb{1}_{\left\{X_i \in D(0,\frac{1}{4}M)\right\}} > \gamma M\right] > \gamma. \tag{2.12}$$

This inequality can be proved by considering the third coordinate. The movements in the third coordinate x_3 can be seen as a one-dimensional simple random walk $\{Y_i\}_{i\geq 0}$ starting from 0. By just calculating the first moment and second moment (first moment is of order $\sqrt{n}$ and second moment is of order n) and then using the Paley-Zygmund inequality, we can prove that there exists some $\eta > 0$ such that for large n,

$$P_0\left[\sum_{i=0}^{n} \mathbb{1}_{\{Y_i=0\}} > \eta\sqrt{n}\right] > \eta. \tag{2.13}$$

In addition, we can take some $c > 0$ such that for large M,

$$P_0[Z] > 1 - \frac{\eta}{2}, \tag{2.14}$$

where Z represents the event that $\xi_{M/4} > cM^2$ and in the first cM^2 moves there are at least $cM^2/4$ ones in the third coordinate.

If the event in (2.13) with $n = [cM^2/4]$ and Z happen at the same time, then the event in (2.12) happens for $\gamma \leq \eta\sqrt{c}/2$. Note that for large M, with more than $\eta/2$ probability the event in (2.13) with $n=[cM^2/4]$ and Z happen at the same time. Let $\gamma=\min\{\eta/2,\eta\sqrt{c}/2\}$ and this completes the proof of (2.12).

By (2.11), (2.12) and the strong Markov property, for any $x\in B(0,M+1)$ and large M,

$$\begin{aligned}&P_x\left[\sum_{i=0}^{\xi_{2M}-1}\mathbb{1}_{\{X_i\in D^+(0,2M)\}}>\gamma M\right]\\ \geq & P_x\left[\tau_{D(y,\frac{1}{4}M)}<\xi_{2M}\right]\cdot P_0\left[\sum_{i=0}^{\xi_{M/4}}\mathbb{1}_{\{X_i\in D(0,M/4)\}}>\gamma M\right]>\mu\cdot\gamma.\end{aligned}$$

In Lemma 2.5, take $\epsilon=\min\{\gamma,\mu\gamma\}$ and N large. We complete the proof. □

Next, we consider the random variable D_M defined by

$$D_M=\sum_{i=0}^{\infty}G(0,X_i)\,\mathbb{1}_{\{X_i\in D^+(0,M)\}}.$$

Lemma 2.6. *For $d=3$, there exist two constants $a,\epsilon>0$ and a positive integer N such that for all $M\geq N$*

$$P_0\left[D_M\geq\epsilon\log(M)\right]\geq 1-\frac{1}{M^a}.$$

Proof: Denote the ϵ and N in Lemma 2.5 by ϵ_1 and N_1. We decompose $[0,\xi_M)$ into k disjoint intervals $[\xi_{N_1},\xi_{2N_1}),[\xi_{2N_1},\xi_{4N_1}),\ldots,\left[\xi_{2^{k-1}N_1},\xi_{2^kN_1}\right)$, where $k=[\log_2(M/N_1)]$. Recall that $G(0,x)\geq C/|x|_\infty$ when $d=3$. Therefore,

$$\begin{aligned}D_M&=\sum_{i=0}^{\infty}G(0,X_i)\,\mathbb{1}_{\{X_i\in D^+(0,M)\}}\\ &\geq\sum_{l=0}^{k-1}\sum_{i=\xi_{2^lN_1}}^{\xi_{2^{l+1}N_1}-1}G(0,X_i)\,\mathbb{1}_{\left\{X_i\in D^+(0,2^{l+1}N_1)\right\}}\\ &\geq\sum_{l=0}^{k-1}\left(\sum_{i=\xi_{2^lN_1}}^{\xi_{2^{l+1}N_1}-1}\frac{C}{2^{l+1}N_1}\mathbb{1}_{\left\{X_i\in D^+(0,2^{l+1}N_1)\right\}}\right).\end{aligned}$$

By the strong Markov property and Lemma 2.5, each of the k terms above is independent and has more than ϵ_1 probability to be more than $C/(2^{l+1}N_1)\cdot\epsilon_1 2^lN_1=C\epsilon_1/2$. Thus, by the Hoeffding's inequality, for large k

$$P_0\left[D_M\geq\frac{C\epsilon_1^2}{4}k\right]\geq 1-e^{-ck}.$$

In Lemma 2.6, take $\epsilon<\frac{C\epsilon_1^2}{4}\log_2 e$, $a<c\log_2 e$ and N large. We complete the proof. □

With the above two lemmas, we can complete the proof of Proposition 2.4.

Proof of Proposition 2.4: We will use the a and ϵ in Lemma 2.6. Take any pair of points x and y in $\partial_i B(0,2M)$. The constants below are all independent of x,y and M. Now, X^1 and X^2 below are two independent simple random walks started at x and y. Define R_M to be

$$R_M=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{1}_{\left\{X_i^1=X_j^2\in\partial_iB(0,M)\right\}}.$$

An easy calculation shows that

$$E_{x,y}\left[R_M\right]\leq C,\text{ where } C \text{ is a constant independent of } x,y \text{ and } M.$$

Let ζ be the stopping time

$$\zeta = \inf\left\{i \geq 0 : X_i^1 \in X^2[0,\infty) \cap \partial_i B(0,M)\right\}.$$

Define σ as

$$\sigma = \inf\left\{j \geq 0 : X_j^2 = X_\zeta^1\right\}.$$

For any vertex z in $\partial_i B(0,M)$, we can find a $M \times M$ disc in $\partial_i B(0,M)$ with z as one of its corners in a deterministic way. We write $D(z)$ for this disc. Define $D_{j,M}$ as

$$D_{j,M} = \sum_{i=0}^{\infty} G\left(X_j^2, X_{j+i}^2\right) \mathbb{1}_{\left\{X_{j+i}^2 \in D(X_j^2)\right\}}$$

The time j is called good if $D_{j,M} \geq \epsilon \log(M)$ and bad otherwise. By the strong Markov property applied to X^1,

$$E_{x,y}\left[R_M | \zeta < \infty, \sigma \text{ is good}\right] \geq \epsilon \log(M).$$

Therefore,

$$\begin{aligned} P_{x,y}\left[\zeta < \infty, \sigma \text{ is good}\right] &\leq E_{x,y}\left[R_M\right]\left[E_{x,y}\left[R_M | \zeta < \infty, \sigma \text{ is good}\right]\right]^{-1} \\ &\leq \frac{C}{\log(M)}. \end{aligned}$$

By Lemma 2.6, symmetry and the Markov property applied to X^2, we have $P_{x,y}[D_{j,M} < \epsilon \log(M)] \leq 1/M^a$ for large M. Thus, for large M

$$\begin{aligned} P_{x,y}[\zeta < \infty, \sigma \text{ is bad}] &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{x,y}\left[X_i^1 = X_j^2 \in \partial_i B(0,M), D_{j,M} < \epsilon \log(M)\right] \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{x,y}\left[X_i^1 = X_j^2 \in \partial_i B(0,M)\right] P_{x,y}\left[D_{j,M} < \epsilon \log(M)\right] \\ &\leq \frac{1}{M^a} E_{x,y}\left[R_M\right] \leq \frac{C}{M^a}. \end{aligned}$$

So,

$$\begin{aligned} P_{x,y}\left[X^1[0,\infty) \cap X^2[0,\infty) \cap \partial_i B(0,M) \neq \emptyset\right] &= P_{x,y}[\zeta < \infty] \\ = P_{x,y}[\zeta < \infty, \sigma \text{ is good}] + P_{x,y}[\zeta < \infty, \sigma \text{ is bad}] &\leq \frac{C}{\log(M)}. \end{aligned}$$

The above inequality holds when M is large. Enlarge C if necessary such that for any M, inequality (2.10) holds. □

3. Percolative properties of $\mathcal{V}$

In this section, we consider the percolative properties of $\mathcal{V}$ and prove Theorem 1.1 which is split into four parts.

