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Extinction time of logistic branching processes in a Brownian environment

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Abstract. In this paper, we study the extinction time of logistic branching processes which are perturbed by an independent random environment driven by a Brownian motion. Our arguments use a Lamperti-type representation which is interesting on its own right and provides a one to one correspondence between the latter family of processes and the family of Feller diffusions which are perturbed by an independent spectrally positive Lévy process. When the independent random perturbation (of the Feller diffusion) is driven by a subordinator then the logistic branching processes in a Brownian environment converges to a specified distribution; otherwise, it becomes extinct a.s. In the latter scenario, and following a similar approach to Lambert (2005), we provide the expectation and the Laplace transform of the absorption time as a functional of the solution to a Ricatti differential equation. In particular, the latter characterises the law of the process coming down from infinity.

1. Introduction and main results.

The prototypical example of continuous state branching processes (or CB-processes) with competition is the so-called logistic Feller diffusion which is defined as the unique strong solution of the following stochastic differential equation (SDE),

$$Y_t = Y_0 + b\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s - c\int_0^t Y_s^2 \mathrm{d}s, \qquad t \geq 0, \tag{1.1}$$

where $b \in \mathbb{R}, c > 0$ and $B = (B_t; t \geq 0)$ is a standard Brownian motion. The logistic Feller diffusion is considered as the random analogue of the so-called logistic growth model, a demographic deterministic model which is well used in ecology. The logistic growth model is an elementary

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combination of geometric growth, such as the Malthusian growth model, for small population sizes and a quadratic density-dependent regulatory mechanism. As we can see from (1.1), the logistic Feller diffusion has such a geometric growth, given by the parameter b and the quadratic regulatory term, when the parameter c is strictly positive. The quadratic regulatory term has a deep ecological meaning as it describes competition between pairs of individuals in a given population and thus deserves special attention.

On the other hand, the logistic Feller diffusion can also be constructed as scaling limits of Bienaymé-Galton-Watson (BGW) processes with competition where the negative interactions between individuals can be observe more clearly. More precisely, BGW processes with competition are continuous time Markov chains where individuals give birth to a random number of offspring independently from one another, but where competitive pressure is also considered, that is to say, each pair of individuals interacts at a fixed rate and one of them is killed as result of such interaction. Such competition pressure is a size-dependence which complies with the deterministic logistic growth model, since it produces a geometric growth coming from the reproduction law of each individual and a quadratic death rate which represents negative interactions between pairs of individuals. For further details about the convergence of BGW processes with competition towards the logistic Feller diffusion, we refer to Section 2.4 in Lambert (2005).

There is a remarkable random time change representation of the logistic Feller diffusion which is quite important for our purposes. Let us consider the random clock

$$C_t := \int_0^t Y_s \mathrm{d}s, \qquad t \geq 0,$$

and $(\eta_t, t \geq 0)$ its right-continuous inverse. The process $(M_t, t \geq 0)$ defined by

$$M_t := \int_0^t \sqrt{Y_s} \mathrm{d}B_s, \qquad t \geq 0,$$

is a local martingale whose quadratic variation is given by $(C_t, t \geq 0)$ (i.e. η represents the right-continuous inverse of the quadratic variation), thus from the extended Dubins-Schwarz Theorem (see for instance Chapter V in Revuz and Yor, 1999 or Theorem 7.1 in Ikeda and Watanabe, 1989) the time-changed process $M \circ \eta$ is a standard Brownian motion. The previous argument clearly implies that $Y \circ \eta$ solves the SDE

$$\mathrm{d}R_t = \mathrm{d}W_t - cR_t \mathrm{d}t, \tag{1.2}$$

where W is a Brownian motion with drift, i.e. $(2\gamma^2)^{-1/2}(W_t - bt)$, for $t \geq 0$, is a standard Brownian motion. Conversely, let R be the unique strong solution of (1.2), T_0^R its first hitting time of 0 and C the right-continuous inverse of

$$\eta_t = \int_0^{t \wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad t \geq 0, \tag{1.3}$$

then $Y = R \circ C$ is a diffusion killed when it hits 0 and solves (1.1). This type of random time-change is known as Lamperti's representation and the unique strong solution of (1.2) is known as the Ornstein-Uhlenbeck diffusion.

Using the above random time-change argument, Lambert (2005) generalised the logistic Feller diffusion by replacing the Ornstein-Uhlenbeck diffusion by a Ornstein-Uhlenbeck process driven by a Lévy process. Before we explain Lambert's construction, let us first introduce one of the basic objects in our study. Let $X = (X_t, t \geq 0)$ be a spectrally positive Lévy process, i.e. a càdlàg stochastic process with independent and stationary increments with no negative jumps. We denote by $\mathbf{P}_x$ for the law of X started from $x \in \mathbb{R}$ and for simplicity, we let $\mathbf{P} = \mathbf{P}_0$. It is known that the law of any spectrally positive Lévy process X is completely characterised by its Laplace exponent ψ which is defined as $\psi(\lambda) = \log \mathbf{E}[e^{-\lambda X_1}]$, for $\lambda \geq 0$, and satisfies the so-called Lévy-Khintchine formula

$$\psi(\lambda) = -b\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \Big(e^{-\lambda u} - 1 + \lambda u \mathbf{1}_{\{u<1\}}\Big)\mu(\mathrm{d}u), \tag{1.4}$$

where $b \in \mathbb{R}$, $\gamma \geq 0$ and μ is a Radon measure concentrated on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge u^2)\mu(\mathrm{d}u) < \infty. \tag{1.5}$$

It is also known that the triplet (b, γ, μ) characterises the law of X.

In Lambert (2005), the author considered an Ornstein-Uhlenbeck driven by the Lévy process X starting from $x > 0$, i.e. the unique strong solution of

$$\mathrm{d}R_t = \mathrm{d}X_t - cR_t\mathrm{d}t. \tag{1.6}$$

The process R is also known as the generalised Ornstein-Uhlenbeck process and defines a time homogeneous Markov process whose transitions $\{p_t(x,\cdot), t \geq 0\}$ are determined by

$$\int_{\mathbb{R}} e^{izy} p_t(x, \mathrm{d}y) = \exp\left\{ixze^{-ct} + \int_0^t \psi(e^{-cs}z)\mathrm{d}s\right\}, \qquad z \in \mathbb{R},$$

see for instance section 17 in Sato (1999). According to Theorem 17.5 in Sato (1999), the following log-moment condition

$$\mathbf{E}\Big[\log^+ X_1\Big] < \infty,$$

is necessary and sufficient for the generalised Ornstein-Uhlenbeck process R to possess an invariant distribution. From Theorem 25.3 in Sato (1999), the previous log-moment condition is equivalent to

$$\int_1^\infty \log(u)\mu(\mathrm{d}u) < \infty. \tag{1.7}$$

For further details on Lévy and generalised Ornstein-Uhlenbeck processes, we refer to the monograph of Sato (1999).

Let T_0^R denotes the first hitting time of 0 of the generalised Ornstein-Uhlenbeck process R, i.e. $T_0^R := \inf\{s : R_s = 0\}$, and consider the random clock $(\eta_t, \geq 0)$ defined by (1.3) in this setting and its right-continuous inverse C. According to Lambert (2005), the logistic branching process is defined as follows

$$Y_t = \begin{cases} R_{C_t} & \text{if } 0 \leq t < \eta_\infty \\ 0 & \text{if } \eta_\infty < \infty \text{ and } t \geq \eta_\infty. \end{cases} \tag{1.8}$$

When $c = 0$, the process Y is the so-called CB-process and the previous random time change relationship is known as the Lamperti transform which was established by Lamperti (1967). In other words, a CB-process is associated with a spectrally positive Lévy process and in particular with its Laplace exponent ψ which takes the role of the offspring generating function in the compound Poisson case. Formally speaking, we shall refer to all ψ consistent with Definition (1.4) as branching mechanisms.

The logistic Feller diffusions and their extensions have been studied by several authors, see for instance Berestycki et al. (2018), Foucart (2019), Lambert (2005), Ma (2015), Pardoux (2016) and the references therein.

As it was observed by Foucart (2019), the definition of the logistic branching process Y, given by (1.8), is inconsistent with the fact that the process R is positive, drifts to ∞ and $\eta_\infty < \infty$, a.s. The latter may occur when

$$\mathcal{E} := \int_0^\theta \frac{1}{x} \exp\left\{\frac{2}{c}\int_x^\theta \frac{\psi(u)}{u}\mathrm{d}u\right\}\mathrm{d}x < \infty, \qquad \text{for some} \qquad \theta > 0, \tag{1.9}$$

according to Lemma 4 in Foucart (2019). Actually, condition (1.9) is necessary and sufficient for the logistic branching process Y to explode with positive probability. We also point out that the process Y does not explode a.s., if the log-moment condition (1.7) holds since it implies that $\mathcal{E} = \infty$. The latter follows from the fact that

$$\int_{0+} \frac{\psi(z)}{z} \mathrm{d}z < \infty \qquad \text{is equivalent to} \qquad \int^{\infty} \log(u)\mu(\mathrm{d}u) < \infty,$$

see for instance Corollary 3.21 in Li (2011).

In Foucart (2019), the author is interested in studying the long term behaviour of the extension of the logistic branching process Y on $[0, \infty]$ where the state ∞ might be an entrance, reflecting or an exit boundary. In particular, Foucart improved the results of Lambert (2005) for such extension. In this paper, we are not interested in the extension of Y, so that we continue our exposition below in the setting of Lambert (2005).

It is important to note that the logistic branching process Y can also be defined (up to the time of explosion) as the unique strong solution of a SDE which can also be extended to more general competition mechanisms. To be more precise, let us consider a general competition mechanism g which is a non-decreasing continuous function on $[0, \infty)$ with $g(0) = 0$, hence the branching process with competition satisfies the following SDE

$$\begin{aligned} Y_t = Y_0 + b\int_0^t Y_s \mathrm{d}s - \int_0^t g(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} \\ + \int_0^t \int_{(1,\infty)} \int_0^{Y_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{1.10}$$

up to explosion, where $B^{(b)}$ is a standard Brownian motion which is independent of the Poisson random measure $N^{(b)}$ which is defined on $\mathbb{R}^3_+$, with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that μ satisfies (1.5) and $\widetilde{N}^{(b)}$ denotes its compensated version. The first line in (1.10) represents a size-dependent diffusion with branching. The size dependent mechanism is given by the mapping $x \mapsto bx - g(x)$ and when $g(x) = cx^2$ with $c > 0$, the first three terms of (1.10) represents the logistic Feller diffusion. The Poisson integrals in (1.10) have the following interpretation: the role of the second coordinate of the Poisson measure $N^{(b)}$ is to mark jumps in order to have them occur only if this mark is below the path of Y; thus the jumps with size in $(z, z + \mathrm{d}z)$ occur at a rate equals $Y_t\mu(\mathrm{d}z)$, that is the branching process jumps at a rate which is linear in the population size similarly to the discrete space state setting.

The SDE in (1.10) was considered by Ma (2015) (see also Berestycki et al., 2018) in the particular case when ψ satisfies (1.4) with

$$\int_{(0,\infty)} (u \wedge u^2)\mu(\mathrm{d}u) < \infty, \tag{1.11}$$

or equivalently $|\psi'(0+)| < \infty$. Such assumption simplifies the previous SDE by modifying the linear and the jump structure terms as follows

$$Y_t = Y_0 - \psi'(0+)\int_0^t Y_s \mathrm{d}s - \int_0^t g(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Moreover, under condition (1.11) the previous SDE does not explode a.s. The term $\psi'(0+)$ represents the Malthusian parameter of the geometric growth.

Our aim is to study the time to extinction of a generalized version of the logistic branching process which includes an extra randomness coming from an independent Brownian motion which can be interpreted as a random environment. To be more precise, we consider the following SDE

$$\begin{aligned} Z_t = Z_0 + \int_0^t \left(bZ_s - cZ_s^2\right)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \sigma\int_0^t Z_s \mathrm{d}B_s^{(e)} \\ + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{1.12}$$

up to explosion, with b, γ, the Brownian motion $B^{(b)}$ and the Poisson random measure $N^{(b)}$ being as before and where $c, \sigma \geq 0$ and $B^{(e)}$ is a standard Brownian motion independent of $B^{(b)}$ and $N^{(b)}$. The SDE (1.12) has a unique non-negative strong solution which satisfies the Markov property, see for instance Theorem 1 in Palau and Pardo (2018).

When $c = 0$, the family of processes described by (1.12) was introduced independently by He et al. (2018) and by Palau and Pardo (2018) with $B^{(e)}$ replaced by a Lévy process under the name of CB-processes in a Lévy random environment. In this particular case, the process Z satisfies the branching property conditionally on the environment $B^{(e)}$ (quenched branching property). This particular case (i.e. $c = 0$) was studied by in Palau and Pardo (2017) where the probability of survival and non-explosion is explicitly determined when the branching mechanism is stable, i.e. $\psi(\lambda) = c_\alpha \lambda^\alpha$, for $\lambda > 0$, with $\alpha \in (0, 1) \cup (1, 2]$ and $c_\alpha < 0$ or $c_\alpha > 0$ accordingly as $\alpha \in (0, 1)$ or $\alpha \in (1, 2]$. The latter events can be computed in a closed-form in this case, since the Laplace transform of Z is explicit, a property which is derived from the quenched branching property of Z. We point out that in Palau and Pardo (2017) there are not necessary and sufficient conditions for CB-processes in a Brownian random environment to explode or become extinct. Under the finite moment condition (1.11), CB-processes in a Lévy random environment do not explode (see for instance Lemma 7 in Bansaye et al., 2021) and moreover, according to He et al. (2018) Grey's condition, i.e.

$$\int^{\infty} \frac{\mathrm{d}z}{\psi(z)} < \infty, \tag{1.13}$$

is a necessary and sufficient condition for the process to become extinct with positive probability, see Theorem 4.1 in He et al. (2018). In the particular case when the random environment is driven by a Brownian motion with drift, the associated CB-process in random environment becomes extinct at finite time a.s. if the drift term is not positive, see Corollary 4.4 in He et al. (2018).

We also observe that the linear drift case, i.e $\psi(u) = -bu$ for $u \geq 0$, when $c > 0$ corresponds to the monomorphic model of a single population living in a patchy environment which was studied recently in Evans et al. (2015).

Interesting path properties of the process Z are known under different conditions on the parameters $(b, c, \gamma, \mu, \sigma)$. Most of them are due to Lambert (2005) when $\sigma = 0$, i.e. when there is no random environment, and to Evans et al. (2015) when $\gamma = 0$ and $\mu \equiv 0$.

