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Two families of improper stochastic integrals with respect to Lévy processes

Ken-iti Sato

Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074 Japan
E-mail address: `ken-iti.sato@nifty.ne.jp`

Abstract. Let $-\infty < \beta < \alpha < \infty$. Denote the inverse functions of $s = g_\alpha(t) = \int_t^\infty u^{-\alpha-1}e^{-u}du$, $t \in (0,\infty)$, and

$$s = g_{\beta,\alpha}(t) = (\Gamma(\alpha-\beta))^{-1}\int_t^1 (1-u)^{\alpha-\beta-1}u^{-\alpha-1}du,$$

$t \in (0,1)$, by $t = f_\alpha(s)$ and $t = f_{\beta,\alpha}(s)$, respectively. Improper stochastic integrals $\int_0^{\infty-} f_\alpha(s)dX_s^{(\mu)}$ and $\int_0^{\infty-} f_{\beta,\alpha}(s)dX_s^{(\mu)}$ with respect to Lévy processes $X^{(\mu)} = \{X_t^{(\mu)}: t \geqslant 0\}$ on $\mathbb{R}^d$ having distribution μ at time 1 are studied. Denote the distributions of these improper integrals by $\Psi_\alpha(\mu)$ and $\Phi_{\beta,\alpha}(\mu)$, respectively. Thus operators Ψ_α and $\Phi_{\beta,\alpha}$ from μ to $\Psi_\alpha(\mu)$ and $\Phi_{\beta,\alpha}(\mu)$, respectively, are defined. The domains of Ψ_α and $\Phi_{\beta,\alpha}$ and the ranges of Ψ_α are described. They are subclasses of the class $ID(\mathbb{R}^d)$ of infinitely divisible distributions on $\mathbb{R}^d$. The ranges of Ψ_α constitute a decreasing family which includes the Goldie–Steutel–Bondesson class $B(\mathbb{R}^d)$ for $\alpha = -1$ and the Thorin class $T(\mathbb{R}^d)$ for $\alpha = 0$. The relation $\Psi_\alpha = \Psi_\beta\Phi_{\beta,\alpha} = \Phi_{\beta,\alpha}\Psi_\beta$, with the equality of the domains, is established. The improper stochastic integrals in two generalized senses (compensated and essential) and in one restricted sense (absolutely definable) are also studied.

1. Introduction

Let $ID(\mathbb{R}^d)$ be the class of infinitely divisible distributions on $\mathbb{R}^d$. For each $\mu \in ID(\mathbb{R}^d)$ denote by $X^{(\mu)} = \{X_t^{(\mu)}: t \geqslant 0\}$ a Lévy process on $\mathbb{R}^d$ with distribution μ at time 1. Given a real-valued nonrandom function $f(s)$ of $s \in [0,\infty)$, we denote by $\Phi_f(\mu)$ or $\Phi_f\mu$ the distribution of $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ when this improper stochastic integral is definable. This Φ_f can be considered as an operator with domain $\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_f;\mathbb{R}^d)$ being the class of $\mu \in ID(\mathbb{R}^d)$ for which $\int_0^{\infty-} f(s)dX_s^{(\mu)}$

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is definable and range $\mathfrak{R}(\Phi_f) = \mathfrak{R}(\Phi_f; \mathbb{R}^d) = \{\Phi_f(\mu) \colon \mu \in \mathfrak{D}(\Phi_f; \mathbb{R}^d)\}$. The range is again a subclass of $ID(\mathbb{R}^d)$. We are concerned with two families of Φ_f which generalize the following three examples:

1. If $f(s) = e^{-s}$, then $\mathfrak{D}(\Phi_f) = ID_{\log}(\mathbb{R}^d)$, the class of infinitely divisible distributions on $\mathbb{R}^d$ with finite log-moment, and $\mathfrak{R}(\Phi_f) = L(\mathbb{R}^d)$, the class of self-decomposable distributions on $\mathbb{R}^d$ (Wolfe (1982), Gravereaux (1982), Jurek and Vervaat (1983), Sato and Yamazato (1983)). We denote this Φ_f by Φ. The stationary distribution of the Ornstein–Uhlenbeck type process driven by Lévy process $X^{(\mu)}$ is equal to $\Phi(\mu)$.

2. Let $\log^+\theta = 0 \vee \log\theta$ for $\theta > 0$. If $f(s) = \log^+(1/s)$, then Φ_f is denoted by Υ and $\mathfrak{R}(\Upsilon) = B(\mathbb{R}^d)$, the Goldie-Steutel-Bondesson class on $\mathbb{R}^d$, with $\mathfrak{D}(\Upsilon) = ID(\mathbb{R}^d)$ (Barndorff-Nielsen and Thorbjørnsen (2002a,b), Barndorff-Nielsen et al. (2006)).

3. Define $f(s)$ by $s = \int_{f(s)}^{\infty} u^{-1}e^{-u}du$. Then Φ_f is denoted by Ψ_0, $\mathfrak{D}(\Psi_0) = ID_{\log}(\mathbb{R}^d)$, and $\mathfrak{R}(\Psi_0) = T(\mathbb{R}^d)$, the Thorin class on $\mathbb{R}^d$ (Barndorff-Nielsen et al. (2006)).

The three operators Φ, Υ, and Ψ_0 are related as

$$\Psi_0 = \Upsilon\Phi = \Phi\Upsilon, \tag{1.1}$$

which is shown in Barndorff-Nielsen et al. (2006).

Let $-\infty < \beta < \alpha < \infty$. Let

$$g_\alpha(t) = \int_t^\infty u^{-\alpha-1}e^{-u}du \quad \text{for } t \in (0,\infty), \tag{1.2}$$

and let $a_\alpha = g_\alpha(0+)$, which equals $\Gamma(-\alpha)$ for $\alpha < 0$ and ∞ for $\alpha \geqslant 0$. Denote by $f_\alpha(s) = t$ for $0 < s < a_\alpha$ the inverse function of $s = g_\alpha(t)$ for $0 < t < \infty$. Let

$$g_{\beta,\alpha}(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_t^1 (1-u)^{\alpha-\beta-1}u^{-\alpha-1}du \quad \text{for } 0 < t \leqslant 1, \tag{1.3}$$

and let $a_{\beta,\alpha} = g_{\beta,\alpha}(0+)$, which equals $\Gamma(-\alpha)/\Gamma(-\beta)$ for $\alpha < 0$ and ∞ for $\alpha \geqslant 0$. Denote by $f_{\beta,\alpha}(s) = t$ for $0 < s < a_{\beta,\alpha}$ the inverse function of $s = g_{\beta,\alpha}(t)$ for $0 < t < 1$. The function $f_\alpha(s)$ strictly decreases from ∞ to 0 as s goes from 0 to a_α; $f_{\beta,\alpha}(s)$ strictly decreases from 1 to 0 as s goes from 0 to $a_{\beta,\alpha}$. For $\mu \in ID(\mathbb{R}^d)$, we denote

$$\Psi_\alpha(\mu) = \Phi_{f_\alpha}(\mu) = \mathcal{L}\left(\int_0^{\infty-} f_\alpha(s)dX_s^{(\mu)}\right) \quad \text{for } \alpha \geqslant 0,$$

$$\Psi_\alpha(\mu) = \Phi_{f_\alpha}(\mu) = \mathcal{L}\left(\int_0^{\Gamma(-\alpha)} f_\alpha(s)dX_s^{(\mu)}\right) \quad \text{for } \alpha < 0,$$

$$\Phi_{\beta,\alpha}(\mu) = \Phi_{f_{\beta,\alpha}}(\mu) = \mathcal{L}\left(\int_0^{\infty-} f_{\beta,\alpha}(s)dX_s^{(\mu)}\right) \quad \text{for } \alpha \geqslant 0,\ \beta < \alpha,$$

$$\Phi_{\beta,\alpha}(\mu) = \Phi_{f_{\beta,\alpha}}(\mu) = \mathcal{L}\left(\int_0^{\Gamma(-\alpha)/\Gamma(-\beta)} f_{\beta,\alpha}(s)dX_s^{(\mu)}\right) \quad \text{for } \beta < \alpha < 0,$$

whenever the integral or the improper integral is defined. Here, for an $\mathbb{R}^d$-valued random variable Y, $\mathcal{L}(Y)$ denotes the distribution of Y. The operator Ψ_0 coincides with that of the third example above; $\Phi_{-1,0}$ and Ψ_{-1} coincide with Φ and Υ in the first and the second example above, respectively. Furthermore,

$$g_{\alpha-1,\alpha}(t) = (1/\alpha)(t^{-\alpha} - 1), \quad f_{\alpha-1,\alpha}(s) = (1+\alpha s)^{-1/\alpha} \quad \text{for } \alpha \neq 0$$

and

$$g_{\beta,-1}(t) = (1/\Gamma(-\beta))(1-t)^{-\beta-1}, \quad f_{\beta,-1}(s) = 1 - (\Gamma(-\beta)s)^{1/(-\beta-1)} \quad \text{for } \beta < -1.$$

In this paper we study, first, the domains of Ψ_α and $\Phi_{\beta,\alpha}$, second, the relation between Ψ_α and $\Phi_{\beta,\alpha}$, and third, the ranges of Ψ_α.

The following asymptotic behaviors of f_α and $f_{\beta,\alpha}$ are proved by standard techniques.

Proposition 1.1. *We have*

$$f_\alpha(s) \sim \log(1/s) \quad \text{as } s \downarrow 0 \text{ for } \alpha \in \mathbb{R}, \tag{1.4}$$

and, as $s \to \infty$,

$$f_\alpha(s) \sim e^{c_1} e^{-s} \quad \text{for } \alpha = 0, \tag{1.5}$$

$$f_\alpha(s) \sim (\alpha s)^{-1/\alpha} \quad \text{for } \alpha > 0, \tag{1.6}$$

$$f_1(s) = s^{-1} - s^{-2}\log s + o(s^{-2}\log s), \tag{1.7}$$

$$f_{\beta,0}(s) \sim e^{c_2} e^{-\Gamma(-\beta)s} \quad \text{for } \beta < 0, \tag{1.8}$$

$$f_{\beta,\alpha}(s) \sim (\alpha\Gamma(\alpha-\beta)s)^{-1/\alpha} \quad \text{for } \alpha > 0 \text{ and } \beta < \alpha, \tag{1.9}$$

$$f_{\beta,1}(s) = (\Gamma(1-\beta))^{-1}s^{-1} + \beta(\Gamma(1-\beta))^{-2}s^{-2}\log s + o(s^{-2}\log s) \text{ for } \beta < 1, \tag{1.10}$$

where

$$c_1 = \int_1^\infty u^{-1}e^{-u}du - \int_0^1 u^{-1}(1-e^{-u})du$$

and $c_2 = (\beta+1)\int_0^1 (1-u)^{-\beta-2}\log(1/u)du$.

In Section 2 we study the domains of Φ_f when $f(s)$ has an asymptotic behavior for $s \to \infty$ slightly more general than that of $f_\alpha(s)$ and $f_{\beta,\alpha}(s)$ in Proposition 1.1. We assume that f is locally square-integrable on $[0,\infty)$. Then $\int_0^t f(s)dX_s^{(\mu)}$ is defined for all finite $t > 0$ and for all $\mu \in ID(\mathbb{R}^d)$, as shown in Sato (2005). The improper stochastic integral $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is defined as the limit in probability of $\int_0^t f(s)dX_s^{(\mu)}$ as $t \to \infty$, whenever the limit exists. The domain $\mathfrak{D}(\Phi_f;\mathbb{R}^d)$ is the class of $\mu \in ID(\mathbb{R}^d)$ for which $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is definable in this sense. Two extensions, compensated and essential, of the improper stochastic integrals are introduced in Sato (2005), together with two extended domains $\mathfrak{D}_{\mathrm{c}}(\Phi_f;\mathbb{R}^d)$ and $\mathfrak{D}_{\mathrm{e}}(\Phi_f;\mathbb{R}^d)$. In this paper we introduce a restricted domain $\mathfrak{D}^0(\Phi_f;\mathbb{R}^d)$, which is the class of $\mu \in ID(\mathbb{R}^d)$ such that $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is absolutely definable. We will study $\mathfrak{D}^0$, $\mathfrak{D}$, $\mathfrak{D}_{\mathrm{c}}$, and $\mathfrak{D}_{\mathrm{e}}$ for Φ_f on $\mathbb{R}^d$. When $f = f_1$, the description of $\mathfrak{D}$, $\mathfrak{D}_{\mathrm{c}}$, and $\mathfrak{D}_{\mathrm{e}}$ is already given in Sato (2005). It is noteworthy that, for a class of f which includes f_1 and $f_{\beta,1}$, the classes $\mathfrak{D}^0$, $\mathfrak{D}$, $\mathfrak{D}_{\mathrm{c}}$, and $\mathfrak{D}_{\mathrm{e}}$ for Φ_f are independent of f and different from each other.

Section 3 proves the relation

$$\Psi_\alpha = \Psi_\beta\Phi_{\beta,\alpha} = \Phi_{\beta,\alpha}\Psi_\beta \quad \text{for } -\infty < \beta < \alpha < \infty, \tag{1.11}$$

which generalizes (1.1). In general, given improper stochastic integral operators Φ_f and Φ_g, we define the domain of the composite operator $\Phi_g\Phi_f$ as

$$\mathfrak{D}(\Phi_g\Phi_f;\mathbb{R}^d)=\{\mu\in ID(\mathbb{R}^d)\colon \mu\in\mathfrak{D}(\Phi_f;\mathbb{R}^d)\text{ and }\Phi_f\mu\in\mathfrak{D}(\Phi_g;\mathbb{R}^d)\}.$$

The assertion (1.11) includes the equality of the domains. The proof of (1.11) is complicated when $\alpha=1$ and we treat it separately.

In Section 4 a description of the ranges of Ψ_α is provided. We obtain a new decreasing family of classes $\mathfrak{R}(\Psi_\alpha)$, $-\infty<\alpha<2$, satisfying $\mathfrak{R}(\Psi_{-1})=B(\mathbb{R}^d)$ and $\mathfrak{R}(\Psi_0)=T(\mathbb{R}^d)$. The relations of $\mathfrak{R}(\Psi_\alpha)$ with the class $\mathfrak{S}_\alpha$ of α-stable distributions on $\mathbb{R}^d$, the class $\mathfrak{S}_\alpha^0$ of strictly α-stable distributions on $\mathbb{R}^d$, and the class of tempered α-stable distributions in the sense of Rosiński (2004) are established. In particular, for $0<\alpha\leqslant 1$, $\mathfrak{S}_\alpha\subset\bigcap_{\beta<\alpha}\mathfrak{R}(\Psi_\beta)$ but $\mathfrak{S}_\alpha\not\subset\mathfrak{R}(\Psi_\alpha)$; for $1<\alpha\leqslant 2$, $\mathfrak{S}_\alpha^0\subset\bigcap_{\beta<\alpha}\mathfrak{R}(\Psi_\beta)$ but $\mathfrak{S}_\alpha^0\not\subset\mathfrak{R}(\Psi_\alpha)$. The moment properties of $\mathfrak{R}(\Psi_\alpha)$ are also discussed.

In the final section we will give some comments on remaining problems.

In the direction contrary to ours, Jurek (1985) is interested in obtaining stochastic integral representations for given classes of limit distributions derived from sequences of independent random variables. He shows in Jurek (1983) that the decreasing sequence of the classes $L_m(\mathbb{R}^d)$, $m=0,1,2,\ldots$, beginning with $L_0(\mathbb{R}^d)=L(\mathbb{R}^d)$, has representation

$$L_m(\mathbb{R}^d)=\mathfrak{R}(\Phi_{h_m})\text{ with }\mathfrak{D}(\Phi_{h_m})=\left\{\mu\in ID(\mathbb{R}^d)\colon \int_{\mathbb{R}^d}(\log^+|x|)^{m+1}\mu(dx)<\infty\right\},$$

where $h_m(s)=e^{-s^{1/(m+1)}}$, although his expression is different. See Jurek (1983), Rocha-Arteaga and Sato (2003), or Sato (1980) for the definition of $L_m(\mathbb{R}^d)$.

This work is much influenced by discussions with M. Maejima, V. Pérez-Abreu, and O.E. Barndorff-Nielsen in Yokohama, Nagoya, London, and in e-mails. Barndorff-Nielsen and Pérez-Abreu have studied Ψ_α in the level of Lévy measures as a one-parameter generalization of the Lévy measure transformation connected with Υ. Some of their results are given in Barndorff-Nielsen and Pérez-Abreu (2005). Those three and the author have been discussing extensions of the paper Barndorff-Nielsen et al. (2006) for long time. Some ideas in this paper are in fact jointly developed. The author expresses his sincere thanks to them.

2. Domains of the stochastic integral operators

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. Elements of $\mathbb{R}^d$ are column d-vectors $x=(x_j)_{1\leqslant j\leqslant d}$; the inner product is $\langle x,y\rangle=\sum_{j=1}^d x_jy_j$ and the norm is $|x|=\langle x,x\rangle^{1/2}$. For $\mu\in ID(\mathbb{R}^d)$, $\widehat{\mu}(z)$, $z\in\mathbb{R}^d$, is the characteristic function of μ and $C_\mu(z)$ is the cumulant function of μ, that is, $C_\mu(z)$ is the unique continuous function satisfying $\widehat{\mu}(z)=e^{C_\mu(z)}$ and $C_\mu(0)=0$. Each $\mu\in ID(\mathbb{R}^d)$ corresponds to a unique Lévy–Khintchine triplet (A,ν,γ) in the sense that

$$C_\mu(z)=-\frac{1}{2}\langle z,Az\rangle+\int_{\mathbb{R}^d}g(z,x)\nu(dx)+i\langle\gamma,z\rangle, \tag{2.1}$$

$$g(z,x)=e^{i\langle z,x\rangle}-1-\frac{i\langle z,x\rangle}{1+|x|^2}, \tag{2.2}$$

where A is a $d \times d$ symmetric nonnegative-definite matrix, called the Gaussian covariance matrix of μ, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$, called the Lévy measure of μ, and γ is an element of $\mathbb{R}^d$, called the location parameter of μ. Sometimes we denote $\mu = \mu_{(A,\nu,\gamma)}$. As in Sato (1999), $C_\sharp = C_\sharp(\mathbb{R}^d)$ is the class of bounded continuous functions on $\mathbb{R}^d$ vanishing on a neighborhood of the origin. Let $C_\sharp^+$ be the class of nonnegative functions in $C_\sharp$. Denote by $\mathcal{B}(\mathbb{R}^d)$ the class of Borel sets in $\mathbb{R}^d$, and by p-lim the limit in probability.

We use the stochastic integrals of nonrandom functions with respect to natural additive processes or independently scattered random measures, developed by Urbanik and Woyczyński (1967), Rajput and Rosinski (1989), Kwapień and Woyczyński (1992), and Sato (2004, 2005). In particular, see the definition of local $X^{(\mu)}$-integrability in Sato (2005). The following fact is found in Sato (2005).

Proposition 2.1. *Fix d. Let $f(s)$ be an $\mathbb{R}$-valued measurable function on $[0,\infty)$. Then, $f(s)$ is locally $X^{(\mu)}$-integrable for all $\mu \in ID(\mathbb{R}^d)$ if and only if $f(s)$ is locally square-integrable on $[0,\infty)$, that is, $\int_0^t f(s)^2 ds < \infty$ for every $t \in [0,\infty)$.*

Two extensions of improper stochastic integrals are introduced in Sato (2005). We say that the compensated improper integral of f with respect to $X^{(\mu)}$ is definable if there is a nonrandom vector $q \in \mathbb{R}^d$ such that $\int_0^{\infty-} f(s)dX_s^{(\mu*\delta_{-q})}$ is definable. Here δ_{-q} is the distribution concentrated at $-q$. Let

$$\mathfrak{D}_{\mathrm{c}}(\Phi_f;\mathbb{R}^d) = \{\mu \in ID(\mathbb{R}^d)\colon \text{ compensated improper integral of } f \text{ with respect to } X^{(\mu)} \text{ is definable}\}. \tag{2.3}$$

We say that the essential improper integral of f with respect to X is definable if there is a nonrandom $\mathbb{R}^d$-valued function q_t on $[0,\infty)$ such that $\int_0^t f(s)dX_s - q_t$ is convergent in probability as $t \to \infty$. Let

$$\mathfrak{D}_{\mathrm{e}}(\Phi_f;\mathbb{R}^d) = \{\mu \in ID(\mathbb{R}^d)\colon \text{ essential improper integral of } f \text{ with respect to } X^{(\mu)} \text{ is definable}\}. \tag{2.4}$$

In order to introduce the concept of absolute definability, let us recall the following fact.