The proof of Theorem 1.1 a) is an adaptation of the Burton-Keane argument (see Theorem 2 of Burton and Keane, 1989, Corollary 2.3 of Sznitman, 2010 and Theorem 1.1 of Teixeira, 2009). The proof presented here is a streamlined version of that in Theorem 1.1 of Teixeira (2009). Also, the maps ϕ and Φ defined below are adapted from (2.23) to (2.24) in Sznitman (2010) and (3.2), (3.4), (3.11) in Teixeira (2009). The idea is to change the situation in a finite box by local surgeries.

Proof of Theorem 1.1 a): Recall that random interlacements are translation-invariant and ergodic as shown in Theorem 2.1 of Sznitman (2010). It follows that the total number N of infinite connected components of $\mathcal{V}$ is a.s. a constant, possibly infinite. The proof contains two parts. The first step is to argue that

$$\text{for } 1 < k < \infty,\ \mathbb{P}[N = k] = 0.$$

Suppose that in contrast for some $k \in (1, \infty)$, there are k infinite clusters a.s. Then, there exists a large constant M such that $\mathbb{P}[A] > 0$, where A denotes the event that all the k infinite connected components of $\mathcal{V}$ intersect the box $K = [-M, M]^d$.

We first prove that $\mathbb{P}[A_1] = 0$, where A_1 denotes the event that $\mathcal{V}\backslash K$ contains more than k infinite connected components. We consider the following map $\phi_1 : W^* \to W^*$. Recall that W_K^* is the set of the paths in W^* that intersect K. For a path ω in W_K^* , whenever ω enters K, the map ϕ_1 adds to ω a subpath that covers K and leaves K at that point. For the paths in $(W_K^*)^c$, the map ϕ_1 is an identity map. ϕ_1 can induce a natural map $\Phi_1 : \Omega \to \Omega$, where Ω is the configuration space of two independent interlacements defined in Section 2. The map Φ_1 is defined as:

$$\Phi_1(\psi) = \sum_{i,j} \delta_{(\phi(\omega_i),\phi(\omega'_j))},\ \text{for } \psi = \sum_{i,j} \delta_{(\omega_i,\omega'_j)} \in \Omega.$$

Note that $\Phi_1(A_1) \subset \{N > k\}$ (because the situation outside K is unchanged and K is occupied) and $\Phi_1 \circ \mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}$. Thus,

$$\begin{aligned}\mathbb{P}[A_1] &\leq \mathbb{P}[\Phi_1^{-1}\{N > k\}] \\ &= \Phi_1 \circ \mathbb{P}[N > k] = 0.\end{aligned}$$

Therefore $\mathbb{P}[A\backslash A_1] = \mathbb{P}[A] > 0$. If the event A happens, then we can find two vertices z_1 and z_2 in $\partial_i K$ such that they are vacant and contained in two distinct infinite components of $\mathcal{V}$. Since the number of choices is finite, there exist z_1 and z_2 in $\partial_i K$ such that $\mathbb{P}[B] > 0$, where B represents the event that $\{A\backslash A_1$, vertices z_1 and z_2 satisfy the above conditions$\}$. Choose a set $U \subset K$ containing a simple path joining z_1 and z_2. We also demand that $U \cap \partial_i K = \{z_1, z_2\}$. Consider the following map $\phi_2 : W^* \to W^*$. For a path ω in W_K^*, whenever ω enters K, the map ϕ_2 replaces the subpath of ω until it leaves K by a subpath that bypasses U and exits K at the same point as ω (If ω enters or leaves K at z_1, then the subpath need not bypass z_1, and the same is true for z_2). For the paths in $(W_K^*)^c$, the map ϕ_2 is an identity map. Then we can define Φ_2 induced by ϕ_2 as before. Note that no path passes through $U\backslash\{z_1, z_2\}$ under ϕ_2 and all the paths passing through z_1 or z_2 must pass through z_1 or z_2 in the preimage of ϕ_2. In addition, the situation outside K remains unchanged. Thus, $\Phi_2(B) \subset \{N < k\}$, since $\mathcal{V}\backslash K$ contains at most k infinite components and two of them are connected under the map Φ_2. So,

$$\begin{aligned}\mathbb{P}[B] &\leq \mathbb{P}\left[\Phi_2^{-1}\{N < k\}\right] \\ &= \Phi_2 \circ \mathbb{P}[N < k] = 0.\end{aligned}$$

We get a contradiction. Hence, for $1 < k < \infty$, $\mathbb{P}[N = k] = 0$.

The second step is to reject that $\mathbb{P}[N = \infty] > 0$. We assume the opposite happens. Then there exists M such that $\mathbb{P}[C] > 0$, where C denotes the event that at least 100^d distinct infinite components in $\mathcal{V}$ intersect $K = [-M, M]^d$. We can find three vacant vertices y_1,y_2 and y_3 in $\partial_i K$ such that they are at least distance 10 from each other and all the corners of the box, and belong to three distinct infinite components. Since there are finitely many choices, $\mathbb{P}[D] > 0$ for some y_1,y_2 and y_3 in $\partial_i K$, where D represents the event that C happens and y_1, y_2, y_3 satisfy the above conditions. We can find a subset $U \subset K$ such that (1). $U\backslash\{0\}$ contains three disjoint simple paths from 0 to y_1, y_2 and y_3; (2). $U \cap \partial_i K = \{y_1, y_2, y_3\}$. We consider the following map $\phi_3 : W^* \to W^*$. For a path ω in W_K^*, whenever ω enters K, the map ϕ_3 replaces the subpath of ω until it leaves K with a subpath that bypasses U, fills $K\backslash U$ and exits K at the same point as ω (If ω enters or leaves K at y_1, then the subpath need not bypass y_1, and the same is true for y_2 and y_3). For the paths in $(W_K^*)^c$, the map ϕ_3 is an identity map. We can define Φ_3 induced by ϕ_3 as before. Note that no path passes $U\backslash\{y_1, y_2, y_3\}$ under ϕ_3 and all the paths passing through y_1, y_2 or y_3

must pass it in the preimage. Besides, $K\backslash U$ is occupied and the situation outside K has not been changed. Thus, in $\Phi_3(D)$, the vertex 0 is a trifurcation point meaning that 0 belongs to an infinite component of $\mathcal{V}$ which is split into three distinct components by deleting 0. By definition, $\Phi_3 \circ \mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}$. This together with $0 < \mathbb{P}[D] \leq \Phi_3 \circ \mathbb{P}[\Phi_3(D)]$ implies that $\mathbb{P}[\Phi_3(D)] > 0$. Thus, $\mathbb{P}[0 \text{ is a trifurcation point}] \geq \mathbb{P}[\Phi_3(D)] > 0$. This is impossible due to the Burton-Keane argument as shown in Theorem 2 of Burton and Keane (1989).

