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The geometry of the space-time Martin boundary is different than the spatial Martin boundary

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Abstract. We consider a nearest neighbour random walk on the integers with absorption at 0 and constant jump probabilities to the left and right of zero. The associated spectral radius is ρ and the spatial ρ-Martin exit boundary comprises two extremal points and associated minimal excessive functions $h_{-\infty}$ and $h_{+\infty}$ associated with sequences x_n converging to $-\infty$ and $+\infty$ respectively. We construct a space-time sequence (x_n, n) with $x_n \to -\infty$ converging to a point in the space-time ρ-Martin exit boundary whose associated space time harmonic function $h(x,t)$ is minimal and of the form $\rho^{-t}h_{+\infty}(x)$ not $\rho^{-t}h_{-\infty}(x)$ as might have been hoped.

1. Introduction

Let K be a substochastic, irreducible matrix with elements $K(x,y)$ where x and y are elements of a countably infinite state space S. We assume that there is at least one $x \in S$ with $K(x,S) < 1$. We think of K as the part of the transition matrix of a Markov chain of $X = \{X_0, X_1, \dots\}$ that describes the evolution of X among the states in S. Since we will not be interested in X after exiting S, we can simply append an additional state δ that is absorbing. The probability $1 - K(x,S)$ can be thought of as the probability of jumping from x to the absorbing state δ. Let $\zeta = \max\{n : X_n \in S\}$ denote the terminal time in S. Clearly, $K^n(x,S) = P_x(\zeta \geq n)$ where the subscript x denotes that we are also conditioning on $X_0 = x$.

The positive θ-harmonic functions h satisfying $\sum_z K(x,z)h(z) = \theta h(x)$, normalized so $h(x_0) = 1$ for some x_0, form a convex set $\mathcal{C}$ of considerable practical interest in potential theory. Martin boundary theory (see Dynkin, 1969) was developed to describe $\mathcal{C}$. If the chain is R-transient we may construct a Green's function or potential by

$$G_{x,y}(T) \equiv \sum_{n\geq 0} K^n(x,y)T^n \tag{1.1}$$

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and $T \le R = 1/\rho$ where $R \ge 1$ is the common radius of convergence of the potential; i.e. independent of x, y. Martin's observation was $k(x, y; T) = G_{x,y}(T)/G_{x_0,y}(T)$ as a function of x satisfies $\sum_z K(x, z)k(z, y; T) = \theta k(x, y) + \delta_y(x)/(TG_{x_0,y}(T))$ where $\theta = 1/T$; that is $k(x, y; T)$ is θ-harmonic except at $x = y$. We can now pick a subsequence y_n which leaves any compact set so that $\delta(x, y_n) = 0$ for n large enough (depending on x) and such that $k(x, y_n; T) \to s(x)$ for each x. By Fatou's Lemma $\sum_z K(x, z)s(z) \le \theta s(x)$; i.e. s is θ-superharmonic and even θ-harmonic if K has finite range.

We can compactify the space S to S^* by adding boundary points $\sigma \in \partial S$ in the T-Martin boundary so that $y_n \to \sigma$ where σ is associated with a θ-superharmonic function such that $k(\cdot, y_n; T) \to s(\cdot)$. See Dynkin (1969) for the associated metric on the compacification. Now consider an extremal point σ in the boundary ∂S associated with a minimal θ-harmonic function h.

The space-time chain (X_n, T_n) where $T_n = t+n$ is a transient Markov chain on $S \times \mathcal{Z}$ with kernel $K(x, t; y, t + 1) = K(x, y)$. The associated Martin kernel is $k(x, t; y, m) = K^{m-t}(x, y)/K^m(x_0, y)$ and as above we may compactify space-time by adding the Martin boundary. Nonterminating trajectories of (X_n, T_n) will converge to points in the space-time exit boundary corresponding to minimal space-time harmonic functions. Under the uniform aperiodicity condition given below, Proposition 1.1 implies a point $\tilde{\sigma}$ in the space-time exit boundary corresponds to $\tilde{h}$ of the form $\tilde{h}(x, t) = h'(x)\theta^{-t}$ where h' is a minimal, θ-harmonic function. Suppose $(y_m, m) \to \tilde{\sigma}$ and $y_m \to \sigma$ in the spatial T-Martin boundary. Suppose $k(x, y_m; T) \to h(x)$. It is reasonable to conjecture that $h' = h$ but the point of this paper is to show that this is not true in general.

1.1. *Consequences of uniform aperiodicity.* We use the uniform aperiodicity condition introduced in Kesten (1995)

> There exist constants $\delta_1 > 0$ and $N < \infty$, and for each $i \in S$, there exist integers $1 \le k_1, \cdots, k_r \le N$, with $k_j = k_j(i)$ and $r = r(i)$ such that $K^{k_s}(i, i) \ge \delta_1$ for $1 \le s \le r$, and g.c.d.$(k_1, \ldots, k_r) = 1$. (Condition [1])

As remarked in Kesten (1995), uniformly in x, there exist some $N < \infty$ and $\delta(d) > 0$ independent of x such that $K^d(x, x) \ge \delta(d)$ for $d \ge N$.

Proposition 1.1. *If K is uniformly aperiodic then a positive, minimal space-time harmonic function $\tilde{h}$ is of the form $h(x)T^t$ where h is a minimal θ-harmonic function $\theta = 1/T$.*

The following proof follows Theorem 3.1 in Lamperti and Snell (1963).

Proof: From the definition of uniform aperiodicity $K^d(x, x) \ge \delta(d) > 0$ for $d \ge N$. Now, $g(x, t) = \tilde{h}(x, t+d)$ is space-time harmonic and since $\tilde{h}(x, t) = \sum_y K^d(x, y)\tilde{h}(y, t+d) \ge K^d(x, x)\tilde{h}(x, t+d) \ge \delta(d)g(x, t)$ we conclude, using minimality, that $\delta(d)g(x, t)$ is proportional to $h(x, t)$; i.e. $\tilde{h}(x, t+d) = c_d\tilde{h}(x, t)$ for $d \ge N$.

Hence $\tilde{h}(x, Nd) = c_d^N\tilde{h}(x, 0) = c_N^d\tilde{h}(x, 0)$ so $c_d = c_N^{d/N} = T^d$ where $T = c_N^{1/N}$. Hence, $\tilde{h}(x, t+d) = T^d\tilde{h}(x, t)$ for all t and $d \ge N$. For $0 \le d' < N$,

$$\begin{aligned}\tilde{h}(x, t + d') &= \tilde{h}(x, t - N + d' + N) = T^{d'+N}\tilde{h}(x, t - N) \\ &= T^{d'+N}T^{-N}\tilde{h}(x, t) = T^{d'}\tilde{h}(x, t).\end{aligned}$$

Hence, $\tilde{h}(x, \ell) = T^\ell\tilde{h}(x, 0)$ for all ℓ..

Since $\tilde{h}$ is space-time harmonic it follows that

$$h(x) := \tilde{h}(x, 0) = \sum_y K(x, y)\tilde{h}(y, 1) = T\sum_y K(x, y)h(y);$$

i.e. h is θ-harmonic where $\theta = 1/T$. h is also minimal for if $g \leq h$ is θ-harmonic then $g(x,t) = T^t g(x)$ is space-time harmonic and $g(x,t) \leq \tilde{h}(x,t)$. By minimality $g(x,t)$ is proportional to $\tilde{h}(x,t)$ so $g(x)$ is proportional to $h(x)$ and it follows that $h(x)$ is minimal. $\square$

Lemma 1.2. *Let K be an irreducible, kernel K on a countable state space S with spectral radius ρ. Assuming Condition [1] then for all $x, y \in S$*

$$\lim_{n\to\infty} \frac{K^{n+1}(x,y)}{K^n(x,y)} = \rho, \tag{1.2}$$

which in turn implies that $\lim_{n\to\infty} K^n(x,y)^{1/n} = \rho$.

This is Lemma 4 in Kesten (1995) and the last statement holds since the ratio test is a corollary of the root test.

We will need the following extension. The argument is similar to that in Lemma 2 in Foley and McDonald (2017).

Lemma 1.3. *Let K be an irreducible kernel on a countable state space S with spectral radius ρ. Suppose that the sequence y_n converges to a point in the spatial T-Martin boundary; i.e. for some index state x_0 and $T = 1/\theta$,*

$$\lim_{n\to\infty} \frac{G_{z,y_n}(T)}{G_{x_0,y_n}(T)} = h(z). \tag{Condition [2]}$$

Moreover, let $\tilde{K}(x,y) = \theta^{-1}K(x,y)h(y)/h(x)$ and assume

$$\liminf_{y_n\to\infty} \left(\tilde{K}^n(x,y_n)\right)^{1/n} = 1, \tag{Condition [3]}$$

for some x (and hence all x).