Proposition 2.2. *Let f be locally square-integrable on $[0,\infty)$. Then*

$$\int_0^{\infty-} f(s)dX_s^{(\mu)}$$

is definable if and only if

$$\lim_{t\to\infty} \int_0^t C_\mu(f(s)z)ds \text{ exists in } \mathbb{C} \text{ for all } z \in \mathbb{R}^d. \tag{2.5}$$

If $Y = \int_0^{\infty-} f(s)dX_s^{(\mu)}$ is definable, then

$$C_Y(z) = \int_0^{\infty-} C_\mu(f(s)z)ds \quad \text{for } z \in \mathbb{R}^d. \tag{2.6}$$

Proof. For any bounded measurable set B, the formula (4.7) of Sato (2004) says that

$$C_{Y(B)}(z) = \int_B C_\mu(f(s)z)ds \quad \text{for } Y(B) = \int_B f(s)dX_s^{(\mu)}. \tag{2.7}$$

Assume (2.5). Let $Y_{t,u} = \int_0^u f(s)dX_s^{(\mu)} - \int_0^t f(s)dX_s^{(\mu)} = \int_t^u f(s)dX_s^{(\mu)}$ for $0 < t < u$. Then

$$Ee^{i\langle z, Y_{t,u}\rangle} = \exp \int_t^u C_\mu(f(s)z)ds \to 1 \text{ as } t, u \to \infty.$$

Thus, for $\varepsilon > 0$, $P[|Y_{t,u}| > \varepsilon] \to 0$ as $t, u \to \infty$. Hence $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is definable.

Conversely, if $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is definable, then $\mathcal{L}\left(\int_0^t f(s)dX_s^{(\mu)}\right)$ tends to an infinitely divisible distribution as $t \to \infty$ and thus its characteristic function converges to a nonzero continuous function, which implies (2.5).

The last assertion in the proposition is now obvious. □

Corollary 2.3. *If*

$$\int_0^\infty |C_\mu(f(s)z)|ds < \infty \textit{ for all } z \in \mathbb{R}^d, \tag{2.8}$$

then $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ *is definable.*

If (2.8) holds, we say that the improper integral $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is *absolutely definable.* Let

$$\mathfrak{D}^0(\Phi_f; \mathbb{R}^d) = \left\{\mu \in ID(\mathbb{R}^d) \colon \int_0^{\infty-} f(s)dX_s^{(\mu)} \text{ is absolutely definable}\right\}. \tag{2.9}$$

We usually write $\mathfrak{D}(\Phi_f)$, $\mathfrak{D}_{\mathrm{c}}(\Phi_f)$, $\mathfrak{D}_{\mathrm{e}}(\Phi_f)$, $\mathfrak{D}^0(\Phi_f)$, suppressing to write $\mathbb{R}^d$. Clearly,

$$\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}_{\mathrm{c}}(\Phi_f) \subset \mathfrak{D}_{\mathrm{e}}(\Phi_f). \tag{2.10}$$

For the last inclusion, recall that

$$\int_0^t f(s)dX_s^{(\mu * \delta_{-q})} = \int_0^t f(s)dX_s^{(\mu)} - \int_0^t f(s)ds\, q.$$

We restrict $f(s)$ by its behavior as $s \to \infty$ and study the domains in (2.10) more explicitly. For two functions f and g, we write $f(s) \asymp g(s)$, $s \to \infty$, if there are positive constants a_1 and a_2 such that $0 < a_1 g(s) \leqslant f(s) \leqslant a_2 g(s)$ for all large s.

Theorem 2.4. *Let* $\alpha \in [0, 1) \cup (1, \infty)$. *Suppose that* φ_α *is locally square-integrable on* $[0, \infty)$ *and satisfies*

$$\varphi_0(s) \asymp e^{-cs} \textit{ as } s \to \infty \textit{ with some } c > 0, \tag{2.11}$$

$$\varphi_\alpha(s) \asymp s^{-1/\alpha} \textit{ as } s \to \infty \textit{ for } \alpha \in (0, 1) \cup (1, \infty). \tag{2.12}$$

(i) *If* $\alpha = 0$, *then*

$$\begin{aligned} \mathfrak{D}^0(\Phi_{\varphi_0}) &= \mathfrak{D}(\Phi_{\varphi_0}) = \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_0}) = \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_0}) \\ &= \left\{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{\mathbb{R}^d} \log^+ |x| \nu(dx) < \infty\right\} \\ &= \left\{\mu \in ID(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty\right\}, \end{aligned} \tag{2.13}$$

which we denote by $ID_{\log}(\mathbb{R}^d)$.

(ii) *If* $0<\alpha<1$, *then*

$$\begin{aligned}\mathfrak{D}^0(\Phi_{\varphi_\alpha}) &= \mathfrak{D}(\Phi_{\varphi_\alpha}) = \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) = \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_\alpha}) \\ &= \left\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1} |x|^\alpha \nu(dx)<\infty\right\} \\ &= \left\{\mu\in ID(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx)<\infty\right\}.\end{aligned} \tag{2.14}$$

(iii) *If* $1<\alpha<2$, *then*

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) \subsetneqq \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) = \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_\alpha}), \tag{2.15}$$

$$\begin{aligned}\mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) &= \left\{\mu=\mu_{(A,\nu,\gamma)}\colon \int_{|x|>1} |x|^\alpha \nu(dx)<\infty\right\} \\ &= \left\{\mu\in ID(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx)<\infty\right\},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathfrak{D}(\Phi_{\varphi_\alpha}) &= \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) \cap \left\{\mu=\mu_{(A,\nu,\gamma)}\colon \gamma = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}\right\} \\ &= \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) \cap \left\{\mu\in ID(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} x\,\mu(dx)=0\right\}.\end{aligned} \tag{2.17}$$

(iv) *If* $\alpha\geqslant 2$, *then*

$$\mathfrak{D}^0(\Phi_{\varphi_\alpha}) = \mathfrak{D}(\Phi_{\varphi_\alpha}) = \{\delta_0\} \subsetneqq \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha}) = \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_\alpha}) = \{\delta_\gamma\colon \gamma\in\mathbb{R}^d\}. \tag{2.18}$$

We prepare two propositions and a lemma.

Proposition 2.5. *Let f be a measurable function and let $X^{(\mu)}$ be a Lévy process on $\mathbb{R}^d$ with $\mu=\mu_{(A,\nu,\gamma)}$. Then f is locally $X^{(\mu)}$-integrable if and only if*

$$\int_0^t f(s)^2(\operatorname{tr} A)ds<\infty, \tag{2.19}$$

$$\int_0^t ds\int_{\mathbb{R}^d}(|f(s)x|^2\wedge 1)\nu(dx)<\infty, \tag{2.20}$$

$$\int_0^t ds\left|f(s)\gamma+f(s)\int_{\mathbb{R}^d}\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right|<\infty \tag{2.21}$$

for all $t\in(0,\infty)$.

This follows from Corollary 2.19 and Theorem 3.1 of Sato (2005).

Proposition 2.6. *Let $X^{(\mu)}$ be a Lévy process on $\mathbb{R}^d$ with $\mu=\mu_{(A,\nu,\gamma)}$. Let f be locally $X^{(\mu)}$-integrable. Let $Y_t=\int_0^t f(s)dX_s^{(\mu)}$ and let $(A^{Y_t},\nu^{Y_t},\gamma^{Y_t})$ be the triplet of $\mathcal{L}(Y_t)$. Then the following are true.*

(i) *For all $t\in(0,\infty)$ and $z\in\mathbb{R}^d$,*

$$\int_0^t |C_\mu(f(s)z)|ds<\infty \quad \textit{and} \quad C_{Y_t}(z)=\int_0^t C_\mu(f(s)z)ds, \tag{2.22}$$

$$A^{Y_t}=\int_0^t f(s)^2 A ds, \tag{2.23}$$

$$\nu^{Y_t}(B)=\int_0^t ds\int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) \quad \textit{for } B\in\mathcal{B}(\mathbb{R}^d) \textit{ satisfying } 0\notin B, \tag{2.24}$$

$$\gamma^{Y_t} = \int_0^t f(s)ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right) \nu(dx)\right). \tag{2.25}$$

(ii) *We have $\mu \in \mathfrak{D}(\Phi_f)$ if and only if the following three conditions are satisfied:*

$$\int_0^\infty f(s)^2 (\operatorname{tr} A) ds < \infty, \tag{2.26}$$

$$\int_0^\infty ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) < \infty, \tag{2.27}$$

$$\gamma^{Y_t} \text{ is convergent in } \mathbb{R}^d \text{ as } t \to \infty. \tag{2.28}$$

(iii) *We have $\mu \in \mathfrak{D}_{\mathrm{e}}(\Phi_f)$ if and only if* (2.26) *and* (2.27) *are satisfied.*

(iv) *If $\mu \in \mathfrak{D}(\Phi_f)$, then the triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ of $\mathcal{L}(Y_{\infty-})$ for $Y_{\infty-} = \int_0^{\infty-} f(s)dX_s^{(\mu)}$ is given by*

$$\widetilde{A} = \int_0^\infty f(s)^2 A ds, \tag{2.29}$$

$$\widetilde{\nu}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d) \text{ satisfying } 0 \notin B, \tag{2.30}$$

$$\widetilde{\gamma} = \lim_{t\to\infty} \gamma^{Y_t}. \tag{2.31}$$

The proofs of these results are given in Propositions 2.17, 5.5, 5.6, and Corollary 2.19 of Sato (2005).

Lemma 2.7. *For $\alpha \in [0,1) \cup (1,\infty)$ let φ_α be as in Theorem 2.4. Let φ_1 be locally square-integrable on $[0,\infty)$ and satisfies $\varphi_1(s) \asymp s^{-1}$ as $s \to \infty$. Let ν be a measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. Then*

$$\int_0^\infty ds \int_{\mathbb{R}^d} (|\varphi_\alpha(s)x|^2 \wedge 1)\nu(dx) < \infty, \tag{2.32}$$

if and only if

$$\begin{cases} \int_{\mathbb{R}^d} \log^+ |x| \nu(dx) < \infty & \text{when } \alpha = 0, \\ \int_{|x|>1} |x|^\alpha \nu(dx) < \infty & \text{when } 0 < \alpha < 2, \\ \nu = 0 & \text{when } \alpha \geqslant 2. \end{cases} \tag{2.33}$$

Proof. Since φ_α is locally square-integrable, $f = \varphi_\alpha$ satisfies (2.20) by Propositions 2.1 and 2.5. Let $0 < \alpha < 2$. There are $s_0 > 0$ and $0 < c_1 \leqslant c_2$ such that $c_1 s^{-1/\alpha} \leqslant \varphi_\alpha(s) \leqslant c_2 s^{-1/\alpha}$ for $s \geqslant s_0$. Then

$$\begin{aligned} \int_{s_0}^\infty ds \int_{\mathbb{R}^d} (|\varphi_\alpha(s)x|^2 \wedge 1)\nu(dx) &\leqslant \int_{s_0}^\infty ds \int_{\mathbb{R}^d} (|c_2 s^{-1/\alpha} x|^2 \wedge 1)\nu(dx) \\ &\leqslant c_3 \int_{s_0}^\infty ds \int_{\mathbb{R}^d} (|s^{-1/\alpha} x|^2 \wedge 1)\nu(dx) \quad \text{(with some } c_3\text{)} \\ &= c_3 \int_{s_0}^\infty ds \int_{|s^{-1/\alpha}x| \leqslant 1} |s^{-1/\alpha} x|^2 \nu(dx) + c_3 \int_{s_0}^\infty ds \int_{|s^{-1/\alpha}x|>1} \nu(dx) \\ &= I_1 + I_2 \quad \text{(say)}, \end{aligned}$$

$$I_1 = c_3 \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_{s_0 \vee |x|^\alpha} s^{-2/\alpha} ds$$

$$
\begin{aligned}
&= c_3 \int_{|x|\leqslant s_0^{1/\alpha}} |x|^2\nu(dx) \int_{s_0}^{\infty} s^{-2/\alpha} ds + c_3 \int_{|x|>s_0^{1/\alpha}} |x|^2\nu(dx) \int_{|x|^\alpha}^{\infty} s^{-2/\alpha} ds \\
&\leqslant c_4 \int_{|x|\leqslant s_0^{1/\alpha}} |x|^2\nu(dx) + c_5 \int_{|x|>s_0^{1/\alpha}} |x|^\alpha \nu(dx) \quad \text{(with some } c_4, c_5),
\end{aligned}
$$

$$
I_2 = c_3 \int_{\mathbb{R}^d} \nu(dx) \int_{s_0}^{s_0\vee|x|^\alpha} ds = c_3 \int_{|x|>s_0^{1/\alpha}} (|x|^\alpha - s_0)\nu(dx),
$$

and similar estimates from below are possible with c_2 replaced by c_1. Thus (2.32) and (2.33) are equivalent if $0<\alpha<2$.

If $\alpha = 0$, then essentially same estimates with $\log^+|x|$ in place of $|x|^\alpha$ yield the equivalence. If $\alpha \geqslant 2$, then (2.32) implies $\nu = 0$, as is seen from a modification of the discussion above. □

Proof of Theorem 2.4. (i) The proof is similar to that of (ii).

(ii) Let $0<\alpha<1$. There are $s_0>0$ and $0<c_1<c_2$ as in the proof of Lemma 2.7. Thus $\int_0^\infty \varphi_\alpha(s)^2 ds < \infty$ and $f(s) = \varphi_\alpha(s)$ satisfies (2.26). Lemma 2.7 says that it satisfies (2.27) if and only if

$$
\int_{|x|>1} |x|^\alpha \nu(dx) < \infty. \tag{2.34}
$$

This property (2.34) is equivalent to the property that $\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty$, which is a special case of Theorem 25.3 of Sato (1999). If (2.34) holds, then (2.28) also holds for $f = \varphi_\alpha$. Indeed, we have (2.21), $\int_{s_0}^\infty \varphi_\alpha(s)ds < \infty$, and

$$
\int_{s_0}^\infty \varphi_\alpha(s)ds \int_{\mathbb{R}^d} |x| \left| \frac{1}{1+|\varphi_\alpha(s)x|^2} - \frac{1}{1+|x|^2} \right| \nu(dx) < \infty, \tag{2.35}
$$

because, letting $\varphi_\alpha(s) < 1$ for $s \geqslant s_0$,

$$
\begin{aligned}
&\int_{s_0}^\infty \varphi_\alpha(s)ds \int_{\mathbb{R}^d} |x| \left| \frac{1}{1+|\varphi_\alpha(s)x|^2} - \frac{1}{1+|x|^2} \right| \nu(dx) \\
&\quad \leqslant \int_{s_0}^\infty \varphi_\alpha(s)ds \int_{\mathbb{R}^d} \frac{|x|^3\nu(dx)}{(1+|\varphi_\alpha(s)x|^2)(1+|x|^2)} \\
&\quad \leqslant c_2 \int_{s_0}^\infty s^{-1/\alpha}ds \int_{\mathbb{R}^d} \frac{|x|^3\nu(dx)}{(1+|c_1 s^{-1/\alpha}x|^2)(1+|x|^2)} \\
&\quad \leqslant c_2 \int_{s_0}^\infty s^{-1/\alpha}ds \left(\int_{|x|\leqslant 1} |x|^2\nu(dx) + \int_{|x|>1,\, |s^{-1/\alpha}x|\leqslant 1} |x|\nu(dx) \right. \\
&\qquad \left. + \int_{|x|>1,\, |s^{-1/\alpha}x|>1} \frac{|x|\nu(dx)}{c_1^2 s^{-2/\alpha}|x|^2} \right) \\
&\quad \leqslant c_2 \int_{s_0}^\infty s^{-1/\alpha}ds \int_{|x|\leqslant 1} |x|^2\nu(dx) + c_2 \int_{|x|>1} |x|\nu(dx) \int_{s_0\vee|x|^\alpha}^\infty s^{-1/\alpha}ds \\
&\qquad + c_2 c_1^{-2} \int_{|x|>1} |x|^{-1}\nu(dx) \int_{s_0}^{s_0\vee|x|^\alpha} s^{1/\alpha}ds \\
&\quad \leqslant c_3 \int_{|x|\leqslant 1} |x|^2\nu(dx) + c_4 \int_{|x|>1} |x|^\alpha \nu(dx)
\end{aligned}
$$

with some c_3, c_4. Further, if (2.34) holds, then (2.8) holds for $f=\varphi_\alpha$, since

$$\begin{aligned}C_\mu(\varphi_\alpha(s)z) &= -\frac{1}{2}\varphi_\alpha(s)^2\langle z, Az\rangle + \int_{\mathbb{R}^d} g(z,\varphi_\alpha(s)x)\nu(dx)\\ &\quad + i\left\langle z, \varphi_\alpha(s)\gamma + \varphi_\alpha(s)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|\varphi_\alpha(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right\rangle\end{aligned} \tag{2.36}$$

from (2.1) and since

$$|g(z,\varphi_\alpha(s)x)| \leqslant c_z(|\varphi_\alpha(s)x|^2 \wedge 1) \tag{2.37}$$

with c_z depending on z. Combining these considerations with (2.10), we obtain (2.14).

(iii) Let $1<\alpha<2$. We have $\int_{s_0}^\infty \varphi_\alpha(s)^2 ds < \infty$ but $\int_{s_0}^\infty \varphi_\alpha(s)ds = \infty$. If $\mu \in \mathfrak{D}(\Phi_{\varphi_\alpha})$, then we have (2.34) by Lemma 2.7, and

$$\gamma = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}, \tag{2.38}$$

which follows from (2.28) since

$$\begin{aligned}\int_{\mathbb{R}^d} x\left(\frac{1}{1+|\varphi_\alpha(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx) &= \int_{\mathbb{R}^d} x\frac{|x|^2-|\varphi_\alpha(s)x|^2}{(1+|\varphi_\alpha(s)x|^2)(1+|x|^2)}\nu(dx)\\ &\to \int_{\mathbb{R}^d}\frac{x|x|^2\nu(dx)}{1+|x|^2}\end{aligned}$$

as $s\to\infty$. The property (2.38) is equivalent to the property $\int_{\mathbb{R}^d} x\mu(dx)=0$, as is seen from differentiation of $C_\mu(z)$. Conversely, if (2.34) and (2.38) hold, then $\mu \in \mathfrak{D}(\Phi_{\varphi_\alpha})$, since

$$\begin{aligned}&\int_{s_0}^t \varphi_\alpha(s)ds\left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|\varphi_\alpha(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\\ &\qquad = -\int_{s_0}^t \varphi_\alpha(s)ds\int_{\mathbb{R}^d} x\frac{|\varphi_\alpha(s)x|^2}{1+|\varphi_\alpha(s)x|^2}\nu(dx)\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}&\int_{s_0}^\infty \varphi_\alpha(s)ds\int_{\mathbb{R}^d}\frac{|x|\,|\varphi_\alpha(s)x|^2\nu(dx)}{1+|\varphi_\alpha(s)x|^2}\\ &\leqslant c_2\int_{s_0}^\infty s^{-1/\alpha}ds\left(\int_{|x|\leqslant 1}|x|^3c_2^2s^{-2/\alpha}\nu(dx) + \int_{|x|>1,\,|s^{-1/\alpha}x|\leqslant 1}|x|^3c_2^2s^{-2/\alpha}\nu(dx)\right.\\ &\quad \left. + \int_{|x|>1,\,|s^{-1/\alpha}x|>1}\frac{|x|^3c_2^2s^{-2/\alpha}\nu(dx)}{c_1^2s^{-2/\alpha}|x|^2}\right)\\ &\leqslant c_2^3\int_{s_0}^\infty s^{-3/\alpha}ds\int_{|x|\leqslant 1}|x|^3\nu(dx) + c_2^3\int_{|x|>1}|x|^3\nu(dx)\int_{s_0\vee|x|^\alpha}^\infty s^{-3/\alpha}ds\\ &\quad + c_2^3c_1^{-2}\int_{|x|>1}|x|\nu(dx)\int_{s_0}^{s_0\vee|x|^\alpha} s^{-1/\alpha}ds\\ &\leqslant c_3\int_{|x|\leqslant 1}|x|^2\nu(dx) + c_4\int_{|x|>1}|x|^\alpha\nu(dx) < \infty\end{aligned}$$

with some c_3, c_4. If (2.34) and (2.38) hold, then we have $\mu\in\mathfrak{D}^0(\Phi_{\varphi_\alpha})$, using (2.36) and (2.37). If (2.34) holds, then $\mu\in\mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_\alpha})$ by virtue of Proposition 2.6 (iii), and

$\mu \in \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha})$ since $\mu * \delta_{-q} \in \mathfrak{D}(\Phi_{\varphi_\alpha})$ for $q = \gamma + \int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$. Thus we obtain (2.15)–(2.17).

(iv) Let $\alpha \geqslant 2$. Both $\int_{s_0}^\infty \varphi_\alpha(s)ds$ and $\int_{s_0}^\infty \varphi_\alpha(s)^2 ds$ are infinite. Use Proposition 2.6 and Lemma 2.7. If $\mu \in \mathfrak{D}(\Phi_{\varphi_\alpha})$, then $\mu = \delta_0$. If $\mu \in \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_\alpha})$, then $\mu = \delta_\gamma$. If $\mu = \delta_\gamma$, then clearly $\mu \in \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_\alpha})$. □

Next we turn our attention to the case $\alpha = 1$. This case is delicate and interesting.