In conclusion, either $N = 0$ a.s. or $N = 1$ a.s. □

Proof of Theorem 1.1 b): This is obvious since $\mathcal{V} = \mathcal{V}_1^{u_1} \cup \mathcal{V}_2^{u_2}$ and by (2.3) $\mathcal{V}^u$ a.s. has an infinite cluster when $u < u_*$. □

Proof of Theorem 1.1 c): Take $h = (u_1 + u_{**})/2 > u_{**}$. We define our seed event G_x as the event that $B(x, L_0)$ is connected with $\partial_i B(x, 2L_0)$ in $\mathcal{V}$, where L_0 is an integer to be determined later. Then, G_x is measurable with respect to the configuration in $B(x, 2L_0)$, shift-invariant and decreasing. G_x is contained in the union of the following two events: (1). in $\mathcal{V}_1^{u_1}$, the box $B(x, L_0)$ is connected to $\partial_i B(x, 2L_0)$; (2). $\mathcal{V}_2^{u_2} \cap B(x, 2L_0) \neq \emptyset$. Since $h > u_{**}$, by Theorem 3.1 of Popov and Teixeira (2015) there exist constants $c, C > 0$ depending on h such that

$$\mathbb{P}\left[B(x, L_0) \xleftrightarrow{\mathcal{V}_1^h} \partial_i B(x, 2L_0)\right] \leq Ce^{-{L_0}^c}.$$

Thus, by the above inequality and (2.2),

$$\begin{aligned}
\mathbb{P}\left[\mathcal{K}^{h,u_2} \in G_x\right] &\leq \mathbb{P}\left[B(x, L_0) \xleftrightarrow{\mathcal{V}_1^h} \partial_i B(x, 2L_0)\right] + \mathbb{P}\left[\mathcal{V}_2^{u_2} \cap B(x, 2L_0) \neq \emptyset\right] \\
&\leq Ce^{-{L_0}^c} + \sum_{y \in B(x, 2L_0)} \mathbb{P}\left[y \in \mathcal{V}_2^{u_2}\right] \\
&= Ce^{-{L_0}^c} + (4L_0 + 1)^d e^{-u_2 \cdot \mathrm{cap}(0)}.
\end{aligned}$$

Next, we will use the decoupling inequalities, i.e., Proposition 2.1. In Proposition 2.1, take $\epsilon > 0$ such that $u_1 = (1 + \epsilon)h$. In (2.7), we take $u_1 = h, u_1^+ = u_1, u_2 = g, u_2^+ = (1 + \epsilon)g$, where g is a constant to be determined later. Therefore, for $n \geq 0$ and $l_0 \geq A$ ($A = A(d, \epsilon)$ is the integer in Proposition 2.1),

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{K}^{u_1,(1+\epsilon)g} \in G_{0,n}\right] \\
\leq &(2l_0 + 1)^{d \cdot 2^{n+1}} \left[\mathbb{P}\left[\mathcal{K}^{h,g} \in G_0\right] + \epsilon(h, L_0, l_0) + \epsilon(g, L_0, l_0)\right]^{2^n} \\
\leq &(2l_0 + 1)^{d \cdot 2^{n+1}} \left[Ce^{-{L_0}^c} + (4L_0 + 1)^d \cdot e^{-g \cdot \mathrm{cap}(0)} + \epsilon(h, L_0, l_0) + \epsilon(g, L_0, l_0)\right]^{2^n}.
\end{aligned} \tag{3.1}$$

Recall that by (2.8),

$$\epsilon(u, L_0, l_0) = \frac{2e^{-uL_0^{d-2} l_0^{\frac{d-2}{2}}}}{1 - e^{-uL_0^{d-2} l_0^{\frac{d-2}{2}}}}.$$

Take $l_0 = A$. There exists a large integer B such that for $L_0 = g = B$ the right-hand side of (3.1) is smaller than 2^{-2^n}, or equivalently

$$(2A + 1)^{2d} \left(Ce^{-B^c} + (4B + 1)^d \cdot e^{-B \cdot \mathrm{cap}(0)} + \epsilon(h, B, A) + \epsilon(B, B, A)\right) < \frac{1}{2}.$$

We claim that if 0 is connected to $\partial_i B(0, 2L_n)$ in $\mathcal{V}$, then the event $G_{0,n}$ happens (see (2.5) for the definition of $G_{0,n}$). We can prove this claim by induction. For $n = 0$, this holds immediately. If for $n = k$ it holds, we consider the case $n = k + 1$. For a simple path connecting 0 to $\partial_i B(0, 2L_{k+1})$, it must pass $\partial_i B(0, L_{k+1}/3)$ and $\partial_i B(0, 2L_{k+1}/3)$. By the induction hypothesis, we can prove that the L_k boxes first passed by the path in $\partial_i B(0, L_{k+1}/3)$ and $\partial_i B(0, 2L_{k+1}/3)$ satisfying $G_{x,k}$ and their distance is by definition larger than $L_{k+1}/100$. Hence, the claim holds for $n = k+1$. By induction, it holds for all n. Recall that $\mathcal{V}$ is decreasing in u_2. Thus, for $L_0 = B$ and $q \geq (1+\epsilon)B$,

$$\mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,q}}{\longleftrightarrow} \partial_i B(0,2L_n)\right] \leq \mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,(1+\epsilon)B}}{\longleftrightarrow} \partial_i B(0,2L_n)\right]$$
$$\leq \mathbb{P}\left[\mathcal{K}^{u_1,(1+\epsilon)B} \in G_{0,n}\right] \leq 2^{-2^n}.$$

As n tends to ∞, the right-hand side tends to zero. Thus,

$$\mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,q}}{\longleftrightarrow} \infty\right] = 0, \text{ for all } q \geq (1+\epsilon)B.$$

Take $C(u_1) = (1+\epsilon)B$ in Theorem 1.1 c) and the proof is completed. □

Remark 3.1. Although the connectivity function we obtain here has stretched exponential decay, by the method in Section 7 of Popov and Teixeira (2015) we can greatly improve the bound to exponential decay in $d \geq 4$ and exponential decay with a logarithmic correction in $d = 3$. In other words, for $d \geq 4$, $u_1 > u_{**}$ and $u_2 > C(u_1)$, there exist two constants $c = c(u_1, u_2, d)$ and $C = C(u_1, u_2, d)$ such that

$$\mathbb{P}\left[0 \overset{\mathcal{V}}{\longleftrightarrow} x\right] \leq Ce^{-c|x|_1}.$$

For $d = 3$, $u_1 > u_{**}$, $u_2 > C(u_1)$ and any $\epsilon > 0$, there exist two constants $c = c(u_1, u_2, d, \epsilon)$ and $C = C(u_1, u_2, d, \epsilon)$ such that

$$\mathbb{P}\left[0 \overset{\mathcal{V}}{\longleftrightarrow} x\right] \leq Ce^{-c\frac{|x|_1}{\log^{3+\epsilon}|x|_1}}.$$

The proof is similar to Section 7 in Popov and Teixeira (2015) despite that we have to change both u_1 and u_2 and produce two error terms in the sprinkling process. For the sake of brevity, we will not give a complete proof here.

Proof of Theorem 1.1 d): Take $\Gamma(x) = \inf\{u : \mathbb{P}\left[0 \overset{\mathcal{V}^{x,u}}{\longleftrightarrow} \infty\right] = 0\}$ (by convention, the infimum of an empty set is ∞). By monotonicity, $\Gamma(x)$ is a decreasing function. Let $u^+ = \inf\{x : \Gamma(x) < \infty\}$. By Theorem 1.1 b) and c), we have $u_* \leq u^+ \leq u_{**}$. It follows from the symmetry of u_1 and u_2 that $\Gamma(x) \geq u^+$ when $x \geq u^+$. When $u_1 < u^+$, we have $\Gamma(u_1) = \infty$ and thus $\mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,u_2}}{\longleftrightarrow} \infty\right] > 0$ which implies that $\mathcal{V}$ percolates. When $u_1 \geq u^+$ and $u_2 < \Gamma(u_1)$, we have $\mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,u_2}}{\longleftrightarrow} \infty\right] > 0$ which implies that $\mathcal{V}$ percolates. When $u_1 \geq u^+$ and $u_2 > \Gamma(u_1)$, we have $\mathbb{P}\left[0 \overset{\mathcal{V}^{u_1,u_2}}{\longleftrightarrow} \infty\right] = 0$ and thus $\mathcal{V}$ doesn't percolate. □

Remark 3.2. Here, Γ is the phase curve of $\mathcal{V}$. We conjecture that it is continuous and strictly decreasing. A much more difficult problem is what happens to $\mathcal{V}$ on this curve, since we do not even know whether $u_* = u_{**}$ and what happens to $\mathcal{V}^{u_*}$.