Lambert divided his study in Lambert (2005) into two cases depending on properties of the measure μ. Both cases always assume that the log-moment condition (1.7) holds and $\sigma = 0$. In the first scenario, by also assuming that

$$\int_{(0,\infty)} (1 \wedge z)\mu(\mathrm{d}z) < \infty \qquad \text{and} \qquad \delta := b - \int_{(0,1)} z\mu(\mathrm{d}z) \geq 0,$$

Lambert shows that if either $\delta \neq 0$, $\mu(0, \infty) = \infty$ or $c < \mu(0, \infty) < \infty$, then the process Z is positive recurrent on $(\delta/c, \infty)$ and possesses a stationary distribution which can be computed explicitly. Moreover if none of the latter conditions are satisfied, then the process Z is null recurrent in $(0, \infty)$ and converges to 0 in probability (see Theorem 3.4 in Lambert, 2005). When the Lévy measure μ satisfies

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty,$$

then the process Z goes to 0 a.s. Moreover, the process Z gets extinct in finite time a.s. accordingly as Grey's condition (1.13) is fulfilled. Let T_0^Z denotes the time to extinction of the process Z, i.e $T_0^Z := \inf\{t \geq 0 : Z_t = 0\}$. In Lambert (2005), under Grey's condition, the Laplace transform of T_0^Z was computed explicitly and the law of the process coming down from infinity, i.e. that Z starts at ∞ and immediately after starting it takes finite values, was also determined.

Evans et al. (2015) consider the longterm behaviour of Z when $\gamma = 0$ and $\mu \equiv 0$. In particular, the authors in Evans et al. (2015) found that starting from a strictly positive state then $Z_t > 0$ for all $t \geq 0$, a.s. Moreover

(a) if $2b < \sigma^2$ then the process Z goes to 0 a.s.,

(b) if $2b = \sigma^2$ then

$$\liminf_{t\to\infty} Z_t = 0 \quad \text{and} \quad \limsup_{t\to\infty} Z_t = \infty \qquad \text{almost surely,}$$

(c) if $2b > \sigma^2$, then Z has a unique stationary distribution.

Recently, Leman and Pardo (2021) studied the event of extinction and the property of coming down from infinity of CB-processes with general competition mechanisms in a Lévy environment under the assumption that the branching mechanism satisfies the first moment condition (1.11). In particular in Leman and Pardo (2021) it is proved, under the so-called Grey's condition together with the assumption that the Lévy environment does not drift towards infinity, that for any starting point the process becomes extinct in finite time a.s. Moreover if the condition on the Lévy environment is replaced by an integrability condition on the competition mechanism then the process comes down from infinity.

In this paper, we study the particular case when the competition mechanism is logistic, where more explicit results about the extinction time can be provided. In particular, when the branching mechanism is associated to a subordinator, i.e. when $\psi(u) < 0$, we provide conditions under which 0 is polar, i.e the process never becomes extinct. Moreover, when the process does not become extinct, we provide conditions for the process to be recurrent or transient and give a description of the invariant measure when it exists.

In order to establish our results, we introduce the following notation. Let us denote by $\mathbb{P}_x$, the law of Z starting from $x > 0$, and define the first hitting time to 0 of Z as follows

$$T_0 = \inf\{t \geq 0, Z_t = 0\},$$

with the convention that $\inf\{\emptyset\} = \infty$. Hence, 0 is polar for Z if and only if $\mathbb{P}_x(T_0 < \infty) = 0$ for all $x > 0$. We adopt the following definition of recurrence and transience (see for instance Chapter X of Revuz and Yor (1999) or Definition 1 in Duhalde et al. (2014))

Definition 1.1. Assume that 0 is polar, the process Z is said to be recurrent if there exists $x > 0$ such that

$$\mathbb{P}_x\left(\liminf_{t\to\infty} |Z_t - x| = 0\right) = 1.$$

On the other hand, the process is said to be transient if

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = \infty\right) = 1, \quad \text{for every} \quad x > 0.$$

Observe that if the property of recurrence is satisfied for a particular $x > 0$, it is also true for all $x > 0$. We also point out that in Definition 1 of Duhalde et al. (2014), the authors did not assume the polarity of 0, since they studied a process with positive immigration. In that case, contrary to ours, the process may grow again after extinction and thus it is either recurrent for all $x \geq 0$, or transient.

For clarity of exposition, we split our results in two cases depending on the form of the branching mechanism ψ, the subordinator case and what we call the general case which is nothing but the cases where the branching mechanism is associated with a subordinator with negative drift or with an unbounded variation spectrally positive Lévy process. Both cases use different techniques also. Indeed in the subordinator case we use the Lamperti-type representation since the law of the underlying process is known and implicity many path properties can be established. Unfortunately, this technique cannot be applied in the general case since the law of the underlying process seems to be not so easy to be determined. Instead, we use a similar approach as in Lambert (2005) where the knowledge of the infinitesimal generator is relevant.

1.1. *Subordinator case.* Let us assume that the branching mechanism is associated to the Laplace transform of a subordinator, that is to say

$$\psi(z) = -\delta z - \int_{(0,\infty)} (1 - e^{-zu})\mu(\mathrm{d}u), \tag{1.14}$$

where

$$\int_{(0,\infty)} (1 \wedge u)\mu(\mathrm{d}u) < \infty \qquad \text{and} \qquad \delta := b - \int_{(0,1)} u\mu(\mathrm{d}u) \geq 0.$$

We also introduce, the function

$$\omega(x) = cx + \frac{\sigma^2 x^2}{2},$$

with $\sigma > 0$ and $c \geq 0$. Notice that we are also considering the case without competition (i.e. $c = 0$) and implicitly we will obtain (up to our knowledge) some unknown path properties for CB-processes in a Brownian environment with branching mechanism given by (1.14).

Our first result provides a necessary and sufficient condition under which the process Z is conservative, i.e. that Z does not explode at finite time a.s.

Theorem 1.2. *Assume that $\sigma > 0$ and $c \geq 0$. The process Z, the unique strong solution of* (1.12) *with branching mechanism given by* (1.14)*, is conservative if and only if*

$$\mathcal{I} := \int_0^1 \frac{1}{\omega(z)} \exp\left\{\int_z^1 \frac{\psi(u)}{\omega(u)}\mathrm{d}u\right\} \mathrm{d}z = \infty.$$

Moreover, if $\sigma^2 > 2\delta$, then the process Z converges to 0 with positive probability, i.e

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = 0\right) > 0, \qquad \textit{for} \quad x > 0.$$

In particular, if we also assume that $\mathcal{I} = \infty$, then the process converges to 0 a.s.

For instance, when the branching mechanism is such that $\psi(z) = -c_\alpha z^\alpha$, for $z \geq 0$, with $\alpha \in (0, 1)$ and $c_\alpha > 0$, that is to say the negative of a stable subordinator, straightforward computations lead to $\mathcal{I}$ is finite or infinite accordingly as $c = 0$ or $c > 0$. In other words, if there is presence of competition the associated process Z is conservative and moreover the process becomes extinct a.s., since σ^2 is always positive. If there is no competition, the process Z explodes with positive probability. The latter case was studied in Palau and Pardo (2017) where the rate of explosion was determined explicitly.

In this setting, we also have the following identity for the total population size of the process Z up to time $T_a = \inf\{t \geq 0 : Z_t \leq a\}$, the first hitting time of Z at a. Let us define

$$f_\lambda(x) := \int_0^\infty \frac{\mathrm{d}z}{\omega(z)} \exp\left\{-xz + \int_\ell^z \frac{\lambda - \psi(u)}{\omega(u)}\mathrm{d}u\right\}, \qquad x \geq 0,$$

where ℓ is an arbitrary constant larger than 0.

Proposition 1.3. *Assume that $\sigma > 0$ and $c \geq 0$. For every $\lambda > 0$ and $x \geq a \geq 0$, we have*

$$\mathbb{E}_x\left[\exp\left\{-\lambda \int_0^{T_a} Z_s \mathrm{d}s\right\}\right] = \frac{f_\lambda(x)}{f_\lambda(a)}. \tag{1.15}$$

Similarly to the case when the environment is fixed (i.e. $\sigma^2 = 0$), treated by Lambert (2005), we observe that when $c > 0$, the process Z may have an invariant distribution which can be described explicitly. In order to do so, we introduce the following notation. Let

$$m(\lambda) := \int_0^\lambda \frac{\psi(u)}{\omega(u)}\mathrm{d}u, \qquad \text{for} \quad \lambda \geq 0, \tag{1.16}$$

which is well defined under the log-moment condition (1.7) and $c > 0$ (see for instance Corollary 3.21 in Li, 2011).

The next Lemma is necessary for the description of the invariant distribution of Z, whenever it exists.

Lemma 1.4. *Assume that $\sigma^2, c > 0$ and that the branching mechanism ψ, given by (1.14), satisfies the log-moment condition (1.7). Then the following identity holds*

$$-m(\lambda) = \frac{2}{\sigma^2}\int_0^\infty \left(1-e^{-\lambda z}\right)\frac{e^{-\frac{2c}{\sigma^2}z}}{z}\left(\delta + \int_0^z e^{\frac{2c}{\sigma^2}u}\bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}z, \tag{1.17}$$

where $\bar{\mu}(x) = \mu(x,\infty)$, and

$$\int_{(0,\infty)} e^{-\lambda z}\nu(\mathrm{d}z) = e^{m(\lambda)}, \qquad \lambda \geq 0,$$

defines a unique probability measure ν on $(0,\infty)$ which is infinitely divisible. In addition, it is self-decomposable whenever $\bar{\mu}(0) \leq \delta$.

We recall that self-decomposable distributions on $(0,\infty)$ is a subclass of infinitely divisible distributions whose Lévy measures have densities which are decreasing on $(0,\infty)$. We refer to Sato (1999) for further details on self-decomposable distributions.

In order to introduce the limiting distribution associated to Z, whenever it exists, we first provide conditions under which $\int_{(0,\infty)} s^{-1}\nu(\mathrm{d}s)$ is finite. For any z sufficiently small, we define two sequences of functions as follows

$$l^{(1)}(z) = |\ln(z)| \qquad \text{and} \qquad l^{(k)}(z) = \ln(l^{(k-1)}(z)), \qquad k \in \mathbb{Z}_+, k \geq 2,$$
$$I^{(1)}(z) = l^{(1)}(z)\int_0^z \bar{\mu}(w)\mathrm{d}w \quad \text{and} \quad I^{(k)}(z) = l^{(k)}(z)\left(I^{(k-1)}(z) - \frac{\sigma^2}{2}\right), \quad k \in \mathbb{Z}_+, k \geq 2.$$

Observe that for any $k \in \mathbb{Z}_+$, $I^{(k)}(z)$ is well defined for z sufficiently small. On the other hand $l^{(k)}(z)$ is well defined for both, z sufficiently small and large. Then, for any continuous function f taking values in $\mathbb{R}$, we set

$$\mathtt{Adh}(f) = \left[\liminf_{z\to 0} f(z), \limsup_{z\to 0} f(z)\right] \subset \mathbb{R}.$$

We are now ready to establish the following two conditions which will give the behaviour of Z under the particular setting when $2\delta = \sigma^2$:

(∂) There exists $n \in \mathbb{Z}_+$ s.t. $\inf(\mathtt{Adh}(I^{(n)})) > \dfrac{\sigma^2}{2}$ and $\mathtt{Adh}(I^{(k)}) = \left\{\dfrac{\sigma^2}{2}\right\}$, $\forall k \in \{1,..,n-1\}$,

($\eth$) There exists $n \in \mathbb{Z}_+$ s.t. $\sup(\mathtt{Adh}(I^{(n)})) < \dfrac{\sigma^2}{2}$ and $\mathtt{Adh}(I^{(k)}) = \left\{\dfrac{\sigma^2}{2}\right\}$, $\forall k \in \{1,..,n-1\}$.

For instance if $\bar{\mu}(0) < \infty$ (i.e. ψ is the Laplace exponent of a compound Poisson process) condition ($\eth$) holds. These two conditions are exclusive conditions under which the process is either positive recurrent or null recurrent, that is to say the process is recurrent and either it has an invariant probability measure or not.

Theorem 1.5. *Assume that $2\delta \geq \sigma^2 > 0$, $c > 0$. Then the point 0 is polar, i.e. $\mathbb{P}_x(T_0 < \infty) = 0$ for all $x > 0$.*

Moreover if

$$\int_0^1 \frac{\mathrm{d}z}{z}\exp\left\{-\int_z^1\int_0^\infty \frac{(1-e^{-us})}{\omega(u)}\mu(\mathrm{d}s)\mathrm{d}u\right\} = \infty \tag{1.18}$$

Z is recurrent. Additionally,

a) *if $2\delta > \sigma^2$ then the process Z is positive recurrent. Its invariant distribution ρ has a finite expected value if and only if* (1.7) *holds. If the latter holds, then ρ is the size-biased distribution of ν, in other words*

$$\rho(\mathrm{d}z) = \left(\int_{(0,\infty)} s^{-1}\nu(\mathrm{d}s)\right)^{-1} z^{-1}\nu(\mathrm{d}z), \qquad z > 0, \tag{1.19}$$

b) *if $2\delta = \sigma^2$ and* (1.7) *holds, together*
   - b.1) *with condition (∂), then Z is positive recurrent and its invariant probability is defined by* (1.19)*,*
   - b.2) *or with condition $(\eth)$, then the process Z is null recurrent and converges to 0 in probability.*

Finally, if (1.18) *is not satisfied, then Z explode at finite time a.s.*

It is important to note that (1.18) is satisfied as soon as (1.7) holds. We also point out that the previous results are consistent with the behaviours found in Proposition 2.1 in Evans et al. (2015) where $\psi(z) = -bz$.

1.2. *General case.* Finally, we consider the case when the process X is not a subordinator, in other words the branching mechanism ψ satisfies that there exist $\vartheta \geq 0$ such that $\psi(z) > 0$ for any $z \geq \vartheta$. For simplicity, we say that the branching mechanism ψ is general if it satisfies the previous assumptions.

In the sequel we assume that $c > 0$ and that the Lévy measure associated to the general branching mechanism ψ satisfies the log-moment condition (1.7). Our main result in this section provides a complete characterization of the Laplace transform of the stopping times

$$T_a = \inf\{t \geq 0 : Z_t \leq a\}, \quad \text{for} \quad a \geq 0,$$

as long as T_0 is finite a.s. To this aim, we introduce the functional

$$\mathtt{I}(\lambda) := \int_0^\lambda e^{m(u)}\mathrm{d}u, \qquad \text{for} \quad \lambda \geq 0, \tag{1.20}$$

where m is defined by (1.16) and well posed under the log-moment (1.7). Observe from our assumptions that m is increasing on (ϑ, ∞) implying that $\mathtt{I}(\cdot)$ is a bijection from $\mathbb{R}_+$ into itself. We denote its inverse by φ and a simple computation provides

$$\varphi'(z) = \exp(-m \circ \varphi(z)). \tag{1.21}$$

The formulation of the Laplace transform of T_a will be written in terms of the solution to a Ricatti equation. Similarly to Lemma 2.1 in Lambert (2005), we deduce the following Lemma on the Ricatti equation of our interest.