Then, the following condition holds

$$\lim_{n\to\infty} \frac{K^{n+t}(x,y_n)}{K^n(x,y_n)} = \theta^t, \tag{Condition [4]}$$

for some x (and hence all x).

Proof: First, if Condition [3] holds for x then for any z

$$K^n(z,y_n) \geq K^m(z,x)K^{n-m}(x,y_n)$$

for some m so

$$\liminf_{n\to\infty} \left(K^n(z,y_n)h(y_n)\right)^{1/n} \geq \liminf_{n\to\infty} \left(K^n(x,y_n)h(y_n)\right)^{1/n} \geq \theta.$$

Hence, Condition [3] holds for all x.

We combine elements of the proof of (5) in Theorem 1.1 in Kingman and Orey (1964) or Theorem 2.1 in Orey (1971). First note h is θ-super harmonic; i.e. $Kh \leq \theta h$. Let $\tilde{K}$ be the associated h-transform; i.e. $\tilde{K}(x,y) = K(x,y)h(y)/(\theta h(x))$. We remark that $\tilde{K}$ is uniformly aperiodic so there exists a N such that $\tilde{K}^d(x,x) > 2\delta(d) > 0$ for $d \geq N$ uniformly in x. We point out that $\tilde{K}$ may be substochastic.

We prove the analogue of (2.14) in Kesten (1995) by showing:

$$\frac{K^{n+t}(x,y_n)}{\theta^t K^n(x,y_n)} = \frac{\tilde{K}^{n+t}(x,y_n)}{\tilde{K}^n(x,y_n)} \to 1.$$

As in Kesten (1995) take $d \geq N$ and define $\delta(d) = \delta$ and $\hat{Q} \equiv \hat{Q}_d = (\tilde{K}^d - \delta)/(1-\delta)$ so $\tilde{K}^d = \delta I + (1-\delta)\hat{Q}$. Note that $\hat{Q}$ is still irreducible and uniformly aperiodic.

For $n = rd+s$, $0 \le s < d$, define $\rho(n; x, y_n) = \sum_z (\hat{Q})^r(x,z)\tilde{K}^s(z,y_n) \le 1$. As in (2.15) in Kesten (1995),

$$\begin{aligned}
\tilde{K}^n(x,y_n) &= \sum_{\ell=0}^{r} \binom{r}{\ell} \delta^\ell (1-\delta)^{r-\ell} \sum_z (\hat{Q})^{r-\ell}(x,z)\tilde{K}^s(z,y_n) \\
&= \sum_{\ell=0}^{r} B(r,\ell)\rho((r-\ell)d+s; x, y_n) \qquad (1.3)
\end{aligned}$$

where $B(r,\ell) = \binom{r}{\ell} \delta^\ell(1-\delta)^{r-\ell}$ and

$$\begin{aligned}
\tilde{K}^{n+d}(x,y_n) &= \sum_{\ell=0}^{r+1} B(r+1,\ell)\rho((r+1-\ell)d+s; x, y_n) \\
&= \sum_{\ell=-1}^{r} B(r+1,\ell+1)\rho((r-\ell)d+s; x, y_n). \qquad (1.4)
\end{aligned}$$

$\tilde{K}^n(x,y_n)$ decays slowly by hypothesis. This allows us to follow (2.2) in Orey (1971). We split the sums in (1.4) and (1.3) into a part close to the mean and a part a large deviation away from the mean, we throw away the large deviation part and then show the ratio of the central part of (1.4) divided by the central part of (1.3) tends to one.

More specifically following Orey (1971) let $\sum'$ denote summation over ℓ satisfying $|\ell - \delta r| \le \epsilon r$ while $\sum''$ denotes summation over ℓ satisfying $|\ell - \delta r| > \epsilon r$. Therefore

$$\begin{aligned}
\frac{\tilde{K}^{n+d}(x,y_n)}{\tilde{K}^n(x,y_n)} &= \frac{\sum' B(r+1,\ell+1)\rho((r-\ell)d+s;x,y_n)}{\sum_{\ell=0}^{r} B(r,\ell)\rho((r-\ell)d+s;x,y_n)} \\
&+ \frac{\sum'' B(r+1,\ell+1)\rho((r-\ell)d+s;x,y_n)}{\tilde{K}^n(x,y_n)}.
\end{aligned}$$

The numerator of the last term approaches zero at an exponential rate while the denominator decays slower than $\exp(-an)$ where $a > 0$ is arbitrarily small so the last term is negligible.

Now split the denominator of the first term into sums $\sum'$ and $\sum''$. For the same reason we may throw away the sum $\sum''$. We conclude

$$\frac{\tilde{K}^{n+d}(x,y_n)}{\tilde{K}^n(x,y_n)} \sim \frac{\sum' B(r+1,\ell+1)\rho((r-\ell)d+s;x,y_n)}{\sum' B(r,\ell)\rho((r-\ell)d+s;x,y_n)}.$$

Now for $|\ell - \delta r| \le \epsilon r$, $B(r+1,\ell+1)/B(r,\ell) = \delta(r+1)/(\ell+1)$ is between $\frac{\delta}{\delta+\epsilon}(1+\mathcal{O}(1/r))$ and $\frac{\delta}{\delta-\epsilon}$. Since ϵ is arbitrarily small it follows that $\tilde{K}^{n+d}(x,y_n)/\tilde{K}^n(x,y_n) \to 1$ as $n \to \infty$ for $d \ge N$.

The same argument with $r+1$ replaced by $r-1$ implies $\tilde{K}^{n-d}(x,y_n)/\tilde{K}^n(x,y_n) \to 1$ as $n \to \infty$ for $d \ge N$ so for instance $\tilde{K}^{n-N}(x,y_n)/\tilde{K}^n(x,y_n) \to 1$.

Next for $0 \le k < N$,

$$\frac{\tilde{K}^{n+k}(x,y_n)}{\tilde{K}^n(x,y_n)} \sim \frac{\tilde{K}^{n+k}(x,y_n)}{\tilde{K}^{n-N}(x,y_n)} = \frac{\tilde{K}^{n-N+d}(x,y_n)}{\tilde{K}^{n-N}(x,y_n)} \qquad (1.5)$$

where $d = N+k$. Now repeat the above argument by representing $n - N = rd + s$ where $0 \le s < d$. It works because $\tilde{K}^{n-N}(x,y_n) \sim \tilde{K}^n(x,y_n)$ is not negligible. A similar argument for $-N < k < 0$ gives, for all k as $n \to \infty$

$$\frac{\tilde{K}^{n+k}(x,y_n)}{\tilde{K}^n(x,y_n)} \to 1.$$

□

1.2. *Spatial Martin exit boundaries.* As in Dynkin (1969), the space of θ-harmonic functions is described by the space of exits E inside the θ-Martin exit boundary; i.e. points in the completion corresponding to limits $h_e(x) = \lim_{z\to e} k(x,z;T)$ where $k(x,z;T) = G_{x,z}(T)/G_{x_0,z}(T)$ $(T = 1/\theta)$ and where h_e is θ-harmonic and minimal and x_0 is some reference point.

The h-transform with respect to a θ-harmonic function h gives a chain $\tilde{X}^h$ having a probability transition kernel $\tilde{K}(x,y) \equiv \tilde{K}^h(x,y) = K(x,y)h(y)/(\theta h(x))$ and associated probability measure $\tilde{P}_{x_0}$. The associated potential is given by

$$\tilde{G}^h_{x,y}(1) = \sum_{n=0}^{\infty} \tilde{K}^n(x,y) = \frac{h(y)}{h(x)} \sum_{n=0}^{\infty} K^n(x,y) T^n = \frac{h(y)}{h(x)} G_{x,y}(T).$$

The associated Martin kernel is

$$\tilde{k}^h(x,y;1) = \frac{\tilde{G}^h_{x,y}(T)}{\tilde{G}^h_{x_0,y}(T)} = \frac{h(x_0)}{h(x)} \frac{G_{x,y}(T)}{G_{x_0,y}(T)} = \frac{h(x_0)}{h(x)} k(x,y;T).$$

Moreover there exists a random variable $\tilde{X}_\infty$ taking values in E such that $\tilde{P}_{x_0}(\tilde{X}_n \to \tilde{X}_\infty) = 1$ (see Theorem 4 in Dynkin, 1969 applied to $\tilde{X}^h$ and the remarks around (46) in Dynkin, 1969).

Now suppose $h = h_e$ is a minimal θ-harmonic and for the rest of this section we mean h_e-transform when we use a tilde. Then, $\tilde{X}_n$ converges a.s. to e, in the Martin topology because e is in the exit boundary (see Theorem 5 in Dynkin (1969)). This means the tail field is trivial and $\tilde{k}^h(x, \tilde{X}_n; 1) \to 1$ a.s.