4. Percolative properties of $\mathcal{K}$

In this section, we study the percolative properties of $\mathcal{K}$ and prove Theorem 1.2. The original model, random interlacements, is almost surely connected and contains a unique infinite component as shown in Corollary 2.3 of Sznitman (2010). Furthermore, in Ráth and Sapozhnikov (2012); Procaccia and Tykesson (2011), it has been shown that any pair of vertices in random interlacements can be connected via at most $\left\lceil\frac{d-1}{2}\right\rceil + 1$ trajectories and this value is optimal. However, the intersection of two independent random interlacements is not connected (this can be easily proved) and may even have no infinite components. Theorem 1.2 is split into four parts.

Proof of Theorem 1.2 a): We use the same method as in the proof of Theorem 1.1 a). We denote by N the number of infinite components. Then, N is a constant a.s., possibly infinite. The proof contains two parts.

The first step is to prove that $\mathbb{P}[1 < N < \infty] = 0$. We assume that on the contrary there exists an integer $k \in (1, \infty)$ such that $\mathbb{P}[N = k] = 1$. Then, there exists M such that $\mathbb{P}[A] > 0$, where A denotes the event that all the k infinite components of $\mathcal{K}$ intersect $K = [-M, M]^d$. We introduce a map $\phi_4 : W^* \to W^*$, which is an identity map on $(W^*_K)^c$. For $\omega \in W^*_K$, the map ϕ_4 adds a subpath that fills the box K when the first time ω enters K. The map ϕ_4 induces a map $\Phi_4 : \Omega \to \Omega$ and $\Phi_4 \circ \mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}$. Note that in $\Phi_4(A)$ there is exactly one infinite component in $\mathcal{K}$, since the situation outside K is unchanged and all the points in K are occupied. Thus, $\mathbb{P}[N = 1] \geq \mathbb{P}[\Phi_4(A)] > 0$, which is a contradiction. Therefore, $\mathbb{P}[1 < N < \infty] = 0$.

The second step is to prove that $\mathbb{P}[N = \infty] = 0$. We assume that in contrast $\mathbb{P}[N = \infty] = 1$. We can find $K = [-M, M]^d$ and $x_1, x_2, x_3 \in \partial_i K$ such that $\mathbb{P}[B] > 0$ where B represents the event that K intersects three distinct infinite components of $\mathcal{K}$ at x_1, x_2 and x_3. We consider the maps ϕ_4 and Φ_4 as before. Then, $\mathbb{P}[\Phi_4(B)] > 0$. In $\Phi_4(B)$, the vertex 0 is a M-trifurcation point (meaning that $B(0, M)$ intersects an infinite component and this infinite component is split into at least three disjoint clusters if we close all the vertices in $B(0, M)$). Thus, the vertex 0 has a positive probability to be a M-trifurcation point. By the Burton-Keane argument, this is impossible. Hence, $\mathbb{P}[N = \infty] = 0$.

In conclusion, either N equals 0 a.s. or N equals 1 a.s. □

The proof of Theorem 1.2 b) is similar to that of Theorem 1 in Ráth and Sapozhnikov (2013). We will use the strong connectivity property of random interlacements as shown in Lemma 3.1 of Ráth and Sapozhnikov (2013), which says that all the occupied vertices in a finite box are connected in a slightly larger box with high probability. The idea of this proposition is that in a $[-N, N]^d$ box, every component has larger than $1 - \exp(-N^\delta)$ probability to have capacity larger than $N^{(d-2)(1-\delta)}$. Thus, the simple random walks started from them have larger than $1 - \exp(-N^\epsilon)$ probability to intersect in a slightly larger box $[-(1+\epsilon)N, (1+\epsilon)N]^d$. The rigorous statement is as follows.

Proposition 4.1. *Let $d \geq 3$, $\epsilon > 0$ and $u > 0$. There exist constants $c = c(d, u, \epsilon) > 0$ and $C = C(d, u, \epsilon) > 0$ such that for all $R \geq 1$,*

$$P\left[\bigcap_{x,y \in \mathcal{I}^u \cap B(0,R)} x \longleftrightarrow y \text{ in } \tilde{\mathcal{I}}^u \cap B(0, (1+\epsilon)R)\right] \geq 1 - C\exp(-cR^{1/6}), \tag{4.1}$$

where $\tilde{\mathcal{I}}^u$ denotes the graph induced by the edges visited by random interlacements on $\mathbb{Z}^d$.

To prove Theorem 1.2 b), we first need to define seed events and prove estimates about them. Unfortunately, the above event is not monotone. Thus, it cannot be defined as the seed event directly. We will separate it into three monotone events. There will be three seed events: $\overline{E}_x, \overline{F}_x$ and $\overline{G}_x$. When we choose appropriate constants, each of them has small probability to happen.

E_x is defined as the intersection of the following two events: (1). for all $e \in \{0,1\}^d$, the graph $\tilde{\mathcal{I}}_1^{u_1} \cap (x + eL_0 + [0, L_0)^d)$ contains a connected component with at least $3m(u_1){L_0}^d/4$ vertices, where $m(u_1) = \mathbb{P}[0 \in \mathcal{I}_1^{u_1}] = 1 - e^{-u_1 \cdot \mathrm{cap}(0)}$; (2). all of the above 2^d components are connected in the graph $\tilde{\mathcal{I}}^{u_1} \cap (x + [0, 2L_0)^d)$. The event F_x is defined as: for all $e \in \{0,1\}^d$, $\mathcal{I}_1^{u_1} \cap (x + eL_0 + [0, L_0)^d)$ contains at most $5m(u_1){L_0}^d/4$ vertices. G_x is defined as: $\mathcal{V}_2^{u_2} \cap (x + [0, 2L_0)^d) = \emptyset$. The events E_x and G_x are increasing and F_x is decreasing. Thus, their complements $\overline{E}_x$, $\overline{G}_x$ are decreasing and $\overline{F}_x$ is increasing. Furthermore, $\overline{E}_x, \overline{G}_x$ and $\overline{F}_x$ are measurable with respect to the configuration in $B(x, 2L_0)$ and shift-invariant. Let $\overline{E}_x, \overline{G}_x, \overline{F}_x$ be our seed events. We need the following proposition to prove Theorem 1.2 b).

Proposition 4.2. *Given $u_1 > 0$, there exist integers l_0, L_0 and a constant $C = C(u_1) > 0$ such that for all $n \geq 0$ and $u_2 > C$,*

$$\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{E}_{0,n}\right], \mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{F}_{0,n}\right], \mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{G}_{0,n}\right] \leq 2^{-2^n}. \tag{4.2}$$

See (2.5) for the definition of $\overline{E}_{0,n}$. One note is that here $\overline{E}_{0,n}$ is the hierarchical event defined by $\overline{E}_x$, not the complement of $E_{0,n}$.