Lemma 1.6. *For any $\lambda > 0$, there exists a unique non-negative solution y_λ to the equation*

$$y' = y^2 - \lambda r^2, \tag{1.22}$$

where $r(z) = \frac{\varphi'(z)}{\sqrt{\omega(\varphi(z))}}$ such that it vanishes at ∞. Moreover, y_λ is positive on $(0,\infty)$, and for any z sufficiently small or large, $y_\lambda(z) \leq \sqrt{\lambda}r(z)$. As a consequence, y_λ is integrable at 0, and it decreases initially and ultimately.

We now state our last result. Recall that the infinitesimal generator $\mathcal{U}$ of the process Z satisfies that for any $f \in C^2$,

$$\begin{aligned}\mathcal{U}f(x) = (bx - cx^2)f'(x) + \left(\gamma^2 x + \frac{\sigma^2}{2}x^2\right) f''(x) \\ + x\int_{(0,\infty)} \left(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\right)\mu(\mathrm{d}z),\end{aligned} \tag{1.23}$$

see for instance Theorem 1 in Palau and Pardo (2018).

Theorem 1.7. *Let $c > 0$ and assume that the branching mechanism ψ is general and its associated Lévy measure satisfies the log-moment condition (1.7). Hence the function*

$$h_\lambda(x) := 1 + \lambda \int_0^\infty \frac{e^{-xz-m(z)}}{\omega(z)} e^{-\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v} \int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v} \mathrm{d}u\mathrm{d}z \tag{1.24}$$

is well defined and positive for any $x > 0$ and $\lambda > 0$ and it is a non-increasing C^2-function on $(0,\infty)$. Moreover it solves

$$\mathcal{U}h_\lambda(x) = \lambda h_\lambda(x), \text{ for any } x > 0. \tag{1.25}$$

Furthermore, if $\mathbb{P}_x(T_0 < \infty) = 1$, for any $x > 0$ then h_λ is also well-defined at 0 with

$$h_\lambda(0) = \exp\left\{\int_0^\infty y_\lambda(v)dv\right\} < \infty,$$

and, for any $x \geq a \geq 0$,

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right] = \frac{h_\lambda(x)}{h_\lambda(a)}. \tag{1.26}$$

In particular, for any $x > 0$,

$$\mathbb{E}_x[T_0] = \int_0^\infty \mathrm{d}u\, e^{m(u)} \int_u^\infty \frac{e^{-m(z)}}{\omega(z)}(1 - e^{-zx})\mathrm{d}z. \tag{1.27}$$

It is important to note that under the assumptions of Theorem 1.2 in Leman and Pardo (2021), the previous result can be applied. To be more precise, according to Theorem 1.2 in Leman and Pardo (2021) if ψ satisfies Grey's condition (1.13) together with (1.11) (i.e. $|\psi'(0+)| < \infty$), then $\mathbb{E}_x[T_0] < \infty$, for any starting point $x \geq 0$. In other words, under these assumptions, the results of Theorem 1.7 apply and moreover the logistic branching process in a Brownian environment Z is Feller and comes down from infinity as it is stated in the following Corollary. Formally, we define the property of *coming down from infinity* in the sense that ∞ is a *continuous entrance point*, i.e.

$$\lim_{a\to\infty} \lim_{x\to\infty} \mathbb{P}_x(T_a < t) = 1 \qquad \text{for all} \quad t > 0,$$

and the original process can be extended into a Feller process on $[0,\infty]$ (see for instance Theorem 20.13 in Kallenberg, 1997 for the diffusion case or Definition 2.2 for Feller processes in Döring and Kyprianou, 2020).

Corollary 1.8. *Assume that $c > 0$ and that the branching mechanism ψ is general and satisfies (1.11) and Grey's condition (1.13). Then the logistic branching process in a Brownian environment Z is Feller and the boundary point ∞ is a continuous entrance point. Moreover, the process Z can be extended into a Feller process on $[0,\infty]$ and, in particular, we have*

$$\mathbb{E}_\infty\left[e^{-\lambda T_a}\right] = \frac{1}{h_\lambda(a)} \qquad \textit{and} \qquad \mathbb{E}_\infty[T_0] = \int_0^\infty \mathrm{d}u\, e^{m(u)} \int_u^\infty \frac{e^{-m(z)}}{\omega(z)}\mathrm{d}z.$$

We believe that T_0 is finite a.s., under much weaker conditions (including the case $-\psi'(0+) = \infty$) than those stated in Corollary 1.8 but in order to deduce such result the knowledge of the underlying process in the Lamperti-type representation is necessary. Under such weaker conditions we can also expect that the process Z must be Feller which can be extended to $[0, \infty]$.

The remainder of this paper is organised as follows. In Section 2, we deal with a Lamperti-type representation which is established for more general competition mechanisms than the logistic case. Such random time change representation is very useful for the proofs of the subordinator case which are presented in Section 3. Section 4 is devoted to the proof of the results presented for the general case which uses the solution of Ricatti differential equation that appears in Lemma 1.6. Finally, in Section 5 we discuss the case when the competition mechanism is more general and the process possesses continuous paths. We call this case branching diffusions with interactions in a Brownian random environment, since the competition mechanism may take negative and positive values. We study this case separately since the techniques we use here are based on the theory of scale functions for diffusions. This allow us to provide a necessary and sufficient condition for extinction and moreover, the Laplace transform of hitting times is computed explicitly in terms of a Ricatti equation. Such results seems complicated to obtain with the presence of jumps coming from the branching mechanism.

2. Lamperti-type transform for CB-processes with competition in a Brownian environment.

Let g be a continuous function on $[0, \infty)$ with $g(0) = 0$ and consider the following SDE

$$\begin{aligned} Z_t = Z_0 + b\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \sigma\int_0^t Z_s \mathrm{d}B_s^{(e)} \\ + \int_0^t\int_{[1,\infty)}\int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t\int_{(0,1)}\int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{2.1}$$

with $\sigma \geq 0$. It is important to note that Proposition 1 in Palau and Pardo (2018) guarantees that the above SDE has a unique strong positive solution up to explosion and by convention here it is identically equal to $+\infty$ after the explosion time.

The main result in this section is the Lamperti-type representation of a CB-process with competition in a Brownian environment. Such random time change representation will be very useful to study path properties of the logistic case. In order to state the Lamperti-type representation, we introduce the family of processes which are involved in the time change.

Let $X = (X_t, t \geq 0)$ be a spectrally positive Lévy process with characteristics $(-b, \gamma, \mu)$ and such that its Lévy measure μ satisfies (1.5). We also consider $W = (W_t, t \geq 0)$ a standard Brownian motion independent of X and assume that g is a continuous function on $[0, \infty)$ with $g(0) = 0$ and such that $\lim_{x\to 0} x^{-1}g(x)$ exists. According to Proposition 1 in Palau and Pardo (2018) for each $x > 0$, there is a unique strong solution to

$$\mathrm{d}R_t = \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\mathrm{d}X_t - \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\frac{g(R_t)}{R_t}\mathrm{d}t + \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\sigma\sqrt{R_t}\mathrm{d}W_t, \tag{2.2}$$

with $R_0 = x$. The assumption that $\lim_{x\to 0} x^{-1}g(x)$ exists, is not necessary but it implies that we can use directly Proposition 1 of Palau and Pardo (2018). We can relax this assumption but further explanations are needed. Indeed a similar approach to Theorems 2.1 and 2.3 in Ma (2015) will guarantee that the SDE defined above for a more general competition mechanism g has a unique strong solution.

It is important to note that in the logistic-case i.e. $g(x) = cx^2$, for $x \geq 0$ and some constant $c > 0$, the process R is a Feller diffusion which is perturbed by the Lévy process X. Moreover if the Lévy process X is a subordinator, then the process R turns out to be a CB-process with immigration.

We now state the Lamperti-type representation of CB-processes with competition in a Brownian environment.

Theorem 2.1. *Let $R = (R_t, t \geq 0)$ be the unique strong solution of (2.2) and $T_0^R = \sup\{s : R_s = 0\}$. We also let C be the right-continuous inverse of η, where*

$$\eta_t = \int_0^{t\wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad t > 0,$$

that is, $C_t := \inf\{s \geq 0, \eta_s > t\}$, for any $t \in [0, +\infty)$. Hence the process defined by

$$Z_t = \begin{cases} R_{C_t}, & \text{if } 0 \leq t < \eta_\infty \\ 0, & \text{if } \eta_\infty < \infty, T_0^R < \infty \text{ and } t \geq \eta_\infty, \\ +\infty, & \text{if } \eta_\infty < \infty, T_0^R = \infty \text{ and } t \geq \eta_\infty, \end{cases}$$

satisfies the SDE (2.1).

Reciprocally, let Z be the unique strong solution to (2.1) with $Z_0 = x$ and let

$$C_t = \int_0^t Z_s \mathrm{d}s, \qquad t > 0.$$

If η denotes the right-continuous inverse of C, then the process defined by

$$R_t = Z_{\eta_t \wedge T_0} \qquad \text{for} \quad t \geq 0.$$

satisfies the SDE (2.2).

Proof of Theorem 2.1: Since X is a spectrally positive Lévy process and $R_{t-} = 0$ implies $R_t = 0$, we get $R_{t-} > 0$ if and only if $t \in [0, T_0^R)$. We also observe that X can be written as follows

$$X_t = bt + \sqrt{2}\gamma B_t + \int_0^t \int_{(0,1)} z\widetilde{M}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{[1,\infty)} zM(\mathrm{d}s, \mathrm{d}z),$$

where B is a standard Brownian motion and M is a Poisson random measure with intensity $\mathrm{d}s\mu(\mathrm{d}z)$ and $\widetilde{M}$ denotes its compensated version. Then from the latter identity and (2.2), we have

$$\begin{aligned} Z_t = x + b\int_0^{C_t\wedge T_0^R} \mathrm{d}s - \int_0^{C_t\wedge T_0^R} \frac{g(R_s)}{R_s}\mathrm{d}s + \sqrt{2}\gamma \int_0^{C_t\wedge T_0^R} \mathrm{d}B_s + \int_0^{C_t\wedge T_0^R} \sigma\sqrt{R_s}\mathrm{d}W_s \\ + \int_0^{C_t\wedge T_0^R} \int_{(0,1)} z\mathbf{1}_{\{R_{s-}>0\}}\widetilde{M}(\mathrm{d}s, \mathrm{d}z) + \int_0^{C_t\wedge T_0^R} \int_{[1,\infty)} z\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0. \end{aligned}$$

On the one hand, by straightforward computations we deduce

$$C_t \wedge T_0^R = \int_0^t Z_s \mathrm{d}s,$$

implying that

$$\int_0^{C_t\wedge T_0^R} \frac{g(R_s)}{R_s}\mathrm{d}s = \int_0^t g(Z_s)\mathrm{d}s,$$

and

$$L_t^{(1)} = \sqrt{2}\gamma \int_0^{C_t\wedge T_0^R} \mathrm{d}B_s \qquad \text{and} \qquad L_t^{(2)} = \sigma \int_0^{C_t\wedge T_0^R} \sqrt{R_s}\mathrm{d}W_s,$$

are independent continuous local martingales with increasing processes

$$\langle L^{(1)}\rangle_t = 2\gamma^2 \int_0^t Z_s \mathrm{d}s \qquad \text{and} \qquad \langle L^{(2)}\rangle_t = \sigma^2 \int_0^t Z_s^2 \mathrm{d}s.$$

On the other hand, we define the random measure $N(\mathrm{d}s,\mathrm{d}z)$ on $(0,\infty)^2$ as follows

$$N((0,t]\times\Lambda)=\int_0^{C_t\wedge T_0^R}\int_{(0,\infty)}\mathbf{1}_\Lambda(z)\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s,\mathrm{d}z).$$

Then $N(\mathrm{d}s,\mathrm{d}z)$ has predictable compensator

$$Z_{s-}\mathrm{d}s\mu(\mathrm{d}z).$$

By Theorems 7.1 and 7.4 in Ikeda and Watanabe (1989), on an extension of the original probability space there exist two independent Brownian motions, $B^{(1)}$ and $B^{(2)}$, and a Poisson random measure $N(\mathrm{d}s,\mathrm{d}u,\mathrm{d}z)$ on $(0,\infty)^3$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that for any $t\geq 0$,

$$\int_0^{C_t\wedge T_0^R}\int_{[1,\infty)}z\mathbf{1}_{\{R_{s-}>0\}}M(\mathrm{d}s,\mathrm{d}z)=\int_0^t\int_{[1,\infty)}\int_0^{Z_{s-}}zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

$$\int_0^{C_t\wedge T_0^R}\int_{(0,1)}z\mathbf{1}_{\{R_{s-}>0\}}\widetilde{M}(\mathrm{d}s,\mathrm{d}z)=\int_0^t\int_{(0,1)}\int_0^{Z_{s-}}z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

$$L_t^{(1)}=\int_0^t\sqrt{2\gamma^2Z_s}\mathrm{d}B_s^{(1)}\qquad\text{and}\qquad L_t^{(2)}=\sigma\int_0^tZ_s\mathrm{d}B_s^{(2)}.$$

Putting all pieces together, we deduce that $(Z_t,t\geq 0)$ is a solution of (2.1) up to explosion.