1.3. *Space-time Martin boundaries.* The following result is a slightly modification of Theorem 1.4 in Chapter 3 of Orey (1971).

Theorem 1.4. *Consider a Markov chain Z_m defined on $(\Omega, \mathcal{F}, P_\alpha)$ taking values in a countable state space S with kernel Q and initial probability distribution α whose support is S. Further suppose Z_m has trivial tail field. Define*

$$h_m(z) = \frac{\beta Q^{m+d}(z)}{\alpha Q^m(z)}, m \geq \max\{0, -d\}$$

where β is any probability on S, then $h_m(Z_m)$ converges almost surely to 1 with respect P_α.

Proof: We repeat the proof found in Foley and McDonald (2017).

$$\begin{aligned} E_\alpha\left[\frac{\beta Q^{m+d}(Z_m)}{\alpha Q^m(Z_m)} | Z_{m+1} = y\right] &= \sum_{x\in S}\left[\frac{\alpha Q^m(x)\beta Q^{m+d}(x)Q(x,y)}{\alpha Q^m(x)\cdot \alpha Q^{m+1}(y)}\right] \\ &= \frac{\beta Q^{m+d+1}(y)}{\alpha Q^{m+1}(y)} = h_{m+1}(y). \end{aligned}$$

Hence, $h_m(Z_m)$ is a positive backward martingale with respect to $\sigma(Z_m, Z_{m+1}, \ldots)$. Moreover,

$$\begin{aligned} E_\alpha\left[h_m(Z_m)\right] &= \sum_{x\in S} \alpha Q^m(x) \frac{\beta Q^{m+d}(x)}{\alpha Q^m(x)} \\ &= \sum_{x\in S} \beta Q^{m+d}(x) = 1. \end{aligned}$$

By the backward martingale theorem $(h_m(Z_m))_m$ is uniformly integrable, and converges in $L^1(P_\alpha)$ and almost surely to H where $EH = E_\alpha h_m(Z_m) = 1$. Moreover, since H is measurable with respect to the tail field it is constant and therefore $H = 1$. □

Consider the h-transformed kernel $\tilde{K}$ of K with respect to h_e, the minimal θ-harmonic associated with a point e in the θ-Martin exit boundary. We normalize h_e so $h_e(x_0) = 1$. We denote by $\tilde{X}^{h_e} \equiv \tilde{X}$ the associated chain, and $\tilde{P}_{x_0}$ the associated probability measure with starting point x_0. Then, by the results in Dynkin (1969), we have $\tilde{X}_n \to e$ a.s. $\tilde{P}_{x_0}$, that is,

$$\tilde{k}^{h_e}(x, e) = \lim_{n\to\infty} \frac{\tilde{G}^{h_e}_{x,\tilde{X}_n}(1)}{\tilde{G}^{h_e}_{x_0,\tilde{X}_n}(1)} = 1 \text{ a.s. } \tilde{P}_{x_0},$$

or equivalently

$$\lim_{n\to\infty} \frac{G_{x,\tilde{X}_n}(T)}{G_{x_0,\tilde{X}_n}(T)} = \frac{h_e(x)}{h_e(x_0)} = h_e(x) \text{ holds a.s. } \tilde{P}_{x_0}.$$

Now apply Theorem 1.4 to $\tilde{K}$ taking $\alpha = \delta_{x_0}$, $\beta = \delta_x$. The tail field of $\tilde{X}$ is trivial w.r.t. P_{x_0}, so by Theorem 1.4, as $n \to \infty$

$$\frac{\tilde{K}^{n-t}(x, \tilde{X}_n)}{\tilde{K}^n(x_0, \tilde{X}_n)} \to 1 \text{ a.s. } \tilde{P}_{x_0}, \tag{1.6}$$

or equivalently

$$\frac{K^{n-t}(x, \tilde{X}_n)}{K^n(x_0, \tilde{X}_n)} \to \theta^{-t} \frac{h_e(x)}{h_e(x_0)} \text{ a.s. } \tilde{P}_{x_0}. \tag{1.7}$$

We can reinterpret (1.6) as a description of the space-time exit boundary. Define the space-time Martin kernel $k((x,t);(y,n)) = \frac{K^{n-t}(x,y)}{K^n(x_0,y)}$, so the above result shows

$$k((x,t);(\tilde{X}_n, n)) \to \theta^{-t} h_e(x) \text{ a.s. } \tilde{P}_{x_0},$$

that is, $(\tilde{X}_n, n) \to \tilde{e}$ a.s.-$\tilde{P}_{x_0}$, where $\tilde{e}$ is an extremal in the space-time Martin exit boundary. Note moreover than $\tilde{X}^{h_e}$ satisfies the hypotheses in Proposition 1.5 below.

Given the above result it is natural to wonder if other (possibly nonrandom) sequences $y_n \to e$ in the spatial θ-Martin exit boundary also converge as space-time sequences; i.e. if $(y_n, n) \to \tilde{e}$.

Proposition 1.5. *Assume Condition [1] and along a sequence (y_n, n) such that $y_n \to e \in E$ Condition [4] holds. Also suppose that for all x the support $\{y : K(x,y) > 0\}$ is finite. Then there exists a subsequence n_i such that*

$$k(x, t; y_{n_i}, n_i) = K^{n_i - t}(x, y_{n_i})/K^{n_i}(x_0, y_{n_i})$$

converges to a space-time harmonic function $\theta^{-t} g(x)$ where g is θ-harmonic.

Proof: Take a subsequence n_i such that (y_{n_i}, n_i) converges to a point $\tilde{e}$ in the space-time entrance boundary; i.e.

$$k(x, t; y_{n_i}, n_i) = \frac{K^{n_i - t}(x, y_{n_i})}{K^{n_i}(x_0, y_{n_i})} \to g(x,t) \text{ for all } x \text{ and } t.$$

By Condition [4]

$$\lim_{n\to\infty} \frac{K^{n+m}(x, y_n)}{K^n(x, y_n)} = \theta^m \text{ for all } m,\ x.$$

It follows that $K^{n+m-1}(x,y_n)/K^{n+m}(x,y_n) \to \theta^{-1}$. Hence

$$\begin{aligned} \frac{g(x,t+1)}{g(x,t)} &= \lim_{n_i\to\infty} \frac{k(x,t+1;y_{n_i},n_i)}{k(x,t;y_{n_i},n_i)} \\ &= \lim_{n_i\to\infty} \frac{K^{n_i-(t+1)}(x,y_{n_i})}{K^{n_i-t}(x,y_{n_i})} = \theta^{-1}. \end{aligned}$$

Therefore, $g(x,t+1) = \theta^{-1} g(x,t)$ so $g(x,t) = \theta^{-t} g(x,0) \equiv \theta^{-t} g(x)$.

Finally, remark that $k(x,t;y_n,n)$ is harmonic on space-time for $t \leq n$. Thus, $g(x,t)$ is subharmonic but by hypothesis the support $\{y : K(x,y) > 0\}$ is finite, so in fact $g(x,t)$ is harmonic on space-time. Therefore

$$\theta^{-t} g(x) = g(x,t) = \sum_y K(x,y) g(y,t+1) = \sum_y K(x,y) \theta^{-(t+1)} g(y).$$

Then, we conclude that $\sum_y K(x,y)g(y) = \theta g(x)$ and g is θ-harmonic as we wanted to prove. □

1.4. *A conjecture.* $\tilde{X}_n \to e$ and satisfies the hypotheses in Proposition 1.5 and in fact $k(x,t;\tilde{X}_n,n) \to \theta^{-t} h_e(x) a.s - \tilde{P}_{x_0}$; i.e. $(\tilde{X}_n, n) \to \tilde{e}$. So it is natural to conjecture that if the sequence $y_n \to e$ in the spatial θ-Martin exit boundary topology and if the Conditions of Proposition 1.5 are satisfied then g above satisfies $g = h_e$. Then all subsequences $k(x,t;y_{n_i},n_i)$ above would converge to the same limit so $(y_n,n) \to \tilde{e}$ associated with $\theta^{-t} h_e(x)$.

Unfortunately as we see in the next section this conjecture is false and g may be the θ-harmonic function associated with a different point on the spatial ρ-Martin boundary. This leaves the interesting question: what are the properties of the sequences $\tilde{X}_n$ that force $(\tilde{X}_n, n) \to \tilde{e}$?

2. Random walk on the integers

In this section we show the conjecture proposed above is false. The first example is a case where things work out as we hope. The spatial ρ-Martin exit boundary exactly matches trajectories to the space-time boundary corresponding to the space-time paths of the h-transform by a ρ-harmonic function h in the spatial ρ-Martin exit boundary. The second example gives a counterexample to the conjecture.