Theorem 2.8. *Suppose that φ_1 is a locally square-integrable function on $[0,\infty)$ such that*

$$\varphi_1(s) \asymp s^{-1} \text{ as } s \to \infty \tag{2.40}$$

and, for some $s_0 > 0$, $c > 0$, and $\psi(s)$,

$$\varphi_1(s) = s^{-1}\psi(s) \text{ for } s \geqslant s_0 \text{ with } \int_{s_0}^\infty s^{-1}|\psi(s) - c|ds < \infty. \tag{2.41}$$

Then

$$\mathfrak{D}^0(\Phi_{\varphi_1}) \subsetneqq \mathfrak{D}(\Phi_{\varphi_1}) \subsetneqq \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_1}) \subsetneqq \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_1}), \tag{2.42}$$

$$\begin{aligned}\mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_1}) &= \left\{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{|x|>1} |x|\,\nu(dx) < \infty\right\} \\ &= \left\{\mu \in ID(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|\,\mu(dx) < \infty\right\},\end{aligned} \tag{2.43}$$

$$\begin{aligned}\mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_1}) = \mathfrak{D}_{\mathrm{e}}(\Phi_{\varphi_1}) \cap \{\mu = \mu_{(A,\nu,\gamma)} \colon \\ \lim_{t\to\infty} \int_{s_0}^t s^{-1}ds \int_{|x|>s} x\nu(dx) \text{ exists in } \mathbb{R}^d\},\end{aligned} \tag{2.44}$$

$$\begin{aligned}\mathfrak{D}(\Phi_{\varphi_1}) &= \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_1}) \cap \left\{\mu = \mu_{(A,\nu,\gamma)} \colon \gamma = -\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}\right\} \\ &= \mathfrak{D}_{\mathrm{c}}(\Phi_{\varphi_1}) \cap \left\{\mu \in ID(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} x\,\mu(dx) = 0\right\},\end{aligned} \tag{2.45}$$

$$\mathfrak{D}^0(\Phi_{\varphi_1}) = \mathfrak{D}(\Phi_{\varphi_1}) \cap \left\{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{s_0}^\infty s^{-1}ds \left|\int_{|x|>s} x\nu(dx)\right| < \infty\right\}. \tag{2.46}$$

Moreover,

$$\mathfrak{D}(\Phi_{\varphi_1}) \cap \left\{\mu = \mu_{(A,\nu,\gamma)} \colon \int_{s_0}^\infty s^{-1}ds \int_{|x|>s} |x|\nu(dx) < \infty\right\} \subsetneqq \mathfrak{D}^0(\Phi_{\varphi_1}). \tag{2.47}$$

Remark to Theorem 2.8. Suppose that $\int_{|x|>1} |x|\nu(dx) < \infty$. Then the condition that

$$\lim_{t\to\infty} \int_{s_0}^t s^{-1}ds \int_{|x|>s} x\nu(dx) \text{ exists in } \mathbb{R}^d \tag{2.48}$$

is equivalent to the condition that

$$\lim_{t\to\infty} \int_{|x|>1} x\log(|x| \wedge t)\nu(dx) \text{ exists in } \mathbb{R}^d. \tag{2.49}$$

This is a kind of *balancing condition* for the Lévy measure ν. The condition of finiteness of $\int_{s_0}^{\infty} s^{-1}ds \int_{|x|>s} |x|\nu(dx)$ is equivalent to the condition that

$$\int_{\mathbb{R}^d} |x| \log^+ |x| \nu(dx) < \infty.$$

Proof of Theorem 2.8. For brevity we write $\varphi = \varphi_1$ in this proof. Suppose that $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\varphi)$. Then,

$$\int_{|x|>1} |x|\, \nu(dx) < \infty \tag{2.50}$$

and (2.38) by the same reason as in the proof of (iii) of Theorem 2.4. We have also (2.28) from Proposition 2.6. Let us show that (2.28) is equivalent to the condition (2.48) under the conditions (2.38) and (2.50). First,

$$\begin{aligned}
&\int_{s_0}^{t} \varphi(s)ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\varphi(s)x|^2} - \frac{1}{1+|x|^2}\right) \nu(dx)\right) \\
&\quad = \int_{s_0}^{t} \varphi(s)ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\varphi(s)x|^2} - 1\right) \nu(dx) \\
&\quad = \int_{s_0}^{t} \psi(s)s^{-1}ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\psi(s)s^{-1}x|^2} - 1\right) \nu(dx) \\
&\quad = \int_{s_0}^{t} cs^{-1}ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|cs^{-1}x|^2} - 1\right) \nu(dx) + I_1(t) + I_2(t), \\
&I_1(t) = \int_{s_0}^{t} \psi(s)s^{-1}ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\psi(s)s^{-1}x|^2} - \frac{1}{1+|cs^{-1}x|^2}\right) \nu(dx), \\
&I_2(t) = \int_{s_0}^{t} (\psi(s)-c)s^{-1}ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|cs^{-1}x|^2} - 1\right) \nu(dx).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{s_0}^{\infty} \psi(s)s^{-1}ds \int_{\mathbb{R}^d} \frac{|x|\ \big||cs^{-1}x|^2 - |\psi(s)s^{-1}x|^2\big|}{(1+|\psi(s)s^{-1}x|^2)(1+|cs^{-1}x|^2)} \nu(dx) \\
&\quad \leqslant c_1 \int_{s_0}^{\infty} s^{-1}|c-\psi(s)|ds \int_{\mathbb{R}^d} \frac{|x|\,|s^{-1}x|^2 \nu(dx)}{(1+|\psi(s)s^{-1}x|^2)(1+|cs^{-1}x|^2)} \\
&\quad \leqslant c_2 \int_{s_0}^{\infty} s^{-1}|c-\psi(s)|ds \left(\int_{|x|\leqslant 1} |x|^2\nu(dx) + \int_{|x|>1} |x|\nu(dx)\right) < \infty
\end{aligned}$$

with some c_1 and c_2, $I_1(t)$ is convergent as $t \to \infty$. Since

$$\int_{s_0}^{\infty} |\psi(s)-c|s^{-1}ds \int_{\mathbb{R}^d} \frac{|x|\,|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2}$$

is finite in the same way, $I_2(t)$ is convergent as $t \to \infty$. Hence (2.28) is equivalent to the convergence of

$$\int_{s_0}^{t} s^{-1}ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|cs^{-1}x|^2} - 1\right) \nu(dx).$$

This is equivalent to the convergence of

$$\int_{s_0}^{t} s^{-1}ds \int_{|x|>1} x \left(\frac{1}{1+|cs^{-1}x|^2} - 1 \right) \nu(dx), \tag{2.51}$$

since

$$\int_{s_0}^{\infty} s^{-1}ds \int_{|x|\leqslant 1} \frac{|x|\,|cs^{-1}x|^2 \nu(dx)}{1+|cs^{-1}x|^2} \leqslant \frac{c}{2} \int_{s_0}^{\infty} s^{-2}ds \int_{|x|\leqslant 1} |x|^2 \nu(dx) < \infty,$$

noting that $\theta/(1+\theta^2) \leqslant 1/2$ for $\theta \geqslant 0$. Further, an equivalent condition is the convergence of

$$\int_{|x|>1} x \log \frac{1+c^2 s_0^{-2}|x|^2}{1+t^{-2}|x|^2} \nu(dx) \tag{2.52}$$

as $t \to \infty$, since

$$(2.51) = -\int_{|x|>1} x\nu(dx) \int_{s_0}^{t} \frac{c^2 s^{-3}|x|^2 ds}{1+c^2 s^{-2}|x|^2} = \frac{1}{2} \int_{|x|>1} x \log \frac{1+c^2 t^{-2}|x|^2}{1+c^2 s_0^{-2}|x|^2} \nu(dx).$$

This is equivalent to the condition that

$$\int_{|x|>1} x \log \frac{|x|}{1+t^{-1}|x|} \nu(dx) \text{ is convergent as } t \to \infty, \tag{2.53}$$

since we see that

$$\lim_{t\to\infty} \int_{|x|>1} x \left(\log \frac{1+c^2 s_0^{-2}|x|^2}{1+t^{-2}|x|^2} - 2\log \frac{|x|}{1+t^{-1}|x|} \right) \nu(dx) \text{ exists,}$$

noting that

$$1 \leqslant \frac{(1+t^{-1}|x|)^2}{1+t^{-2}|x|^2} \leqslant 2 \quad \text{and} \quad c^2 s_0^{-2} \leqslant \frac{1+c^2 s_0^{-2}|x|^2}{|x|^2} \leqslant 1 + c^2 s_0^{-2}.$$

Now

$$\lim_{t\to\infty} \left(\int_{|x|>1} x \log \frac{|x|}{1+t^{-1}|x|} \nu(dx) - \int_{|x|>1} x \log(|x| \wedge t) \nu(dx) \right) = 0,$$

since

$$\lim_{t\to\infty} x \left(\log \frac{|x|}{1+t^{-1}|x|} - \log(|x| \wedge t) \right) = 0$$

and

$$\begin{aligned} &\left| \log \frac{|x|}{1+t^{-1}|x|} - \log(|x| \wedge t) \right| \\ &\qquad = 1_{\{1<|x|\leqslant t\}}(x) \left| \log \frac{1}{1+t^{-1}|x|} \right| + 1_{\{|x|>t\}}(x) \left| \log \frac{t^{-1}|x|}{1+t^{-1}|x|} \right| \leqslant \log 2 \end{aligned}$$

for $|x| > 1$. Thus (2.28) is equivalent to (2.49). Finally, (2.49) is equivalent to (2.48), because

$$\begin{aligned} \int_{s_0}^{t} s^{-1}ds \int_{|x|>s} x\nu(dx) &= \int_{|x|>s_0} x\nu(dx) \int_{s_0}^{|x|\wedge t} s^{-1}ds \\ &= \int_{|x|>s_0} x \log(|x| \wedge t) \nu(dx) - \int_{|x|>s_0} x\nu(dx) \log s_0. \end{aligned}$$

Conversely, if (2.50), (2.38), and (2.48) are satisfied, then $\mu \in \mathfrak{D}(\Phi_f)$, as is seen from the discussion above.

Assertion (2.43) on $\mathfrak{D}_{\mathrm{e}}(\Phi_\varphi)$ follows from Proposition 2.6 (iii) and Lemma 2.7.

If $\mu=\mu_{(A,\nu,\gamma)}\in\mathfrak{D}_{\mathrm{c}}(\Phi_\varphi)$, then $\mu=\mu_{(A,\nu,\gamma')}\in\mathfrak{D}(\Phi_\varphi)$ for some γ' and hence (2.50) and (2.48) are fulfilled. Conversely, if we have (2.50) and (2.48), then $\mu=\mu_{(A,\nu,\gamma')}\in\mathfrak{D}(\Phi_\varphi)$ for $\gamma'=-\int x|x|^2(1+|x|^2)^{-1}\nu(dx)$ and $\mu\in\mathfrak{D}_{\mathrm{c}}(\Phi_\varphi)$. This shows (2.44).

Let us prove assertion (2.46). Suppose that μ satisfies (2.50), (2.38), and

$$\int_{s_0}^{\infty}s^{-1}ds\left|\int_{|x|>s}x\nu(dx)\right|<\infty. \tag{2.54}$$

In order to show that $\mu\in\mathfrak{D}^0(\Phi_\varphi)$, it is enough to show finiteness of

$$\int_{s_0}^{\infty}\varphi(s)ds\left|\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|\varphi(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right|, \tag{2.55}$$

since we have (2.36) and (2.37). We may assume $s_0>1$. We have

$$\begin{aligned}
(2.55)&=\int_{s_0}^{\infty}\varphi(s)ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|\varphi(s)x|^2}-1\right)\nu(dx)\right|\\
&=\int_{s_0}^{\infty}\psi(s)s^{-1}ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|\psi(s)s^{-1}x|^2}-1\right)\nu(dx)\right|\\
&\leqslant\int_{s_0}^{\infty}cs^{-1}ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|cs^{-1}x|^2}-1\right)\nu(dx)\right|+J_1+J_2,\\
J_1&=\int_{s_0}^{\infty}\psi(s)s^{-1}ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|\psi(s)s^{-1}x|^2}-\frac{1}{1+|cs^{-1}x|^2}\right)\nu(dx)\right|,\\
J_2&=\int_{s_0}^{\infty}|\psi(s)-c|s^{-1}ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|cs^{-1}x|^2}-1\right)\nu(dx)\right|.
\end{aligned}$$

The finiteness of J_1, J_2, and $\int_{s_0}^{\infty}s^{-1}ds\left|\int_{|x|\leqslant 1}x((1+|cs^{-1}x|^2)^{-1}-1)\nu(dx)\right|$ can be checked as before. Next,

$$\begin{aligned}
&\int_{s_0}^{\infty}s^{-1}ds\left|\int_{|x|>1}x\left(\frac{1}{1+|cs^{-1}x|^2}-1\right)\nu(dx)\right|\leqslant\int_{s_0}^{\infty}s^{-1}ds\left|\int_{|x|>s}x\nu(dx)\right|+J_3+J_4,\\
&\quad J_3=\int_{s_0}^{\infty}s^{-1}ds\left|\int_{|x|>s}\frac{x\nu(dx)}{1+|cs^{-1}x|^2}\right|,\\
&\quad J_4=\int_{s_0}^{\infty}s^{-1}ds\left|\int_{1<|x|\leqslant s}x\left(\frac{1}{1+|cs^{-1}x|^2}-1\right)\nu(dx)\right|,
\end{aligned}$$

and J_3 and J_4 are finite because

$$\begin{aligned}
&\int_{s_0}^{\infty}s^{-1}ds\int_{|x|>s}\frac{|x|\nu(dx)}{1+|cs^{-1}x|^2}=\int_{|x|>s_0}|x|\nu(dx)\int_{s_0}^{|x|}\frac{s^{-1}ds}{1+|cs^{-1}x|^2}\\
&\quad=\int_{|x|>s_0}|x|\left(-\log s_0+\log|x|-\frac{1}{2}\log(1+s_0^{-2}c^2|x|^2)+\frac{1}{2}\log(1+c^2)\right)\nu(dx),\\
&\int_{s_0}^{\infty}s^{-1}ds\int_{1<|x|\leqslant s}\frac{|x|\,|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2}=\int_{|x|>1}|x|\nu(dx)\int_{s_0\vee|x|}^{\infty}\frac{s^{-3}c^2|x|^2ds}{1+s^{-2}c^2|x|^2}\\
&\quad=\frac{1}{2}\int_{1<|x|\leqslant s_0}|x|\log(1+s_0^{-2}c^2|x|^2)\nu(dx)+\frac{1}{2}\int_{|x|>s_0}|x|\log(1+c^2)\nu(dx).
\end{aligned}$$

Conversely, suppose that $\mu \in \mathfrak{D}^0(\Phi_\varphi)$. Then we have (2.50) and (2.38), since $\mu \in \mathfrak{D}(\Phi_\varphi)$. We have

$$\int_0^\infty |\mathrm{Im}\, C_\mu(\varphi(s)z)| ds < \infty$$

and

$$\begin{aligned}
\mathrm{Im}\, C_\mu(\varphi(s)z) &= \int_{\mathbb{R}^d} \left(\sin\langle z, \varphi(s)x\rangle - \frac{\langle z, \varphi(s)x\rangle}{1+|\varphi(s)x|^2} \right) \nu(dx) \\
&\quad + \left\langle z, \varphi(s)\gamma + \varphi(s) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\varphi(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right\rangle,
\end{aligned}$$

$$\begin{aligned}
&\int_{s_0}^\infty ds \int_{\mathbb{R}^d} \left| \sin\langle z, \varphi(s)x\rangle - \frac{\langle z, \varphi(s)x\rangle}{1+|\varphi(s)x|^2} \right| \nu(dx) \\
&\quad \leqslant \int_{s_0}^\infty ds \int_{|\varphi(s)x|\leqslant 1} \left(\frac{1}{6} |\langle z, \varphi(s)x\rangle|^3 + |\langle z, \varphi(s)x\rangle| \frac{|\varphi(s)x|^2}{1+|\varphi(s)x|^2} \right) \nu(dx) \\
&\quad\quad + c_3 \int_{s_0}^\infty ds \int_{|\varphi(s)x|>1} \nu(dx) \\
&\quad \leqslant c_4 \int_{s_0}^\infty ds \int_{|\varphi(s)x|\leqslant 1} |\varphi(s)x|^3 \nu(dx) + c_3 \int_{s_0}^\infty ds \int_{|\varphi(s)x|>1} \nu(dx),
\end{aligned}$$

$$\begin{aligned}
\int_{s_0}^\infty ds \int_{|\varphi(s)x|\leqslant 1} |\varphi(s)x|^3 \nu(dx) &\leqslant c_2^3 \int_{s_0}^\infty ds \int_{c_1|s^{-1}x|\leqslant 1} |s^{-1}x|^3 \nu(dx) \\
&= c_2^3 \int_{\mathbb{R}^d} |x|^3 \nu(dx) \int_{s_0 \vee (c_1|x|)}^\infty s^{-3} ds < \infty,
\end{aligned}$$

$$\int_{s_0}^\infty ds \int_{|\varphi(s)x|>1} \nu(dx) \leqslant \int_{s_0}^\infty ds \int_{c_2|s^{-1}x|>1} \nu(dx) = \int_{\mathbb{R}^d} \nu(dx) \int_{s_0}^{s_0 \vee (c_2|x|)} ds < \infty,$$

where $c_1 s^{-1} \leqslant \varphi(s) \leqslant c_2 s^{-1}$ for $s \geqslant s_0$ and c_3 and c_4 are constants depending on z. Hence

$$\int_{s_0}^\infty ds \left| \left\langle z, \varphi(s)\gamma + \varphi(s) \int_{\mathbb{R}^d} x \left(\frac{1}{1+|\varphi(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right\rangle \right| < \infty.$$

Since z is arbitrary, it follows that (2.55) is finite. Then we obtain (2.54) similarly to the discussion that we made after (2.55). Thus we have proved (2.46). This also yields (2.47) except the strictness of the inclusion.

It remains to show the strictness of the inclusions in (2.42) and (2.47). Example 2.9 will show that $\mathfrak{D}^0(\Phi_\varphi) \neq \mathfrak{D}(\Phi_\varphi)$. It is evident from (2.44) and (2.45) that $\mathfrak{D}(\Phi_\varphi) \neq \mathfrak{D}_{\mathrm{c}}(\Phi_\varphi)$. In order to see that $\mathfrak{D}_{\mathrm{c}}(\Phi_\varphi) \neq \mathfrak{D}_{\mathrm{e}}(\Phi_\varphi)$, consider a measure

$$\nu(B) = \int_{|\xi|=1} \lambda(d\xi) \int_2^\infty 1_B(u\xi) \frac{du}{u^2 (\log u)^{1+p}}, \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where $0 < p \leqslant 1$ and λ is a finite measure on the unit sphere in $\mathbb{R}^d$ such that $\int_{|\xi|=1} \xi \lambda(d\xi) \neq 0$. Then $\int_{|x|>1} |x| \nu(dx) < \infty$ but (2.49) does not hold, since

$$\int_{|x|>1} x \log(|x| \wedge t) \nu(dx) = \int_{|\xi|=1} \xi \lambda(d\xi) \int_2^\infty \log(u \wedge t) \frac{du}{u (\log u)^{1+p}}$$

and $\int_2^\infty \log(u \wedge t) u^{-1} (\log u)^{-1-p} du \to \infty$ as $t \to \infty$. In order to see the strictness of the inclusion in (2.47), consider a symmetric Lévy measure ν such that $\int_{|x|>1} |x| \nu(dx) < \infty$ and $\int_{\mathbb{R}^d} |x| \log^+ |x| \nu(dx) = \infty$. $\square$

Example 2.9. Let φ_1 be as in Theorem 2.8. Define a measure ν on $\mathbb{R}^d$ as

$$\nu(B) = \int_{S_0} \lambda(d\xi) \sum_{n\in\mathbb{Z}} 1_B(n\xi) a_n, \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{2.56}$$

where S_0 is a nonempty Borel set on the unit sphere $\{|\xi| = 1\}$ satisfying $S_0 \cap (-S_0) = \emptyset$, λ is a finite measure on S_0 satisfying $\int_{S_0} \xi\lambda(d\xi) \neq 0$, $\mathbb{Z}$ is the class of all integers, and a_n, $n \in \mathbb{Z}$, are such that $a_0 = a_1 = a_{-1} = 0$ and, for positive integers n, m,

$$\begin{aligned}
&a_n = \frac{1}{n}\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right), \ a_{-n} = 0 \text{ for } 2^{m^2} < n < 2^{(m+1)^2}, \ m \text{ odd},\\
&a_n = 0, \ a_{-n} = \frac{1}{n}\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right) \text{ for } 2^{m^2} < n < 2^{(m+1)^2}, \ m \text{ even},\\
&a_n = \frac{1}{n}\left(\frac{1}{\log n} + \frac{1}{\log(n+1)}\right), \ a_{-n} = 0 \text{ for } n = 2^{m^2}, \ m \text{ even},\\
&a_n = 0, \ a_{-n} = \frac{1}{n}\left(\frac{1}{\log n} + \frac{1}{\log(n+1)}\right) \text{ for } n = 2^{m^2}, \ m \text{ odd}.
\end{aligned}$$

Then

$$\int_{|x|>1} |x|\nu(dx) = \int_{S_0} \lambda(d\xi) \sum_{n\in\mathbb{Z}} |n| a_n < \infty, \tag{2.57}$$

$$\int_2^\infty s^{-1} ds \left| \int_{|x|>s} x\nu(dx) \right| = \left| \int_{S_0} \xi\lambda(d\xi) \right| \int_2^\infty s^{-1} ds \left| \sum_{|n|>s} n a_n \right| = \infty, \tag{2.58}$$

$$\int_2^t s^{-1} ds \int_{|x|>s} x\nu(dx) = \left(\int_{S_0} \xi\lambda(d\xi) \right) \int_2^t s^{-1} ds \sum_{|n|>s} n a_n \text{ is convergent as } t \to \infty. \tag{2.59}$$

Consequently, if $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$ has this ν as Lévy measure and $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$, then $\mu \in \mathfrak{D}(\Phi_{\varphi_1}) \setminus \mathfrak{D}^0(\Phi_{\varphi_1})$.