Proof of Proposition 4.2: Take $\epsilon = 1/2$ in Proposition 2.1. Let $l_0 = A$ (A is the integer in Proposition 2.1). We begin with $\overline{E}_{0,n}$. By an appropriate ergodic theorem, e.g., Theorem 8.6.9. in Dunford and Schwartz (1988),

$$\lim_{L_0\to\infty}\left[\left|\frac{1}{L_0^d}|\mathcal{I}^u \cap [0,L_0)^d| - m(u)\right| > \delta\right] = 0, \text{ for all } \delta > 0. \tag{4.3}$$

By Proposition 4.1 and (4.3), we can prove that for fixed u_1 and u_2,

$$\lim_{L_0\to\infty}\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{E}_x\right] = 0. \tag{4.4}$$

For more details, one can see Lemma 4.2 in Ráth and Sapozhnikov (2013). Inserting $u_1^+ = u_1, u_2^+ = u_2, u_1 = 2u_1/3, u_2 = 2u_2/3$ into (2.7), we get that

$$\begin{aligned}&\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{E}_{0,n}\right]\\ \leq&(2A+1)^{d\cdot 2^{n+1}}\left[\mathbb{P}\left[\mathcal{K}^{\frac{2}{3}u_1,\frac{2}{3}u_2} \in \overline{E}_0\right] + \epsilon\left(\frac{2}{3}u_1, L_0, A\right) + \epsilon\left(\frac{2}{3}u_2, L_0, A\right)\right]^{2^n}.\end{aligned}$$

By (2.8) and (4.4), there exists $\Gamma = \Gamma(u_1, u_2) > 0$ such that for $L_0 > \Gamma$,

$$(2A+1)^{2d}\left(\mathbb{P}\left[\mathcal{K}^{\frac{2}{3}u_1,\frac{2}{3}u_2} \in \overline{E}_0\right] + \epsilon\left(\frac{2}{3}u_1, L_0, A\right) + \epsilon\left(\frac{2}{3}u_2, L_0, A\right)\right) < \frac{1}{2}.$$

Combining the above two inequalities, we have for $L_0 > \Gamma$,

$$\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{E}_{0,n}\right] \leq 2^{-2^n}.$$

Note that $\overline{E}_{0,n}$ is independent of $\mathcal{I}_2^{u_2}$. Thus, we can let Γ independent of u_2.

For $\overline{F}_{0,n}$, by (4.3), $\lim_{L_0\to\infty}\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{F}_x\right] = 0$. Inserting $u_1^- = u_1, u_2^- = u_2, u_1 = 2u_1, u_2 = 2u_2$ into (2.6), we get that

$$\begin{aligned}&\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{F}_{0,n}\right]\\ \leq&(2A+1)^{d\cdot 2^{n+1}}\left[\mathbb{P}\left[\mathcal{K}^{2u_1,2u_2} \in \overline{F}_0\right] + \epsilon(u_1, L_0, A) + \epsilon(u_2, L_0, A)\right]^{2^n}.\end{aligned}$$

Thus, we can take $\Lambda = \Lambda(u_1, u_2)$ such that for $L_0 \geq \Lambda$,

$$\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{F}_{0,n}\right] \leq 2^{-2^n}.$$

Note that $\overline{F}_{0,n}$ is independent of $\mathcal{I}_2^{u_2}$. So, we can let Λ independent of u_2.

For $\overline{G}_{0,n}$, by (2.2),

$$\begin{aligned}\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{G}_x\right] &= \mathbb{P}\left[\mathcal{V}_2^{u_2} \cap (x + [0, 2L_0)^d) \neq \emptyset\right]\\ &\leq \sum_{y\in(x+[0,2L_0)^d)}\mathbb{P}\left[y \in \mathcal{V}_2^{u_2}\right] = (2L_0)^d e^{-u_2\cdot\mathrm{cap}(0)}.\end{aligned}$$

Inserting $u_1^+ = u_1, u_2^+ = u_2, u_1 = 2u_1/3, u_2 = 2u_2/3$ into (2.7), we get that

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in \overline{G}_{0,n}\right] \\
\le&(2A+1)^{d\cdot 2^{n+1}}\left[\mathbb{P}\left[\mathcal{K}^{\frac{2}{3}u_1,\frac{2}{3}u_2} \in \overline{G}_0\right] + \epsilon\left(\frac{2}{3}u_1, L_0, A\right) + \epsilon\left(\frac{2}{3}u_2, L_0, A\right)\right]^{2^n} \\
\le&(2A+1)^{d\cdot 2^{n+1}}\left[(2L_0)^d e^{-\frac{2}{3}u_2\cdot\mathrm{cap}(0)} + \epsilon\left(\frac{2}{3}u_1, L_0, A\right) + \epsilon\left(\frac{2}{3}u_2, L_0, A\right)\right]^{2^n}.
\end{aligned} \tag{4.5}$$

There exists an integer $H > \min\{\Gamma, \Lambda\}$ such that

$$(2A+1)^{2d}\left((2H)^d e^{-\frac{2}{3}H\cdot\mathrm{cap}(0)} + \epsilon\left(\frac{2}{3}u_1, H, A\right) + \epsilon\left(\frac{2}{3}H, H, A\right)\right) < \frac{1}{2}.$$

Take $L_0 = H$ and $C(u_1) = H$ in Proposition 4.2. Note that the left-hand side of (4.5) is decreasing in u_2 when n, l_0 and L_0 are fixed. Thus, for $n \ge 0$, $l_0 = A$, $L_0 = H$ and $u_2 > C(u_1)$, we have $P\left[\mathcal{K}^{u_1,u_2} \in \overline{G}_{0,n}\right] \le 2^{-2^n}$. This completes the proof of (4.2). □

Proof of Theorem 1.2 b): Take the l_0, L_0 and $C(u_1)$ in Proposition 4.2. We call a vertex x good if $E_x \cap F_x \cap G_x$ occurs, otherwise bad. We claim that if there exists a nearest neighbor path of good vertices with infinite length, then there exists an infinite component of $\mathcal{K}$ along this path. By G_x, along this path $\mathcal{V}_2^{u_2}$ is empty. It follows from E_x and F_x that there is only one component with more than $3m(u_1)L_0^d/4$ vertices in each L_0 box along the path and all these components are connected with the neighboring ones. Therefore, there is an infinite component in $\mathcal{K}$. If there are no nearest neighbor paths of good vertices with infinite length, then by the dual argument there are infinitely many *-neighbor circuits of bad vertices surrounding 0.

By an induction argument (see Lemma 5.2 of Ráth and Sapozhnikov, 2013),

$$\begin{aligned}
&\mathbb{P}\left[x \text{ is connected to } \partial_i B(x, 2L_n) \text{ by a *-neighbor path of bad vertices}\right] \\
\le&4^d\left[\mathbb{P}\left[\overline{E}_{0,n}\right] + \mathbb{P}\left[\overline{F}_{0,n}\right] + \mathbb{P}\left[\overline{G}_{0,n}\right]\right] \le 12^d \cdot 2^{-2^n}.
\end{aligned}$$

Therefore,

$$\sum_{x\in\mathbb{Z}^2\times\{0\}^{d-2}} \mathbb{P}\left[Z_x\right] < \infty,$$

where Z_x represents the event that x is passed by a *-neighbor circuit of bad vertices surrounding 0. Hence, there are finitely many such circuits. Therefore, an infinitely long path of good vertices exists a.s., which implies that $\mathcal{K}$ percolates a.s. when $u_2 > C(u_1)$. □

Proof of Theorem 1.2 c): Given K a finite subset of $\mathbb{Z}^d$, we write N_K^1 and N_K^2 for the number of paths passing through K in $\mathcal{I}_1^{u_1}$ and $\mathcal{I}_2^{u_2}$, i.e., N_K in (2.1). We need the following lemma.