For the reciprocal, we first observe that since Z has no negative jumps and $Z_{t-}=0$ implies $Z_t=0$, we get $Z_{t-}>0$ if and only if $Z_t>0$ for $t\in[0,T_0)$. Thus $R_{t-}>0$ if and only if $R_t>0$ for $t\in[0,C_{T_0})$, then for any $t\in[0,C_{T_0})$, the equation (2.2) is equivalent to

$$R_t=\mathrm{d}X_t-\frac{g(R_t)}{R_t}\mathrm{d}t+\sigma\sqrt{R_t}\mathrm{d}W_t.\tag{2.3}$$

Since the process Z satisfies the SDE (2.1) and $R_t=Z_{\eta_t\wedge T_0}$, we have

$$\begin{aligned}R_t=&Z_0+b\int_0^{\eta_t\wedge T_0}Z_s\mathrm{d}s+\int_0^{\eta_t\wedge T_0}\sqrt{2\gamma^2Z_s}\mathrm{d}B_s+\sigma\int_0^{\eta_t\wedge T_0}Z_s\mathrm{d}B_s^{(e)}-\int_0^{\eta_t\wedge T_0}g(Z_s)\mathrm{d}s\\&+\int_0^{\eta_t\wedge T_0}\int_{[1,\infty)}\int_0^{Z_{s-}}zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)+\int_0^{\eta_t\wedge T_0}\int_{(0,1)}\int_0^{Z_{s-}}z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).\end{aligned}\tag{2.4}$$

On the one hand, by straightforward computations we deduce

$$\int_0^{\eta_t\wedge T_0}Z_s\mathrm{d}s=t\wedge C_{T_0},\qquad\text{and}\qquad\int_0^{\eta_t\wedge T_0}g(Z_s)\mathrm{d}s=\int_0^{t\wedge C_{T_0}}\frac{g(R_s)}{R_s}\mathrm{d}s.$$

The latter identities imply

$$M_t^{(1)}=\int_0^{\eta_t\wedge T_0}\sqrt{2\gamma^2Z_s}\mathrm{d}B_s\qquad\text{and}\qquad M_t^{(2)}=\sigma\int_0^{\eta_t\wedge T_0}Z_s\mathrm{d}B_s^{(e)},$$

are independent continuous local martingales with increasing processes

$$\langle M^{(1)}\rangle_t=2\gamma^2\int_0^{\eta_t\wedge T_0}Z_s\mathrm{d}s=2\gamma^2(t\wedge C_{T_0})\quad\text{and}\quad\langle M^{(2)}\rangle_t=\sigma^2\int_0^{\eta_t\wedge T_0}Z_s^2\mathrm{d}s=\sigma^2\int_0^{t\wedge C_{T_0}}R_s\mathrm{d}s.$$

By Theorems 7.1 and 7.4 in Ikeda and Watanabe (1989), on an extension of the original probability space there exist two independent Brownian motions, $B^{(1)}$ and $B^{(2)}$, such that for any $t\geq 0$,

$$M_t^{(1)}=B_{t\wedge C_{T_0}}^{(1)}\qquad\text{and}\qquad M_t^{(2)}=\sigma\int_0^{t\wedge C_{T_0}}\sqrt{R_s}\mathrm{d}B_s^{(2)}.\tag{2.5}$$

On the other hand, we define the random measure $M(\mathrm{d}s, \mathrm{d}z)$ on $(0,\infty)^2$ as follows

$$\begin{aligned} M((0,t]\times\Lambda) = &\int_0^{\eta_t\wedge T_0}\int_{(0,\infty)}\int_0^{Z_{s-}} \mathbf{1}_\Lambda(z) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_{C_{T_0}}^t \int_{(0,\infty)}\int_0^1 \mathbf{1}_\Lambda(z)\mathbf{1}_{\{t>C_{T_0}\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{2.6}$$

Then $M(\mathrm{d}s, \mathrm{d}z)$ has predictable compensator $\mathrm{d}s\mu(\mathrm{d}z)$. Hence, $M(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure on $(0,\infty)^2$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)$. Putting all the pieces together, we deduce that (2.3) holds for $t \in [0, C_{T_0})$. Recall that $Z_{T_0-} = Z_{T_0} = 0$. Then on $\{C_{T_0} < \infty\}$ by using (2.4)-(2.5), we deduce that the right hand side of (2.3) is equal to 0 for $t = C_{T_0}$ and then for all $t \geq C_{T_0}$. □

3. Proofs of the subordinator case

In this part, we provide the proofs of Theorems 1.2 and 1.5. Their proof relies on the Lamperti-type representation in the discussed in the previous section. Unfortunatelly, the same techniques cannot be used in the general case since a deep understanding of the process R is required such as its marginal laws and path properties as recurrence and transience which seems not so clear to deduce.

In the particular case when the spectrally positive Lévy process X is a subordinator in the Lamperti-type representation in Theorem 2.1, the process R turns out to be a Feller diffusion with immigration. In other words, it is the unique positive strong solution of the following SDE up to the first hitting time of 0:

$$R_t = R_0 + X_t - c\int_0^t R_s \mathrm{d}s + \int_0^t \sqrt{\sigma^2 R_s}\mathrm{d}W_s. \tag{3.1}$$

The branching mechanism ω and the immigration mechanism ϕ associated to the process R, are given by

$$\omega(z) = cz + \frac{\sigma^2 z^2}{2} \qquad \text{and} \qquad \phi(z) = -\psi(z) = \delta z + \int_{(0,\infty)} (1-e^{-zu})\mu(\mathrm{d}u),$$

respectively and where

$$\int_{(0,\infty)} (1\wedge u)\mu(\mathrm{d}u) < \infty \qquad \text{and} \qquad \delta = b - \int_{(0,1)} u\mu(\mathrm{d}u) \geq 0.$$

We denote by $\mathbb{Q}_x$, for the law of the Feller diffusion with immigration R starting from $x > 0$.

This type of processes have been studied recently by many authors, see for instance the papers of Keller-Ressel and Mijatović (2012) and Duhalde et al. (2014) and the references therein. In Keller-Ressel and Mijatović (2012), the authors were interested in the invariant distribution associated to the process R and Duhalde et al. (2014) studied first passage times problems and provide necessary and sufficient conditions for polarity and recurrence.

Lemma 3.1. *Let $R = (R_t; t \geq 0)$ be the Feller diffusion with immigration described by (3.1) with branching and immigration mechanisms given by ω and ϕ, respectively. The point 0 is polar, i.e. $T_0^R = \infty$ almost surely, if and only if $2\delta \geq \sigma^2$.*

Proof: According to Theorem 2 in Duhalde et al. (2014), the point 0 is polar for the Feller diffusion with immigration R, accordingly as

$$\int_1^\infty \frac{\mathrm{d}\lambda}{\omega(\lambda)} \exp\left\{\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z\right\} = \infty. \tag{3.2}$$

Let $K := 2c/\sigma^2$, which is equal to 0 if $c = 0$. Then for any $\lambda > 1$ and $x_0 > 0$, we have

$$\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z = \frac{2}{\sigma^2}\int_1^\lambda \left(\frac{\delta z}{Kz+z^2} + \frac{1}{Kz+z^2}\int_0^\infty (1-e^{-zu})\mu(\mathrm{d}u)\right)\mathrm{d}z. \tag{3.3}$$

Since all terms in (3.3) are positive, we can separate the above integral into two terms and study each of them independently. Then

$$\begin{aligned}\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z &= \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\int_0^\infty \mu(\mathrm{d}u)\int_1^\lambda \frac{1-e^{-zu}}{Kz+z^2}\mathrm{d}z\\ &\le \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\int_0^{x_0} \mu(\mathrm{d}u)\int_1^\lambda \frac{zu}{Kz+z^2}\mathrm{d}z + \frac{2}{\sigma^2}\int_{x_0}^\infty \mu(\mathrm{d}u)\int_1^\lambda \frac{1}{z^2}\mathrm{d}z\\ &\le \frac{2\delta}{\sigma^2}\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\left(\int_0^{x_0} u\mu(\mathrm{d}u)\right)\ln\left(\frac{K+\lambda}{K+1}\right) + \frac{2}{\sigma^2}\bar{\mu}(x_0),\end{aligned}$$

where we used Fubini-Tonelli's theorem to obtain the first equality. The above inequality holds for any $x_0 > 0$, hence for any $\varepsilon > 0$, we can choose $x_0 > 0$ such that

$$\int_0^{x_0} u\mu(\mathrm{d}u) \le \frac{\sigma^2}{2}\varepsilon.$$

Then for any $\lambda > 1$, the following inequalities hold

$$K_1(x_0)\frac{(K+\lambda)^{\frac{2\delta}{\sigma^2}}}{\lambda^2} \le \frac{1}{\omega(\lambda)}\exp\left\{\int_1^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z\right\} \le K_2(x_0)\frac{(K+\lambda)^{\frac{2\delta}{\sigma^2}+\varepsilon}}{\lambda^2},$$

where $K_1(x_0)$ and $K_2(x_0)$ are positive constants which are independent from λ. Therefore we conclude that (3.2) holds if and only if $2\delta \ge \sigma^2$. □

Proof of Theorem 1.2: We first treat the case $\sigma^2 > 2\delta$. From Lemma 3.1, we observe that 0 is not polar, meaning that the Feller diffusion with immigration R hits 0 with positive probability. From Theorem 2.1, we then deduce

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = 0\right) \ge \mathbb{Q}_x(T_0^R < \infty) > 0, \qquad x > 0.$$

In other words, with positive probability, the process Z does not explode. Moreover, if $\mathcal{I} = \infty$, Theorem 3 in Duhalde et al. (2014) implies that the process R, the unique strong solution to (3.1), is recurrent in the sense of Duhalde et al. (2014) (i.e. without assuming the polarity of 0, cf. remark after Definition 1.1). In other words, since 0 is not polar, R hits 0 at finite time a.s. Since we are interested in the unique strong solution of (3.1) up to the first hitting time of 0, the latter probability equals 1, i.e. the process Z converges to 0 a.s.

Next, we assume $2\delta \ge \sigma^2$. From Lemma 3.1, we know that $T_0^R = \infty$ a.s. and thus $\eta_t = \int_0^t \frac{1}{R_s}\mathrm{d}s$ for any $t \ge 0$. If we also assume that $\mathcal{I} = \infty$, then the solution to (3.1) is recurrent and 0 is polar. Let us thus prove that the limit η_∞ of $(\eta_t, t \ge 0)$ is ∞ a.s. If we define recursively the sequences of finite stopping times as follows $\tau_0^+ = 0$, and for any $k \ge 1$,

$$\tau_{k+1}^- = \inf\{t \ge \tau_k^+, R_s \le 1\} \qquad \text{and} \qquad \tau_{k+1}^+ = \inf\{t \ge \tau_{k+1}^-, R_s \ge 2\},$$

we deduce that, since $\{\tau_k^+ - \tau_k^-, k \ge 1\}$ is an infinite sequence of strictly positive i.i.d random variables,

$$\eta_\infty = \int_0^\infty \frac{1}{R_s}\mathrm{d}s \ge \sum_{k\ge 1}\frac{1}{2}(\tau_k^+ - \tau_k^-) = \infty, \qquad \text{a.s.} \tag{3.4}$$

This implies that C_t, the right inverse of η_t, is well defined on $(0,\infty)$ and that $Z_t = R_{C_t}$ for any $t \ge 0$. In other words, the process Z is conservative.

If $\mathcal{I} < \infty$, then the process R is transient according to Theorem 3 in Duhalde et al. (2014). Recall that the Laplace transform of R_t satisfies

$$\mathbb{Q}_x[e^{-\lambda R_t}] = \exp\left\{-xv_t(\lambda) - \int_0^t \phi(v_s(\lambda))\mathrm{d}s\right\}, \qquad \text{for} \quad \lambda \geq 0,$$

where $v_t(\lambda)$ is solution of

$$\frac{\partial}{\partial t} v_t(\lambda) = -\omega(v_t(\lambda)), \qquad \text{with} \qquad v_0(\lambda) = \lambda. \tag{3.5}$$

From the form of the branching mechanims ω and the previous identity, we deduce

$$v_t(\lambda) = \frac{\lambda e^{-ct}}{1 + \frac{\sigma^2 \lambda}{2c}(1 - e^{-ct})}, \qquad \text{for} \quad t, \lambda \geq 0.$$

Therefore, by Tonelli's Theorem, identity (3.5), the fact that $v_\infty(\lambda) = 0$ and using twice the change of variables $y = v_t(\lambda)$, we deduce that for $\theta > 0$

$$\begin{aligned}\mathbb{Q}_x\left[\int_0^\infty \frac{1 - e^{-\theta R_s}}{R_s}\mathrm{d}s\right] &= \int_0^\theta \mathrm{d}\lambda \int_0^\infty \mathrm{d}s\, \mathbb{Q}_x\left[e^{-\lambda R_s}\right] \\ &= \int_0^\theta \mathrm{d}\lambda \int_0^\lambda \frac{\mathrm{d}u}{\omega(u)} \exp\left\{-xu - \int_u^\lambda \frac{\phi(y)}{\omega(y)}\mathrm{d}y\right\},\end{aligned}$$

which is clearly finite from our hypothesis. Since the Feller diffusion with immigration R is transient, it is clear that

$$\lim_{s\to\infty} e^{-\theta R_s} = 0, \qquad \mathbb{Q}_x\text{-a.s.},$$

implying that

$$\mathbb{Q}_x\left[\int_0^\infty \frac{1}{R_s}\mathrm{d}s\right] < \infty,$$

and implicitly the process Z explodes at finite time a.s. This completes the proof. □

We now proceed with the proofs of Proposition 1.3, Lemma 1.4 and Theorem 1.5 where it is assumed that $c > 0$.