2.1. *Homogeneous, nearest neighbor random walk.* We begin with an example given in Example 2, Chapter 10, Section 13 in Kemeny et al. (1976) where everything can be calculated. Consider a transient random walk on the integers with transition kernel $K(x,x+1) = p$ and $K(x,x-1) = q$ with $p + q = 1$ and $p > q$. To avoid periodicity issues we restrict n, x, t and y_n to be even.

$$K^n(0,0) = \binom{n}{\frac{n}{2}} (pq)^{n/2} \sim \frac{2^n}{\sqrt{n\pi/2}} (pq)^{n/2}.$$

Since $(K^n(0,0))^{1/n} \to 2\sqrt{pq}$ it follows that the spectral radius is $\rho = 2\sqrt{pq}$, and the chain is R-recurrent for $R = 1/\rho$. Moreover, for $T < R$ the chain is T-transient. In what follows we denote by $\theta = 1/T$.

The space-time Martin exit boundary is given by limits along sequences $(y_n, n) \to \tilde{e}$ of

$$k(x,t;y_n,n) = \frac{K^{n-t}(x,y_n)}{K^n(0,y_n)}.$$

In particular, we have

$$\begin{aligned} k(x,0;y_n,n) = \frac{K^n(x,y_n)}{K^n(0,y_n)} &= \frac{\binom{n}{\frac{n+y_n-x}{2}} p^{(n+y_n-x)/2} q^{(n-y_n+x)/2}}{\binom{n}{\frac{n+y_n}{2}} p^{(n+y_n)/2} q^{(n-y_n)/2}} \\ &= \frac{\left(\frac{y_n+n}{2}\right)! \left(\frac{n-y_n}{2}\right)!}{\left(\frac{n+y_n-x}{2}\right)! \left(\frac{n-y_n+x}{2}\right)!} p^{-x/2} q^{x/2} \\ &= \frac{((y_n+n)/2)\,((y_n+n)/2-1)\cdots((y_n+n-x)/2+1)}{((n-y_n+x)/2)\,((n-y_n+x)/2-1)\cdots((n-y_n)/2+1)} \left(\frac{q}{p}\right)^{x/2}. \end{aligned}$$

Now if $y_n/n \to \alpha$ for $-1 < \alpha < 1$ then dividing the above top and bottom through by $n^{|x|/2}$ we see that

$$k(x,0;y_n,n) = \frac{K^n(x,y_n)}{K^n(0,y_n)} \to h_\theta(x,0) = h_\theta(x) = \left(\frac{1+\alpha}{1-\alpha}\right)^{x/2} \left(\frac{q}{p}\right)^{x/2}.$$

Note that h_θ is an eigenvector for K (on the integers) with eigenvalue

$$\theta = \sqrt{pq}\left(\left(\frac{1+\alpha}{1-\alpha}\right)^{1/2} + \left(\frac{1-\alpha}{1+\alpha}\right)^{1/2}\right) = \frac{2\sqrt{pq}}{(1-\alpha)^{1/2}(1+\alpha)^{1/2}};$$

i.e. h_θ is θ-harmonic. Hence $h_\theta(x,t) = \theta^{-t}h_\theta(x)$ is space-time harmonic for K and $k(x,t;y_n,n) \to h_\theta(x,t)$.

We can also obtain the extremals in the spatial θ-Martin boundary. Take $T = 1/\theta$ and define

$$h_\theta^+(x) = \lim_{y\to+\infty} \frac{G_{x,y}(T)}{G_{0,y}(T)} \text{ and } h_\theta^-(x) = \lim_{y\to-\infty} \frac{G_{x,y}(T)}{G_{0,y}(T)}.$$

For $x > 0$ and $\alpha > 0$,

$$\lim_{y\to+\infty} \frac{G_{x,y}(T)}{G_{0,y}(T)} = \frac{1}{F_{0,x}(T)} = \frac{1}{F_{0,1}(T)^x}.$$

However, $F_{0,1}(z) = \frac{1-\sqrt{1-4pqz^2}}{2zq}$ is the z-transform of the first passage time from 0 to 1, and a bit of calculation shows $\sqrt{1-4pqT^2} = |\alpha|$ so

$$\frac{1}{F_{0,1}(T)} = \frac{1+\alpha}{2pT} = \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2}\left(\frac{q}{p}\right)^{1/2},$$

that is, $h_\theta^+(x) = h_\theta(x)$, when $\alpha > 0$. The same result holds for $x \le 0$.

Now take $x > 0$ and $\alpha < 0$

$$\lim_{y\to-\infty} \frac{G_{x,y}(T)}{G_{0,y}(T)} = F_{x,0}(T) = F_{1,0}(T)^x.$$

However, this time $F_{1,0}(z) = \frac{1-\sqrt{1-4pqz^2}}{2pz}$ is the z-transform of the first passage time from 1 to 0, so

$$F_{1,0}(T) = \frac{1-|\alpha|}{2pT} = \frac{1+\alpha}{2p}\theta = \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2}\left(\frac{q}{p}\right)^{1/2},$$

that is, $h_\theta^-(x) = h_\theta(x)$, when $\alpha < 0$. The same holds true for $x \le 0$.

It is interesting to note that the drift of the h_θ-transform is

$$\frac{1}{\theta}\left(p\left(\frac{1+\alpha}{1-\alpha}\right)^{1/2}\left(\frac{q}{p}\right)^{1/2} - q\left(\frac{1+\alpha}{1-\alpha}\right)^{-1/2}\left(\frac{q}{p}\right)^{-1/2}\right) = \frac{2\alpha pq}{(1-\alpha)^{1/2}(1+\alpha)^{1/2}} = \alpha.$$

Consequently the associated h_θ-transformed process $\tilde{X}_n^{h_\theta}$ satisfies $\tilde{X}_n^{h_\theta}/n \to \alpha$ so $k(x,0;\tilde{X}_n^{h_\theta},n) \to h_\theta(x)$ but we already knew this from (1.7). We now see that for $\alpha \neq 0$, the spatial θ-Martin boundary has the same geometry as the space-time θ-Martin boundary. Moreover, in this example, the spatial θ-Martin boundary is the same as the geometric boundary.

Note also that if $y_n = \alpha n$; i.e. y_n drifts like a trajectory of $\tilde{X}_n^{h_\theta}$, then

$$K^n(x,y_n) \sim K^n(0,\alpha n) = \binom{n}{(n+\alpha n)/2} p^{(n+\alpha n)/2} q^{(n-\alpha n)/2}.$$

Using Stirling's formula and a bit of calculation

$$\begin{aligned}(K^n(0,y_n))^{1/n} &\sim \frac{2\sqrt{pq}}{(1+\alpha)^{1/2}(1-\alpha)^{1/2}}\left(\frac{p}{q}\right)^{\alpha/2}\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha/2}\\ &= \frac{\theta}{h_\theta(y_n)^{1/n}}.\end{aligned}$$

Hence, Condition [3] holds and by Proposition 1.5 there exists a subsequence n_i such that

$$k(x,t;y_{n_i},n_i) = K^{n_i-t}(x,y_{n_i})/K^{n_i}(x_0,y_{n_i})$$

converges to a space-time harmonic function $\theta^{-t}g(x)$ where g is θ-harmonic. But here $g = h_\theta$ since the spatial and space-time boundaries coincide and everything works as expected.

The above discussion is a bit different when $\alpha = 0$. In this situation, $\theta = \rho = 2\sqrt{pq}$ is the spectral radius. But even though the chain is ρ-recurrent the ρ-Martin kernel $F_{x,y}(R)/F_{0,y}(R)$ still exists where $F_{x,y}$ is the transform of the first passage time from x to y. In this case $F_{x,y}(R)/F_{0,y}(R) = h_\theta^+(x) = h_\theta^-(x) = \left(\frac{q}{p}\right)^{x/2}$ which is the unique ρ-harmonic function. Consequently the spatial ρ-Martin boundary has only one point. This point corresponds to the one point in the space-time boundary associated with the sequence $(0,n)$. In this sense the spatial and space-time boundaries have the same geometry. We note that h_ρ-transform $\tilde{X}_n^{h_\rho}$ is a random walk with kernel $\tilde{K}(x,x+1) = \tilde{K}(x,x-1) = 1/2$. So, while $\tilde{X}_n^{h_\rho}$ converges almost surely in the Martin topology to a single point in the ρ-Martin boundary, it doesn't converge to the geometric boundary $\{+\infty,-\infty\}$. In this case, the Martin boundary does not match with the geometric boundary.

2.2. *Two point Martin boundary.* We now consider the two-sided example which provides our counterexample. Consider a nearest neighbour random walk on the integers where, for $x > 0$,

$$K(x,x+1) = p, K(x,x-1) = q,$$
$$K(-x,-x+1) = a, K(-x,-x-1) = b$$

and $K(0,1) = p$, $K(0,-1) = b$. We assume $p+q = 1$, $p < q$ and $a+b = 1$, $b < a$, that is, there is killing at 0. We also assume $\rho = 2\sqrt{pq} > 2\sqrt{ab}$ which implies $b < p < 1/2$. Set $\Gamma = \sqrt{1-\frac{ab}{pq}} < 1$ and for $|z| < \frac{pq}{ab}$ set $\Gamma(z) = \sqrt{1-\frac{ab}{pq}z}$.