Proof of (2.57)–(2.59) is as follows. Since $n(a_n + a_{-n})$ is either $= (\log n)^{-1} - (\log(n+1))^{-1} \sim n^{-1}(\log n)^2$ or $= (\log 2^{m^2})^{-1} + (\log(2^{m^2}+1))^{-1} \sim 2m^{-2}\log 2$, we see that $\sum_{k\in\mathbb{Z}} |k| a_k < \infty$, that is, (2.57). Since $\int_{|x|>s} x\nu(dx) = \int_{|x|\geqslant s} x\nu(dx)$ for almost every s, we consider $\int_{|x|\geqslant s} x\nu(dx)$. For some $b \in \mathbb{R}$ we have

$$\sum_{|k|\geqslant s} k a_k = \begin{cases} b + (\log n)^{-1} & \text{for } n-1 < s \leqslant n, \ 2 < n \leqslant 2^4,\\ b - (\log n)^{-1} & \text{for } n-1 < s \leqslant n, \ 2^4 < n \leqslant 2^9,\\ b + (\log n)^{-1} & \text{for } n-1 < s \leqslant n, \ 2^9 < n \leqslant 2^{16},\\ \cdots . \end{cases}$$

Since $\sum_{|k|\geqslant s} k a_k \to 0$ as $s \to \infty$, we see that $b = 0$. Hence $\left|\sum_{|k|\geqslant n} k a_k\right| = (\log n)^{-1}$ for $n = 3, 4, \ldots$, and

$$\int_2^\infty s^{-1} ds \left| \sum_{|k|\geqslant s} k a_k \right| \geqslant \frac{1}{3\log 3} + \frac{1}{4\log 4} + \cdots = \infty,$$

that is, (2.58). Next, let us show (2.59). Denote

$$p_m = \sum_{2^{m^2} \leqslant n < 2^{(m+1)^2}} \int_n^{n+1} \frac{ds}{s \log(n+1)},$$

$$q_m = \int_{2^{m^2}}^{2^{(m+1)^2}} \frac{ds}{s \log s} = 2 \log\left(1 + \frac{1}{m}\right).$$

Then $\sum_{m=1}^{\infty} (-1)^{m+1} q_m$ is convergent. Also $\sum_{m=1}^{\infty} |p_m - q_m| < \infty$, because

$$\begin{aligned} 0 \leqslant q_m - p_m &\leqslant \sum_{2^{m^2} \leqslant n < 2^{(m+1)^2}} \int_n^{n+1} s^{-1} ds \left(\frac{1}{\log n} - \frac{1}{\log(n+1)} \right) \\ &\leqslant \text{const} \sum_{2^{m^2} \leqslant n < 2^{(m+1)^2}} \frac{1}{n^2 (\log n)^2}. \end{aligned}$$

We have

$$\int_{2^{m^2}}^{2^{(m+1)^2}} s^{-1} ds \sum_{|k| \geqslant s} k a_k = (-1)^{m+1} p_m.$$

Hence

$$\sum_{m=1}^{M} \int_{2^{m^2}}^{2^{(m+1)^2}} s^{-1} ds \sum_{|k| \geqslant s} k a_k = \sum_{m=1}^{M} (-1)^{m+1} q_m + \sum_{m=1}^{M} (-1)^{m+1} (p_m - q_m),$$

which is convergent as $M \to \infty$. Thus we obtain (2.59).

Remark 2.10. *As is remarked immediately after Theorem 2.8,* (2.59) *is equivalent to* (2.49)*, that is, to convergence of* $\int_{|x|>1} x \log(|x| \wedge t) \nu(dx)$ *as* $t \to \infty$*. But this is not equivalent to convergence of* $\int_{1<|x|<t} x \log |x| \, \nu(dx)$ *as* $t \to \infty$*. Indeed,*

$$\int_{|x|>1} x \log(|x| \wedge t) \nu(dx) = \int_{1<|x|<t} x \log |x| \, \nu(dx) + (\log t) \int_{|x| \geqslant t} x \nu(dx)$$

and in this example $(\log n) \sum_{|k| \geqslant n} k a_k$ *oscillates, taking values* 1 *and* -1*.*

3. Relation between the two families

We study the relation between the families $\{\Psi_\alpha\}$ and $\{\Phi_{\beta,\alpha}\}$ defined from the functions f_α and $f_{\beta,\alpha}$ in Section 1. If needed, we define $f_\alpha(s) = 0$ for $s \geqslant a_\alpha$ and $f_{\beta,\alpha}(s) = 0$ for $s \geqslant a_{\beta,\alpha}$. In this way we will consider $\mathfrak{D}$, $\mathfrak{D}^0$, $\mathfrak{D}_{\mathrm{c}}$, $\mathfrak{D}_{\mathrm{e}}$ for Ψ_α with $\alpha < 0$ also and for $\Phi_{\beta,\alpha}$ with $\beta < \alpha < 0$ also. Propositions 1.1 and 2.1 guarantee that, for $\alpha \in \mathbb{R}$, f_α is locally $X^{(\mu)}$-integrable for all Lévy processes $X^{(\mu)}$. Thus, for $\alpha \geqslant 0$, Theorems 2.4 and 2.8 are applicable in the description of the domains of Ψ_α and $\Phi_{\beta,\alpha}$. In particular,

$$\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}(\Phi_{\beta,\alpha}) \quad \text{for } \alpha \geqslant 0,\ \beta < \alpha, \tag{3.1}$$

and the same equalities with $\mathfrak{D}^0$, $\mathfrak{D}_{\mathrm{c}}$, $\mathfrak{D}_{\mathrm{e}}$ in place of $\mathfrak{D}$ hold. If $\alpha \geqslant 2$, then $\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}(\Phi_{\beta,\alpha}) = \{\delta_0\}$.

Theorem 3.1. *Let* $-\infty < \beta < \alpha < \infty$*. Then*

$$\Psi_\alpha = \Psi_\beta \Phi_{\beta,\alpha} = \Phi_{\beta,\alpha} \Psi_\beta, \tag{3.2}$$

including the equality of the domains.

In order to prove the theorem, we use the expressions

$$C_{\Psi_\alpha\mu}(z)=\int_0^{a_\alpha-}C_\mu(f_\alpha(s)z)ds=\int_{0+}^{\infty}C_\mu(uz)u^{-\alpha-1}e^{-u}du \tag{3.3}$$

for $\mu\in\mathfrak{D}(\Psi_\alpha)$ and

$$\begin{aligned}&C_{\Phi_{\beta,\alpha}\mu}(z)\\&=\int_0^{a_{\beta,\alpha}-}C_\mu(f_{\beta,\alpha}(t)z)dt=\frac{1}{\Gamma(\alpha-\beta)}\int_{0+}^{1}C_\mu(vz)(1-v)^{\alpha-\beta-1}v^{-\alpha-1}dv\end{aligned} \tag{3.4}$$

for $\mu\in\mathfrak{D}(\Phi_{\beta,\alpha})$.

Lemma 3.2. *Let $-\infty<\beta<\alpha<\infty$. We have*

$$\int_0^{a_{\beta,\alpha}}dt\int_0^{a_\beta}|C_\mu(f_\beta(s)f_{\beta,\alpha}(t)z)|ds<\infty \quad \textit{for } z\in\mathbb{R}^d \tag{3.5}$$

if and only if $\mu\in\mathfrak{D}^0(\Psi_\alpha)$. If $\mu\in\mathfrak{D}^0(\Psi_\alpha)$, then $\mu\in\mathfrak{D}^0(\Psi_\beta)\cap\mathfrak{D}^0(\Phi_{\beta,\alpha})$, $\Psi_\beta\mu\in\mathfrak{D}^0(\Phi_{\beta,\alpha})$, $\Phi_{\beta,\alpha}\mu\in\mathfrak{D}^0(\Psi_\beta)$, and

$$\Psi_\alpha\mu=\Psi_\beta\Phi_{\beta,\alpha}\mu=\Phi_{\beta,\alpha}\Psi_\beta\mu. \tag{3.6}$$

Proof. Assume that $\alpha\geqslant 0$. Using Theorem 2.4, Theorem 2.8, and Remark to it, we see that $\mathfrak{D}^0(\Psi_\alpha)=\mathfrak{D}^0(\Phi_{\beta,\alpha})\subset\mathfrak{D}^0(\Psi_\beta)$. Using them for $f_\alpha(s)=1_{[1,\infty)}(s)s^{-1/\alpha}$ (for $\alpha>0$) or $f_0(s)=e^{-s}$ (for $\alpha=0$), we also see that $\mu\in\mathfrak{D}^0(\Psi_\alpha)$ if and only if

$$\int_1^\infty|C_\mu(s^{-1/\alpha}z)|ds=\alpha\int_0^1|C_\mu(uz)|u^{-\alpha-1}du<\infty$$

(for $\alpha>0$) or $\int_0^\infty|C_\mu(e^{-s}z)|ds=\int_0^1|C_\mu(uz)|u^{-1}du<\infty$ (for $\alpha=0$). Let us prove that $\mu\in\mathfrak{D}^0(\Psi_\alpha)$ if and only if (3.5) holds. Denote by I the double integral in (3.5). Then

$$I=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-v)^{\alpha-\beta-1}v^{-\alpha-1}dv\int_0^\infty|C_\mu(uvz)|u^{-\beta-1}e^{-u}du.$$

Let $c_1,c_2,\dots$ denote positive constants. If $\int_0^1|C_\mu(uz)|u^{-\alpha-1}du<\infty$, then $I<\infty$, because

$$\begin{aligned}I&\leqslant c_1\int_{1/2}^1(1-v)^{\alpha-\beta-1}dv\int_0^\infty|C_\mu(uvz)|u^{-\beta-1}e^{-u}du\\&\quad+c_2\int_0^{1/2}v^{-\alpha-1}dv\int_0^\infty|C_\mu(uvz)|u^{-\beta-1}e^{-u}du=I_1+I_2,\quad\text{(say)}\\I_1&\leqslant c_3\int_{1/2}^1(1-v)^{\alpha-\beta-1}dv\int_0^\infty|C_\mu(uz)|u^{-\beta-1}e^{-u/v}du\\&=c_3\int_0^\infty|C_\mu(uz)|u^{-\beta-1}e^{-u}du\int_0^u v^{\alpha-\beta-1}(u+v)^{-\alpha+\beta}e^{-v}dv\\&\leqslant c_4\int_0^\infty|C_\mu(uz)|u^{-\alpha-1}e^{-u}du\int_0^u v^{\alpha-\beta-1}e^{-v}dv\\&\leqslant c_5+c_6\int_0^1|C_\mu(uz)|u^{-\beta-1}du<\infty,\\I_2&=c_2\int_0^{1/2}v^{\beta-\alpha-1}dv\int_0^\infty|C_\mu(uz)|u^{-\beta-1}e^{-u/v}du\end{aligned}$$

$$
\begin{aligned}
&= c_2 \int_0^\infty |C_\mu(uz)| u^{-\alpha-1} du \int_{2u}^\infty v^{\alpha-\beta-1} e^{-v} dv \\
&\leqslant c_7 + c_8 \int_0^1 |C_\mu(uz)| u^{-\alpha-1} du < \infty.
\end{aligned}
$$

If $\int_0^1 |C_\mu(uz)| u^{-\alpha-1} du = \infty$, then $I \geqslant c_9 I_2 = \infty$. Now, if $\mu \in \mathfrak{D}^0(\Psi_\alpha)$, then $\Psi_\beta \mu \in \mathfrak{D}^0(\Phi_{\beta,\alpha})$ and $\Phi_{\beta,\alpha}\mu \in \mathfrak{D}^0(\Psi_\beta)$, since

$$
\begin{aligned}
\int_0^\infty |C_{\Psi_\beta \mu}(f_{\beta,\alpha}(t)z)| dt &\leqslant \int_0^\infty dt \int_0^{a_\beta} |C_\mu(f_\beta(s) f_{\beta,\alpha}(t) z)| ds = I < \infty, \\
\int_0^\infty |C_{\Phi_{\beta,\alpha}\mu}(f_\beta(s)z)| ds &\leqslant \int_0^{a_\beta} ds \int_0^\infty |C_\mu(f_{\beta,\alpha}(t) f_\beta(s) z)| ds = I < \infty.
\end{aligned}
$$

Further, if $\mu \in \mathfrak{D}^0(\Psi_\alpha)$, then (3.6) holds, since it follows from (3.5) that

$$
\begin{aligned}
C_{\Phi_{\beta,\alpha}\Psi_\beta \mu}(z) &= \int_0^\infty dt \int_0^{a_\beta} C_\mu(f_\beta(s) f_{\beta,\alpha}(t) z) ds \\
&= \int_0^{a_\beta} ds \int_0^\infty C_\mu(f_{\beta,\alpha}(t) f_\beta(s) z) dt = C_{\Psi_\beta \Phi_{\beta,\alpha}\mu}(z)
\end{aligned}
$$

and that

$$
\begin{aligned}
C_{\Psi_\beta \Phi_{\beta,\alpha}\mu}(z) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty u^{-\beta-1} e^{-u} du \int_0^1 C_\mu(uvz)(1-v)^{\alpha-\beta-1} v^{-\alpha-1} dv \\
&= \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty e^{-u} du \int_0^u C_\mu(vz)(u-v)^{\alpha-\beta-1} v^{-\alpha-1} dv \\
&= \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty C_\mu(vz) v^{-\alpha-1} dv \int_v^\infty e^{-u}(u-v)^{\alpha-\beta-1} du \\
&= \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty C_\mu(vz) v^{-\alpha-1} dv \int_0^\infty e^{-u-v} u^{\alpha-\beta-1} du \\
&= \int_0^\infty C_\mu(vz) v^{-\alpha-1} e^{-v} dv = C_{\Psi_\alpha \mu}(z).
\end{aligned}
$$

This proves the lemma under the assumption that $\alpha \geqslant 0$.

If $\alpha < 0$, then $I < \infty$ for all $\mu \in ID(\mathbb{R}^d)$ and we have $\mathfrak{D}^0(\Psi_\alpha) = \mathfrak{D}^0(\Psi_\beta) = \mathfrak{D}^0(\Phi_{\beta,\alpha}) = ID(\mathbb{R}^d)$, which makes similar discussion easier. $\square$

Proof of Theorem 3.1 for $\alpha \neq 1$. Assume that $\alpha \geqslant 2$. Then $\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}(\Phi_{\beta,\alpha}) = \{\delta_0\}$. For δ_0, (3.2) is trivial. Further, $\mathfrak{D}(\Psi_\beta \Phi_{\beta,\alpha}) = \{\delta_0\}$. To see $\mathfrak{D}(\Phi_{\beta,\alpha}\Psi_\beta) = \{\delta_0\}$, notice that if $\Psi_\beta \mu = \mu_{(A',\nu',\gamma')} = \delta_0$ for $\mu = \mu_{(A,\nu,\gamma)}$, then $A = 0$ from $A' = 0$ by (2.29), $\nu = 0$ from $\nu' = 0$ by

$$
0 = \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu'(dx) = \int_0^\infty u^{-\beta-1} e^{-u} du \int_{\mathbb{R}^d} (|ux|^2 \wedge 1) \nu(dx),
$$

and $\gamma = 0$ from $\gamma' = 0$ combined with $\nu = 0$ by (2.25) and (2.31).

Assume that $\alpha \in (-\infty, 1) \cap (1, 2)$. We have $\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}^0(\Psi_\alpha) = \mathfrak{D}(\Phi_{\beta,\alpha}) = \mathfrak{D}^0(\Phi_{\beta,\alpha})$ from Theorem 2.4. Hence, if $\mu \in \mathfrak{D}(\Psi_\alpha)$, then $\mu \in \mathfrak{D}^0(\Psi_\beta) \cap \mathfrak{D}(\Phi_{\beta,\alpha})$, $\Psi_\beta \mu \in \mathfrak{D}(\Phi_{\beta,\alpha})$, $\Phi_{\beta,\alpha}\mu \in \mathfrak{D}^0(\Psi_\beta)$, and (3.6) holds by virtue of Lemma 3.2. Hence $\mathfrak{D}(\Psi_\alpha) = \mathfrak{D}(\Psi_\beta \Phi_{\beta,\alpha})$ and $\mathfrak{D}(\Psi_\alpha) \subset \mathfrak{D}(\Phi_{\beta,\alpha}\Psi_\beta)$. In order to see $\mathfrak{D}(\Psi_\alpha) \supset \mathfrak{D}(\Phi_{\beta,\alpha}\Psi_\beta)$, suppose that $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\beta)$ and $\mu' = \Psi_\beta \mu = \mu_{(A',\nu',\gamma')} \in$

$\mathfrak{D}(\Phi_{\beta,\alpha})$. We claim that $\mu \in \mathfrak{D}(\Psi_\alpha)$. This is trivial if $\alpha < 0$. Now recall Theorem 2.4. If $\alpha = 0$, then

$$\begin{aligned}\infty > \int \log^+ |x|\nu'(dx) &= \int_0^{a_\beta} ds \int_{|f_\beta(s)x|>1} \log|f_\beta(s)x|\nu(dx)\\ &= \int_0^\infty t^{-\beta-1}e^{-t}dt \int_{|tx|>1} \log|tx|\nu(dx)\\ &= \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty t^{-\beta-1}e^{-t}\log t\, dt + \int_{\mathbb{R}^d} \log|x|\nu(dx) \int_{1/|x|}^\infty t^{-\beta-1}e^{-t}dt\\ &= c_1 + c_2 \int_{|x|>1} \log|x|\nu(dx)\end{aligned}$$

with $c_1, c_2 > 0$ and hence $\mu \in \mathfrak{D}(\Psi_0)$. If $0 < \alpha < 1$, then, similarly,

$$\infty > \int_{|x|>1} |x|^\alpha \nu'(dx) = \int_{\mathbb{R}^d} |x|^\alpha \nu(dx) \int_{1/|x|}^\infty t^{\alpha-\beta-1}e^{-t}dt \geqslant c_3 \int_{|x|>1} |x|^\alpha \nu(dx).$$

If $1 < \alpha < 2$, then $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ in the same way and moreover $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$; this is from $\mu \in \mathfrak{D}(\Psi_\beta)$ if $\beta \geqslant 1$ or from the two expressions

$$\begin{aligned}\gamma' &= -\int_{\mathbb{R}^d} \frac{x|x|^2\nu'(dx)}{1+|x|^2} = -\int_0^\infty t^{-\beta-1}e^{-t}dt \int_{\mathbb{R}^d} \frac{tx|tx|^2\nu(dx)}{1+|tx|^2}\\ &= \int_0^\infty t^{-\beta}e^{-t}dt \left(-\int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2} + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|tx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right),\\ \gamma' &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty t^{-\beta}e^{-t}dt \left(\gamma + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|tx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\end{aligned}$$

if $\beta < 1$. □

Let us prepare a technical lemma for the proof of Theorem 3.1 for $\alpha = 1$.

Lemma 3.3. *Let φ be a function satisfying the conditions on φ_1 imposed in Theorem 2.8. Let $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\varphi)$. Let $x_n = \int_{|x|>n} x\nu(dx)$ and let $\mu_n = \mu_{(A_n,\nu_n,\gamma_n)}$ be such that $A_n = A$,*

$$\nu_n(dx) = \begin{cases} 1_{\{|x|\leqslant n\}}(x)\nu(dx) + n^{-1}|x_n|\delta_{n|x_n|^{-1}x_n}(dx) & \text{if } x_n \neq 0,\\ 1_{\{|x|\leqslant n\}}(x)\nu(dx) & \text{if } x_n = 0,\end{cases}$$

and $\gamma_n = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu_n(dx)$. Then, $\mu_n \in \mathfrak{D}(\Phi_\varphi)$ and, as $n \to \infty$, $\mu_n \to \mu$ and $\Phi_\varphi\mu_n \to \Phi_\varphi\mu$.