Proof of Proposition 1.3: The proof of this result is a direct consequence of the Lamperti-type representation (Theorem 2.1) and Theorem 1 in Duhalde et al. (2014). □

Proof of Lemma 1.4: We first recall that m, introduced in (1.16), is well defined under the log-moment condition (1.7). Then, similarly to (3.3), we have

$$-m(\lambda) = \int_0^\lambda \frac{\phi(z)}{\omega(z)}\mathrm{d}z = \frac{2}{\sigma^2}\int_0^\lambda \frac{\delta z}{\frac{2c}{\sigma^2}z + z^2}\mathrm{d}z + \int_0^\lambda \left(\frac{1}{\frac{2c}{\sigma^2}z + z^2}\int_0^\infty (1 - e^{-zu})\mu(\mathrm{d}u)\right)\mathrm{d}z. \tag{3.6}$$

For simplicity in exposition, we study the two last integrals independently. For the first integral of (3.6), we observe

$$\int_0^\lambda \frac{\delta z}{Kz + z^2}\mathrm{d}z = \delta \int_0^\lambda \int_0^\infty e^{-v(z+K)}\mathrm{d}v\mathrm{d}z = \int_0^\infty (1 - e^{-\lambda v})\frac{\delta e^{-Kv}}{v}\mathrm{d}v,$$

where $K := 2c/\sigma^2$ and the last equality follows from an application of Fubini-Tonelli's theorem. For the second integral of (3.6), we use again Fubini-Tonelli's theorem, to deduce

$$\int_0^\lambda \frac{1}{Kz + z^2}\left(\int_0^\infty (1 - e^{-zu})\mu(\mathrm{d}u)\right)\mathrm{d}z = \frac{1}{K}\int_0^\infty \left(\int_0^\lambda \frac{K(1 - e^{-zu})}{Kz + z^2}\mathrm{d}z\right)\mu(\mathrm{d}u).$$

Now, we fix $u > 0$ and study the integral inside the brackets. Since the map $z \mapsto (1-e^{-zu})/z$ is integrable at 0, we have

$$\begin{aligned}
\int_0^\lambda \frac{K(1-e^{-zu})}{Kz+z^2}\mathrm{d}z &= \int_0^\lambda \left(\frac{1-e^{-zu}}{z} - \frac{1-e^{-zu}}{K+z}\right)\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - \int_K^{K+\lambda} \frac{1-e^{Ku}e^{-zu}}{z}\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_K^{K+\lambda}\frac{1}{z}\mathrm{d}z - e^{Ku}\int_K^{K+\lambda}\frac{1-e^{-zu}}{z}\mathrm{d}z \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_K^{K+\lambda}\int_0^\infty e^{-zv}\mathrm{d}v\mathrm{d}z \\
&\qquad + e^{Ku}\left(\int_0^K \frac{1-e^{-zu}}{z}\mathrm{d}z - \int_0^{K+\lambda}\frac{1-e^{-zu}}{z}\mathrm{d}z\right) \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}\mathrm{d}v - (1-e^{Ku})\int_0^\infty \frac{e^{-Kv}}{v}(1-e^{-\lambda v})\mathrm{d}v \\
&\qquad - e^{Ku}\int_0^u \frac{e^{-Kv}}{v}(1-e^{-\lambda v})\mathrm{d}v \\
&= \int_0^u \frac{1-e^{-\lambda v}}{v}(1-e^{-Kv})\mathrm{d}v + (e^{Ku}-1)\int_u^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\mathrm{d}v
\end{aligned}$$

where the second identity follows from the change of variables $zu = \lambda v$, the third identity is obtained by adding and subtracting e^{Ku}, the fifth identity follows from Fubini-Tonelli's Theorem and the change of variables $Kv = zu$ and $(K+\lambda)v = zu$ and finally, the last identity follows by adding and subtracting

$$\int_0^u \frac{1-e^{-\lambda v}}{v}e^{-Kv}\mathrm{d}v.$$

In other words, we get

$$\int_0^\lambda \frac{K(1-e^{-zu})}{Kz+z^2}\mathrm{d}z = \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}(e^{K(v\wedge u)}-1)\mathrm{d}v = \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^{v\wedge u} Ke^{Kz}\mathrm{d}z\right)\mathrm{d}v.$$

Putting all pieces together and using twice Fubini-Tonelli's theorem, we obtain the following expression for the second integral of (3.6)

$$\begin{aligned}
\int_0^\lambda \frac{1}{Kz+z^2}\left(\int_0^\infty (1-e^{-zu})\mu(\mathrm{d}u)\right)\mathrm{d}z &= \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^\infty \left(\int_0^{v\wedge u} e^{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u)\right)\mathrm{d}v \\
&= \int_0^\infty \frac{1-e^{-\lambda v}}{v}e^{-Kv}\left(\int_0^v e^{Kz}\bar{\mu}(z)\mathrm{d}z\right)\mathrm{d}v. \qquad (3.7)
\end{aligned}$$

Finally from identity (3.6) and the previous computations, we find (1.17).

Next, we define the positive measure $\Pi(\mathrm{d}z)$ as follows

$$\Pi(\mathrm{d}z) = \frac{2e^{-Kz}}{\sigma^2 z}\left(\delta + \int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\right)\mathrm{d}z, \qquad (3.8)$$

and prove that $\int_{(0,\infty)}(1\wedge z)\Pi(\mathrm{d}z)$ is finite. To this aim, we observe that the following three inequalities hold true,

$$\int^\infty \frac{\bar{\mu}(w)}{w}\mathrm{d}w < \infty, \qquad \int_0 \bar{\mu}(w)\mathrm{d}w < \infty \quad \text{and} \quad \int_u^\infty \frac{e^{-Kz}}{z}\mathrm{d}z \le \frac{e^{-Ku}}{Ku}. \qquad (3.9)$$

Indeed, the finiteness of the first two integral follows from Fubini-Tonelli's theorem, since

$$\int_1^\infty \frac{\bar{\mu}(w)}{w}\mathrm{d}w = \int_1^\infty \ln(z)\mu(z)\mathrm{d}z \qquad \text{and} \qquad \int_0^1 \bar{\mu}(w)\mathrm{d}w = \int_0^\infty (1\wedge z)\mu(z)\mathrm{d}z.$$

With this in mind, we observe

$$\int_0^1 z\Pi(\mathrm{d}z) \le \frac{2}{\sigma^2}\left(\delta + e^K\int_0^1 \bar{\mu}(v)\mathrm{d}v\right) < \infty.$$

Moreover,

$$\begin{aligned}\int_1^\infty \Pi(\mathrm{d}z) &= \frac{2}{\sigma^2}\left(\delta\int_1^\infty \frac{e^{-Kz}}{z}\mathrm{d}z + \int_1^\infty \frac{e^{-Kz}}{z}\int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\mathrm{d}z\right)\\ &\le \frac{2}{\sigma^2}\left(\frac{\delta}{K}e^{-K} + \int_0^\infty e^{Kv}\bar{\mu}(v)\int_{v\vee 1}^\infty \frac{e^{-Kz}}{z}\mathrm{d}z\mathrm{d}v\right)\\ &\le \frac{2}{\sigma^2}\left(\frac{\delta}{K}e^{-K} + \frac{1}{K}\int_0^1 \bar{\mu}(v)\mathrm{d}v + \frac{1}{K}\int_1^\infty \frac{\bar{\mu}(v)}{v}\mathrm{d}v\right) < \infty.\end{aligned}$$

In other words, the probability measure ν is infinitely divisible with support on $(0,\infty)$ and with Laplace exponent $-m$. Finally, if $\bar{\mu}(0) \le b$, a simple computation guarantees that k defined by

$$k(z) = \frac{2e^{-Kz}}{\sigma^2}\left(\delta + \int_0^z e^{Kv}\bar{\mu}(v)\mathrm{d}v\right),$$

is non-increasing and Theorem 15.10 in Sato (1999) implies the self-decomposability of ν. □

Proof of Theorem 1.5: Recall from Theorem 3 in Duhalde et al. (2014) that the solution to (3.1) is recurrent if and only if $\mathcal{I} = \infty$. From the definition of functions ϕ and ω and the fact that $2\delta \ge \sigma^2$ and $c > 0$, we deduce that $\mathcal{I} = \infty$ if and only if (1.18) is satisfied.

In other words, under the assumption that $2\delta \ge \sigma^2$ and (1.18) hold, we have that 0 is polar and that R is recurrent. From the proof of (3.4), we deduce that C_t, the right inverse of η_t, is well defined on $(0,\infty)$ and that $Z_t = R_{C_t}$ for any $t \ge 0$. That is to say Z is also recurrent, $T_0 = \infty$ a.s. and has an invariant measure that we denote by ρ.

Next, we characterise the invariant measure ρ below. In order to do so, we use the infinitesimal generator $\mathcal{U}$ of Z, i.e. ρ is an invariant measure for Z if and only if

$$\int_0^\infty \mathcal{U}f(z)\rho(\mathrm{d}z) = 0,$$

for any f in the domain of $\mathcal{U}$. According to Palau and Pardo (2018), the infinitesimal generator $\mathcal{U}$ satisfies for any $f \in C_b^2(\mathbb{R}_+)$,

$$\mathcal{U}f(x) = x\mathcal{A}f(x) - cx^2f'(x) + \frac{\sigma^2}{2}x^2f''(x),$$

where $\mathcal{A}$ represents the generator of the spectrally positive Lévy process associated to the branching mechanism ψ. For the particular choice of $f(x) = e^{-\lambda x}$, for $\lambda > 0$, we observe $\mathcal{A}f(x) = \psi(\lambda)e^{-\lambda x}$ implying that

$$0 = \int_0^\infty \mathcal{U}f(z)\rho(\mathrm{d}z) = \int_0^\infty \Big(\psi(\lambda) + \omega(\lambda)z\Big)ze^{-\lambda z}\rho(\mathrm{d}z).$$

Then, similarly as in Lambert (2005), we denote the Laplace transform of $z\rho(\mathrm{d}z)$ by χ and performing the previous identity, we observe that χ satisfies the ordinary differential equation

$$\psi(\lambda)\chi(\lambda) - \omega(\lambda)\chi'(\lambda) = 0 \qquad \text{on} \qquad (0,\infty).$$

Straightforward computations implies that χ satisfies

$$\chi(\lambda) = K_0\exp\left\{\int_\theta^\lambda \frac{\psi(u)}{\omega(u)}\mathrm{d}u\right\}, \tag{3.10}$$

for some constants $K_0 > 0$ and $\theta \geq 0$. We can now prove the cases (a) and (b).

Let us assume that (1.7) is satisfied or equivalently the integrability of ψ/ω at 0. We take $\theta = 0$ in (3.10) and deduce that $\chi(\lambda) = K_0 \exp(m(\lambda))$ for some constant $K_0 > 0$. In other words, we have for $z \geq 0$

$$\rho(\mathrm{d}z) = K_0 \frac{1}{z} \nu(\mathrm{d}z),$$

with a possible Dirac mass at 0, and where ν is defined in Lemma 1.4. We can conclude as soon as we prove that $\varrho := \int_0^\infty z^{-1}\nu(\mathrm{d}z)$ is finite if $2\delta > \sigma^2$ or if $2\delta = \sigma^2$ and condition (∂) holds and that ρ is infinite if $2\delta = \sigma^2$ and condition $(\eth)$ holds. Indeed, if $\varrho < \infty$, ρ defined by (1.19) is the unique invariant probability measure of Z and consequently it is positive recurrent. If $\varrho = \infty$, then all invariant measures of Z are non-integrable at 0, so that Z_t converges to 0 in probability and since Z oscillates in $(0, \infty)$ then it is null-recurrent.

Therefore, it remains to verify whether ϱ is finite or not. Note that formally,

$$\int_0^\infty e^{m(\lambda)} \mathrm{d}\lambda = \int_{(0,\infty)} z^{-1}\nu(\mathrm{d}z) = \varrho.$$

Hence, ϱ is finite if and only if $e^{m(\lambda)}$ is integrable at ∞. From the proof of Lemma 1.4 (see (3.6) and (3.7)), we deduce

$$-m(\lambda) = \frac{2\delta}{\sigma^2} \ln\left(1 + \frac{\lambda}{K}\right) + \int_0^{+\infty} \frac{(1 - e^{-\lambda z})}{z} h(z)\mathrm{d}z, \tag{3.11}$$

where we recall that $K = 2c/\sigma^2$, and

$$h(z) = \frac{2}{\sigma^2} e^{-Kz} \int_0^z e^{Kw} \bar{\mu}(w)\mathrm{d}w.$$

With all this in mind, we study the integral in the right-hand side of (3.11) for λ large enough following a similar approach to the proofs of Theorem 53.6 in Sato (1999) or Theorem 3.4 in Lambert (2005). We take $x > 0$ and $\lambda > 1$, and split the interval $(0, \infty)$ into $(0, x/\lambda]$, $(x/\lambda, x]$ and (x, ∞). From (3.9),we deduce

$$\begin{aligned}\int_x^\infty \frac{h(z)}{z}\mathrm{d}z &= \frac{2}{\sigma^2} \int_0^\infty e^{Kw} \bar{\mu}(w) \left(\int_{x\vee w}^\infty \frac{e^{-Kz}}{z}\mathrm{d}z\right) \mathrm{d}w \\ &\leq \frac{2}{K\sigma^2} \left(\frac{1}{x} \int_0^x \bar{\mu}(w)\mathrm{d}w + \int_x^\infty \frac{\bar{\mu}(w)}{w}\mathrm{d}w\right) < \infty,\end{aligned}$$

which guarantees, together with the Dominated Convergence Theorem, that

$$\int_x^\infty (1 - e^{-\lambda z}) \frac{h(z)}{z}\mathrm{d}z \quad \text{converges as} \quad \lambda \to \infty.$$

On the other hand, we observe

$$\begin{aligned}\int_0^{x/\lambda} (1 - e^{-\lambda z}) \frac{h(z)}{z}\mathrm{d}z &= \frac{\sigma^2}{2} \int_0^x \frac{(1 - e^{-z})}{z} e^{-\frac{Kz}{\lambda}} \left(\int_0^{z/\lambda} e^{Kw} \bar{\mu}(w)\mathrm{d}w\right) \mathrm{d}z \\ &\leq \frac{\sigma^2}{2} e^{Kx} \int_0^x \frac{(1 - e^{-z})}{z}\mathrm{d}z \int_0^x \bar{\mu}(w)\mathrm{d}w < \infty,\end{aligned}$$

which implies the convergence of

$$\int_0^{x/\lambda} (1 - e^{-\lambda z}) \frac{h(z)}{z}\mathrm{d}z \qquad \text{when} \quad \lambda \to \infty.$$

A similar change of variables lead us to deduce

$$\int_{x/\lambda}^{x} e^{-\lambda z} h(z)/z \mathrm{d}z$$

converges when λ grows to ∞. Putting the pieces together in (3.11), we deduce that for any $x > 0$ and for λ large enough

$$-m(\lambda) = \frac{2\delta}{\sigma^2} \ln\left(1 + \frac{\lambda}{K}\right) + \int_{x}^{x/\lambda} \frac{h(z)}{z} \mathrm{d}z + K_1(x) + o(1),$$

where $K_1(x)$ is a non-negative constant. Hence, for λ large enough and for any $x > 0$,

$$e^{m(\lambda)} = \frac{K_2(x)}{(1+\lambda)^{2\delta/\sigma^2}} \exp\left\{ -\int_{x}^{x/\lambda} \frac{h(z)}{z} \mathrm{d}z + o(1) \right\}, \tag{3.12}$$

where $K_2(x)$ is a positive constant.

It thus remains to study the integral term in (3.12). Since h is positive, we can find $K_3(x) > 0$ such that for any λ large enough, $e^{m(\lambda)} \leq K_3(x)\lambda^{-2\delta/\sigma^2}$, and we conclude as soon as $2\delta > \sigma^2$. This implies part (a), when (1.7) holds.