The z-transform of the recurrence time from $x > 0$ to 0 is $\left(\frac{1-\sqrt{1-4pqz^2}}{2zp}\right)^x$ while the recurrence time from $-x < 0$ to 0 is $\left(\frac{1-\sqrt{1-4abz^2}}{2zb}\right)^x$. The z-transform of the recurrence time to 0 for the K kernel is

$$\begin{aligned}F_{0,0}(z) &= zpF_{1,0}(z) + zbF_{-1,0}(z)\\ &= zp\frac{(1-\sqrt{1-4pqz^2})}{2zp} + zb\frac{(1-\sqrt{1-4abz^2})}{2zb}\end{aligned}$$

as in Seneta and Vere-Jones (1966). Since $F_{0,0}(z)$ becomes singular at $z = R = (2\sqrt{pq})^{-1}$ and takes the value $1/2 + (1-\sqrt{1-ab/pq})/2 < 1$ there, we conclude the spectral radius of K is $\rho = 2\sqrt{pq}$ and K is R-transient. Moreover

$$G_{0,0}(z) = \frac{1}{1-F_{0,0}(z)} = \frac{\sqrt{1-4abz^2}-\sqrt{1-4pqz^2}}{2(pq-ab)z^2}.$$

Consider $f(s) = bs^2 - 2\sqrt{pq}s + a = 0$. The roots of $f(s) = 0$ are

$$t_0 = \sqrt{pq}(1-\sqrt{1-ab/pq})/b \text{ and } t_1 = \sqrt{pq}(1+\sqrt{1-ab/pq})/b.$$

Both roots are real since $ab < pq$. The mid point between the roots is $\sqrt{pq}/b > \sqrt{pq/ab} > 1$. Since $f(0) = a$, $f'(0) < 0$ and $f(1) = 1 - 2\sqrt{pq} > 0$ it follows that both roots are greater than one. Notice that for either root t: $at^{-1} + bt = (a + bt^2)/t = 2\sqrt{pq} = \rho$; i.e. t^{-x} is ρ-harmonic on $x < 0$ if $t = t_0$ or $t = t_1$.

Consider positive ρ-harmonic functions of the form

$$h(x) = \begin{cases} (1 + cx)\sqrt{\frac{q}{p}}^{x} & \text{if } x > 0 \\ f_0 t_0^{-x} + f_1 t_1^{-x} \text{ where } f_1 \geq 0 \text{ and } f_0 + f_1 = 1 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \end{cases}$$

where to be positive ρ-harmonic at $x = 0$ requires

$$\rho = p(1 + c)\sqrt{q/p} + b(f_0 t_0 + f_1 t_1),$$

that is, $c = (f_0 - f_1)\Gamma \geq 0$.

Setting $f_0 = 1$, and therefore $f_1 = 0$, gives the ρ-harmonic function

$$h_{+\infty}(x) = \begin{cases} (1 + \Gamma x)\sqrt{\frac{q}{p}}^{x} \text{ where} & \text{if } x > 0 \\ t_0^{-x} & \text{if } x < 0 \\ 1 & \text{if } x = 0. \end{cases}$$

In fact $h_{+\infty}$ is the associated extremal positive ρ-harmonic function for sequences $z_n \to +\infty$ associated with the Martin boundary point $\{+\infty\}$:

$$\lim_{z_n \to +\infty} \frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = \begin{cases} \lim_{z_n \to +\infty} \frac{G_{x,z_n}(R)}{F_{0,x}(R) G_{x,z_n}(R)} = \frac{1}{F_{0,x}(R)} & \text{if } x > 0 \\ \lim_{z_n \to +\infty} \frac{F_{x,0}(R) G_{0,z_n}(R)}{G_{0,z_n}(R)} = F_{x,0}(R) & \text{if } x < 0 \\ 1 & \text{if } x = 0. \end{cases}$$

For $x < 0$, we have $F_{x,0}(R) = \left(\frac{1 - \sqrt{1 - 4abR^2}}{2Rb}\right)^{-x} = t_0^{-x}$, so for $x < 0$

$$\lim_{z_n \to +\infty} \frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = t_0^{-x}.$$

Now, t_0^{-x} on $x < 0$ extends uniquely to $h_{+\infty}$ on $x \geq 0$ and since the Martin kernel converges to a positive ρ-harmonic function, we conclude that

$$\lim_{z_n \to +\infty} \frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = h_{+\infty}(x) \text{ for all } x.$$

Thus, $h_{+\infty}$ is indeed an extremal, positive ρ-harmonic function associated with $\{+\infty\}$ in the Martin boundary. Another consequence of this convergence is that for $x > 0$

$$\frac{1}{F_{0,x}(R)} = (1 + \Gamma x)\sqrt{\frac{q}{p}}^{x}.$$

The other extremal positive ρ-harmonic function $h_{-\infty}$ occurs when $c = 0$, in which case $f_0 = f_1 = \frac{1}{2}$. Hence,

$$h_{-\infty}(x) = \begin{cases} \sqrt{\frac{q}{p}}^{x} & \text{if } x > 0 \\ \frac{1}{2} t_0^{-x} + \frac{1}{2} t_1^{-x} & \text{if } x < 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Indeed, for sequences $z_n \to -\infty$ associated with the Martin boundary point $\{-\infty\}$ the associated extremal positive ρ-harmonic function is $h_{-\infty}$:

$$\lim_{z_n\to-\infty}\frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = \begin{cases} \lim_{z_n\to-\infty}\frac{F_{x,0}(R)G_{0,z_n}(R)}{G_{0,z_n}(R)} = F_{x,0}(R) & \text{if } x>0\\ \lim_{z_n\to-\infty}\frac{G_{x,z_n}(R)}{F_{0,x}(R)G_{x,z_n}(R)} = \frac{1}{F_{0,x}(R)} & \text{if } x<0\\ 1 & \text{if } x=0. \end{cases}$$

For $x>0$, we have $F_{x,0}(R) = \left(\frac{1-\sqrt{1-4pqR^2}}{2Rp}\right)^x = \sqrt{\frac{q}{p}}^x$, and so for $x>0$ it holds

$$\lim_{z_n\to-\infty}\frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = \sqrt{\frac{q}{p}}^x.$$

Now, $\sqrt{\frac{q}{p}}^x$ on $x>0$ extends uniquely to $h_{-\infty}$ on $x\le 0$ and since the Martin kernel converges to a positive ρ-harmonic function then

$$\lim_{z_n\to-\infty}\frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = h_{-\infty}(x),$$

holds for all x and $h_{-\infty}$ is indeed an extremal, positive ρ-harmonic function associated with $\{-\infty\}$ in the Martin boundary. Another consequence of this convergence is that for $x<0$

$$\frac{1}{F_{0,x}(R)} = \frac{1}{2}t_0^{-x} + \frac{1}{2}t_1^{-x}.$$

2.3. *The counterexample.* We now come to the main result. We have seen above that

$$\lim_{z_n\to-\infty}\frac{G_{x,z_n}(R)}{G_{0,z_n}(R)} = h_{-\infty}(x) \text{ for all } x.$$

The following theorem therefore comes as surprise because taking $z_n = -y_n$ we see the space-time limit associated with sequences $(z_n, 2n)$ is $h_{+\infty}$ not $h_{-\infty}$ as we would hope!

Theorem 2.1. *If x and y_n are even, where $y_n \to \infty$ but sufficiently slowly such that $y_n = \log(o(n))/\alpha$ where $\alpha = \log(1/(1-\Gamma)) > 0$ then*

$$\lim_{n\to\infty}\left(K^{2n}(x,-y_n)\right)^{1/(2n)} = 2\sqrt{pq} \text{ and}$$

$$\lim_{n\to\infty}\frac{K(x,0;-y_n,2n)}{K(0,0;-y_n,2n)} = \lim_{n\to\infty}\frac{K^{2n}(x,-y_n)}{K^{2n}(0,-y_n)} = h_{+\infty}(x).$$

For an intuitive explanation of this paradox, we remark that

$$\mu_{-\infty}(x) = h_{-\infty}(x)\gamma(x)/\gamma(0)$$

is ρ-invariant ($\rho = 2\sqrt{pq}$), where γ is a reversibility measure for the chain (see (2.1) below). We can then check that the conditions of Lemma 2 in Foley and McDonald (2017) hold along the sequence $-y_n$. First $K^{2n}(-y_n, x) = \frac{\gamma(x)}{\gamma(-y_n)}K^{2n}(x,-y_n)$ so $\left(K^{2n}(-y_n,x)\right)^{1/(2n)} \sim 2\sqrt{pq}$. Similarly $(\mu_{-\infty}(-y_n))^{1/(2n)} \to 1$.