Proof. Notice that

$$\begin{aligned}\int_{|x|>1} x\log|x|\nu_n(dx) &= \int_{1<|x|\leqslant n} x\log|x|\nu(dx) + x_n \log n\\ &= \int_{|x|>1} x\log(|x|\wedge n)\nu(dx).\end{aligned} \tag{3.7}$$

It is clear from Theorem 2.8 that $\mu_n \in \mathfrak{D}(\Phi_\varphi)$. We have $|x_n| \to 0$, $\int h(x)\nu_n(dx) \to \int h(x)\nu(dx)$ for each $h \in C_\sharp$, and

$$\gamma_n = -\int_{|x|\leqslant n} \frac{x|x|^2}{1+|x|^2}\nu(dx) - x_n\frac{n^2}{1+n^2} \to -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx) = \gamma.$$

Moreover, $\nu_n = \nu$ on $|x| < 1$. Hence we have $\mu_n \to \mu$, using Theorem 8.7 of Sato (1999). Let $\widetilde{\mu} = \Phi_\varphi \mu = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ and $\widetilde{\mu}_n = \Phi_\varphi \mu_n = \mu_{(\widetilde{A}_n,\widetilde{\nu}_n,\widetilde{\gamma}_n)}$. Let us prove that $\widetilde{\mu}_n \to \widetilde{\mu}$. We use s_0, c, and $\psi(s)$ in (2.41) and $0 < c_1 \leqslant c_2$ such that $c_1 s^{-1} \leqslant \varphi(s) \leqslant c_2 s^{-1}$ for $s \geqslant s_0$. First, $\widetilde{A}_n = \widetilde{A}$ from (2.29). Next, from (2.30),

$$\int_{\mathbb{R}^d} h(x)\widetilde{\nu}_n(dx) = \int_0^\infty ds \int_{|x|\leqslant n} h(\varphi(s)x)\nu(dx) + \int_0^\infty h(\varphi(s)n|x_n|^{-1}x_n)ds\, n^{-1}|x_n|$$
$$\to \int_0^\infty ds \int_{\mathbb{R}^d} h(\varphi(s)x)\nu(dx) = \int_{\mathbb{R}^d} h(x)\widetilde{\nu}(dx)$$

for $h \in C_\sharp$, since, for any $\varepsilon > 0$,

$$\int_{s_0}^\infty 1_{\{|x|>\varepsilon\}}(\varphi(s)n|x_n|^{-1}x_n)ds\, n^{-1}|x_n| = \int_{s_0}^\infty 1_{\{\varphi(s)n>\varepsilon\}} ds\, n^{-1}|x_n|$$
$$\leqslant \int_{s_0}^\infty 1_{\{c_1 s^{-1} n>\varepsilon\}} ds\, n^{-1}|x_n| \leqslant \frac{c_1}{\varepsilon}|x_n| \to 0.$$

Further we claim that

$$\lim_{\varepsilon\downarrow 0} \int_{|x|\leqslant\varepsilon} |x|^2\widetilde{\nu}_n(dx) = 0 \quad \text{uniformly with respect to } n. \tag{3.8}$$

Indeed,

$$\begin{aligned}\int_{|x|\leqslant\varepsilon} |x|^2\widetilde{\nu}_n(dx) &= \int_0^\infty ds \int_{|\varphi(s)x|\leqslant\varepsilon} |\varphi(s)x|^2 \nu_n(dx)\\
&= \int_{\mathbb{R}^d} |x|^2\nu_n(dx) \int_{(s_0,\infty)\cap\{|\varphi(s)x|\leqslant\varepsilon\}} \varphi(s)^2 ds\\
&\quad + \int_{\mathbb{R}^d} |x|^2\nu_n(dx) \int_{(0,s_0)\cap\{|\varphi(s)x|\leqslant\varepsilon\}} \varphi(s)^2 ds = I_1 + I_2 \quad \text{(say)}\end{aligned}$$

and

$$\begin{aligned}I_1 &\leqslant \int_{\mathbb{R}^d} |x|^2\nu_n(dx) \int_{(s_0,\infty)\cap\{c_1 s^{-1}|x|\leqslant\varepsilon\}} c_2^2 s^{-2} ds\\
&= c_2^2 \int_{|x|\leqslant\varepsilon s_0/c_1} |x|^2\nu_n(dx) \int_{s_0}^\infty s^{-2}ds + c_2^2 \int_{|x|>\varepsilon s_0/c_1} |x|^2\nu_n(dx) \int_{c_1|x|/\varepsilon}^\infty s^{-2}ds\\
&= c_2^2 s_0^{-1} \int_{|x|\leqslant\varepsilon s_0/c_1} |x|^2\nu(dx) + c_2^2 \int_{\varepsilon s_0/c_1<|x|\leqslant 1} \varepsilon c_1^{-1}|x|\nu(dx)\\
&\quad + \varepsilon c_2^2 c_1^{-1}\left(\int_{1<|x|\leqslant n} |x|\nu(dx) + |x_n|\right),\end{aligned}$$

where, as $\varepsilon \downarrow 0$, the first and the third terms tend to 0 uniformly in n (recall that $\int_{|x|>1} |x|\nu(dx) < \infty$), and the second term also tends to 0 since the integrand is bounded by $s_0^{-1}|x|^2$ and tends to 0. Further,

$$\begin{aligned}I_2 &\leqslant \int_{|x|\leqslant 1} |x|^2\nu(dx) \int_{(0,s_0)\cap\{|\varphi(s)x|\leqslant\varepsilon\}} \varphi(s)^2 ds\\
&\quad + \left(\varepsilon \int_{1<|x|\leqslant n} |x|\nu(dx) + \varepsilon|x_n|\right) \int_0^{s_0} |\varphi(s)|ds,\end{aligned}$$

which tends to 0 uniformly in n. This proves (3.8).

We prove that $\widetilde{\gamma}_n \to \widetilde{\gamma}$ as $n \to \infty$. This will finish the proof that $\widetilde{\mu}_n \to \widetilde{\mu}$, using Theorem 8.7 of Sato (1999). From (2.25), (2.31), and (2.38) we have

$$\widetilde{\gamma} = -\lim_{t\to\infty} J^1(t), \qquad J^1(t) = \int_0^t \varphi(s)ds \int_{\mathbb{R}^d} \frac{x|\varphi(s)x|^2}{1+|\varphi(s)x|^2}\nu(dx),$$

$$\widetilde{\gamma}_n = -\lim_{t\to\infty} J_n^1(t), \qquad J_n^1(t) = \int_0^t \varphi(s)ds \int_{\mathbb{R}^d} \frac{x|\varphi(s)x|^2}{1+|\varphi(s)x|^2}\nu_n(dx).$$

Consider the following chain of approximations of $\widetilde{\gamma}$:

$$J^2(t) = \int_{s_0}^t \varphi(s)ds \int_{\mathbb{R}^d} \frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}, \qquad J^3(t) = \int_{s_0}^t \varphi(s)ds \int_{|x|>1} \frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2},$$

$$J^4(t) = \int_{s_0}^t \varphi(s)ds \int_{|x|>1} \frac{x|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2}, \quad J^5(t) = \int_{s_0}^t cs^{-1}ds \int_{|x|>1} \frac{x|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2},$$

$$J^6(t) = c\int_{|x|>1} x\log\frac{cs_0^{-1}|x|}{1+ct^{-1}|x|}\nu(dx), \qquad J^7(t) = c\int_{|x|>1} x\log(|x|\wedge t)\nu(dx).$$

For each n, denote by $J_n^j(t)$, $j = 2,\ldots,7$, the functions $J^j(t)$ with ν_n in place of ν. Notice that $\lim_{t\to\infty} J_n^7(t) = c\int_{|x|>1} x\log|x|\nu_n(dx)$ and hence

$$\lim_{n\to\infty}\lim_{t\to\infty} J_n^7(t) = \lim_{t\to\infty} J^7(t)$$

by virtue of (3.7) and condition (2.49). We claim that

$$\lim_{t\to\infty}(J^j(t) - J^{j+1}(t)) \text{ exists in } \mathbb{R}^d, \tag{3.9}$$

$$\lim_{t\to\infty}(J_n^j(t) - J_n^{j+1}(t)) \text{ exists in } \mathbb{R}^d, \tag{3.10}$$

$$\lim_{n\to\infty}\lim_{t\to\infty}(J_n^j(t) - J_n^{j+1}(t)) = \lim_{t\to\infty}(J^j(t) - J^{j+1}(t)) \tag{3.11}$$

for $j = 1,\ldots,6$. Then, since we have

$$J^1(t) = \sum_{j=1}^{6}(J^k(t) - J^{j+1}(t)) + J^7(t)$$

and the corresponding identity for $J_n^1(t)$, it follows that $\widetilde{\gamma}_n \to \widetilde{\gamma}$. The proof of (3.9)–(3.11) is as follows. For $j = 1$ we have

$$J^1(t) - J^2(t) = \int_0^{s_0} \varphi(s)ds \int_{\mathbb{R}^d} \frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2},$$

which is independent of t. Notice that

$$\begin{aligned}&\int_0^{s_0} |\varphi(s)|ds \int_{\mathbb{R}^d} \frac{|x|\,|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}\\ &\quad\leqslant \frac{1}{2}\int_0^{s_0} \varphi(s)^2ds \int_{|x|\leqslant 1} |x|^2\nu(dx) + \int_0^{s_0} |\varphi(s)|ds \int_{|x|>1} |x|\nu(dx) < \infty\end{aligned}$$

and

$$\begin{aligned}J_n^1(t) - J_n^2(t) &= \int_0^{s_0} \varphi(s)ds \int_{|x|\leqslant n} \frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2} + x_n\int_0^{s_0} \varphi(s)\frac{\varphi(s)^2n^2}{1+\varphi(s)^2n^2}ds\\ &\to J^1(t) - J^2(t) \quad \text{as } n \to \infty.\end{aligned}$$

For $j=2$ notice that

$$
\begin{aligned}
J^2(t)-J^3(t) &= \int_{s_0}^{t}\varphi(s)ds\int_{|x|\leqslant 1}\frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}\\
&\to \int_{s_0}^{\infty}\varphi(s)ds\int_{|x|\leqslant 1}\frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}\quad \text{as } t\to\infty,
\end{aligned}
$$

$$
\int_{s_0}^{\infty}\varphi(s)ds\int_{|x|\leqslant 1}\frac{|x|\,|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}\leqslant\int_{s_0}^{\infty}c_2^3s^{-3}ds\int_{|x|\leqslant 1}|x|^3\nu(dx)<\infty,
$$

$$
J_n^2(t)-J_n^3(t)=\int_{s_0}^{t}\varphi(s)ds\int_{|x|\leqslant 1}\frac{x|\varphi(s)x|^2\nu(dx)}{1+|\varphi(s)x|^2}=J^2(t)-J^3(t)\quad\text{for } n\geqslant 2.
$$

For $j=3$

$$
J^3(t)-J^4(t)=\int_{s_0}^{t}s^{-1}\psi(s)ds\int_{|x|>1}\frac{x(|s^{-1}\psi(s)x|^2-|cs^{-1}x|^2)\nu(dx)}{(1+|cs^{-1}x|^2)(1+|s^{-1}\psi(s)x|^2)},
$$

$$
\begin{aligned}
&\int_{s_0}^{\infty}s^{-1}\psi(s)ds\int_{|x|>1}\frac{|x|\,\big||s^{-1}\psi(s)x|^2-|cs^{-1}x|^2\big|\,\nu(dx)}{(1+|cs^{-1}x|^2)(1+|s^{-1}\psi(s)x|^2)}\\
&\quad=\int_{s_0}^{\infty}s^{-2}\psi(s)|\psi(s)-c|ds\int_{|x|>1}\frac{|x|^2(|s^{-1}\psi(s)x|^2+|cs^{-1}x|^2)\nu(dx)}{(1+|cs^{-1}x|^2)(1+|s^{-1}\psi(s)x|^2)}\\
&\quad\leqslant\int_{s_0}^{\infty}s^{-2}\psi(s)|\psi(s)-c|ds\left(\frac{c^{-1}s}{4}+\frac{\psi(s)^{-1}s}{4}\right)\int_{|x|>1}|x|\nu(dx)<\infty
\end{aligned}
$$

by (2.41) and

$$
\begin{aligned}
\lim_{t\to\infty}(J_n^3(t)-J_n^4(t)) &= \int_{s_0}^{\infty}s^{-1}\psi(s)ds\int_{1<|x|\leqslant n}\frac{x(|s^{-1}\psi(s)x|^2-|cs^{-1}x|^2)\nu(dx)}{(1+|cs^{-1}x|^2)(1+|s^{-1}\psi(s)x|^2)}\\
&\quad+x_n\int_{s_0}^{\infty}\frac{s^{-1}\psi(s)((s^{-1}\psi(s)n)^2-(cs^{-1}n)^2)ds}{(1+(cs^{-1}n)^2)(1+(s^{-1}\psi(s)n)^2)}.
\end{aligned}
$$

As $n\to\infty$, the first term tends to $\lim_{t\to\infty}(J^3(t)-J^4(t))$ and the second term tends to zero. For $j=4$,

$$
J^4(t)-J^5(t)=\int_{s_0}^{t}s^{-1}(\psi(s)-c)ds\int_{|x|>1}\frac{x|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2},
$$

$$
\begin{aligned}
&\int_{s_0}^{\infty}s^{-1}|\psi(s)-c|ds\int_{|x|>1}\frac{|x|\,|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2}\\
&\quad\leqslant\int_{s_0}^{\infty}s^{-1}|\psi(s)-c|ds\int_{|x|>1}|x|\nu(dx)<\infty,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t\to\infty}(J_n^4(t)-J_n^5(t)) &= \int_{s_0}^{\infty}s^{-1}(\psi(s)-c)ds\int_{1<|x|\leqslant n}\frac{x|cs^{-1}x|^2\nu(dx)}{1+|cs^{-1}x|^2}\\
&\quad+x_n\int_{s_0}^{\infty}s^{-1}(\psi(s)-c)\frac{(cs^{-1}n)^2ds}{1+(cs^{-1}n)^2}.
\end{aligned}
$$

As $n\to\infty$, the first term tends to $\lim_{t\to\infty}(J^4(t)-J^5(t))$ and the second term tends to zero. For $j=5$ notice that

$$
J^5(t)=\int_{|x|>1}x\nu(dx)\int_{s_0}^{t}cs^{-1}\frac{c^2s^{-2}|x|^2ds}{1+c^2s^{-2}|x|^2}=\frac{c}{2}\int_{|x|>1}x\log\frac{1+c^2s_0^{-2}|x|^2}{1+c^2t^{-2}|x|^2}\nu(dx),
$$

$$
\begin{aligned}
J^5(t)-J^6(t) &= \frac{c}{2}\int_{|x|>1} x\log\frac{(1+c^2s_0^{-2}|x|^2)(1+ct^{-1}|x|)^2}{(1+c^2t^{-2}|x|^2)(cs_0^{-1}|x|)^2}\nu(dx)\\
&\to \frac{c}{2}\int_{|x|>1} x\log\frac{1+c^2s_0^{-2}|x|^2}{c^2s_0^{-2}|x|^2}\nu(dx)\quad \text{as } t\to\infty,
\end{aligned}
$$

by a discussion similar to that concerning (2.53), and

$$
\lim_{t\to\infty}(J_n^5(t)-J_n^6(t)) = \frac{c}{2}\int_{1<|x|\leqslant n} x\log\frac{1+c^2s_0^{-2}|x|^2}{c^2s_0^{-2}|x|^2}\nu(dx)+\frac{c}{2}x_n\log\frac{1+c^2s_0^{-2}n^2}{c^2s_0^{-2}n^2},
$$

which tends to $\lim_{t\to\infty}(J^5(t)-J^6(t))$ as $n\to\infty$. Finally, for $j=6$,

$$
J^6(t)-J^7(t) = c\int_{|x|>1} x\left(\log\frac{cs_0^{-1}|x|}{1+ct^{-1}|x|}-\log(|x|\wedge t)\right)\nu(dx),
$$

$$
\begin{aligned}
&\int_{|x|>1}|x|\left|\log\frac{cs_0^{-1}|x|}{1+ct^{-1}|x|}-\log(|x|\wedge t)\right|\nu(dx)\\
&\qquad=\int_{1<|x|\leqslant t}|x|\left|\log\frac{cs_0^{-1}}{1+ct^{-1}|x|}\right|\nu(dx)+\int_{|x|>t}|x|\left|\log\frac{cs_0^{-1}t^{-1}|x|}{1+ct^{-1}|x|}\right|\nu(dx),
\end{aligned}
$$

$$
\left|\log\frac{cs_0^{-1}}{1+ct^{-1}|x|}\right|\leqslant|\log(cs_0^{-1})|+\log(1+c)\quad\text{on }\{1<|x|\leqslant t\},
$$

$$
\left|\log\frac{cs_0^{-1}t^{-1}|x|}{1+ct^{-1}|x|}\right|\leqslant|\log(s_0^{-1})|+\left|\log\frac{c}{1+c}\right|\quad\text{on }\{|x|>t\}.
$$

Thus

$$
\begin{aligned}
&\lim_{t\to\infty}(J^6(t)-J^7(t)) = c\int_{|x|>1}x\log(cs_0^{-1})\nu(dx),\\
&\lim_{t\to\infty}(J_n^6(t)-J_n^7(t)) = c\int_{1<|x|\leqslant n}x\log(cs_0^{-1})\nu(dx)+cx_n\log(cs_0^{-1})\\
&\qquad\to\lim_{t\to\infty}(J^6(t)-J^7(t))
\end{aligned}
$$

as $n\to\infty$. □

Proof of Theorem 3.1 for $\alpha=1$. Let $-\infty<\beta<1$. We use $c_1,c_2,\dots$ for positive constants.

Step 1. Given $\mu\in\mathfrak{D}(\Psi_1)=\mathfrak{D}(\Phi_{\beta,1})$, we prove that $\Phi_{\beta,1}\mu\in\mathfrak{D}(\Psi_\beta)$ and $\Psi_\beta\Phi_{\beta,1}\mu=\Psi_1\mu$. Consider μ_n in Lemma 3.3. Since $\mu_n\in\mathfrak{D}^0(\Psi_1)$ by virtue of Theorem 2.8, we can apply Lemma 3.2. Thus $\mu_n\in\mathfrak{D}^0(\Phi_{\beta,1})$, $\Phi_{\beta,1}\mu\in\mathfrak{D}^0(\Psi_\beta)$, and

$$
\Psi_\beta\Phi_{\beta,1}\mu_n=\Psi_1\mu_n. \tag{3.12}
$$

Lemma 3.3 says that $\mu_n\to\mu$, $\Psi_1\mu_n\to\Psi_1\mu$, and $\Phi_{\beta,1}\mu_n\to\Phi_{\beta,1}\mu$. We claim that

$$
\Phi_{\beta,1}\mu\in\mathfrak{D}(\Psi_\beta)\quad\text{and}\quad\Psi_\beta\Phi_{\beta,1}\mu_n\to\Psi_\beta\Phi_{\beta,1}\mu. \tag{3.13}
$$

Denote $\widetilde{\mu}=\Psi_1\mu$, $\mu'=\Phi_{\beta,1}\mu$, $\mu''=\Psi_\beta\mu'$, $\widetilde{\mu}_n=\Psi_1\mu_n$, $\mu_n'=\Phi_{\beta,1}\mu_n$, and $\mu_n''=\Psi_\beta\mu_n'$. The triplets of μ, $\widetilde{\mu}$, μ', μ'', μ_n, $\widetilde{\mu}_n$, μ_n', and μ_n'' are written as (A,ν,γ), $(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})$, and similarly. If $\beta<0$, then $\mathfrak{D}(\Psi_\beta)=ID(\mathbb{R}^d)$,

$$
C_{\mu_n''}(z)=\int_0^{\Gamma(-\beta)}C_{\mu_n'}(f_\beta(s)z)ds,
$$

and

$$|C_{\mu'_n}(f_\beta(s)z)| \leqslant \frac{1}{2}(\operatorname{tr} A'_n)|f_\beta(s)z|^2 + 3(1+|f_\beta(s)z|^2)\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu'_n(dx) + |\gamma'_n|\,|f_\beta(s)z|,$$

where $A'_n = A'$, $\sup_n \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu'_n(dx) < \infty$, and $\sup_n |\gamma'_n| < \infty$ (recall the proof of Lemma 3.3). Since $C_{\mu'_n}(z) \to C_{\mu'}(z)$ for all z, we see that $C_{\mu''_n}(z) \to C_{\mu''}(z)$, that is, (3.13) holds if $\beta < 0$. Next, assume that $0 < \beta < 1$. In order to prove that $\mu' \in \mathfrak{D}(\Psi_\beta)$, it is enough to show that $\int_{|x|>1}|x|^\beta \nu'(dx) < \infty$ (Theorem 2.4). We have

$$\begin{aligned}\int_{|x|>1}|x|^\beta \nu'(dx) &= \frac{1}{\Gamma(1-\beta)}\int_0^1 (1-v)^{-\beta}v^{-2}dv \int_{|vx|>1} v^\beta |x|^\beta \nu(dx)\\ &= \frac{1}{\Gamma(1-\beta)}\int_{|x|>1}|x|^\beta \nu(dx)\int_{1/|x|}^1 (1-v)^{-\beta}v^{-2}dv = \frac{1}{\Gamma(1-\beta)}(I_1+I_2),\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{1<|x|\leqslant 2}|x|^\beta \nu(dx)\int_{1/|x|}^1 (1-v)^{-\beta}v^{-2}dv \leqslant c_1\int_{1<|x|\leqslant 2}|x|^\beta\nu(dx) < \infty,\\ I_2 &= c_2 + \int_{|x|>2}|x|^\beta\nu(dx)\int_{1/|x|}^{1/2}(1-v)^{-\beta}v^{-2}dv\\ &\leqslant c_2 + c_3\int_{|x|>2}|x|^\beta\nu(dx)\int_{1/|x|}^{1/2}v^{-2}dv \leqslant c_2 + c_4\int_{|x|>2}|x|\nu(dx) < \infty.\end{aligned}$$