Next, we prove part (b), i.e. we assume that $2\delta = \sigma^2$ and that (1.7) holds. For the sake of brevity, we concentrate on the case (∂), the case $(\eth)$ uses similar arguments. Under condition (∂), there exists $n \in \mathbb{Z}_+$ such that $\inf(\mathtt{Adh}(I^{(n)})) > \sigma^2/2$ and $\mathtt{Adh}(I^{(k)}) = \{\sigma^2/2\}$, for any $k \in \{1, .., n-1\}$. Let us define by recurrence the collection of functions $\bar{I}$ such that

$$\bar{I}^{(1)}(z) = l^{(1)}(z)h(z) \qquad \text{and} \qquad \bar{I}^{(k)}(z) = l^{(k)}(z)\left[\bar{I}^{(k-1)}(z) - 1\right], \qquad k \in \mathbb{Z}_+,\ k \geq 2.$$

Note that the sequences $\{\bar{I}^{(k)}\}_{k\leq n}$ and $\{I^{(k)}\}_{k\leq n}$ satisfy similar recurrence relations but start on different values. From the definition of h and a recurrence argument, it is straightforward to compute that for any $k \in \mathbb{Z}_+$, and for z small enough,

$$\frac{2}{\sigma^2} e^{-Kz} I^{(k)}(z) + (e^{-Kz} - 1) \sum_{j=2}^{k} \prod_{i=j}^{k} l^{(i)}(z) \leq \bar{I}^{(k)}(z) \leq \frac{2}{\sigma^2} I^{(k)}(z).$$

Since $(e^{-Kz} - 1)$ behaves as $-Kz$, for z small enough, the second term of the left hand side converges to 0 when z converges to 0 and we deduce that the sequences of functions $\{\bar{I}^{(k)}\}_{k\leq n}$ and $\{I^{(k)}\}_{k\leq n}$ satisfy similar assumptions, which are $\inf(\mathtt{Adh}(\bar{I}^{(n)})) = A > 1$ and $\mathtt{Adh}(\bar{I}^{(k)}) = \{1\}$, for any $k \in \{1, .., n-1\}$. Let us fix $\varepsilon > 0$ such that $A - \varepsilon > 1$ and $x > 0$ such that $\bar{I}^{(n)}(x) \geq A - \varepsilon$. Using the definition of $\{\bar{I}^{(k)}\}_{k\geq 0}$ and a recurrence argument, we obtain that for any z sufficiently small,

$$h(z) = \frac{\bar{I}^{(n)}(z)}{\prod_{i=1}^{n} l^{(i)}(z)} + \sum_{j=1}^{n-1} \frac{1}{\prod_{i=1}^{j} l^{(i)}(z)}.$$

Hence,

$$\int_{x/\lambda}^{x} \frac{h(z)}{z} \mathrm{d}z \geq (A - \varepsilon) \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{n} l^{(i)}(z)} + \sum_{j=1}^{n-1} \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{j} l^{(i)}(z)}. \tag{3.13}$$

Moreover from the definition of $l^{(j)}$, we have for any $j \in \mathbb{Z}_+$,

$$\begin{aligned} \int_{x/\lambda}^{x} \frac{\mathrm{d}z}{z \prod_{i=1}^{j} l^{(i)}(z)} &= l^{(j+1)}(x) - l^{(j+1)}\left(\frac{x}{\lambda}\right) \\ &= l^{(j+1)}(x) + l^{(j+1)}(\lambda) - R^{(j+1)}(x, \lambda) \qquad \text{as} \quad \lambda \to \infty, \end{aligned}$$

where the sequence $\{R^{(k)}\}_{k\geq 2}$ satisfies the following recurrence relation: for any x small enough and λ large enough,

$$R^{(2)}(x,\lambda) = \ln\left(1+\frac{l^{(1)}(x)}{l^{(1)}(\lambda)}\right) \qquad \text{and} \qquad R^{(j)}(z) = \ln\left(1+\frac{R^{(j-1)}(x,\lambda)}{l^{(j-1)}(\lambda)}\right), \quad j\in\{3,..,n+1\}.$$

Hence, we deduce that $R^{(j)}(x,\lambda)$ converges to 0 when λ increases to ∞, for all $j\in\{3,..,n+1\}$. In addition with (3.13), as soon as λ is sufficiently large, we have

$$\int_{x/\lambda}^{x}\frac{h(z)}{z}\mathrm{d}z \geq (A-\varepsilon)l^{(n+1)}(\lambda) + \sum_{j=1}^{n-1} l^{(j+1)}(\lambda) + K_4(x),$$

where $K_4(x)$ is a finite constant. Hence using (3.12), we deduce that for λ sufficiently large there exist a finite constant $K_5(x)>0$ such that

$$e^{m(\lambda)} \leq \frac{K_5(x)}{\lambda\prod_{i=1}^{n-1} l^{(i)}(\lambda)(l^{(n)}(\lambda))^{A-\varepsilon}}. \tag{3.14}$$

Since $A-\varepsilon>1$, the right hand side of (3.14) is integrable at ∞. Indeed, for any z,y sufficiently large such that $l^{(n)}(y)>0$ and $l^{(n)}(z)>0$, with the change of variables $u=l^n(\lambda)$, we have

$$\int_z^y \frac{1}{\lambda\prod_{i=1}^{n-1} l^{(i)}(\lambda)(l^{(n)}(\lambda))^{A-\varepsilon}}\mathrm{d}\lambda = \int_{l^{(n)}(z)}^{l^{(n)}(y)}\frac{1}{u^{A-\varepsilon}}\mathrm{d}u \underset{b\to\infty}{\longrightarrow} \int_{l^{(n)}(z)}^{\infty}\frac{1}{u^{A-\varepsilon}}\mathrm{d}u<\infty.$$

Finally, we have proved that under condition (∂),

$$\int^{\infty} e^{m(\lambda)}\mathrm{d}\lambda < \infty.$$

This completes the proof of part (b) and the cases when condition (1.7) is satisfied.

Now, we deal with the case when the log-moment condition (1.7) does not hold and $2\delta>\sigma^2$. Under this assumption we show that Z is still positive recurrent but its invariant distribution has an infinite expected value. Recall that condition $2\delta>\sigma^2$ guarantees that Z is recurrent with an invariant distribution ρ satisfying (3.10). However in this case, ψ/ω is not integrable at 0 and we can not take $\theta=0$ in (3.10), instead we let $\theta=1$. Formally, the following identity still holds

$$\int_0^\infty \chi(\lambda)\mathrm{d}\lambda = \int_0^\infty \rho(\mathrm{d}z).$$

Our aim is thus to prove that the latter identity is finite but the expected value of ρ is infinite.

On the one hand, recalling that $K=2c/\sigma^2$ and taking λ smaller than 1, we use the definition of ψ and Fubini-Tonnelli's Theorem to deduce

$$\begin{aligned}
-\int_\lambda^1\frac{\psi(z)}{\omega(z)}\mathrm{d}z &= \frac{2\delta}{\sigma^2}\ln\left(\frac{K+1}{K+\lambda}\right) + \frac{2}{\sigma^2}\int_0^\infty\left(\int_\lambda^1\frac{1-e^{-zu}}{Kz+z^2}\mathrm{d}z\right)\mu(\mathrm{d}u)\\
&\leq \frac{2\delta}{\sigma^2}\ln\left(1+\frac{1}{K}\right) + \frac{2}{\sigma^2}\int_0^A\left(\int_\lambda^1\frac{zu}{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u) + \frac{2}{\sigma^2}\int_A^\infty\left(\int_\lambda^1\frac{1}{Kz}\mathrm{d}z\right)\mu(\mathrm{d}u)\\
&\leq \frac{2\delta}{\sigma^2}\ln\left(1+\frac{1}{K}\right) + \frac{2}{K\sigma^2}\int_0^A u\mu(\mathrm{d}u) - \frac{2}{K\sigma^2}\ln(\lambda)\bar{\mu}(A),
\end{aligned}$$

for any $A > 0$. Thus, we take $A > 0$ in such a way that $\bar{\mu}(A) \leq K\sigma^2/4$. Implying that for any $\lambda \leq 1$, we get

$$\chi(\lambda) \leq K_0 \frac{e^{K(A)}}{\lambda^{1/2}},$$

with K_0 and $K(A)$ two positive constants which are independent from λ. In other words, χ is integrable near 0. On the other hand, since

$$\int_1^\lambda \frac{\psi(z)}{\omega(z)} \mathrm{d}z \leq -\frac{2b}{\sigma^2} \ln\left(\frac{K+1}{K+\lambda}\right),$$

we also have

$$\chi(\lambda) \leq K_0 \left(\frac{K+1}{K+\lambda}\right)^{\frac{2b}{\sigma^2}},$$

implying that

$$\int_0^\infty \chi(\lambda)\mathrm{d}\lambda < \infty,$$

since $2b > \sigma^2$. In other words Z has a finite invariant measure and is positive recurrent. Moreover, since the log-moment condition (1.7) does not hold, a straightforward computation gives

$$\int_0^\infty z\rho(\mathrm{d}z) = \lim_{\lambda\to 0} \int_0^\infty e^{-\lambda z} z\rho(\mathrm{d}z) = \lim_{\lambda\to 0} \chi(\lambda) = \infty.$$

Finally, if condition (1.18) does not hold then $\mathcal{I} < \infty$ and from Theorem 1.2 the process Z explodes in finite time a.s. □

4. General case

For the proof of Theorem 1.7 recall that the associated Lévy process X which appears in (2.2) is general, that is to say, there exist $\vartheta \geq 0$ such that $\psi(z) > 0$ for any $z \geq \vartheta$ and the log-moment condition (1.7) is satisfied.

Proof of Theorem 1.7: Let us fix $\lambda > 0$, and denote by Φ the function

$$\Phi(z) := \frac{e^{-m(z)}}{\omega(z)} \exp\left\{-\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v\right\} \int_0^z \exp\left\{m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\} \mathrm{d}u,$$

in other words, we have

$$h_\lambda(x) = 1 + \lambda \int_0^\infty e^{-xz}\Phi(z)\mathrm{d}z,$$

which was defined by (1.24). For simplicity in exposition, we split the proof in six steps.

Step 1: We first prove that h_λ is well defined on $(0,\infty)$ or equivalently, we prove that $z \mapsto e^{-xz}\Phi(z)$ is integrable on $(0,\infty)$ as soon as x is positive. From the definitions of m and $\mathtt{I}$ (see (1.16) and (1.20), respectively), it is straightforward that

$$\exp\left\{m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\} \to 1, \qquad \text{as} \quad u \to 0, \tag{4.1}$$

implying

$$e^{-xz}\Phi(z) \sim \frac{z}{\omega(z)} \sim \frac{1}{c}, \qquad \text{as} \quad z \to 0, \tag{4.2}$$

hence the integrability at 0.

Concerning the neighbourhood of ∞, we see from Lemma 1.6 that $y_\lambda(z) \leq \sqrt{\lambda}\frac{\varphi'(z)}{\sqrt{\omega(\varphi(z))}}$ which is equivalent to $\sqrt{2\lambda}\frac{\varphi'(z)}{\sigma\varphi(z)}$. In addition with (1.21), we deduce

$$\int_0^{\mathtt{I}(z)} y_\lambda(u)\mathrm{d}u = \mathrm{O}\left(\ln(z)\right) \qquad \text{and} \qquad \frac{\psi(z)}{\omega(z)} + \mathtt{I}'(z)y_\lambda(\mathtt{I}(z)) \geq 0 \qquad \text{as} \quad z \to \infty. \tag{4.3}$$

Then, for any $x > 0$ and for u sufficiently large, we have

$$\left| \frac{\exp\left\{m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\}}{\left(\frac{x}{2} + \frac{\psi(u)}{\omega(u)} + \mathtt{I}'(u)y_\lambda(\mathtt{I}(u))\right)\exp\left\{\frac{xu}{2} + m(u) + \int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\}} \right| \leq \frac{2\exp\left\{-\frac{xu}{2} + \int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v\right\}}{x},$$

which converges to 0 as u goes to ∞. In other words,

$$\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}z = \mathrm{o}\left(\exp\left\{\frac{xz}{2} + m(z) + \int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v\right\}\right), \quad \text{as} \quad z \to \infty. \tag{4.4}$$

Finally from the definition of Φ, we obtain

$$e^{-zx}\Phi(z) = \mathrm{o}\left(\frac{1}{\omega(z)}e^{-\frac{xz}{2}}\right), \qquad \text{as} \quad z \to \infty, \tag{4.5}$$

implying the integrability of $z \mapsto e^{-zx}\Phi(z)$ at ∞. It is important to note that (4.2) and (4.5), also imply that the mappings $z \mapsto ze^{-xz}\Phi(z)$ and $z \mapsto z^2e^{-zx}\Phi(z)$ are integrable on $(0,\infty)$ and that h_λ is a C^2-function on $(0,\infty)$.

Step 2: Now, we prove (1.25). The infinitesimal generator of Z satisfies (1.23), i.e. for any $f \in C_b^2(\mathbb{R}_+)$

$$\mathcal{U}f(x) = x\mathcal{A}f(x) - cx^2f'(x) + \frac{\sigma^2}{2}x^2f''(x) \tag{4.6}$$

where $\mathcal{A}$ is the generator of the spectrally positive Lévy process associated to branching mechanism ψ. Since, for $f(x) = e^{-zx}$, $\mathcal{A}f(x) = \psi(z)e^{-zx}$ with $z \geq 0$, we deduce using integrations by parts (twice) that

$$\begin{aligned} \mathcal{U}h_\lambda(x) - \lambda h_\lambda(x) &= \lambda\int_0^\infty \Big(x\psi(z) + x^2\omega(z) - \lambda\Big)\Phi(z)e^{-zx}\mathrm{d}z - \lambda \\ &= \lambda\Bigg(\int_0^\infty \Big((\psi\Phi)'(z) + (\omega\Phi)''(z) - \lambda\Phi(z)\Big)e^{-xz}\mathrm{d}z - 1 \\ &\quad - xw(z)\Phi(z)e^{-xz}\Big|_{z=0}^{z=\infty} + \Big(\psi(z)\Phi(z) + (\omega\Phi)'(z)\Big)e^{-xz}\Big|_{z=0}^{z=\infty}\Bigg). \end{aligned} \tag{4.7}$$

Let us prove that the right-hand side of the latter expression equals 0. Recall that $m'(z) = \frac{\psi(z)}{\omega(z)}$ and $\mathtt{I}'(z) = e^{m(z)}$, then

$$(\omega\Phi)'(z) = -\psi(z)\Phi(z) - y_\lambda(\mathtt{I}(z))e^{-\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v}\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u + e^{\int_0^{\mathtt{I}(z)} y_\lambda(v)\mathrm{d}v}. \tag{4.8}$$

In addition with the fact that y_λ is solution to (1.22), we deduce that $(\omega\Phi)''(z) = -(\psi\Phi)'(z)+\lambda\Phi(z)$ for any $z \geq 0$. On the other hand, using (4.2) and (4.5), we have that

$$xw(z)\Phi(z)e^{-xz}\Big|_{z=0}^{z=\infty} = 0,$$

and from (4.8), together with (4.3) and (4.4), we deduce

$$\lim_{z\to\infty}(\psi(z)\Phi(z) + (\omega\Phi)'(z))e^{-xz} = 0,$$

as soon as $x>0$. Therefore, it remains to study the previous limit but when z goes to 0. According to (4.8),

$$\lim_{z\to 0}(\psi(z)\Phi(z)+(\omega\Phi)'(z))e^{-xz} = 1 - \lim_{z\to 0} y_\lambda(\mathtt{I}(z))\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u. \tag{4.9}$$

By Lemma 1.6 and (4.1), we deduce

$$y_\lambda(\mathtt{I}(z))\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u \le \sqrt{\frac{\lambda}{\omega(z)}}e^{-m(z)}\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(v)\mathrm{d}v}\mathrm{d}u \sim \sqrt{\frac{\lambda}{cz}}z, \quad \text{as} \quad z\to 0,$$

which implies that the right-hand side of (4.9) equals 1. In other words, the right-hand side of (4.7) equals 0, meaning that $\mathcal{U}h_\lambda(x)=\lambda h_\lambda(x)$ for any $x>0$ and that (1.25) holds.