Lemma 4.3. *For $d \ge 3$, $u_1, u_2 > 0$ and any integer $M \ge 1$, we have*

$$\lim_{L_0\to\infty} \mathbb{P}\left[\partial_i B(0, L_0) \overset{\mathcal{I}_1^{u_1}\cap\mathcal{I}_2^{u_2}}{\longleftrightarrow} \partial_i B(0, 2L_0) \middle| N^1_{B(0,2L_0)} = N^2_{B(0,2L_0)} = M\right] = 0. \tag{4.6}$$

Proof: Denote the probability in the left-hand side by I. The proof is fairly simple for $d \ge 4$. Equation (2.1) implies that conditional on $\left\{N^1_{B(0,2L_0)} = M\right\}$, interlacements $\mathcal{I}_1^{u_1}$ in $B(0, 2L_0)$ can be seen as M independent simple random walks started at some points in $\partial_i B(0, 2L_0)$. The same is true for $N^2_{B(0,2L_0)}$ and $\mathcal{I}_2^{u_2}$. To connect $\partial_i B(0, L_0)$ with $\partial_i B(0, 2L_0)$, at least one vertex of $\partial_i B(0, L_0)$ should be occupied. There are at most $C{L_0}^{d-1}$ many vertices but the probability of a vertex to be occupied is at most $CM^2 \cdot 1/{L_0}^{d-2} \cdot 1/{L_0}^{d-2}$. Thus,

$$I \le CM^2 {L_0}^{d-1} \cdot \frac{1}{{L_0}^{d-2}} \cdot \frac{1}{{L_0}^{d-2}} = C\frac{1}{L_0^{d-3}}.$$

For $d = 3$, we use Proposition 2.4 which is a more powerful estimate. By Proposition 2.4,

$$\begin{aligned} I &\le M^2 \max_{x,y\in\partial_i B(0,2L_0)} P_{x,y}\left[X^1[0,\infty)\cap X^2[0,\infty)\cap \partial_i B(0,L_0)\neq\emptyset\right] \\ &< \frac{C}{\log(L_0)}. \end{aligned}$$

Let L_0 tend to ∞. Then, both terms above tend to 0 and we complete the proof. □

We define the seed event G_x as the event that $\partial_i B(x, L_0)$ is connected to $\partial_i B(x, 2L_0)$ in $\mathcal{K}$. Thus, G_x is measurable with respect to the configuration in $B(x, 2L_0)$, shift-invariant and increasing.

Let $u_1 = u_2$ and $M = u_1 \cdot {L_0}^{d-2} = u_2 \cdot {L_0}^{d-2}$. Observe that there exists B such that $\mathrm{cap}(B(0, 2L_0)) \le BL_0^{d-2}$ and $N_{B(x,2L_0)} \sim \mathrm{Poisson}(u \cdot \mathrm{cap}(B(0, 2L_0)))$. Thus, by the Hoeffding's inequality, there exist two constants δ and γ such that

$$\begin{aligned} \mathbb{P}\left[G_x\right] &\le \mathbb{P}\left[N^1_{B(x,2L_0)} > 2BM\right] + \mathbb{P}\left[N^2_{B(x,2L_0)} > 2BM\right] \\ &\quad + \mathbb{P}\left[G_x \middle| N^1_{B(x,2L_0)} \le 2BM, N^2_{B(x,2L_0)} \le 2BM\right] \\ &\le \delta e^{-\gamma M} + \mathbb{P}\left[G_x \middle| N^1_{B(x,2L_0)} = [2BM], N^2_{B(x,2L_0)} = [2BM]\right] \end{aligned}$$

($[x]$ represents the largest integer $\le x$). Take $\epsilon = 1/2$ in Proposition 2.1 and $l_0 = A$ (A is the integer in Proposition 2.1). Inserting $u_1^- = u_1/2, u_1 = u_1, u_2^- = u_2/2, u_2 = u_2$ into (2.6), toghther with (2.8) we get that

$$\begin{aligned} &\mathbb{P}\left[\mathcal{K}^{\frac12 u_1,\frac12 u_2} \in G_{0,n}\right] \\ \le&(2A+1)^{d\cdot 2^{n+1}}\left[\mathbb{P}\left[\mathcal{K}^{u_1,u_2}\in G_0\right] + \epsilon\left(\frac12 u_1, L_0, A\right) + \epsilon\left(\frac12 u_2, L_0, A\right)\right]^{2^n} \\ \le&(2A+1)^{d\cdot 2^{n+1}}\left[\delta e^{-\gamma M} + 4\frac{e^{-\frac12 MA^{\frac{d-2}{2}}}}{1-e^{-\frac12 MA^{\frac{d-2}{2}}}}\right. \\ &\left. \qquad\qquad + \mathbb{P}\left[G_x \middle| N^1_{B(x,2L_0)} = [2BM], N^2_{B(x,2L_0)} = [2BM]\right]\right]^{2^n}. \end{aligned}$$

Next, pick a large M such that

$$(2A+1)^{2d}\left(\delta e^{-\gamma M} + 4\frac{e^{-\frac12 MA^{\frac{d-2}{2}}}}{1-e^{-\frac12 MA^{\frac{d-2}{2}}}}\right) < \frac14.$$

Finally, thanks to Lemma 4.3, we can enlarge L_0 such that

$$(2A+1)^{2d}\mathbb{P}\left[G_x \middle| N^1_{B(x,2L_0)} = [2BM], N^2_{B(x,2L_0)} = [2BM]\right] < \frac14.$$

Combining the above three inequalities, we get that there exist certain $u_1 = u_2 > 0$ and L_0, l_0 such that for all $n \ge 0$

$$\mathbb{P}\left[\mathcal{K}^{u_1,u_2} \in G_{0,n}\right] \le 2^{-2^n}. \tag{4.7}$$

We claim that if 0 is connected to $\partial_i B(0, 2L_n)$ in $\mathcal{K}$, then the event $G_{0,n}$ happens (see (2.5) for the definition of $G_{0,n}$). We can prove this claim by induction. For $n = 0$, this holds immediately. If for $n = k$ it holds, then we consider the case $n = k + 1$. For a simple nearest neighbor path connecting 0 to $\partial_i B(0, 2L_{k+1})$, it must pass $\partial_i B(0, L_{k+1}/3)$ and $\partial_i B(0, 2L_{k+1}/3)$. By the induction hypothesis, we can prove that the L_k boxes first passed by the path in $\partial_i B(0, L_{k+1}/3)$ and $\partial_i B(0, 2L_{k+1}/3)$ satisfying $G_{x,k}$ and their distance is by definition larger than $L_{k+1}/100$. Hence, the claim holds for

$n = k+1$. By induction, it holds for all n. So, $\mathbb{P}\left[0 \overset{\mathcal{K}}{\longleftrightarrow} \partial_i B(0, 2L_n)\right] \le \mathbb{P}\left[\mathcal{K} \in G_{0,n}\right]$. Together with (4.7), we have

$$\mathbb{P}\left[0 \overset{\mathcal{K}}{\longleftrightarrow} \partial_i B(0, 2L_n)\right] \le 2^{-2^n}.$$

Let n tend to ∞. Then, we have $\mathbb{P}\left[0 \overset{\mathcal{K}}{\longleftrightarrow} \infty\right] = 0$ for some $u_1 = u_2 > 0$. We take $c(d) = u_1 = u_2$ in Theorem 1.2 c) and get the non-trivial phase transition of $\mathcal{K}$. □

Remark 4.4. In this proof, we obtain the stretched exponential decay of connectivity function. By the method in Section 7 of Popov and Teixeira (2015), we can greatly improve the bound to exponential decay for $d \ge 4$ and exponential decay with a logarithmic correction for $d = 3$ (one can see Remark 3.1 for more details).

We just proved that when both u_1 and u_2 tend to 0, there are no infinite components in $\mathcal{K}$. We conjecture that for $d \ge 3$ if one fix one of u_1 and u_2 and let the other tend to 0, there are no infinite components in $\mathcal{K}$.

We can prove this rigorously for $d \ge 5$. We now present the proof of Theorem 1.2 d), which gives a precise statement of the claim above.