Hence $\mu' \in \mathfrak{D}(\Psi_\beta)$. It follows from (2.29) that $A'_n = A'$ and $A''_n = A''$. For any $h \in C_\sharp^+$ we have $\int h(x)\nu''_n(dx) \to \int h(x)\nu''(dx)$, noticing that, first, $\nu''_n = \widetilde{\nu}_n$ from (3.12), second, $\int h(x)\widetilde{\nu}_n(dx) \to \int h(x)\widetilde{\nu}(dx)$ from Lemma 3.3, and third,

$$\begin{aligned}\int h(x)\nu''(dx) &= \frac{1}{\Gamma(1-\beta)}\int_0^\infty u^{-\beta-1}e^{-u}du\int_0^1(1-v)^{-\beta}v^{-2}dv\int_{\mathbb{R}^d}h(uvx)\nu(dx)\\ &= \int_0^\infty v^{-2}e^{-v}dv\int_{\mathbb{R}^d}h(vx)\nu(dx) = \int h(x)\widetilde{\nu}(dx)\end{aligned}$$

in the same way as in the proof of Lemma 3.2. We have

$$\gamma'' = \int_0^\infty f_\beta(s)ds\gamma' + \lim_{t\to\infty}\int_0^t f_\beta(s)ds\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f_\beta(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu'(dx)$$

and the corresponding expression of γ''_n. Recall that $\gamma'_n \to \gamma'$. We will prove the following three facts:

$$\int_{\mathbb{R}^d}h(x)|x|^\beta\nu'_n(dx) \to \int_{\mathbb{R}^d}h(x)|x|^\beta\nu'(dx) \text{ as } n\to\infty \text{ for } h \in C_\sharp; \tag{3.14}$$

$$\int_0^\infty |f_\beta(s)x|\left|\frac{1}{1+|f_\beta(s)x|^2} - \frac{1}{1+|x|^2}\right|ds \leqslant \begin{cases} c_5|x|^2 & \text{if } |x|\leqslant 1,\\ c_6|x|^\beta & \text{if } |x|>1;\end{cases} \tag{3.15}$$

$$\int_{|x|\leqslant\varepsilon}|x|^2\nu'_n(dx) \to 0 \text{ uniformly in } n \text{ as } \varepsilon\downarrow 0. \tag{3.16}$$

If we prove these facts, then

$$\gamma'' = \int_0^\infty f_\beta(s)ds\gamma' + \int_0^\infty f_\beta(s)ds\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f_\beta(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu'(dx)$$

together with the similar expression of γ_n'' and we will obtain $\gamma_n'' \to \gamma''$. Using (3.16), we will reach the conclusion that $\mu_n'' \to \mu''$. To prove (3.14), it suffices to show

$$\lim_{l\to\infty} \sup_n \int_{|x|>l} |x|^\beta \nu_n'(dx) = 0. \tag{3.17}$$

Notice that

$$\begin{aligned}
\sup_n \int_{|x|>l} |x|^\beta \nu_n'(dx) &= \frac{1}{\Gamma(1-\beta)} \sup_n \int_0^1 (1-v)^{-\beta} v^{-2} dv \int_{|vx|>l} |vx|^\beta \nu_n(dx) \\
&= \frac{1}{\Gamma(1-\beta)} \sup_{n>l} \int_{|x|>l} |x|^\beta \nu_n(dx) \int_{l/|x|}^1 (1-v)^{-\beta} v^{\beta-2} dv \\
&\leqslant c_7 \sup_{n>l} \left(\int_{l<|x|\leqslant n} |x|^\beta \left(\frac{l}{|x|}\right)^{\beta-1} \nu(dx) + n^\beta \frac{|x_n|}{n} \left(\frac{l}{n}\right)^{\beta-1} \right) \\
&= c_7 \sup_{n>l} \left(l^{\beta-1} \int_{l<|x|\leqslant n} |x|\nu(dx) + l^{\beta-1}|x_n| \right) \to 0, \quad l \to \infty.
\end{aligned}$$

Thus (3.17) is true. To see (3.15),

$$\int_0^\infty |f_\beta(s)x| \frac{||x|^2 - |f_\beta(s)x|^2| ds}{(1+|f_\beta(s)x|^2|)(1+|x|^2)} = \int_0^{s_0} + \int_{s_0}^\infty = I_1 + I_2 \quad \text{(say)},$$

$$I_1 \leqslant \int_0^{s_0} |f_\beta(s)x| \frac{(|x|^2 + |f_\beta(s)x|^2) ds}{(1+|f_\beta(s)x|^2|)(1+|x|^2)} \leqslant \begin{cases} 2^{-1}s_0|x|^2 + 2^{-1}|x|^2 \int_0^{s_0} f_\beta(s)^2 ds, \\ 2^{-1}s_0 + 2^{-1}\int_0^{s_0} f_\beta(s) ds, \end{cases}$$

$$I_2 \leqslant c_8|x|^3 \int_{s_0}^\infty s^{-1/\beta} ds \leqslant c_9|x|^2 \quad \text{for } |x| \leqslant 1,$$

$$\begin{aligned}
I_2 &\leqslant c_{10} \int_{s_0}^\infty \frac{s^{-1/\beta}|x|^3 ds}{(1+|c_{11}s^{-1/\beta}x|^2)(1+|x|^2)} \\
&\leqslant c_{10} \int_{s_0\vee|x|^\beta}^\infty s^{-1/\beta}|x| ds + c_{10} \int_{s_0}^{s_0\vee|x|^\beta} \frac{s^{-1/\beta}|x| ds}{c_{11}^2 s^{-2/\beta}|x|^2} \leqslant c_{12}|x|^\beta \quad \text{for } |x| > 1.
\end{aligned}$$

Finally, in order to see (3.16), it suffices to notice that

$$\begin{aligned}
\int_{|x|\leqslant\varepsilon} |x|^2 \nu_n'(dx) &= \frac{1}{\Gamma(1-\beta)} \int_{\mathbb{R}^d} |x|^2 \nu_n(dx) \int_0^{1\wedge\varepsilon|x|^{-1}} (1-v)^{-\beta} dv \\
&\leqslant c_{13} \int_{|x|\leqslant\varepsilon} |x|^2 \nu_n(dx) + c_{14} \int_{|x|>\varepsilon} \varepsilon|x|\nu_n(dx) \\
&\leqslant c_{13} \int_{|x|\leqslant\varepsilon} |x|^2 \nu(dx) + c_{14} \int_{\varepsilon<|x|\leqslant 1} \varepsilon|x|\nu(dx) + c_{14}\varepsilon \int_{|x|>1} |x|\nu_n(dx),
\end{aligned}$$

where the second term tends to zero as $\varepsilon \downarrow 0$ since $\varepsilon|x| < |x|^2$ on $\{\varepsilon < |x| \leqslant 1\}$ and so does the third term since $\int_{|x|>1} |x|\nu_n(dx) \leqslant \int_{|x|>1} |x|\nu(dx)$. Now the proof of (3.13) in the case $0 < \beta < 1$ is complete. Modification in the case $\beta = 0$, which is not difficult, of the proof for $0 < \beta < 1$ is omitted. Thus we have (3.13) in all cases. Combining this with (3.12), we obtain that $\mu'' = \widetilde{\mu}$.

Step 2. Given $\mu \in \mathfrak{D}(\Psi_1) \subset \mathfrak{D}(\Psi_\beta)$, we prove that $\Psi_\beta\mu \in \mathfrak{D}(\Phi_{\beta,1})$ and $\Phi_{\beta,1}\Psi_\beta\mu = \Psi_1\mu$. In this step we denote $\widetilde{\mu} = \Psi_1\mu$, $\mu' = \Psi_\beta\mu$, $\mu'' = \Phi_{\beta,1}\mu'$, and the corresponding triplets in an obvious way. To see that $\mu' \in \mathfrak{D}(\Phi_{\beta,1})$, notice that, first,

$$\int_{|x|>1} |x|\nu'(dx) = \int_{\mathbb{R}^d} |x|\nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} du$$
$$\leqslant \int_{|x|\leqslant 1} |x| e^{-c_1/|x|} \nu(dx) + c_2 \int_{|x|>1} |x|\nu(dx) < \infty,$$

second,

$$\begin{aligned}
\gamma' &= \int_0^{a_\beta} f_\beta(s) ds \gamma + \int_0^{a_\beta} f_\beta(s) ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f_\beta(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \\
&= -\int_0^{\infty} u^{-\beta} e^{-u} du \int_{\mathbb{R}^d} x \left(\frac{|x|^2}{1+|x|^2} - \frac{1}{1+|ux|^2} + \frac{1}{1+|x|^2} \right) \nu(dx) \\
&= -\int_0^{\infty} u^{-\beta} e^{-u} du \int_{\mathbb{R}^d} x \frac{|ux|^2}{1+|ux|^2} \nu(dx) = -\int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu'(dx),
\end{aligned}$$

using (3.15) for $0 < \beta < 1$ and similarly for $\beta \leqslant 0$, and (2.38), and third,

$$\int_{|x|>1} x \log \frac{|x|}{1+t^{-1}|x|} \nu'(dx) \text{ is convergent as } t \to \infty, \tag{3.18}$$

which is equivalent to (2.48) with ν replaced by ν'. The proof of (3.18) is as follows. Denote by $I_1(t)$ the integral in (3.18). Then

$$I_1(t) = \int_{\mathbb{R}^d} x\nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} \log \frac{|ux|}{1+t^{-1}|ux|} du.$$

Let

$$\begin{aligned}
I_2(t) &= \int_{|x|>1} x\nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} \log \frac{|ux|}{1+t^{-1}|ux|} du, \\
I_3(t) &= \int_{|x|>1} x\nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} \log \frac{|ux|}{1+t^{-1}|x|} du, \\
I_4(t) &= \int_{|x|>1} x \log \frac{|x|}{1+t^{-1}|x|} \nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} du, \\
I_5(t) &= \Gamma(1-\beta) \int_{|x|>1} x \log \frac{|x|}{1+t^{-1}|x|} \nu(dx).
\end{aligned}$$

We claim that $I_j(t) - I_{j+1}(t)$ is convergent as $t \to \infty$ for $j = 1,2,3,4$. Since convergence of $I_5(t)$ as $t \to \infty$ is known from $\mu \in \mathfrak{D}(\Psi_1)$, this will prove (3.18). For $j=1$, convergence of $I_1(t) - I_2(t)$ follows from $2^{-1} \leqslant (1+t^{-1})^{-1} \leqslant |ux|(1+t^{-1}|ux|)^{-1} \leqslant |ux|$ for $|ux|>1$ and $t>1$ and from

$$\int_{|x|\leqslant 1} |x|\nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} (\log 2 + \log|ux|) du < \infty.$$

For $j=2$, notice that

$$\left| \log \frac{|ux|}{1+t^{-1}|ux|} - \log \frac{|ux|}{1+t^{-1}|x|} \right| = \left| \log \frac{1+t^{-1}|x|}{1+t^{-1}|ux|} \right| \leqslant |\log u|$$

for all $t>0$, $u>0$, and $x\in\mathbb{R}^d$, since

$$1\geqslant\frac{1+t^{-1}|x|}{1+t^{-1}|ux|}\geqslant\frac{1}{u}\quad\text{for } u>1,$$
$$1\leqslant\frac{1+t^{-1}|x|}{1+t^{-1}|ux|}\leqslant\frac{1}{u}\quad\text{for } 0<u\leqslant 1.$$

For $j=3$, simply note that

$$\int_{|x|>1}|x|\nu(dx)\int_0^\infty u^{-\beta}e^{-u}|\log u|du<\infty.$$

For $j=4$, notice that $2^{-1}\leqslant(1+t^{-1})^{-1}\leqslant|x|(1+t^{-1}|x|)^{-1}\leqslant|x|$ for $t>1$ and $|x|>1$ and that

$$\int_{|x|>1}|x|\log|x|\nu(dx)\int_0^{1/|x|}u^{-\beta}e^{-u}du\leqslant c_3\int_{|x|>1}|x|^\beta\log|x|\nu(dx)<\infty.$$

Hence $\mu'\in\mathfrak{D}(\Phi_{\beta,1})$.

In order to prove $\mu''=\widetilde{\mu}$, it suffices to show that $A''=\widetilde{A}$, $\nu''=\widetilde{\nu}$, and $\gamma''=\widetilde{\gamma}$. We use (2.29)–(2.31). Among them, $A''=\widetilde{A}$ and $\nu''=\widetilde{\nu}$ are proved like the discussion in lines 9–4 above (3.14). To show $\gamma''=\widetilde{\gamma}$, let

$$q(u)=\int_{\mathbb{R}^d}x\frac{|ux|^2}{1+|ux|^2}\nu(dx),\quad q'(u)=\int_{\mathbb{R}^d}x\frac{|ux|^2}{1+|ux|^2}\nu'(dx).$$

These are functions from $[0,\infty)$ into $\mathbb{R}^d$ ($q'(u)$ does not mean the derivative of $q(u)$). We have, using also (2.38) and the like,

$$\begin{aligned}
\widetilde{\gamma}&=-\lim_{t\to\infty}\int_0^t f_1(s)q(f_1(s))ds=-\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty u^{-1}e^{-u}q(u)du,\\
\gamma''&=-\lim_{t\to\infty}\int_0^t f_{\beta,1}(s)q'(f_{\beta,1}(s))ds=\lim_{\varepsilon\downarrow 0}\frac{-1}{\Gamma(1-\beta)}\int_\varepsilon^1(1-v)^{-\beta}v^{-1}q'(v)dv\\
&=\lim_{\varepsilon\downarrow 0}\frac{-1}{\Gamma(1-\beta)}\int_\varepsilon^1(1-v)^{-\beta}v^{-1}dv\int_0^\infty u^{-\beta}e^{-u}q(uv)du\\
&=\lim_{\varepsilon\downarrow 0}\frac{-1}{\Gamma(1-\beta)}\int_0^\infty u^{-\beta}q(u)du\int_\varepsilon^1(1-v)^{-\beta}v^{\beta-2}e^{-u/v}dv\\
&=\lim_{\varepsilon\downarrow 0}\frac{-1}{\Gamma(1-\beta)}\int_0^\infty u^{-1}e^{-u}q(u)du\int_0^{\varepsilon^{-1}(1-\varepsilon)u}v^{-\beta}e^{-v}dv.
\end{aligned}$$

Recall that $\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty u^{-1}e^{-u}q(u)du$ exists. Then, as $\varepsilon\downarrow 0$,

$$\begin{aligned}
&\int_0^\varepsilon u^{-1}e^{-u}q(u)du\int_0^{\varepsilon^{-1}(1-\varepsilon)u}v^{-\beta}e^{-v}dv\\
&\qquad=\int_0^{1-\varepsilon}v^{-\beta}e^{-v}dv\int_{\varepsilon(1-\varepsilon)^{-1}v}^{\varepsilon}u^{-1}e^{-u}q(u)du\to 0,\\
&\int_\varepsilon^\infty u^{-1}e^{-u}q(u)\left(\frac{1}{\Gamma(1-\beta)}\int_0^{\varepsilon^{-1}(1-\varepsilon)u}v^{-\beta}e^{-v}dv-1\right)du\\
&\qquad=\frac{1}{\Gamma(1-\beta)}\int_\varepsilon^\infty u^{-1}e^{-u}q(u)du\int_{\varepsilon^{-1}(1-\varepsilon)u}^\infty v^{-\beta}e^{-v}dv
\end{aligned}$$

$$= \frac{1}{\Gamma(1-\beta)} \int_{1-\varepsilon}^{\infty} v^{-\beta} e^{-v} dv \int_{\varepsilon}^{\varepsilon(1-\varepsilon)^{-1}v} u^{-1} e^{-u} q(u) du \to 0.$$

It follows that $\gamma'' = \widetilde{\gamma}$.

Step 3. We prove that Ψ_1, $\Psi_\beta \Phi_{\beta,1}$, and $\Phi_{\beta,1}\Psi_\beta$ have the identical domain. Recall that $\mathfrak{D}(\Psi_1) = \mathfrak{D}(\Phi_{\beta,1})$. It is already proved that $\mathfrak{D}(\Psi_\beta \Phi_{\beta,1}) = \mathfrak{D}(\Psi_1)$ and that $\mathfrak{D}(\Psi_1) \subset \mathfrak{D}(\Phi_{\beta,1}\Psi_\beta)$. It remains to show that $\mathfrak{D}(\Phi_{\beta,1}\Psi_\beta) \subset \mathfrak{D}(\Psi_1)$. Let $\mu \in \mathfrak{D}(\Phi_{\beta,1}\Psi_\beta)$. That is, let $\mu \in \mathfrak{D}(\Psi_\beta)$ and $\Psi_\beta \mu \in \mathfrak{D}(\Phi_{\beta,1})$. Again we denote $\mu' = \Psi_\beta \mu$. We have $\int_{|x|>1} |x| \nu(dx) < \infty$, because

$$\infty > \int_{|x|>1} |x| \nu'(dx) = \int_{\mathbb{R}^d} |x| \nu(dx) \int_{1/|x|}^{\infty} u^{-\beta} e^{-u} du \geqslant \int_{|x|>1} |x| \nu(dx) \int_1^{\infty} u^{-\beta} e^{-u} du.$$

We also have $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$ because, if not, then lines 7–9 of Step 2 would show that $\gamma' \neq -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu'(dx)$, which contradicts that $\mu' \in \mathfrak{D}(\Phi_{\beta,1})$. Finally, we see the convergence of $\int_{|x|>1} x \log(|x|/(1+t^{-1}|x|))\nu(dx)$ as $t \to \infty$, since the proof of (3.18) in Step 2 shows that this convergence is equivalent to the convergence of $\int_{|x|>1} x \log(|x|/(1+t^{-1}|x|))\nu'(dx)$. Consequently, $\mu \in \mathfrak{D}(\Psi_1)$. □

The proof of Theorem 3.1 is now complete.

4. Ranges of the stochastic integral operators

We study the ranges $\mathfrak{R}(\Psi_\alpha)$. If $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\alpha)$ with $-\infty < \alpha < 2$, then the triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ of $\widetilde{\mu} = \Psi_\alpha \mu$ is expressed as

$$\widetilde{A} = \Gamma(2-\alpha)A, \tag{4.1}$$

$$\widetilde{\nu}(B) = \int_0^{\infty} t^{-\alpha-1} e^{-t} dt \int_{\mathbb{R}^d} 1_B(tx)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{4.2}$$

$$\widetilde{\gamma} = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} t^{-\alpha} e^{-t} \left(\gamma - \int_{\mathbb{R}^d} x \left(\frac{1}{1+|x|^2} - \frac{1}{1+t^2|x|^2} \right) \nu(dx) \right) dt, \tag{4.3}$$

as Proposition 2.6 says. We will repeatedly use this expression.

We begin with establishing the one-to-one property of Ψ_α.

Proposition 4.1. *For each $\alpha \in \mathbb{R}$ the mapping Ψ_α is one-to-one. That is, if $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\alpha)$ and $\Psi_\alpha \mu = \widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$, then (A,ν,γ) is determined by $(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})$. Furthermore, A is determined by $\widetilde{A}$ alone, and ν is determined by $\widetilde{\nu}$ alone.*

Proof. If $\alpha \geqslant 2$, then $\mathfrak{D}(\Psi_\alpha) = \{\delta_0\}$ and the assertion is trivial. Let $\alpha < 2$. Then, A is determined by $\widetilde{A}$ from (4.1). Given $\varphi \in C_\sharp^+$, we have, for any $p > 0$,

$$\int_{\mathbb{R}^d} \varphi(p^{-1}x)\widetilde{\nu}(dx) = \int_0^{\infty} u^{-\alpha-1} e^{-u} du \int_{\mathbb{R}^d} \varphi(p^{-1}ux)\nu(dx)$$

from (4.2). Thus

$$p^{\alpha} \int_{\mathbb{R}^d} \varphi(p^{-1}x)\widetilde{\nu}(dx) = \int_0^{\infty} s^{-\alpha-1} e^{-ps} ds \int_{\mathbb{R}^d} \varphi(sx)\nu(dx).$$

By the uniqueness theorem in Laplace transform theory, we see that, for a.e. s, $s^{-\alpha-1}\int_{\mathbb{R}^d} \varphi(sx)\nu(dx)$ is determined by $\widetilde{\nu}$. Since $s^{-\alpha-1}\int_{\mathbb{R}^d} \varphi(sx)\nu(dx)$ is continuous in $s>0$, it is determined by $\widetilde{\nu}$ for all $s>0$. Letting $s=1$, we see that $\int_{\mathbb{R}^d}\varphi(x)\nu(dx)$ is determined by $\widetilde{\nu}$. If $\alpha<1$, then $\int_0^\infty t^{-\alpha}e^{-t}dt<\infty$ and γ is determined by $\widetilde{\nu}$ and $\widetilde{\gamma}$ from (4.3). If $\alpha\geqslant 1$, then Theorem 2.4 says that γ is determined by ν. $\square$

A distribution $\mu=\mu_{(A,\nu,\gamma)}\in ID(\mathbb{R}^d)$ is said to be *Gaussian* if $\nu=0$; otherwise it is said to be *non-Gaussian*. It is said to be *centered Gaussian* if $\nu=0$ and $\gamma=0$. It is said to be *purely non-Gaussian* if $\nu\neq 0$ and $A=0$.

If $\alpha\geqslant 2$, then $\mathfrak{R}(\Psi_\alpha)=\{\delta_0\}$, which follows from Theorem 2.4 (iv). Let $S=\{\xi\in\mathbb{R}^d\colon|\xi|=1\}$, the unit sphere in $\mathbb{R}^d$. The ranges $\mathfrak{R}(\Psi_\alpha)$ for $\alpha<2$ are described as follows.