Hence by the cited Lemma 2 in Foley and McDonald (2017), we have $K^{2n+1}(-y_n,S)/K^{2n}(-y_n,S) \to 2\sqrt{pq}$. Next, $K^{2n+1}(-y_n,S) = K^{2n}(-y_n,S) - K^{2n}(-y_n,0)\kappa$ where $\kappa = 1-(p+b)$ is the killing probability at zero. Dividing through by $K^{2n}(-y_n,S)$ we get

$$\frac{K^{2n}(-y_n,0)}{K^{2n}(-y_n,S)} \to \frac{1-\rho}{\kappa}.$$

Consequently,

$$\begin{aligned}\frac{K^{2n}(-y_n,x)}{K^{2n}(-y_n,S)} &\sim \frac{1-\rho}{\kappa}\frac{K^{2n}(-y_n,x)}{K^{2n}(-y_n,0)}\\ &= \frac{1-\rho}{\kappa}\frac{\gamma(x)}{\gamma(0)}\frac{K^{2n}(x,-y_n)}{K^{2n}(0,-y_n)}\\ &\sim \frac{1-\rho}{\kappa}\frac{\gamma(x)}{\gamma(0)}h_{+\infty}(x) = \frac{1-\rho}{\kappa}\mu_{+\infty}(x)\end{aligned}$$

where $\mu_{+\infty}(x) = \frac{\gamma(x)}{\gamma(0)}h_{+\infty}(x)$.

Therefore, the ratio limit theorem $K^{2n}(x,-y_n)/K^{2n}(0,-y_n) \to h_{+\infty}(x)$ is equivalent to the Yaglom limit $K^{2n}(-y_n,x)/K^{2n}(-y_n,S) \to \frac{\kappa}{1-\rho}\mu_{+\infty}(x)$. Now as explained in Foley and McDonald (2017) the trajectories contributing to this Yaglom limit start out like the $\hat{h}$-transform where $\hat{h}(x) = \lim_{n\to\infty} K^n(x,S)/K^n(0,S)$ but as is seen in Example 2 in Foley and McDonald (2017) $\hat{h}$ is precisely $h_{+\infty}$. Thus the trajectories of the rare event of starting from $-y_n$ and surviving until time $2n$ are the trajectories of the $h_{+\infty}$-transform. For $y>0$,

$$\tilde{K}(y,y+1) = p\frac{1+\Gamma(y+1)}{1+\Gamma y}\sqrt{q/p}/\rho = \frac{1}{2}\frac{1+\Gamma(y+1)}{1+\Gamma y}$$

while $\tilde{K}(y,y-1) = \frac{1}{2}\frac{1+\Gamma(y-1)}{1+\Gamma y}$. We note this is the kernel of a transient Markov chain drifting slower and slower to infinity. Thus to survive to be at x at time $2n$ the trajectories drift slowly out to $+\infty$ away from the killing at zero and then rapidly return to x at the end.

This is possible because y_n is so small compared to n. For y_n bigger something must break. For trajectories given by the $h_{-\infty}$-transform for $y<0$ large

$$\tilde{K}(y,y-1) \sim \frac{b}{2\sqrt{pq}}t_1 = \frac{1}{2}(1+\sqrt{1-ab/(pq)})$$

while

$$\tilde{K}(y,y+1) \sim \frac{a}{2\sqrt{pq}}\frac{1}{t_1} = \frac{1}{2}(1-\sqrt{1-ab/(pq)}).$$

We conclude that the $h_{-\infty}$-transform trajectories $\tilde{X}_n$ drift linearly to $-\infty$. Moreover, by Theorem 1.4 if $-y_n = \tilde{X}_{2n}$ is such a trajectory then

$$\frac{K^{2n}(x,-y_n)}{K^{2n}(0,-y_n)} \quad\to\quad \frac{h_{-\infty}(x)}{h_{-\infty}(0)};$$

i.e. a completely different ratio limit theorem holds and it is reasonable to conjecture that the above Yaglom limit fails as well. It seems likely that to be at x at time $2n$ starting from from $-y_n$ the trajectories stay negative away from 0 and then return to x at the end.

Proof of theorem 2.1: One advantage of nearest neighbour walks is that starting from a mass $\gamma(0)$ at 0 we can construct a reversibility measure $\gamma(x)$ which satisfies $\gamma(x)K(x,x+1) = \gamma(x+1)K(x+1,x)$. Calculation gives

$$\gamma(x) = \begin{cases}\gamma(0)(p/q)^x & \text{if } x>0\\ \gamma(0)(a/b)^x & \text{if } x<0\\ \gamma(0) & \text{if } x=0\end{cases} \tag{2.1}$$

and $\gamma(x)K^n(x,y) = \gamma(y)K^n(y,x)$ holds for all x,y. It also follows that $\gamma(x)G_{x,y}(z) = \gamma(y)G_{y,x}(z)$. Also recall the fact that, for $|x|\le 1$

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n,$$

where $\binom{1/2}{n} = (-1)^{n-1} A_n$ and

$$A_n := \frac{(2n-3)!}{2^{2n-2} n!(n-2)!} \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}$$

by Stirling's formula. Note that this means

$$\sqrt{1-x} \quad = \quad 1 - \sum_{n=1}^{\infty} A_n x^n. \tag{2.2}$$

We will need the following estimates, where we assume u is a complex number

$$|u| < pq/ab, \qquad \text{(Condition [a])}$$
$$|u-1| < (pq-ab)/(ab). \qquad \text{(Condition [b])}$$

Note that $pq/ab > 1$ and $0 < \Gamma = \Gamma(1) < 1$.

Lemma 2.2. (1) *If Condition [a] holds, then*

$$|\Gamma(u)| \le \sqrt{2},$$
$$|1-\Gamma(u)| < 1.$$

(2) *If Condition [a], Condition [b] hold, then*

$$\left| \frac{1-\Gamma(u)}{1-\Gamma} - 1 \right| \le \frac{\Gamma}{1-\Gamma} \frac{ab}{pq-ab} |u-1|$$
$$\left| \left(\frac{1-\Gamma(u)}{1-\Gamma} \right)^y - 1 \right| \le \frac{ab}{pq-ab} \, |u-1| \left(\frac{1}{1-\Gamma} \right)^y,$$

where y is a positive integer.

Proof: For the first bound just recall the definition of Γ and $\Gamma(u)$, which gives

$$|\Gamma(u)|^2 \quad = \quad \left| 1 - \frac{ab}{pq} u \right| \le 2,$$

so $|\Gamma(u)| \le \sqrt{2}$.

For the second bound, by hypothesis $|u| \le (1-\epsilon)\frac{pq}{ab}$, where $0 < \epsilon < 1/2$, so

$$\begin{aligned} |1-\Gamma(u)| \quad &= \quad \left| 1 - \sqrt{1 - \frac{ab}{pq} u} \right| = \left| \sum_{k=1}^{\infty} K_k \left(\frac{ab}{pq} u \right)^k \right| \\ &\le \quad \sum_{k=1}^{\infty} A_k \left(\frac{ab}{pq} |u| \right)^k \le \sum_{k=1}^{\infty} A_k (1-\epsilon)^k \\ &= \quad 1 - \sqrt{1-(1-\epsilon)} < 1. \end{aligned}$$

For the third bound, we have

$$\begin{aligned}
\left|\left(\frac{1-\Gamma(u)}{1-\Gamma}\right)-1\right| &= \left|\left(\frac{1-\sqrt{1-\frac{ab}{pq}u}}{1-\sqrt{1-\frac{ab}{pq}}}\right)-1\right| \\
&\leq \frac{1}{1-\Gamma}\left|\Gamma-\sqrt{1-\frac{ab}{pq}+\frac{ab}{pq}(1-u)}\right| \\
&= \frac{\Gamma}{1-\Gamma}\left|1-\sqrt{1-\frac{ab}{pq}\frac{(u-1)}{1-\frac{ab}{pq}}}\right| \\
&= \frac{\Gamma}{1-\Gamma}\left|\sum_{k=1}^{\infty}A_k\left(\frac{ab}{pq}\frac{(u-1)}{1-\frac{ab}{pq}}\right)^k\right| \\
&\leq \frac{\Gamma}{1-\Gamma}\sum_{k=1}^{\infty}A_k\left(\frac{ab}{pq}\frac{|u-1|}{1-\frac{ab}{pq}}\right)^k \\
&= \frac{\Gamma}{1-\Gamma}\left(1-\sqrt{1-\frac{ab}{pq-ab}|u-1|}\right) \\
&\leq \frac{\Gamma}{1-\Gamma}\left(1-\left(1-\frac{ab}{pq-ab}|u-1|\right)\right) \\
&= \frac{\Gamma}{1-\Gamma}\frac{ab}{pq-ab}|u-1|,
\end{aligned}$$

which gives the third bound.