Proof of Theorem 1.2 d): It is sufficient to prove a variant of Lemma 4.3 with only one conditioning, i.e., for $d \ge 5$, $u_1, u_2 > 0$ and any interger $M \ge 1$

$$\lim_{L_0 \to \infty} \mathbb{P}\left[\partial_i B(0, L_0) \overset{\mathcal{I}_1^{u_1} \cap \mathcal{I}_2^{u_2}}{\longleftrightarrow} \partial_i B(0, 2L_0) \middle| N^1_{B(0,2L_0)} = M\right] = 0.$$

It follows from (2.1) that contional on $\left\{N^1_{B(0,2L_0)} = M\right\}$, interlacements $\mathcal{I}_1^{u_1}$ in $B(0, 2L_0)$ are M independent simple random walks started at some random points in $\partial_i B(0, 2L_0)$. Note that

$$\lim_{L_0 \to \infty} P[\text{no two simple random walks of these } M \text{ ones intersect}] = 1,$$

for that with high probability their starting points are at least $\sqrt{L_0}$ from each other and random walks started from these points do not intersect when $d \ge 5$. Therefore, we can assume that $M = 1$. By the strong Markov property, $\mathcal{I}_1^{u_1}$ in $B(0, 2L_0)$ can be dominated by two independent simple random walks from some point x in $\partial_i B(0, L_0)$. With high probability, these two random walks do not intersect out a small box, say $B(x, L_0/2)$. So, we only need to prove that

$$\lim_{L_0 \to \infty} \sup_{x \in \partial_i B(0, L_0)} P_x\left[\partial_i B(0, \frac{3}{2} L_0) \overset{X[0,\infty) \cap \mathcal{I}_2^{u_2}}{\longleftrightarrow} \partial_i B(0, 2L_0)\right] = 0. \tag{4.8}$$

The time n is called a cut time if $X[0, n] \cap X(n, \infty) = \emptyset$. Call a time n bad if it is a cut time and $X(i)$ is not occupied by $\mathcal{I}_2^{u_2}$, otherwise good. Given $\delta > 0$, we can choose $N > 0$ and $\epsilon > 0$ such that for all integer $L_0 \ge 1$

$$P\left[\min_{i > NL_0^2} |X(i)|_\infty > 4L_0\right] > 1 - \delta \tag{4.9}$$

and

$$P\left[\max_{|i-j| \le \epsilon L_0^2; i,j \le NL_0^2} |X(i) - X(j)|_\infty < \frac{1}{2} L_0\right] > 1 - \delta. \tag{4.10}$$

Take N large first and then ϵ small. The above two inequalities can be obtained by the central limit theorem and reflection principle. Furthermore, for arbitrary $N, \epsilon > 0$ and L_0 large,

$$P\left[\text{there are no consecutive } \epsilon L_0^2 \text{ good times in } \left[0, NL_0^2\right]\right] > 1 - 2\delta. \tag{4.11}$$

Following is the proof of (4.11). The cut times of X are independent of $\mathcal{I}_2^{u_2}$. Given NL_0^2 different points chronologically, for L_0 large

$$P\left[\text{some consecutive } L_0 \text{ points are all occupied by } \mathcal{I}_2^{u_2}\right] < \delta,$$

since

$$P\left[\mathcal{I}^{u_2} \text{ contains } M \text{ given points}\right] \leq Ce^{-cM^{\frac{d-2}{d}}}.$$

Therefore,

$$P\left[\text{there is a bad time in every consecutive } L_0 \text{ cut times}\right] > 1-\delta.$$

In addition, for a random walk in $d \geq 5$, the density of cut times will converge to a positive constant as shown in (1) of Lawler (1996). Combining these two facts, we can get (4.11).

If the above three events in (4.9), (4.10) and (4.11) occur at the same time, then the event in (4.8) will not happen for that X leaves $B(0, 2L_0)$ after NL_0^2 and the first NL_0^2 steps are cut into disjoint subpaths with diameter smaller than $L_0/2$ by the bad times. Let δ tend to zero and the proof of (4.8) is completed. This completes the proof for the case $d \geq 5$. □

Remark 4.5. For $d = 4$, the bi-infinite simple random walk still has cut times and bad times, so we guess that $\partial_i B(0, L_0)$ and $\partial_i B(0, 2L_0)$ can be disconnected by bad times. However, for $d = 3$, there are no cut times for a doubly-infinite simple random walk, i.e., $\zeta_3 > 1/4$ (see (2) to (3) in Lawler, 1996), and a new approach is required.

5. On the coexistence of infinite clusters

In this section, we will consider the phase diagram of $\mathcal{K}$ and $\mathcal{V}$ put together and prove Theorem 1.3 which is split into two parts.

Proof of Theorem 1.3a): Take $u_1 \in (0, u_*)$ and u_2 sufficiently large. By Theorem 1.1b) and Theorem 1.2b), we conclude that both $\mathcal{K}$ and $\mathcal{V}$ have an infinite component. □

Next, we consider whether there is some region such that neither of $\mathcal{K}$ and $\mathcal{V}$ percolates, or equivalently the occupied vertices and the vacant vertices wrap each other. In general, this problem is difficult in low dimensions except $d = 2$ due to lack of adequate tools. We claim that in high dimensions there does not exist such region, see Theorem 1.3b) for a precise statement.

Proof of Theorem 1.3b): The proof relies on Theorem 1.4 and Theorem 0.1 of Sznitman (2011). By Theorem 0.1 in Sznitman (2011), we know that

$$\liminf_{d\to\infty} \frac{u_*(d)}{\log(d)} \geq 1.$$

Thus, there exists a constant D such that for all $d \geq D$, we have $u_*(d) \geq \log(d)/2$. Theorem 1.4 tells us that when $d \geq D_1$ and $u_1, u_2 > 1$, the intersection $\mathcal{K}$ percolates. Let $D_2 = \min\{D_1, D, 10\}$.

For all $d > D_2$ and $u_1, u_2 > 0$, there are two cases.

(1). $\min\{u_1, u_2\} < u_*(d)$. By Theorem 1.1 b), the vacant set $\mathcal{V}$ percolates.

(2). $\min\{u_1, u_2\} \geq u_*(d) > 1$. By Theorem 1.4 and $d > D_2 \geq D_1$, the intersection $\mathcal{K}$ percolates. □

The following part is devoted to the proof of Theorem 1.4. The proof contains two parts. The first is to prove that in each hypercube $\{0,1\}^d$, with high probability, there is a ubiquitous connected component (meaning that most vertices in the hypercube are connected to it, see below for a rigorous definition) and this ubiquitous component is also connected to those of the neighboring hypercubes (which also exist with high probability). The second step is to prove that such hypercubes with ubiquitous components percolate in the whole space. H represents a hypercube. For $x \in \mathbb{Z}^d$, let $H_x = x + \{0,1\}^d$ be the hypercube at x. Recall that Bernoulli site percolation with parameter

p is a model in which each vertex is independently occupied with probability p and vacant with probability $1-p$, denoted by Bernoulli(p).

Proof of Theorem 1.4: Lemma 2.2 says that $P_0(\tau^+_{\{0,1\}^d} = \infty) > c_1$. Thanks to this Lemma, we claim that the interlacements with intensity 1 in a hypercube H can dominate Bernoulli site percolation with parameter $1 - e^{-c_1^2}$. For a vertex x in H, we only count the paths that pass through H only at x and only once. They have intensity $P_x(\tau_H^+ = \infty) \cdot P_x(\tau_H^+ = \infty) \geq c_1^2$. Hence, conditional on all the other vertices in H, the probability that x is occupied is at least $1 - e^{-c_1^2}$. This completes the proof of the claim. Furthermore, $\mathcal{K}^{1,1}$ can dominate Bernoulli site percolation with parameter $(1-e^{-c_1^2})^2$ in H. Let $3p = (1-e^{-c_1^2})^2$. It is sufficient to prove the increasing properties about $\mathcal{K}^{1,1}$ in H for Bernoulli($3p$).