Step 3: Our next step is to prove that $\int_0^\infty y_\lambda(v)\mathrm{d}v$ is finite as soon as $\mathbb{P}_x(T_0<\infty)=1$, for any $x>0$, actually Lemma 1.6 is not enough to conclude. With this goal in mind, we fix $x>0$ and $\lambda\ge 0$ and set the function $G_{\lambda,x}$ as follows,

$$G_{\lambda,x}(v) := \int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[e^{-vZ_t}\Big]\mathrm{d}t, \qquad \text{for any} \quad v\ge 0.$$

This function is related with the Laplace transform of T_0, indeed

$$\lim_{v\to\infty}\lambda G_{\lambda,x}(v) = \mathbb{E}_x\Big[e^{-\lambda T_0}\Big].$$

The latter is positive since $\mathbb{P}_x(T_0<\infty)=1$. Our aim is to find a second formulation to $G_{\lambda,x}$, related to $\int_0^\infty y_\lambda(v)\mathrm{d}v$, to conclude.

Let us provide some properties of $G_{\lambda,x}$. We first note that for any h belonging to the domain of $\mathcal{U}$, the following identity holds

$$\lambda\int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[h(Z_t)\Big]\mathrm{d}t = h(x) + \int_0^\infty e^{-\lambda t}\mathbb{E}_x\Big[\mathcal{U}h(Z_t)\Big]\mathrm{d}t.$$

By taking $h(x)=e^{-vx}$ together with identity (4.6), we deduce

$$\begin{aligned}\lambda G_{\lambda,x}(v) &= e^{-vx} + \int_0^\infty e^{-\lambda t}\mathbb{E}_x\left[\psi(v)Z_s e^{-vZ_s} + \omega(v)Z_t^2 e^{-vZ_t}\right]\mathrm{d}t\\ &= e^{-vx} - \psi(v)G'_{\lambda,x}(v) + \omega(v)G''_{\lambda,x}(v).\end{aligned}$$

Moreover $\lambda G_{\lambda,x}(0)=1$ and the dominated convergence theorem implies

$$G'_{\lambda,x}(v) = -\int_0^\infty e^{-\lambda t}\mathbb{E}_x[Z_t e^{-vZ_t}\mathbf{1}_{\{Z_t>0\}}]\mathrm{d}t \longrightarrow 0, \quad \text{as} \quad v\to\infty.$$

We now prove that $G_{\lambda,x}$ is the unique solution to $\omega(v)y''(v)-\psi(v)y'(v)-\lambda y(v)=e^{-vx}$ with conditions $\lambda y(0)=1$ and $\lim_{v\to\infty}y'(v)=0$. In order to do so, we will explicit the set of functions that satisfy the equation with condition $\lambda y(0)=1$. First of all, let us prove that the following function, for any $v\ge 0$,

$$k(v) := \frac{1}{\lambda}e^{-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}\left(1+\lambda\int_0^v\int_u^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s+m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u\right) \tag{4.10}$$

satisfies the same conditions as $G_{\lambda,x}$. We first observe that

$$\begin{aligned}&\int_0^v\int_u^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s+m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u\\ &\qquad = \int_0^\infty \frac{e^{-zx}}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s}\left(\int_0^{v\wedge z} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s}\mathrm{d}u\right)\mathrm{d}z\end{aligned} \tag{4.11}$$

is finite according to (4.1). In other words, k is well defined. Moreover, $\lambda k(0) = 1$ and since $\mathtt{I}'(z) = \exp(m(z))$, a straightforward computation gives

$$k'(v) = -e^{m(v)} y_\lambda(\mathtt{I}(v))k(v) + e^{m(v)+\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} \int_v^\infty \frac{e^{-zx}}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s} \mathrm{d}z. \tag{4.12}$$

From (4.4) and (4.11), we deduce that k is bounded by some constant C on $\mathbb{R}$ and from Lemma 1.6, we also see that

$$\left| e^{m(v)} y_\lambda(\mathtt{I}(v))k(v) \right| \leq C\sqrt{\frac{\lambda}{\omega(v)}} \longrightarrow 0, \qquad \text{as} \quad v \to +\infty.$$

For the second term of the right-hand side of (4.12), we use a similar arguments to those used to deduce (4.4) which gives

$$\int_v^\infty \frac{e^{-xz}}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s} \mathrm{d}z = o\left(e^{-m(v)-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s - \frac{xv}{2}} \right), \qquad \text{as} \quad v \to \infty.$$

That is to say that $k'(v)$ converges to 0 when v goes to ∞. Finally, from (4.12), a straightforward computation provides

$$\omega(v)k''(v) = \psi(v)k'(v) + \lambda k(v) - e^{-vx}. \tag{4.13}$$

Putting all pieces together, we prove that k and $G_{\lambda,x}$ satisfy the same differential equation with conditions $\lambda k(0) = 1$ and $\lim_{v\to\infty} k'(v) = 0$.

Furthermore, from this, we deduce that the set of functions that satisfy

$$\omega(v)y''(v) - \psi(v)y'(v) - \lambda y(v) = e^{-vx},$$

with conditions $\lambda y(0) = 1$ is exactly $S := \{k_A, A \in \mathbb{R}\}$, with

$$k_A(v) := k(v) + Ae^{-\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} \int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u,$$

Let us prove that $\lim_{v\to+\infty} k'_A(v) = 0$ if and only if $A = 0$. Indeed,

$$k'_A(v) = k'(v) + Ae^{m(v)+\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} \left[1 - \frac{1}{\alpha(v)} \int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u \right],$$

where

$$\frac{1}{\alpha(v)} := y_\lambda(\mathtt{I}(v)) e^{-2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s}.$$

Using Lemma 1.6, we have

$$\alpha'(v)e^{-m(v)-2\int_0^{\mathtt{I}(v)} y_\lambda(s)\mathrm{d}s} = -\frac{y'_\lambda(\mathtt{I}(v))}{y_\lambda(\mathtt{I}(v))^2} + 2 = 1 + \lambda \frac{r^2(\mathtt{I}(v))}{y_\lambda^2(\mathtt{I}(v))} \geq 2,$$

for any v large enough. In other words, there exist $v_0 > 0$ such that for any $v \geq v_0$,

$$\frac{1}{\alpha(v)} \int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u \leq \frac{1}{2} + \frac{1}{\alpha(v)} \int_0^{v_0} e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u.$$

Since $\lim_{v\to\infty} \alpha(v) = \infty$, the latter inequality guarantees

$$\limsup_{v\to\infty} \frac{1}{\alpha(v)} \int_0^v e^{m(u)+2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}u \leq \frac{1}{2}.$$

In addition to the expression of k'_A, we deduce that $\lim_{v\to\infty} k'_A(v) = 0$ if and only if $A = 0$. Thus there exist a unique function in S that satisfies $\lim_{v\to\infty} k'_A(v) = 0$. Finally, since both k and $G_{\lambda,x}$ belong to S and satisfy the condition, then both functions are equals on $\mathbb{R}$.

Furthermore, with a direct application of Fubini's theorem

$$\lim_{v\to\infty} \int_0^v \int_u^\infty \frac{e^{-zx}}{\omega(z)} e^{-m(z)-\int_0^{\mathtt{I}(z)} y_\lambda(s)\mathrm{d}s + m(u) + 2\int_0^{\mathtt{I}(u)} y_\lambda(s)\mathrm{d}s} \mathrm{d}z\mathrm{d}u = \int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z > 0.$$

In addition with (4.10), we get

$$e^{-\int_0^\infty y_\lambda(s)\mathrm{d}s}\left(1+\lambda\int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z\right)=\lim_{v\to\infty}\lambda k(v)=\lim_{v\to\infty}\lambda G_{\lambda,x}(v)=\mathbb{E}_x\left[e^{-\lambda T_0}\right]>0.$$

We conclude that $\int_0^\infty y_\lambda(v)\mathrm{d}v$ is finite and

$$\mathbb{E}_x\left[e^{-\lambda T_0}\right]=e^{-\int_0^\infty y_\lambda(v)\mathrm{d}v}\left(1+\lambda\int_0^\infty e^{-zx}\Phi(z)\mathrm{d}z\right). \tag{4.14}$$

Step 4: We next prove that $h_\lambda(0)=\exp\{\int_0^\infty y_\lambda(v)\mathrm{d}v\}$. The main issue comes from the fact that we can not make x tend to 0 directly in the formula of h_λ since we do not know the integrability of $\Phi(z)$ near ∞. However, from (4.1) we know that for any $v\in(0,\infty)$,

$$\lambda\int_0^\infty\frac{1}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)}y_\lambda(s)\mathrm{d}s}\int_0^{z\wedge v}e^{m(u)+2\int_0^{\mathtt{I}(u)}y_\lambda(s)\mathrm{d}s}\mathrm{d}u\mathrm{d}z<\infty.$$

The goal is to take v near ∞. Using Fubini's theorem and twice the following change of variables $z\mapsto\mathtt{I}(z)$, we find

$$\begin{aligned}\lambda\int_0^\infty\frac{1}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)}y_\lambda(s)\mathrm{d}s}&\int_0^{z\wedge v}e^{m(u)+2\int_0^{\mathtt{I}(u)}y_\lambda(s)\mathrm{d}s}\mathrm{d}u\mathrm{d}z\\&=\int_0^{\mathtt{I}(v)}e^{2\int_0^u y_\lambda(s)\mathrm{d}s}\int_u^\infty\lambda\frac{e^{-2m(\varphi(z))}}{w(\varphi(z))}e^{-\int_0^z y_\lambda(s)\mathrm{d}s}\mathrm{d}z\mathrm{d}u.\end{aligned}$$

Recalling that, according to Lemma 1.6,

$$\lambda\frac{e^{-2m(\varphi(z))}}{w(\varphi(z))}=\lambda\frac{\varphi'(z)^2}{w(\varphi(z))}=y_\lambda^2(z)-y_\lambda'(z),$$

and using integration by parts on the term $y_\lambda^2(z)e^{-\int_0^z y_\lambda}$, we finally deduce

$$\lambda\int_0^\infty\frac{1}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)}y_\lambda(s)\mathrm{d}s}\int_0^{z\wedge v}e^{m(u)+2\int_0^{\mathtt{I}(u)}y_\lambda(s)\mathrm{d}s}\mathrm{d}u\mathrm{d}z=e^{\int_0^{\mathtt{I}(v)}y_\lambda(s)\mathrm{d}s}-1.$$

Since the integrand is positive, we let v tend to ∞ to find

$$h_\lambda(0)=1+\lambda\int_0^\infty\frac{1}{\omega(z)}e^{-m(z)-\int_0^{\mathtt{I}(z)}y_\lambda(s)\mathrm{d}s}\int_0^z e^{m(u)+2\int_0^{\mathtt{I}(u)}y_\lambda(s)\mathrm{d}s}\mathrm{d}u\mathrm{d}z=e^{\int_0^\infty y_\lambda(s)\mathrm{d}s} \tag{4.15}$$

which is finite according to the previous step.

Step 5: We now prove identity (1.26). First, let us assume that $x\geq a>0$ and define for any $n\in\mathbb{Z}_+$,

$$\theta_n=\inf\{t\geq 0, Z_t\geq n\}.$$

Recalling that $\mathcal{U}h_\lambda = \lambda h_\lambda$ and that the expression of $\mathcal{U}$ is given by (1.23), we deduce from Itô's formula applied to $Z_{t\wedge T_a\wedge\theta_n}$ and C^2-function $(t,u)\mapsto e^{-\lambda t}h_\lambda(u)$ that, for any $n\in\mathbb{Z}_+$,

$$\begin{aligned}
e^{-\lambda t\wedge T_a\wedge\theta_n}h_\lambda(Z_{t\wedge T_a\wedge\theta_n}) = h_\lambda(x) &+ \int_0^{t\wedge T_a\wedge\theta_n} e^{-\lambda s}h_\lambda'(Z_s)\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s \\
&+ \int_0^{t\wedge T_a\wedge\theta_n} \sigma e^{-\lambda s}h_\lambda'(Z_s)Z_s\mathrm{d}B_s^{(e)} \\
&+ \int_0^{t\wedge T_a\wedge\theta_n}\int_{(0,1)}\int_0^{Z_{s-}} e^{-\lambda s}\left(h_\lambda(Z_{s-}+z)-h_\lambda(Z_{s-})\right)\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\
&+ \int_0^{t\wedge T_a\wedge\theta_n}\int_{[1,\infty)}\int_0^{Z_{s-}} e^{-\lambda s}\left(h_\lambda(Z_{s-}+z)-h_\lambda(Z_{s-})\right)N^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\
&- \int_0^{t\wedge T_a\wedge\theta_n} e^{-\lambda s}Z_s\int_{[1,\infty)}\left(h_\lambda(Z_s+z)-h_\lambda(Z_s)\right)\mu(\mathrm{d}z)\mathrm{d}s,
\end{aligned} \tag{4.16}$$

where all terms are well defined since h_λ is positive non-increasing, h_λ' is negative non-decreasing, and $(Z_s, s\le t\wedge T_a\wedge\theta_n)$ take values on $[a,n]$. According to the same arguments, the three first integrals of the r.h.s. of (4.16) are martingales. Since

$$\mathbb{E}_x\left[\int_0^{t\wedge T_a\wedge\theta_n}\int_1^\infty\int_0^{Z_s}\left|e^{-\lambda s}\left(h_\lambda(Z_s+z)-h_\lambda(Z_{s-})\right)\right|\mu(\mathrm{d}z)\mathrm{d}s\mathrm{d}u\right] \le 2h_\lambda(a)tn\int_1^\infty\mu(\mathrm{d}z) < +\infty,$$

the fourth and integrals of the r.h.s. of (4.16) can be written as a martingale by observing that the fifth integral is exactly the compensator. Finally, taking the expectation of (4.16), we obtain for any $n\ge 1$ and $t\ge 0$ that

$$\mathbb{E}_x\left[e^{-\lambda t\wedge T_a\wedge\theta_n}h_\lambda(Z_{t\wedge T_a\wedge\theta_n})\right] = h_\lambda(x).$$

Since h_λ is bounded by $h_\lambda(0)<\infty$, we use the dominated convergence theorem and make n go to 0 and t go to ∞. Since $\mathbb{P}_x(T_a<\infty)=1$ (recall that we are assuming that $\mathbb{P}_x(T_0<\infty)=1$) and thus $Z_{T_a}=a$, $\mathbb{P}_x$-a.s., we deduce (1.26) for $x\ge a>0$. For $a=0$, identity (1.26) has already been obtained in (4.14) and (4.15).