Theorem 4.2. *Let $-\infty<\alpha<2$. Let $\widetilde{\mu}=\mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}\in ID(\mathbb{R}^d)$. In order that $\widetilde{\mu}\in\mathfrak{R}(\Psi_\alpha)$, it is necessary and sufficient that one of the following conditions depending on α is satisfied.*

(i) $(-\infty<\alpha<1)$ *Either $\widetilde{\mu}$ is Gaussian, or $\widetilde{\mu}$ is non-Gaussian and*

$$\widetilde{\nu}(B)=\int_S\widetilde{\lambda}(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha-1}\widetilde{h}_\xi(u)du,\qquad B\in\mathcal{B}(\mathbb{R}^d),\tag{4.4}$$

where

$$\begin{cases}\widetilde{\lambda}\text{ is a measure on }S\text{ and }\widetilde{h}_\xi(u)\text{ is a function measurable}\\ \text{in }\xi\text{ and, for }\widetilde{\lambda}\text{-a. e. }\xi\text{, completely monotone in }u\in(0,\infty),\\ \text{not identically zero, and }\lim_{u\to\infty}\widetilde{h}_\xi(u)=0.\end{cases}\tag{4.5}$$

(ii) $(\alpha=1)$ *Either $\widetilde{\mu}$ is centered Gaussian, or $\widetilde{\mu}$ is non-Gaussian and $\widetilde{\nu}$ has expression* (4.4) *with $\alpha=1$ together with* (4.5), *and*

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty te^{-t}dt\int_{\mathbb{R}^d}\frac{|x|^2x\nu(dx)}{1+t^2|x|^2}\ \text{exists in }\mathbb{R}^d\text{ and equals }-\widetilde{\gamma}\tag{4.6}$$

for the measure ν satisfying (4.2), *which is uniquely constructed from $\widetilde{\nu}$.*

(iii) $(1<\alpha<2)$ *Either $\widetilde{\mu}$ is centered Gaussian, or $\widetilde{\mu}$ is non-Gaussian and $\widetilde{\nu}$ has expression* (4.4) *satisfying condition* (4.5), *and*

$$\int_{\mathbb{R}^d}x\widetilde{\mu}(dx)=0.\tag{4.7}$$

(It follows from (4.4) *and* (4.5) *with $1<\alpha<2$ that $\int_{\mathbb{R}^d}|x|\widetilde{\mu}(dx)<\infty$.)*

The result (i) is already known for $\alpha=-1,0$ in Theorems A and C of Barndorff-Nielsen et al. (2006). In fact it is shown there that $\mathfrak{R}(\Psi_{-1})=B(\mathbb{R}^d)$ and $\mathfrak{R}(\Psi_0)=T(\mathbb{R}^d)$, where $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ are, respectively, the Goldie-Steutel-Bondesson class and the Thorin class on $\mathbb{R}^d$ introduced in Barndorff-Nielsen et al. (2006).

Remark 4.3. *In* (4.5) *it follows from the complete monotonicity of $\widetilde{h}_\xi(u)$ that $\lim_{u\to\infty}\widetilde{h}_\xi(u)$ exists and is nonnegative. If $\alpha\leqslant 0$, then automatically*

$$\lim_{u\to\infty}\widetilde{h}_\xi(u)=0$$

for $\widetilde{\lambda}$-a. e. ξ, since

$$\infty>\int_{|x|>1}\widetilde{\nu}(dx)=\int_S\widetilde{\lambda}(d\xi)\int_1^\infty u^{-\alpha-1}\widetilde{h}_\xi(u)du.$$

We prepare a lemma.

Lemma 4.4. *Let $-\infty < \alpha < 2$. Let $\widetilde{\nu}$ be a measure on $\mathbb{R}^d$ such that $\widetilde{\nu}(\{0\}) = 0$ and $0 < \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\widetilde{\nu}(dx) < \infty$. Then there is a measure ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$,*

$$\begin{cases} \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty & \text{for } \alpha < 0, \\ \int_{\mathbb{R}^d}(|x|^2 \wedge \log(1+|x|))\nu(dx) < \infty & \text{for } \alpha = 0, \\ \int_{\mathbb{R}^d}(|x|^2 \wedge |x|^\alpha)\nu(dx) < \infty & \text{for } 0 < \alpha < 2, \end{cases} \tag{4.8}$$

and condition (4.2) *if and only if $\widetilde{\nu}$ satisfies* (4.4) *together with condition* (4.5).

Proof. *The "only if" part.* Suppose that there is a measure ν satisfying $\nu(\{0\}) = 0$ and conditions (4.2) and (4.8). Then $\nu \neq 0$. Let (λ, ν_ξ) be a polar decomposition of ν, that is,

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \qquad B \in \mathcal{B}(\mathbb{R}^d), \tag{4.9}$$

where λ is a measure on S with $0 < \lambda(S) \leqslant \infty$, ν_ξ is a measure on $(0,\infty)$ with $0 < \nu_\xi((0,\infty)) \leqslant \infty$ and $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}(\mathbb{R}^d)$ (see Lemma 2.1 of Barndorff-Nielsen et al. (2006)). We have

$$\int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) < \infty \tag{4.10}$$

from (4.8). It follows from (4.2) that for any nonnegative measurable function $\varphi(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)\widetilde{\nu}(dx) &= \int_0^\infty t^{-\alpha-1}e^{-t}dt \int_{\mathbb{R}^d} \varphi(tx)\nu(dx) \\ &= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_0^\infty t^{-\alpha-1}e^{-t}\varphi(tr\xi)dt \\ &= \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(dr) \int_0^\infty r^\alpha u^{-\alpha-1}e^{-u/r}\varphi(u\xi)du \\ &= \int_S \lambda(d\xi) \int_0^\infty \varphi(u\xi)u^{-\alpha-1}\widetilde{h}_\xi(u)du, \end{aligned}$$

where

$$\widetilde{h}_\xi(u) = \int_0^\infty r^\alpha e^{-u/r}\nu_\xi(dr). \tag{4.11}$$

This expression of $\widetilde{h}_\xi(u)$ shows that it is measurable in ξ and, for λ-a. e. ξ, completely monotone in $u \in (0,\infty)$, not identically zero, and $\lim_{u\to\infty} \widetilde{h}_\xi(u) = 0$.

The "if" part. Suppose that $\widetilde{\nu}$ satisfies (4.4) and (4.5). Then, for $\widetilde{\lambda}$-a. e. ξ, we can find by Bernstein's theorem a measure $\widetilde{Q}_\xi$ on $(0,\infty)$ such that

$$\widetilde{h}_\xi(u) = \int_{(0,\infty)} e^{-uv}\widetilde{Q}_\xi(dv). \tag{4.12}$$

In general $\widetilde{Q}_\xi$ is a measure on $[0,\infty)$, but it has no point mass at 0 since

$$\lim_{u\to\infty} \widetilde{h}_\xi(u) = 0.$$

Choosing $\widetilde{h}_\xi(u) = 0$ for ξ in the exceptional set of $\widetilde{\lambda}$-measure 0, we define $\widetilde{Q}_\xi$ for all ξ. For each Borel set B in $(0,\infty)$, $\widetilde{Q}_\xi(B)$ is measurable in ξ (see the proof of

Lemma 3.3 of Sato (1980) for the details). If two measures ν_ξ and $\widetilde{Q}_\xi$ satisfy

$$\int_0^\infty \varphi(v)\widetilde{Q}_\xi(dv) = \int_0^\infty \varphi(r^{-1})r^p \nu_\xi(dr) \tag{4.13}$$

for all nonnegative measurable functions φ on $(0,\infty)$, then

$$\int_0^\infty \psi(r)\nu_\xi(dr) = \int_0^\infty \psi(v^{-1})v^p \widetilde{Q}_\xi(dv) \tag{4.14}$$

for all nonnegative measurable functions ψ on $(0,\infty)$. Therefore we define ν_ξ from $\widetilde{Q}_\xi$ by (4.14). Let $\lambda = \widetilde{\lambda}$ and define ν by (4.9). Then

$$\begin{aligned}
\infty > \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\widetilde{\nu}(dx) &= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty (u^2 \wedge 1)u^{-\alpha-1}\widetilde{h}_\xi(u)du \\
&= \int_S \widetilde{\lambda}(d\xi) \int_0^\infty (u^2 \wedge 1)u^{-\alpha-1}du \int_{(0,\infty)} e^{-uv}\widetilde{Q}_\xi(dv) \\
&= \int_S \widetilde{\lambda}(d\xi) \int_{(0,\infty)} a(v)\widetilde{Q}_\xi(dv),
\end{aligned}$$

where

$$\begin{aligned}
a(v) &= \int_0^1 u^{1-\alpha}e^{-uv}du + \int_1^\infty u^{-\alpha-1}e^{-uv}du \\
&= v^{\alpha-2}\int_0^v u^{1-\alpha}e^{-u}du + v^\alpha \int_v^\infty u^{-\alpha-1}e^{-u}du.
\end{aligned}$$

Thus we obtain (4.8), since

$$a(v) \sim \begin{cases} c_1 v^\alpha & \text{as } v \downarrow 0 \text{ for } \alpha < 0, \\ \log(1/v) & \text{as } v \downarrow 0 \text{ for } \alpha = 0, \\ c_2 & \text{as } v \downarrow 0 \text{ for } 0 < \alpha < 2, \\ c_3 v^{\alpha-2} & \text{as } v \uparrow \infty \text{ for } -\infty < \alpha < 2, \end{cases}$$

where c_1, c_2, and c_3 are positive constants. Since we have (4.11), we can repeat the calculation above (4.11), which gives (4.2). □

Remark to Lemma 4.4. The proof of this lemma shows the following. A polar decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ of $\widetilde{\nu}$ is determined from a polar decomposition (λ, ν_ξ) of ν by $\widetilde{\lambda} = \lambda$, (4.4), (4.5), and (4.11). Conversely, given $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$ satisfying (4.4) and (4.5), we can determine a decomposition (λ, ν_ξ) by $\lambda = \widetilde{\lambda}$ and

$$\nu_\xi = \mathrm{Inv}_\alpha(L^{-1}(\widetilde{h}_\xi)). \tag{4.15}$$

Here L is the Laplace transform of measures and L^{-1} is the inverse of L; for two measures ρ and $\widetilde{\rho}$ on $(0,\infty)$ we say that ρ is the α-inversion of $\widetilde{\rho}$ if

$$\int_{(0,\infty)} \psi(r)\rho(dr) = \int_{(0,\infty)} \psi(v^{-1})v^\alpha \widetilde{\rho}(dv) \tag{4.16}$$

for all nonnegative measurable functions ψ and write $\rho = \mathrm{Inv}_\alpha(\widetilde{\rho})$. Note that

$$\mathrm{Inv}_\alpha(\mathrm{Inv}_\alpha(\widetilde{\rho})) = \widetilde{\rho}. \tag{4.17}$$

Proof of Theorem 4.2. (i) $(-\infty < \alpha < 1)$ Assume that $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. If $\widetilde{\mu}$ is non-Gaussian, then Lemma 4.4 assures that (4.4) and (4.5) hold. To see the converse part, if $\widetilde{\mu}$ is Gaussian, then, letting $A = (\Gamma(2-\alpha))^{-1}\widetilde{A}$, $\nu = 0$, and $\gamma = (\Gamma(1-\alpha))^{-1}\widetilde{\gamma}$, we see that $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\alpha)$ and $\widetilde{\mu} = \Psi_\alpha \mu$. If $\widetilde{\nu} \neq 0$ and (4.4) and (4.5) hold, then choose the measure ν in Lemma 4.4, $A = (\Gamma(2-\alpha))^{-1}\widetilde{A}$ and

$$\gamma = (\Gamma(1-\alpha))^{-1}\left(\widetilde{\gamma} + \int_0^\infty t^{-\alpha}e^{-t}dt \int_{\mathbb{R}^d} x\left(\frac{1}{1+|x|^2} - \frac{1}{1+t^2|x|^2}\right)\nu(dx)\right),$$

noting that

$$\int_0^\infty t^{-p}e^{-t}dt \int_{\mathbb{R}^d} |x|\left|\frac{1}{1+|x|^2} - \frac{1}{1+t^2|x|^2}\right|\nu(dx) < \infty,$$

as is checked in Section 3, and see that $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\alpha)$ and $\Psi_\alpha\mu = \widetilde{\mu}$.

(ii) ($\alpha = 1$) Suppose that $\widetilde{\mu} \in \mathfrak{R}(\Psi_1)$ and $\widetilde{\mu} = \Psi_1\mu$ with $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_1)$. If $\widetilde{\mu}$ is Gaussian, then $\nu = 0$ by Proposition 4.1 and $\gamma = 0$ by Theorem 2.8, and consequently $\widetilde{\gamma} = 0$ by (4.3), that is, $\widetilde{\mu}$ is centered. If $\widetilde{\mu}$ is non-Gaussian, then Lemma 4.4 assures us of the expression (4.4) of $\widetilde{\nu}$ with $\alpha = 1$ and the condition (4.5), and it follows from (2.38) and from (4.3) with $\alpha = 1$ that (4.6) is true.

Turning to the converse, if $\widetilde{\mu}$ is centered Gaussian, then $\widetilde{\mu} \in \mathfrak{R}(\Psi_1)$ from Theorem 2.8. Suppose that $\widetilde{\mu}$ is non-Gaussian and satisfies (4.4), (4.5), and (4.6). Then, by Lemma 4.4, there is a measure ν satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d}(|x|^2 \wedge |x|)\nu(dx) < \infty$, and (4.2) with $\alpha = 1$. The proof of Proposition 4.1 shows the uniqueness of such a measure ν. Let $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$, $A = \widetilde{A}$, and $\mu = \mu_{(A,\nu,\gamma)}$. It follows from (4.6) that

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 te^{-t}\left(\int_{|x|>1}\frac{|x|^2 x\nu(dx)}{1+t^2|x|^2}\right)dt \text{ exists in } \mathbb{R}^d, \tag{4.18}$$

because

$$\int_1^\infty te^{-t}dt\int_{\mathbb{R}^d}\frac{|x|^3\nu(dx)}{1+t^2|x|^2} < \infty \text{ and } \int_0^1 te^{-t}dt\int_{|x|\leqslant 1}\frac{|x|^3\nu(dx)}{1+t^2|x|^2} < \infty.$$

Hence $\mu \in \mathfrak{D}(\Psi_1)$ by Theorem 2.8, since the condition (2.48) is rewritten to (4.18) in our case (see also Theorem 5.12 (i) of Sato (2005)). It also follows from (4.6) that

$$\begin{aligned}\widetilde{\gamma} &= -\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty te^{-t}dt\int_{\mathbb{R}^d}\frac{|x|^2x\nu(dx)}{1+t^2|x|^2}\\ &= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty t^{-1}e^{-t}dt\left(-\int_{\mathbb{R}^d}\frac{|x|^2x\nu(dx)}{1+|x|^2} - \int_{\mathbb{R}^d}x\left(\frac{1}{1+|x|^2}-\frac{1}{1+t^2|x|^2}\right)\nu(dx)\right),\end{aligned}$$

which equals the right-hand side of (4.3). Therefore $\Psi_1\mu = \widetilde{\mu}$ and $\widetilde{\mu} \in \mathfrak{R}(\Psi_1)$.

(iii) ($1 < \alpha < 2$) Assume that $\widetilde{\mu} = \Psi_\alpha\mu$ with some $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Psi_\alpha)$. If $\widetilde{\mu}$ is Gaussian, then $\widetilde{\mu}$ is centered by the same reason as in (ii). Next, suppose that $\widetilde{\mu}$ is non-Gaussian. Then Lemma 4.4 says that $\widetilde{\nu}$ has expression (4.4) satisfying (4.5). Since $\mu \in \mathfrak{D}(\Psi_\alpha)$, ν and γ satisfy (2.34) and (2.38). Thus

$$\begin{aligned}\int_0^\infty t^{2-\alpha}e^{-t}dt\int_{\mathbb{R}^d}\frac{|x|^3\nu(dx)}{1+t^2|x|^2} &\leqslant \int_{|x|\leqslant 1}|x|^3\nu(dx)\int_0^\infty t^{2-\alpha}e^{-t}dt\\ &\quad + \int_{|x|>1}|x|^3\nu(dx)\int_0^{1/|x|}t^{2-\alpha}e^{-t}dt + \int_{|x|>1}|x|\nu(dx)\int_{1/|x|}^\infty t^{-\alpha}e^{-t}dt\\ &= I_1 + I_2 + I_3 \qquad \text{(say)}\end{aligned}$$

and $I_1 < \infty$, $I_2 \leqslant c_1 \int_{|x|>1} |x|^\alpha \nu(dx) < \infty$, and $I_3 \leqslant c_2 \int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ with some c_1, c_2. Hence, using (4.2), we see that $\int_{\mathbb{R}^d} |x|^3(1+|x|^2)^{-1}\widetilde{\nu}(dx) < \infty$, that is, $\int_{|x|>1} |x|\widetilde{\nu}(dx) < \infty$. Using (4.3) and (2.38), we obtain

$$\widetilde{\gamma} = -\lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty t^{2-\alpha} e^{-t} dt \int_{\mathbb{R}^d} \frac{|x|^2 x\nu(dx)}{1+t^2|x|^2} = -\int_0^\infty t^{2-\alpha} e^{-t} dt \int_{\mathbb{R}^d} \frac{|x|^2 x\nu(dx)}{1+t^2|x|^2}$$

Hence

$$\widetilde{\gamma} = -\int_{\mathbb{R}^d} \frac{|x|^2 x}{1+|x|^2} \widetilde{\nu}(dx), \tag{4.19}$$

which is equivalent to (4.7).

Let us consider the converse. If $\widetilde{\mu}$ is centered Gaussian, then obviously $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. Let $\widetilde{\mu}$ be non-Gaussian such that (4.4), (4.5), and (4.7) are satisfied. Using Lemma 4.4, we can find a measure ν satisfying $\nu(\{0\}) = 0$, (4.2), and (4.8). We have $\int_{|x|>1} |x|\widetilde{\nu}(dx) < \infty$ by the same reason as in the preceding paragraph. Thus $\int_{\mathbb{R}^d} |x|\widetilde{\mu}(dx) < \infty$ and we have (4.19). Let $\gamma = -\int_{\mathbb{R}^d} x|x|^2(1+|x|^2)^{-1}\nu(dx)$, $A = (\Gamma(2-\alpha))^{-1}\widetilde{A}$, and $\mu = \mu_{(A,\nu,\gamma)}$. Then $\mu \in \mathfrak{D}(\Psi_\alpha)$ by Theorem 2.4. Further,

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty t^{-\alpha} e^{-t} \left(\gamma - \int_{\mathbb{R}^d} x \left(\frac{1}{1+|x|^2} - \frac{1}{1+t^2|x|^2}\right)\nu(dx)\right) dt = -\int_{\mathbb{R}^d} \frac{|x|^2 x}{1+|x|^2}\widetilde{\nu}(dx),$$

which equals $\widetilde{\gamma}$. Hence (4.3) is true. Therefore $\Psi_\alpha \mu = \widetilde{\mu}$ and thus $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. □

Let us study how $\mathfrak{R}(\Psi_\alpha)$ changes with α.

Proposition 4.5. *For* $-\infty < \alpha < 2$

$$\mathfrak{R}(\Psi_\alpha) \supsetneqq \bigcup_{\alpha'>\alpha} \mathfrak{R}(\Psi_{\alpha'}). \tag{4.20}$$

Corollary 4.6. *For* $-\infty < \beta < \alpha \leqslant 2$, $\mathfrak{R}(\Psi_\beta) \supsetneqq \mathfrak{R}(\Psi_\alpha)$.

Proof of Proposition 4.5. *Step 1.* Let us show that $\mathfrak{R}(\Psi_\alpha) \supset \mathfrak{R}(\Psi_{\alpha'})$ for $-\infty < \alpha < \alpha' < \infty$. If $\alpha' \geqslant 2$, then this is evident since $\mathfrak{R}(\Psi_{\alpha'}) = \{\delta_0\}$. Let $-\infty < \alpha < \alpha' < 2$. Let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \mathfrak{R}(\Psi_{\alpha'})$. If $\widetilde{\nu} = 0$, then we get $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$, using Theorem 4.2. Suppose that $\widetilde{\nu} \neq 0$. Then, Lemma 4.4 says that $\widetilde{\nu}$ satisfies (4.4) and (4.5) with α replaced by α'. Thus

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} u^{-(\alpha'-\alpha)} \widetilde{h}_\xi^{(\alpha')} du,$$

where $\widetilde{h}_\xi^{(\alpha')}$ is completely monotone and tends to 0 as $u \to \infty$. Since $u^{-(\alpha'-\alpha)}$ is completely monotone, $u^{-(\alpha'-\alpha)}\widetilde{h}_\xi^{(\alpha')}$ is completely monotone and tends to 0 as $u \to \infty$. Hence, again by Lemma 4.4, there is $\nu^{(\alpha)}$ satisfying $\nu^{(\alpha)}(\{0\}) = 0$, (4.2), and (4.8) with ν replaced by $\nu^{(\alpha)}$. In the case $\alpha < 1$, we have now $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ by Theorem 4.2 (i). In the case $\alpha > 1$, Theorem 4.2 says that $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$ and $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. In the case $\alpha = 1$, we have $\int_{|x|>1} |x|^3(1+|x|^2)^{-1}\widetilde{\nu}(dx) < \infty$ and (4.19) since $1 < \alpha' < 2$. It follows that

$$\int_0^\infty t^{-2} e^{-t} dt \int_{\mathbb{R}^d} \frac{|tx|^3 \nu^{(1)}(dx)}{1+|tx|^2} < \infty \text{ and } \widetilde{\gamma} = -\int_0^\infty t^{-2} e^{-t} dt \int_{\mathbb{R}^d} \frac{|tx|^2 tx \nu^{(1)}(dx)}{1+|tx|^2}.$$

Hence (4.6) is satisfied. This means that $\widetilde{\mu} \in \mathfrak{R}(\Psi_1)$.