Finally, for the fourth bound we use

$$\begin{aligned}
\left|\left(\frac{1-\Gamma(u)}{1-\Gamma}\right)^y-1\right| &= \left|\left(\frac{1-\Gamma(u)}{1-\Gamma}\right)-1\right|\left|\sum_{k=0}^{y-1}\left(\frac{1-\Gamma(u)}{1-\Gamma}\right)^k\right| \\
&\leq \frac{\Gamma}{1-\Gamma}\left(\frac{ab}{pq-ab}\right)|u-1|\sum_{k=0}^{y-1}\left(\frac{1}{1-\Gamma}\right)^k \\
&\leq \left(\frac{ab}{pq-ab}\right)|u-1|\left(\frac{1}{1-\Gamma}\right)^y.
\end{aligned}$$

□

For $x \geq 0$, $K^{2n}(x,-y_n) = [z^{2n}]G_{x,-y_n}(z)$, that is, the coefficient of the z^{2n} term in the power series expansion of $G_{x,-y_n}(z)$. Next

$$\begin{aligned}
G_{x,-y_n}(z) &= F_{x,0}(z)G_{0,-y_n}(z) \\
&= F_{x,0}(z)\frac{\gamma(-y_n)}{\gamma(0)}G_{-y_n,0}(z) \\
&= F_{x,0}(z)\frac{\gamma(-y_n)}{\gamma(0)}F_{-y_n,0}(z)G_{0,0}(z),
\end{aligned}$$

where γ is the reversibility measure. Therefore

$$\frac{K^{2n}(x,-y_n)}{K^{2n}(0,-y_n)} = \frac{[z^{2n}]\left(F_{x,0}(z)F_{-y_n,0}(z)G_{0,0}(z)\right)}{[z^{2n}]\left(F_{-y_n,0}(z)G_{0,0}(z)\right)} \tag{2.3}$$

since $F_{0,0}(z)=1$.

Next

$$[z^{2n}](F_{x,0}(z)F_{-y_n,0}(z)G_{0,0}(z)) = [z^{2n}]\mathcal{C}(z) = [z^{2n+2+x+y_n}]\mathcal{B}(z)$$

where

$$\mathcal{C}(z) = \left(\left(\frac{1-\sqrt{1-4pqz^2}}{2zp}\right)^x \left(\frac{1-\sqrt{1-4abz^2}}{2zb}\right)^{y_n} \frac{\sqrt{1-4abz^2}-\sqrt{1-4pqz^2}}{2(pq-ab)z^2}\right)$$

and

$$\mathcal{B}(z) = \left(\left(\frac{1-\sqrt{1-4pqz^2}}{2p}\right)^x \left(\frac{1-\sqrt{1-4abz^2}}{2b}\right)^{y_n} \frac{\sqrt{1-4abz^2}-\sqrt{1-4pqz^2}}{2(pq-ab)}\right).$$

Setting $w=4pqz^2$ we note the above is $(4pq)^{n+1+x/2+y_n/2}\mathcal{A}_n$ where

$$\mathcal{A}_n = [w^{n+1+x/2+y_n/2}]\left(\left(\frac{1-\sqrt{1-w}}{2p}\right)^x \left(\frac{1-\Gamma(w)}{2b}\right)^{y_n} \frac{\Gamma(w)-\sqrt{1-w}}{2(pq-ab)}\right). \tag{2.4}$$

Hence (2.3) can be rewritten

$$\frac{(4pq)^{x/2}}{(2p)^x}\frac{[w^{n+1+x/2+y_n/2}]\left((1-\sqrt{1-w}\,)^x(1-\Gamma(w))^{y_n}(\Gamma(w)-\sqrt{1-w})\right)}{[w^{n+1+y_n/2}]\left((1-\Gamma(w))^{y_n}(\Gamma(w)-\sqrt{1-w})\right)}. \tag{2.5}$$

We now analyse the asymptotics of the numerator and denominator of the above ratio. To evaluate the numerator remark that

$$\left((1-\sqrt{1-w})^x(1-\Gamma(w))^{y_n}(\Gamma(w)-\sqrt{1-w})\right) = A(w)+B(w)$$

where

$$A(w) \;=\; (1-\Gamma(w))^{y_n}\Gamma(w)\sum_{k=0}^{x}(-1)^k\binom{x}{k}(1-w)^{k/2}$$

and

$$B(w) = -(1-\Gamma(w))^{y_n}\sum_{k=0}^{x}(-1)^k\binom{x}{k}(1-w)^{(k+1)/2}.$$

We show below that

$$\begin{aligned}
&[w^{n+1+x/2+y_n/2}]\,(B(w))\\
&\sim\; -[w^{n+1+x/2+y_n/2}](1-\Gamma)^{y_n}\sum_{k=0}^{x}(-1)^k\binom{x}{k}(1-w)^{(k+1)/2} &(2.6)\\
&\sim\; -[w^{n+1+x/2+y_n/2}](1-\Gamma)^{y_n}(1-w)^{1/2}\\
&\sim\; (1-\Gamma)^{y_n}\frac{1}{2\sqrt{\pi}(n+1+x/2+y_n/2)^{3/2}}. &(2.7)
\end{aligned}$$

Moreover, we show below that

$$\begin{aligned}
&[w^{n+1+x/2+y_n/2}]\,(A(w))\\
&\sim\; [w^{n+1+x/2+y_n/2}]\left((1-\Gamma)^{y_n}\Gamma\sum_{k=1}^{x}(-1)^k\binom{x}{k}(1-w)^{k/2}\right) &(2.8)\\
&\sim\; -[w^{n+1+x/2+y_n/2}]\left((1-\Gamma)^{y_n}\Gamma x(1-w)^{1/2}\right)\\
&\sim\; (1-\Gamma)^{y_n}\Gamma x\,\frac{1}{2\sqrt{\pi}(n+1+x/2+y_n/2)^{3/2}}.
\end{aligned}$$

We conclude that

$$[w^{n+1+x/2+y_n/2}]\left((1-\sqrt{1-w})^x(1-\Gamma(w))^{y_n}(\Gamma(w)-\sqrt{1-w})\right)$$
$$\sim \quad (1+\Gamma x)(1-\Gamma)^{y_n}\frac{1}{2\sqrt{\pi}(n+1+x/2+y_n/2)^{3/2}}. \tag{2.9}$$

Similarly, we have

$$[w^{n+1+y_n/2}]\left((1-\Gamma(w))^{y_n}(\Gamma(w)-\sqrt{1-w})\right)$$
$$\sim \quad (1-\Gamma)^{y_n}(-1)^{n+1+y_n/2}\frac{1}{2\sqrt{\pi}(n+1+y_n/2)^{3/2}}.$$

We conclude that (2.5) is asymptotically the same as

$$\frac{(4pq)^{x/2}}{(2p)^x}(1+\Gamma x)\frac{(n+1+y_n/2)^{3/2}}{(n+1+x/2+y_n/2)^{3/2}}$$
$$\to \quad \left(1+\sqrt{1-\frac{ab}{pq}x}\right)(q/p)^{x/2}=h_{+\infty}(x).$$

The result extends to negative x.

We now check (2.6) and (2.8). To check (2.6) it suffices to show

$$f_n=[w^{n+1+x/2+y_n/2}](f(w))=o(n^{-3/2})$$

where

$$f(w)=\left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}-1\right)\sum_{k=0}^{x}(-1)^k\left(\begin{array}{c}x\\k\end{array}\right)(1-w)^{(k+1)/2}.$$

We follow the steps in Flajolet and Odlyzko (1990). Starting with Cauchy's formula

$$[w^{n+1+x/2+y_n/2}](f(w))=\frac{1}{2i\pi}\int_C f(w)\frac{dw}{w^{n+2+x/2+y_n/2}}$$

where $C=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ and

$$\begin{aligned}
\gamma_1 &= \left\{w:|w-1|=\frac{1}{n}, Arg(1-w)\in[-\pi+\phi,\pi-\phi]\right\}\\
\gamma_2 &= \left\{w:\frac{1}{n}\le|w-1|,|w|\le 1+\nu, Arg(1-w)=-\pi+\phi\right\}\\
\gamma_3 &= \left\{w:|w|=1+\nu, Arg(1-w)\in[-\pi+\phi,\pi-\phi]\right\}\\
\gamma_4 &= \left\{w:\frac{1}{n}\le|w-1|,|w|\le 1+\nu, Arg(1-w)=\pi-\phi\right\},
\end{aligned}$$

where ϕ is a fixed angle in $(0,\pi/2]$. Also we assume that $r=(1+\nu)ab/pq<1$, which is possible for all small positive ν. Here we use the principal branch of $\sqrt{1-w}$, so if $w\in C$ we have $Arg(1-w)\in[-\pi+\phi,\pi-\phi]$.