Step 6: Next, we handle the result on the expectation of T_0, i.e. identity (1.27), using similar arguments as in the proof of Theorem 3.9 in Lambert (2005). We denote $H(t,\lambda)$ for the Laplace transform $\mathbb{E}_x[e^{-\lambda Z_t}]$, and observe

$$\lim_{\lambda\to\infty}\int_0^\infty(1-H(t,\lambda))\mathrm{d}t = \mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{Z_t>0\}}dt\right] = \mathbb{E}_x\Big[T_0\Big].$$

On the other hand, from (4.6), for any $t\ge 0, \lambda>0$,

$$\frac{\partial H}{\partial t}(t,\lambda) = -\psi(\lambda)\frac{\partial H}{\partial\lambda}(t,\lambda)+\omega(\lambda)\frac{\partial^2 H}{\partial\lambda^2}(t,\lambda) = \omega(\lambda)e^{m(\lambda)}\frac{\partial}{\partial\lambda}\left(\frac{\partial H}{\partial\lambda}(t,\lambda)e^{-m(\lambda)}\right),$$

which, by integrating $\frac{\partial H}{\partial\lambda}e^{-m(\lambda)}$ with respect to λ yields to

$$\frac{\partial H}{\partial\lambda}(t,\lambda) = -e^{m(\lambda)}\int_\lambda^\infty\frac{e^{-m(u)}}{\omega(u)}\frac{\partial H}{\partial t}(t,u)\mathrm{d}u$$

and then integrating again with respect to λ and t on $[0,\lambda]\times\mathbb{R}$, we obtain

$$\int_0^\infty(1-H(t,\lambda))\mathrm{d}t = \int_0^\lambda e^{m(u)}\int_u^\infty\frac{e^{-m(z)}}{\omega(z)}(1-e^{-zx})\mathrm{d}z\mathrm{d}u.$$

Letting λ go to ∞, we deduce (1.27). The proof of Theorem 1.7 is now complete. ∎

5. Branching diffusion with interactions in a Brownian environment

We finish this paper with some interesting remarks on branching diffusions with interactions in a Brownian environment. We decide to treat this case separately since the competition mechanism g may take negative and positive values and the techniques we use here are different from the rest of the paper. Our methodology are based on the theory of scale functions for diffusions. This allow us to provide a necessary and sufficient condition for extinction and moreover, the Laplace transform of hitting times is computed explicitly in terms of a Ricatti equation. Such results seems complicated to obtain with the presence of jumps coming from the branching mechanism or the random environment

More general competition mechanisms were considered by Ba and Pardoux (2015) in the case when the branching mechanism is of the form $\psi(u) = \gamma^2 u^2$, for $u \geq 0$, see also Chapter 8 in the monograph of Pardoux (2016). In this case, the CB-process with competition can be written as the unique strong solution of the following SDE

$$Y_t = Y_0 + \int_0^t h(Y_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)},$$

where h is a continuous function satisfying $h(0) = 0$ and such that

$$h(x+y) - h(x) \leq Ky, \qquad x, y \geq 0,$$

for some positive constant K. According to Ba and Pardoux, the process Y gets extinct in finite time if and only if

$$\int_1^\infty \exp\left\{-\frac{1}{2}\int_1^u \frac{h(r)}{r}\mathrm{d}r\right\}\mathrm{d}u = \infty.$$

Here, we focus on the Feller diffusion case and general competition mechanism where more explicit functionals of the process can be computed. In this particular case, the process that we are interested on is defined by the unique strong solution of

$$Z_t = Z_0 + b\int_0^t Z_s\mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t \sigma Z_s \mathrm{d}B_s^{(e)}, \tag{5.1}$$

where g is a real-valued continuous function satisfying the conditions in Proposition 1 in Palau and Pardo (2018).

Our first main result provides a necessary and sufficient condition for the process Z defined by (5.1) to become extinct

Theorem 5.1. *Assume that Z is the unique strong solution of (5.1), then*

$$\mathbb{P}_x\Big(T_0 < \infty\Big) = 1 \qquad \textit{if and only if} \qquad \int^\infty \exp\left\{2\int_1^u \frac{g(z) - bz}{2\gamma^2 z + \sigma^2 z^2}\mathrm{d}z\right\}\mathrm{d}u = \infty. \tag{5.2}$$

Moreover

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = \infty\right) = 1 - \mathbb{P}_x\Big(T_0 < \infty\Big).$$

In particular, we may have the following situations

i) If there exist $z_0 > 0$ and $w < b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \leq wz$, then

$$\mathbb{P}_x(T_0 < \infty) < 1.$$

An example of this situation is the cooperative case, that is to say when $g(z)$ is decreasing and $b > \frac{\sigma^2}{2}$.

ii) If there exist $z_0 > 0$ and $w > b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \geq wz$, then

$$\mathbb{P}_x(T_0 < \infty) = 1.$$

An example of this situation are large competition mechanisms, that is to say for $g(z) \geq bz$ for any z large enough. For instance, the latter holds for the so-called logistic case i.e. $g(z) = cz^2$.

Proof of Theorem 5.1: We first observe from Dubins-Schwarz Theorem, that the law of Z is equal to the law of the following diffusion

$$\mathrm{d}Y_t = (bY_t - g(Y_t))\mathrm{d}t - \sqrt{2\gamma^2 Y_t + \sigma^2 Y_t^2}\mathrm{d}W_t,$$

where W is a standard Brownian motion. Associated to Y, we introduce for any $z \in \mathbb{R}$,

$$\mathtt{b}(z) := g(z) - bz, \qquad \mathtt{d}(z) := \frac{1}{2}\Big(2\gamma^2 z + \sigma^2 z^2\Big),$$

as well as the following functions related with the scale function of Y, for any $x, l \in \mathbb{R}_+$

$$s(l) = \exp\left\{\int_1^l \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}, \quad S(l,x) = \int_l^x s(u)\mathrm{d}u \quad \text{and} \quad \Sigma(l,x) = \int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)s(\eta)}\mathrm{d}\eta\right) s(u)\mathrm{d}u.$$

Observe that for any $x \in \mathbb{R}_+$,

$$S(0,x) = \int_0^x \exp\left\{2\int_1^u \frac{g(z) - bz}{2\gamma^2 z + \sigma^2 z^2}\mathrm{d}z\right\}\mathrm{d}u. \tag{5.3}$$

For simplicity, we denote $S(x) = S(0,x)$.

In order to prove the first statement of this proposition, we follow the approach of Chapter 15 in Karlin and Taylor (1981) which ensures that the equivalence (5.2) follows from the fact that $\lim_{l\to 0}\Sigma(l,x)$ is finite. Indeed, according to Lemma 15.6.3 in Karlin and Taylor (1981), the finiteness of $\lim_{l\to 0}\Sigma(l,x)$ for an $x > 0$ implies the finiteness of $\lim_{l\to 0} S(l,x) = S(0,x)$ for all $x \geq 0$. Then Lemma 15.6.2 in Karlin and Taylor (1981) guarantees that for any $y \geq x$, $T_0 \wedge T_y < \infty$, a.s., and Section 3 of Chapter 15 provides the following formulation

$$\mathbb{P}_x(T_0 < T_y) = \frac{S(x) - S(y)}{S(0) - S(y)}. \tag{5.4}$$

By making y tend to ∞, we find the equivalence (5.2) as required.

Hence let us show that $\lim_{l\to 0}\Sigma(l,x)$ is finite. In order to do so, we fix $\varepsilon > 0$ and $x \in (0,1)$ in such a way that for any $z \leq x$, $|\mathtt{b}(z)| \leq \varepsilon$. Therefore

$$\begin{aligned}\Sigma(l,x) &= \int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\exp\left\{\int_\eta^1 \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}\eta\right)\exp\left\{-\int_u^1 \frac{\mathtt{b}(z)}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}u \\ &\leq C_1(x)\int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\exp\left\{\int_\eta^x \frac{\varepsilon}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}\eta\right)\exp\left\{\int_u^x \frac{\varepsilon}{\mathtt{d}(z)}\mathrm{d}z\right\}\mathrm{d}u \\ &\leq C_2(x)\int_l^x \left(\int_u^x \frac{1}{\mathtt{d}(\eta)}\left(\frac{1 + \frac{\sigma^2}{2\gamma^2}\eta}{\eta}\right)^{\varepsilon/\gamma^2}\mathrm{d}\eta\right)\left(\frac{1 + \frac{\sigma^2}{2\gamma^2}u}{u}\right)^{\varepsilon/\gamma^2}\mathrm{d}u,\end{aligned} \tag{5.5}$$

where $C_1(x)$ and $C_2(x)$ are positive constants that only depend on x. Moreover, in a neighbourhood of 0, we have

$$\frac{1}{\mathtt{d}(\eta)}\left(\frac{1 + \frac{\sigma^2}{2\gamma^2}\eta}{\eta}\right)^{\varepsilon/\gamma^2} \underset{\eta\to 0}{\sim} \frac{1}{2\gamma^2}\frac{1}{\eta^{1+\varepsilon/\gamma^2}},$$

which is not integrable at 0. Hence,

$$\int_u^x \frac{1}{\mathtt{d}(\eta)}\left(\frac{1 + \frac{\sigma^2}{2\gamma^2}\eta}{\eta}\right)^{\varepsilon/\gamma^2}\mathrm{d}\eta \underset{u\to 0}{\sim} C_3(x)\frac{1}{u^{\varepsilon/\gamma^2}},$$

where $C_3(x)$ is a positive constant that only depends on x. This implies that the integrand on the right-hand side of the last inequality in (5.5) is equivalent to $u^{-2\varepsilon/\gamma^2}$ which is integrable at 0 as soon as ε is chosen small enough. The latter implies that $\lim_{l\to 0}\Sigma(l,x)<\infty$ which completes the first statement of this proposition.

In order to finish the proof, note that for any $y>x$,

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z(t)=\infty\right)\geq \mathbb{P}_x(T_y<T_0)=\frac{S(0)-S(x)}{S(y)-S(0)}.$$

Since it holds for any $y\geq x$, we can take y goes to ∞. By writing $S(\infty):=\lim_{y\to\infty}S(y)\in(0,\infty]$, we deduce

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z(t)=\infty\right)\geq \frac{S(0)-S(x)}{S(\infty)-S(0)},$$

and the right-hand side is equal to $1-\mathbb{P}_x(T_0<\infty)$ according to (5.4), whenever $S(\infty)$ is finite or not. This ends the proof. $\square$

Our second result gives a formulation the Laplace transform of the first passage time

$$T_a=\inf\{t: Z_t\leq a\},\qquad \text{for}\quad a\geq 0,$$

by using the solution to the Ricatti equation described in the next Lemma and depending on the scale function S defined by (5.3). The proof of Proposition 5.1 guarantees that S is well-defined. Moreover, it is clear that the function $S:\mathbb{R}_+\to(0,S(\infty))$ is continuous and bijective, and under condition (5.2), $S(\infty)$ equals ∞. We denote by $\bar{\varphi}(x)$ the inverse of S on $(0,S(\infty))$. Following similar arguments to those provided in the proof of Lemma 2.1 in Lambert (2005), we deduce the following properties on the solution to the Ricatti equation that we are interested in.

Lemma 5.2. *For any $\lambda>0$, there exists a unique non-negative solution $\bar{y}_\lambda$ on $(0,S(\infty))$ to the equation*

$$y'=y^2-\lambda\bar{r}^2,$$

where

$$\bar{r}(z)=\frac{\bar{\varphi}'(z)}{\sqrt{\gamma^2\bar{\varphi}(z)+\frac{\sigma^2}{2}(\bar{\varphi}(z))^2}},$$

such that it vanishes at $S(\infty)$. Moreover, $\bar{y}_\lambda$ is positive on $(0,S(\infty))$, and for any z sufficiently small or close to $S(\infty)$, $\bar{y}_\lambda(z)\leq\sqrt{\lambda}\bar{r}(z)$. In particular, $\bar{y}_\lambda$ is integrable at 0 if $\gamma\neq 0$, and it decreases initially and ultimately.

Our next result provides explicitly the Laplace transform of T_a in terms of the function $\bar{y}_\lambda$.

Proposition 5.3. *Assume that $\gamma>0$. Then, for any $x\geq a\geq 0$, and for any $\lambda>0$,*

$$\mathbb{E}_x\left[e^{-\lambda T_a}\right]=\exp\left\{-\int_{S(a)}^{S(x)}\bar{y}_\lambda(u)\mathrm{d}u\right\}. \tag{5.6}$$

Note that if (5.2) is satisfied, then $T_a<\infty$ a.s.

Proof: Let $x\geq a>0$, then $(Z_{t\wedge T_a},t\geq 0)$, under $\mathbb{P}_x$, is a process with values in $[a,\infty)$. For any $y\geq a$, we define

$$f_{\lambda,a}(y)=\exp\left\{-\int_{S(a)}^{S(x)}\bar{y}_\lambda(u)\mathrm{d}u\right\}.$$

A direct computation ensures that $f_{\lambda,a}$ is a C^2-function on $[a,\infty)$, bounded by 1, $f_{\lambda,a}(a)=1$ and such that it solves

$$\mathtt{d}(y)f''(y)-\mathtt{b}(y)f'(y)-\lambda f(y)=0. \tag{5.7}$$

Applying Itô Formula to the function $F(t,y) = e^{-\lambda t} f_{\lambda,a}(y)$ and the process $(Z_{t\wedge T_a}, t \geq 0)$, we obtain by means of (5.7),

$$e^{-\lambda t} f_{\lambda,a}(Z_{t\wedge T_a}) = f_{\lambda,a}(x) + \int_0^{t\wedge T_a} f'_{\lambda,a}(Z_s)\sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \sigma \int_0^{t\wedge T_a} f'_{\lambda,a}(Z_s) Z_s \mathrm{d}B_s^{(e)}.$$

We then use a sequence of stopping time $(T_n, n \geq 1)$ that reduces the two local martingales of the right-hand side and from the optimal stopping theorem, we obtain for any $n \geq 1$

$$\mathbb{E}_x \left[e^{-\lambda T_n \wedge T_a} f_{\lambda,a}(Z_{T_n \wedge T_a}) \right] = f_{\lambda,a}(x).$$

Letting n go to ∞ gives (5.6) for any $x \geq a > 0$. We finally let a go to 0 to deduce the result for $a = 0$ and conclude the proof. □

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