Next, we will define some notation to be used later. We want to mention that all the inequalities below hold when d is larger than a universal constant and all the constants are independent of d. Let $n = 2^d = |H|$. For any subset X of a hypercube H, write $N(X)$ for all the neighbors of X in H and $\overline{X}$ for $X \cup N(X)$. A connected component of H is called an atom if it contains more than d^{100} vertices. Call a connected component A of H a ubiquitous component if $\left|\overline{A}\right| > \left(1 - 1/d^2\right) n$. There is only one ubiquitous component in a hypercube H. Suppose that in contrast H has two distinct ubiquitous components A and B. Since $B \cap \overline{A} = \emptyset$, we have $|B| \leq n - \left|\overline{A}\right| < n/d^2$. Hence,

$$\begin{aligned} |\overline{B}| &= |B| + |N(B)| \\ &\leq |B| + d|B| < \left(1 - 1/d^2\right) n. \end{aligned}$$

B is not a ubiquitous component. Therefore, H has at most one ubiquitous component.

The seed event G_x is defined as the intersection of the following two events: (1). for any $e \in \{(0,0),(0,1),(0,-1),(1,0),(-1,0)\} \times \{0\}^{d-2}$, $\mathcal{K}^{1,1} \cap H_{x+e}$ has a ubiquitous component; (2). all the above five ubiquitous components are connected in $\mathcal{K}^{1,1} \cap B(x,2)$. The event G_x is measurable with respect to the configuration in $B(x,2)$, shift-invariant and increasing. $\overline{G}_x$ is the complement of G_x. We first prove the following inequality:

$$\lim_{d \to \infty} d^3 \mathbb{P}\left[\overline{G}_x\right] = 0. \tag{5.1}$$

It is sufficient to prove this property for Bernoulli($3p$). Now, H_x is a fixed hypercube and H_1, H_2, H_3, H_4 are the four neighboring hypercubes of H_x in the first and second directions. The proof follows three steps. With high probability, (1). in Bernoulli(p), most vertices in H_x have a neighbor in an atom; (2). in Bernoulli(2p), these atoms are connected to a ubiquitous component of H_x; (3). in Bernoulli(3p), this ubiquitous component is connected with those of the neighboring four hypercubes. If the above three events happen simultaneously, then G_x happens.

Consider Bernoulli site percolation on H_x with parameter p. For a fixed vertex of H_x, we can construct a 1000-high tree in H_x in which every node except leaves has more than $d/2000$ descendants. By the Hoeffding's inequality, $P[Y] \geq 1 - e^{-Cd}$, where Y represents the event that each node except leaves has more than $pd/10000$ occupied descendants. When Y happens, this vertex has a neighbor in an atom. Thus, by the Markov's inequality

$$P[Z] \geq 1 - e^{-Cd},$$

where Z represents the event that except n/e^{Cd} vertices, every vertex of H_x has a neighbor in an atom.

Consider, now, the set of atoms obtained in H_x. We open all the vacant vertices independently with probability $q = p/(1-p)$. With these additional open vertices, the atoms in H_x have a large probability to be connected to a ubiquitous component. We claim that with high probability no union of atoms A covering more than n/d^5 vertices can be separated from the union of all the other atoms B, when B also has at least n/d^5 vertices. This follows from the following lemma.

Lemma 5.1. *Assume that Z happens and $A, B \subset H_x$ satisfy the above conditions including $|A|, |B| > n/d^5$ and $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Also, except n/e^{Cd} vertices, every vertex of H_x has a neighbor in A or B. Then, there exist cn/d^7 pairwise disjoint paths connecting A and B, which have length at most three, i.e., in the form $A \leftrightarrow y \leftrightarrow B$ or $A \leftrightarrow y \leftrightarrow z \leftrightarrow B$.*

Proof: There are two cases. Case 1: $|N(A) \cap N(B)| > n/d^7$, then take $A \leftrightarrow y \leftrightarrow B$, where $y \in N(A) \cap N(B)$. Case 2: $|N(A) \cap N(B)| < n/d^7$. Suppose that $\left|\overline{A}\right| \le \left|\overline{B}\right|$, then we can prove that $\left|\overline{A}\right| < 3n/4$. With the isoperimetric inequality, $\left|N\left(\overline{A}\right)\right| > \left|\overline{A}\right| \left(n - \left|\overline{A}\right|\right)/(nd) > cn/d^6$. In addition, by Z, all the vertices in $N\left(\overline{A}\right)$ except n/e^{Cd} ones should have a neighbor in B. Thus, there are at least cn/d^6 different paths in the form $A \leftrightarrow y \leftrightarrow B$ or $A \leftrightarrow y \leftrightarrow z \leftrightarrow B$, where $y \in N(A)$ and $z \in N\left(\overline{A}\right)$. Since y can be in at most d different paths, there are at least cn/d^7 disjoint paths. □

The number of choices of A and B satisfying the above conditions is at most $2^{n/d^{100}}$. By the above lemma, each pair has smaller than $(1-q^2)^{cn/d^7}$ probability to be still disconnected. So, the probability of existing such A and B is at most $(1-q^2)^{cn/d^7} \cdot 2^{n/d^{100}}$. If there are no such pairs of A and B, we obtain a ubiquitous component (because the vertices not in this component is at most $n/e^{Cd} + d \cdot n/d^5$). Therefore,

$$P[U] \le e^{-Cd} + \left(1-q^2\right)^{cn/d^7} 2^{n/d^{100}},$$

where U represents the event that there are no ubiquitous components in H_x. By symmetry,

$$P[V] \le 5\left(e^{-Cd} + \left(1-q^2\right)^{cn/d^7} 2^{n/d^{100}}\right),$$

where V represents the event that there are no ubiquitous components in H_1, H_2, H_3, H_4 or H_x.

Suppose that V does not happen and we obtain five ubiquitous components in these five hypercubes. The final step is to connect the ubiquitous component in H_x with the neighboring four ones. We open all the vacant vertices independently with probability $r = p/(1-2p)$. Note that there are 2^{d-1} common vertices for the neighboring two hypercubes H_x and H_1 and at least $2^{d-1} - 2n/d^2 > cn$ of them are connected to both the two ubiquitous components in H_x and H_1. Thus, with probability more than $1-(1-r)^{cn}$, the two ubiquitous components in H_x and H_1 are connected with each other, the same for H_2, H_3 and H_4.

Summing over all the probabilities of bad events, we have

$$P\left[\overline{G_x}\right] \le 5\left(e^{-Cd} + \left(1-q^2\right)^{cn/d^7} \cdot 2^{n/d^{100}}\right) + 4(1-r)^{cn}.$$

This completes the proof of (5.1).

Remark 5.2. The most costly step is the first one. It is easy to find that (5.1) still holds when $u = \frac{1}{d^{1/2-\epsilon}}$ for any $\epsilon > 0$.

For the second part, we directly use Theorem 2.2 in Sznitman (2011), i.e.,

Proposition 5.3. *G_x is measurable with respect to the configuration in $B(x,2)$ and shift-invariant. If $\limsup_d d^3 \mathbb{P}\left[\overline{G_x}\right] < \infty$, then G_x will percolate in the slab $\mathbb{Z}^2 \times \{0\}^{d-2}$ for some large d meaning that there exists an infinite long nearest neighbor path in which every vertex satisfies G_x.*

The proof of this result uses a direct decoupling inequality rather than the sprinkling version, so we do not even need monotonicity of the seed event. Observe that $u \le d$ also holds here.

Once there exists an infinitely long nearest neighbor path in which every vertex satisfies G_x, there is an infinite component of $\mathcal{K}$ along this path, because every hypercube along this path has a ubiquitous component and all these components are connected with the neighboring four ones. □

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