We proceed by evaluating

$$f_n^{(j)}=\frac{1}{2\pi}\int_{\gamma_j}|f(w)|\frac{|dw|}{|w|^{n+2+x/2+y_n/2}},$$

and we have $|f_n|\le f_n^{(1)}+f_n^{(2)}+f_n^{(3)}+f_n^{(4)}$.

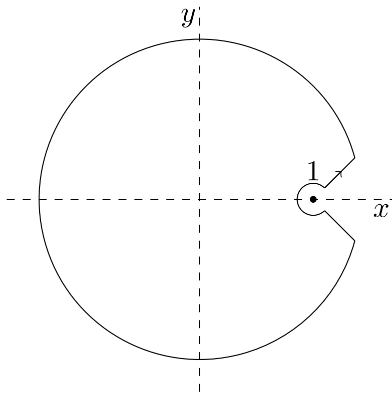

FIGURE 2.1. Integration path.

Smaller circle (γ_1): On the circle $w = 1 + \frac{1}{n}e^{i\theta}$, Condition [a], Condition [b] are satisfied, if n is large enough, so

$$\begin{aligned}\left|\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n} - 1\right| &\leq \frac{ab}{pq-ab}\,|w-1|\left(\frac{1}{1-\Gamma}\right)^{y_n}\\ &= \frac{ab}{pq-ab}\,\frac{1}{n}\left(\frac{1}{1-\Gamma}\right)^{y_n}.\end{aligned}$$

However by hypothesis $y_n = \log(o(n))/\alpha$ which implies

$$\frac{1}{n}\left(\frac{1}{1-\Gamma}\right)^{y_n} = \frac{1}{n}\exp(\alpha y_n) = \frac{1}{n}\exp(\log(o(n))) = o(1).$$

Hence

$$\begin{aligned} f_n^{(1)} &\leq \frac{1}{2\pi}o(1)\sum_{k=0}^{x}\binom{x}{k}\left(\frac{1}{n}\right)^{(k+1)/2}\left(1-\frac{1}{n}\right)^{-(n+2+y_n/2+x)}\frac{2\pi}{n}\\ &= o(n^{-3/2}).\end{aligned}$$

Rectilinear part (γ_2 and γ_4): Set $u = e^{i\phi}$ and perform the change of variable $w = 1 + (ut/n)$, with $1 \leq t \leq n(pq/ab - 1)$. On the Rectilinear part Condition [a], Condition [b] of Lemma 2.2 are satisfied, so

$$\begin{aligned}\left|\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n} - 1\right| &\leq \frac{ab}{pq-ab}|w-1|\left(\frac{1}{1-\Gamma}\right)^{y_n}\\ &\leq \frac{ab}{pq-ab}\,\frac{t}{n}\left(\frac{1}{1-\Gamma}\right)^{y_n}.\end{aligned}$$

From the above fact, we have

$$
\begin{aligned}
f_n^{(2)} &\leq \frac{ab}{pq-ab}\left(\frac{1}{1-\Gamma}\right)^{y_n} \\
&\cdot \frac{1}{2\pi}\sum_{k=0}^{x}\binom{x}{k}\int_1^\infty \left(\frac{t}{n}\right)^{(k+3)/2}\left(1+\frac{t}{n}\right)^{-(n+2+y_n/2+x)}\frac{dt}{n} \\
&\sim \frac{ab}{pq-ab}\left(\frac{1}{1-\Gamma}\right)^{y_n}\frac{1}{2\pi}\int_1^\infty t^{3/2}\left(1+\frac{t}{n}\right)^{-(n+2+y_n/2+x)} dt\,\left(\frac{1}{n}\right)^{5/2} \\
&\sim \frac{ab}{2\pi(pq-ab)}\int_1^\infty t^{3/2}e^{-t}dt\,\left(\frac{1}{1-\Gamma}\right)^{y_n}\left(\frac{1}{n}\right)^{(5/2)}.
\end{aligned}
$$

However $\left(\frac{1}{1-\Gamma}\right)^{y_n} = o(n)$ so $f_n^{(2)} = o(n^{-3/2})$.

Larger circle (γ_3): On the large circle $w = (1+\nu)e^{i\theta}$ so Condition [a] holds. Recall that $r = (1+\nu)ab/pq < 1$.

$$
\begin{aligned}
\left|\frac{1-\sqrt{1-(1+\nu)\frac{ab}{pq}e^{i\theta}}}{1-\sqrt{1-\frac{ab}{pq}}}\right| &= \frac{|1-(1-\sum_{n=1}^\infty \frac{(2n-3)!}{2^{2n-2}n!(n-2)!}r^n e^{i\theta n})|}{1-\sqrt{1-\frac{ab}{pq}}} \\
&\leq \frac{\sum_{n=1}^\infty \frac{(2n-3)!}{2^{2n-2}n!(n-2)!}r^n}{1-\sqrt{1-\frac{ab}{pq}}} = \frac{1-\sqrt{1-r}}{1-\sqrt{1-\frac{ab}{pq}}} \\
&\leq \frac{1-(1-r)}{1-(1-\frac{ab}{2pq})} = 2(1+\nu).
\end{aligned}
$$

Therefore on the big circle

$$
\begin{aligned}
f_n^{(3)} &\leq (1+\nu)\big((2(1+\nu))^{y_n}+1\big)(3+\nu)^x(1+\nu)^{-(n+2+x/2+y_n/2)} \\
&= O\left(\exp\left(y_n\log(2(1+\nu)^{1/2}) - n\log(1+\nu)\right)\right)
\end{aligned}
$$

and this decays exponentially fast since $y_n/n = o(1)$.

To check (2.8) it suffices to show

$$
\begin{aligned}
&[w^{n+1+x/2+y_n/2}]\left(\left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}\left(\frac{\Gamma(w)}{\Gamma}\right)-1\right)\cdot\mathcal{D}(w)\right) \\
= \;& o(n^{-3/2}),
\end{aligned}
$$

where $\mathcal{D}(w) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}(1-w)^{k/2}$.

We do the $k=0$ term above first since this term has no singularity at 1 so

$$
f(w) = \left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}\left(\frac{\Gamma(w)}{\Gamma}\right)-1\right)
$$

is analytic on the entire circle $C = \{z = (1+\nu)e^{i\theta}\}$ where $0 \le \theta < 2\pi$. Moreover Condition [a] holds on C. From Step 3 above

$$\begin{aligned}
&\left|[w^{n+1+x/2+y_n/2}]\left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}\left(\frac{\Gamma(w)}{\Gamma}\right)-1\right)\right| \\
&= \frac{1}{2\pi}\left|\int_C f(w)\frac{dw}{w^{n+2+x/2+y_n/2}}\right| \\
&\le \frac{1}{2\pi}\left(\frac{2}{\Gamma}(2(1+\nu))^{y_n}+1\right)\frac{1}{(1+\nu)^{n+2+x/2+y_n/2}}2\pi(1+\nu)
\end{aligned}$$

which converges exponentially fast to zero since $y_n/n = o(1)$.

For the remaining terms we again follow the steps in Flajolet and Odlyzko (1990). First note

$$\begin{aligned}
&\left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}\left(\frac{\Gamma(w)}{\Gamma}\right)-1\right) \\
&= \left(\left(\frac{1-\Gamma(w)}{1-\Gamma}\right)^{y_n}-1\right)\frac{1}{\Gamma}-\left(\left(\frac{\Gamma(w)}{1-\Gamma}\right)^{y_n+1}-1\right)\frac{1-\Gamma}{\Gamma}.
\end{aligned}$$

In this form we can use all the estimates developed in the first part.

Finally, by (2.4) and (2.9), we get

$$\begin{aligned}
\left(K^{2n}(x,-y_n)\right)^{1/(2n)} &= \left((4pq)^{n+1+x/2+y_n/2}\mathcal{A}_n\right)^{1/(2n)} \\
&\sim 2\sqrt{pq}. \qquad (2.10)
\end{aligned}$$

□

Note that Condition [3] is satisfied in this example because of (2.10) and since $h_{-\infty}(-y_n) = (t_0^{y_n} + t_1^{y_n})/2$ so

$$(h_{-\infty}(-y_n))^{1/(2n)} \sim t_1^{y_n/(2n)} \to 1.$$

By Lemma 1.3, it follows that Condition [4] holds as do the conditions of Proposition 1.5. By Proposition 1.5 then $K^{2n}(x,-y_n)/K^{2n}(0,-y_n)$ has a limiting ρ-harmonic function along a subsequence. However we showed this limit is $h_{+\infty}(x)$ not $h_{-\infty}(x)$ as expected.

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