Step 2. Let $-\infty < \alpha < 2$. Step 1 shows (4.20) except the strictness of the inclusion. In order to see the strictness, let us construct $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ such that $\widetilde{\mu} \notin \bigcup_{\alpha'>\alpha} \mathfrak{R}(\Psi_{\alpha'})$. Let

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} e^{-u} du$$

with $0 < \widetilde{\lambda}(S) < \infty$. Then $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \widetilde{\nu}(dx) < \infty$. We use Theorem 4.2. If $\alpha < 1$, then let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ with arbitrary $\widetilde{A}$ and $\widetilde{\gamma}$, and see that $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. If $1 < \alpha < 2$, then, noting $\int_{|x|>1} |x| \widetilde{\nu}(dx) < \infty$, choose $\widetilde{\gamma}$ satisfying (4.19), let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ with arbitrary $\widetilde{A}$, and see that $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$. If $\alpha = 1$, then choose $\widetilde{\lambda}$ as the uniform measure on S, find that the corresponding ν constructed in the proof of Lemma 4.4 is rotation-invariant, and let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ with $\widetilde{\gamma} = 0$ and $\widetilde{A}$ arbitrary to conclude that $\widetilde{\mu} \in \mathfrak{R}(\Psi_1)$.

In all cases $\widetilde{\mu} \notin \bigcup_{\alpha'>\alpha} \mathfrak{R}(\Psi_{\alpha'})$. Indeed, suppose that, on the contrary, $\widetilde{\mu} \in \mathfrak{R}(\Psi_{\alpha'})$ for some $\alpha < \alpha' < 2$. Then

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}^{(\alpha')}(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha'-1} \widetilde{h}_\xi^{(\alpha')}(u) du,$$

where, for $\widetilde{\lambda}^{(\alpha')}$-a. e. ξ, $\widetilde{h}_\xi^{(\alpha')}(u)$ is completely monotone, is not identically zero, and tends to 0 as $u \to \infty$. Using the uniqueness assertion of polar decompositions in Lemma 2.1 of Barndorff-Nielsen et al. (2006), we can find a measurable function $0 < c(\xi) < \infty$ such that $\widetilde{\lambda}^{(\alpha')}(d\xi) = c(\xi)\widetilde{\lambda}(d\xi)$ and

$$c(\xi) u^{-\alpha'-1} \widetilde{h}_\xi^{(\alpha')}(u) = u^{-\alpha-1} e^{-u} \quad \text{for } \widetilde{\lambda}\text{-a. e. } \xi.$$

It follows that, for $\widetilde{\lambda}$-a. e. ξ, $\lim_{u \downarrow 0} \widetilde{h}_\xi^{(\alpha')}(u) = 0$, which contradicts the complete monotonicity. $\square$

A distribution μ is said to be *trivial* if it is concentrated at a point. Let $\mu \in ID(\mathbb{R}^d)$ and $0 < \alpha \leqslant 2$. We say that μ is *strictly α-stable* if, for any $t > 0$, $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha} z)$, $z \in \mathbb{R}^d$. We say that μ is *α-stable* if, for any $t > 0$, there is $\gamma_t \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^t = \widehat{\mu}(t^{1/\alpha} z) \exp(i\langle \gamma_t, z \rangle)$. (When μ is trivial, this terminology is different from that of Sato (1999).) Let

$$\mathfrak{S}_\alpha^0 = \mathfrak{S}_\alpha^0(\mathbb{R}^d) = \{\mu \in ID(\mathbb{R}^d) \colon \ \mu \text{ is strictly } \alpha\text{-stable}\},$$
$$\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d) = \{\mu \in ID(\mathbb{R}^d) \colon \ \mu \text{ is } \alpha\text{-stable}\}.$$

Proposition 4.7. (i) *Let $0 < \alpha \leqslant 1$. We have*

$$\mathfrak{S}_\alpha \subset \bigcap_{\beta<\alpha} \mathfrak{R}(\Psi_\beta). \tag{4.21}$$

If $\mu \in \mathfrak{S}_\alpha$ and μ is non-trivial, then $\mu \notin \mathfrak{R}(\Psi_\alpha)$.

(ii) *Let $1 < \alpha \leqslant 2$. We have*

$$\mathfrak{S}_\alpha^0 \subset \bigcap_{\beta<\alpha} \mathfrak{R}(\Psi_\beta). \tag{4.22}$$

If $\mu \in \mathfrak{S}_\alpha \setminus \mathfrak{S}_\alpha^0$, then $\mu \notin \bigcup_{\beta>1} \mathfrak{R}(\Psi_\beta)$. If $\mu \in \mathfrak{S}_\alpha$ and $\mu \neq \delta_0$, then $\mu \notin \mathfrak{R}(\Psi_\alpha)$.

Proof. Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$. If $0 < \alpha < 2$, then $\mu \in \mathfrak{S}_\alpha$ if and only if $A = 0$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr, \qquad B \in \mathcal{B}(\mathbb{R}^d) \tag{4.23}$$

with a measure λ on S satisfying $0 \leqslant \lambda(S) < \infty$. Hence, factoring $r^{-\alpha-1} = r^{-\beta-1} r^{-(\alpha-\beta)}$ for $\beta < \alpha$, we obtain (4.21) and (4.22) from Theorem 4.2. The other assertions are proved from the same theorem and the uniqueness assertion of polar decompositions and from $\mathfrak{R}(\Psi_2) = \{\delta_0\}$. □

Proposition 4.8. *If* $0 < \alpha \leqslant 2$, *then*

$$\bigcap_{\beta<\alpha} \mathfrak{R}(\Psi_\beta) \supsetneqq \mathfrak{R}(\Psi_\alpha). \tag{4.24}$$

Proof. Without the strictness, (4.24) follows from Corollary 4.6. The strictness is a consequence of Proposition 4.7. □

Let $0 < \alpha < 2$. Rosiński (2004) calls $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$ *tempered α-stable* if $A = 0$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} h_\xi(r) dr, \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a measure on S with $0 < \lambda(S) < \infty$ and $h_\xi(u)$ is measurable in ξ and completely monotone in u, $\lim_{u\to\infty} h_\xi(u) = 0$, and $\lim_{u\downarrow 0} h_\xi(u) = 1$.

Proposition 4.9. (i) *Let* $0 < \alpha < 1$. *If* $\widetilde{\mu}$ *is tempered* α*-stable in Rosiński's sense, then* $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$.

(ii) *Let* $1 \leqslant \alpha < 2$. *If* $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ *is tempered* α*-stable in Rosiński's sense and if* (4.6) (*for* $\alpha = 1$) *or* (4.7) (*for* $1 < \alpha < 2$) *is satisfied, then* $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$.

(iii) *Let* $0 < \alpha < 2$. *Some purely non-Gaussian* $\widetilde{\mu}$ *in* $\mathfrak{R}(\Psi_\alpha)$ *is not tempered* α*-stable in Rosiński's sense.*

Proof. The assertions (i) and (ii) follow from Theorem 4.2. To show (iii), choose $\widetilde{\mu} \in \mathfrak{S}_{\alpha'}$ for some $\alpha' \in (\alpha, 1)$ if $0 < \alpha < 1$ or choose $\widetilde{\mu} \in \mathfrak{S}^0_{\alpha'}$ for some $\alpha' \in (\alpha, 2)$ if $1 \leqslant \alpha < 2$. Then $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ by virtue of Proposition 4.7 while $\widetilde{\mu}$ is not tempered α-stable in Rosiński's sense. □

Let us study properties of moments of distributions in $\mathfrak{R}(\Psi_\alpha)$.

Proposition 4.10. *Let* $0 < \alpha < 2$.

(i) *If* $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$, *then, for all* $\beta \in (0, \alpha)$, $\int_{\mathbb{R}^d} |x|^\beta \widetilde{\mu}(dx) < \infty$.

(ii) *There is* $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ *such that* $\int_{\mathbb{R}^d} |x|^\alpha \widetilde{\mu}(dx) = \infty$.

(iii) *There is a non-Gaussian* $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ *such that, for all* $\alpha' > 0$,

$$\int_{\mathbb{R}^d} |x|^{\alpha'} \widetilde{\mu}(dx) < \infty.$$

Proof. (i) Let $\mu \in \mathfrak{D}(\Psi_\alpha)$ and $\widetilde{\mu} = \Psi_\alpha \mu$. Let ν and $\widetilde{\nu}$ be their Lévy measures. It follows from (4.2) that

$$\int_{|x|>1} |x|^\beta \widetilde{\nu}(dx) = \int_{\mathbb{R}^d} |x|^\beta \nu(dx) \int_{1/|x|}^\infty t^{-(\alpha-\beta)-1} e^{-t} dt < \infty$$

for $0 < \beta < \alpha$, since $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ from Theorems 2.4, 2.8.

(ii) Define a measure ν by

$$\nu(B) = \int_S \lambda(d\xi) \int_e^\infty 1_B(r\xi) r^{-1-\alpha} (\log r)^{-p} dr$$

with $0 < \lambda(S) < \infty$ and $1 < p \leqslant 2$. Then $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$. Let A be arbitrary. If $\mu = \mu_{(A,\nu,\gamma)}$ with some γ is in $\mathfrak{D}(\Psi_\alpha)$, then the Lévy measure $\widetilde{\nu}$ of $\widetilde{\mu} = \Psi_\alpha(\mu)$ satisfies

$$\int_{|x|>1} |x|^\alpha \widetilde{\nu}(dx) = \int_S \lambda(d\xi) \int_e^\infty r^{-1} (\log r)^{-p} dr \int_{1/r}^\infty t^{-1} e^{-t} dt = \infty,$$

since $\int_{1/r}^\infty t^{-1} e^{-t} dt \sim \log r$ as $r \to \infty$, and hence $\int_{\mathbb{R}^d} |x|^\alpha \widetilde{\mu}(dx) = \infty$. If $0 < \alpha < 1$, then, for any γ, μ is in $\mathfrak{D}(\Psi_\alpha)$. If $\alpha = 1$, then, letting λ be the uniform measure on S and $\gamma = 0$, we get $\mu \in \mathfrak{D}(\Psi_1)$. If $1 < \alpha < 2$, then, choosing γ as in (2.38), we get $\mu \in \mathfrak{D}(\Psi_\alpha)$. We have used Theorems 2.4, 2.8.

(iii) Let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ with $\widetilde{A}$ arbitrary and

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(u\xi) u^{-1-\alpha} e^{-u} du,$$

where $0 < \widetilde{\lambda}(S) < \infty$. Then, for all $\alpha' > 0$, $\int_{|x|>1} |x|^{\alpha'} \widetilde{\nu}(dx) < \infty$ and hence $\int |x|^{\alpha'} \widetilde{\mu}(dx) < \infty$. If $0 < \alpha < 1$, then, for any $\widetilde{\gamma}$, $\widetilde{\mu}$ is in $\mathfrak{R}(\Psi_\alpha)$. If $\alpha = 1$, then, letting $\widetilde{\lambda}$ be the uniform measure on S and $\widetilde{\gamma} = 0$, we get $\mu \in \mathfrak{R}(\Psi_1)$. If $1 < \alpha < 2$, then, choosing $\widetilde{\gamma}$ as in (4.19), we get $\mu \in \mathfrak{R}(\Psi_\alpha)$. We have used Theorem 4.2. □

Remark 4.11. *By the same method as above we can prove that, for $0 < \alpha < 2$, any $\widetilde{\mu} \in \mathfrak{R}(\Psi_\alpha)$ has Lévy measure $\widetilde{\nu}$ satisfying $\int_{|x|\leqslant 1} |x|^\alpha \widetilde{\nu}(dx) = \infty$. Indeed, for $\widetilde{\mu} = \Psi_\alpha \mu$,*

$$\int_{|x|\leqslant 1} |x|^\alpha \widetilde{\nu}(dx) = \int_{\mathbb{R}^d} |x|^\alpha \nu(dx) \int_0^{1/|x|} t^{-1} e^{-t} dt = \infty.$$

Proposition 4.12. *There is $\widetilde{\mu} \in \mathfrak{R}(\Psi_0) = T(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x|^{\alpha'} \widetilde{\mu}(dx) = \infty$ for all $\alpha' > 0$.*

On the other hand, it is clear from Proposition 4.10 that there is a non-Gaussian $\widetilde{\mu} \in \mathfrak{R}(\Psi_0)$ satisfying $\int_{\mathbb{R}^d} |x|^{\alpha'} \widetilde{\mu}(dx) < \infty$ for all $\alpha' > 0$, since $\mathfrak{R}(\Psi_0) \supset \mathfrak{R}(\Psi_\alpha)$ for $\alpha > 0$.

Proof of Proposition 4.12. Let $\widetilde{h}(u) = (\log(2+u))^{-p}$. Then $\widetilde{h}(u)$ is completely monotone on $(0,\infty)$ for $p > 0$. Let

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(u\xi) u^{-1} \widetilde{h}(u) du$$

with $0 < \widetilde{\lambda}(S) < \infty$ and $p > 1$. Then $\int_{|x|\leqslant 1} |x|^2 \widetilde{\nu}(dx) = \widetilde{\lambda}(S) \int_0^1 u \widetilde{h}(u) du < \infty$,

$$\int_{|x|>1} \widetilde{\nu}(dx) = \widetilde{\lambda}(S) \int_1^\infty u^{-1} \widetilde{h}(u) du < \infty,$$

and $\int_{|x|>1} |x|^p \widetilde{\nu}(dx) = \widetilde{\lambda}(S) \int_1^\infty u^{p-1} \widetilde{h}(u) du = \infty$ for all $p > 0$. Use Theorem 4.2. Then any $\widetilde{\mu}$ with Lévy measure $\widetilde{\nu}$ is in $\mathfrak{R}(\Psi_0)$ and has the desired property. □

We introduce the modified ranges of Ψ_α connected to compensated and essential improper integrals and absolute definability, as follows:

$$\mathfrak{R}_{\mathrm{c}}(\Psi_\alpha;\mathbb{R}^d)=\Big\{\mathcal{L}\Big(\int_0^{\infty-} f_\alpha(s)dX_s^{(\mu*\delta_{-q})}\Big)\colon \mu\in ID(\mathbb{R}^d) \text{ and } q\in\mathbb{R}^d$$
$$\text{such that } \int_0^{\infty-} f_\alpha(s)dX_s^{(\mu*\delta_{-q})} \text{ is definable}\Big\},$$
$$\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha;\mathbb{R}^d)=\Big\{\mathcal{L}\Big(\operatorname*{p-lim}_{t\to\infty}\Big(\int_0^t f_\alpha(s)dX_s^{(\mu)}-q_t\Big)\Big)\colon \mu\in ID(\mathbb{R}^d) \text{ and } q_t \text{ from}$$
$$[0,\infty) \text{ into } \mathbb{R}^d \text{ such that } \operatorname*{p-lim}_{t\to\infty}\Big(\int_0^t f_\alpha(s)dX_s^{(\mu)}-q_t\Big) \text{ exists}\Big\},$$
$$\mathfrak{R}^0(\Psi_\alpha;\mathbb{R}^d)=\{\Psi_\alpha(\mu)\colon \mu\in\mathfrak{D}^0(\Psi_\alpha;\mathbb{R}^d)\}.$$

We usually write $\mathfrak{R}_{\mathrm{c}}(\Psi_\alpha)$, $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$, and $\mathfrak{R}^0(\Psi_\alpha)$, omitting $\mathbb{R}^d$. Clearly,

$$\mathfrak{R}^0(\Psi_\alpha)\subset\mathfrak{R}(\Psi_\alpha)\subset\mathfrak{R}_{\mathrm{c}}(\Psi_\alpha)\subset\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha). \tag{4.25}$$

Proposition 4.13. *For any $\alpha\in\mathbb{R}$, $\mathfrak{R}_{\mathrm{c}}(\Psi_\alpha)=\mathfrak{R}(\Psi_\alpha)$.*

Proof. If $\widetilde{\mu}\in\mathfrak{R}_{\mathrm{c}}(\Psi_\alpha)$, then $\widetilde{\mu}=\mathcal{L}\left(\int_0^{\infty-} f_\alpha(s)dX_s^{(\mu*\delta_{-q})}\right)$ for some $\mu\in ID(\mathbb{R}^d)$ and $q\in\mathbb{R}^d$ and thus $\widetilde{\mu}=\Psi_p(\mu*\delta_{-q})\in\mathfrak{R}(\Psi_\alpha)$. ☐

Proposition 4.14. (i) *Let $-\infty<\alpha<1$. Then $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)=\mathfrak{R}(\Psi_\alpha)$.*

(ii) *Let $1\leqslant\alpha<2$. Then $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)\supsetneqq\mathfrak{R}(\Psi_\alpha)$ and $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$ is the class of $\widetilde{\mu}\in ID(\mathbb{R}^d)$ which is either Gaussian or non-Gaussian with Lévy measure $\widetilde{\nu}$ having expression* (4.4) *with* (4.5).

(iii) *Let $\alpha\geqslant 2$. Then $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)=\{\delta_\gamma\colon\gamma\in\mathbb{R}^d\}\supsetneqq\mathfrak{R}(\Psi_\alpha)=\{\delta_0\}$.*

The assertion (ii) has different implications between in the case $1<\alpha<2$ and in the case $\alpha=1$. Indeed, recall Theorem 4.2. For $1<\alpha<2$, any $\widetilde{\mu}\in\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$ can be shifted to a member of $\mathfrak{R}(\Psi_\alpha)$, but, for $\alpha=1$, $\mathfrak{R}_{\mathrm{e}}(\Psi_1)$ is truly larger than $\mathfrak{R}(\Psi_1)$ in the sense that there is $\widetilde{\mu}\in\mathfrak{R}_{\mathrm{e}}(\Psi_1)$ which cannot be shifted to a member of $\mathfrak{R}(\Psi_1)$ as the example in the proof of $\mathfrak{D}_{\mathrm{c}}(\Phi_f)\neq\mathfrak{D}_{\mathrm{e}}(\Phi_f)$ of Theorem 2.8 shows.

Another consequence of the theorem above combined with Theorem 4.2 is that, for all $\alpha\in(-\infty,2)$, $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$ is the class of $\widetilde{\mu}\in ID(\mathbb{R}^d)$ which is either Gaussian or non-Gaussian with Lévy measure $\widetilde{\nu}$ having expression (4.4) with (4.5). It is noteworthy that the class is determined only by properties of Lévy measures.

Proof of Proposition 4.14. (i) Let $-\infty<\alpha<1$. Let $\widetilde{\mu}\in\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$. Then $\widetilde{\mu}$ is the distribution of $\text{p-lim}_{t\to\infty}(\int_0^t f_\alpha(s)dX_s^{(\mu)}-q_t)$ for some $\mu\in\mathfrak{D}_{\mathrm{e}}(\Psi_\alpha)$ and function q_t. Since $\mathfrak{D}_{\mathrm{e}}(\Psi_\alpha)=\mathfrak{D}(\Psi_\alpha)$ (Theorem 2.4), $\int_0^t f_\alpha(s)dX_s^{(\mu)}$ is convergent in probability. Hence q_t is convergent to some $\widetilde{q}$ in $\mathbb{R}^d$. It follows that $\widetilde{\mu}=(\Psi_\alpha\mu)*\delta_{-\widetilde{q}}$. Thus $\widetilde{\mu}\in\mathfrak{R}(\Psi_\alpha)$ by virtue of Theorem 4.2. Hence $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)=\mathfrak{R}(\Psi_\alpha)$.

(ii) Denote by $\mathfrak{R}_{\mathrm{e}}^*(\Psi_\alpha)$ the class of $\widetilde{\mu}\in ID(\mathbb{R}^d)$ which is either Gaussian or non-Gaussian with Lévy measure $\widetilde{\nu}$ having expression (4.4) with (4.5).

Let $1<\alpha<2$. Let $\widetilde{\mu}\in\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$. Then $\widetilde{\mu}$ is the distribution of

$$\text{p-lim}_{t\to\infty}\left(\int_0^t f_\alpha(s)dX_s^{(\mu)}-q_t\right)$$

for some function q_t and some $\mu = \mu_{(A,\nu,\gamma)}$ with $\int_{|x|>1} |x|^\alpha \nu(dx) < \infty$ (Theorem 2.4 (iii)). Choose q' in such a way that $\mu * \delta_{-q'}$ has mean 0. We have

$$\int_0^t f_\alpha(s) dX_s^{(\mu)} - q_t = \int_0^t f_\alpha(s) d(X_s^{(\mu)} - sq') + \left(\int_0^t f_\alpha(s) ds\, q' - q_t \right).$$

As $t \to \infty$, the first term in the right-hand side is convergent in probability (see Theorem 2.4 (iii)). Denote the limit law by ρ. The second term is thus convergent to some $\widetilde{q}$ in $\mathbb{R}^d$. Now we have $\rho \in \mathfrak{R}(\Psi_\alpha)$ and $\widetilde{\mu} = \rho * \delta_{\widetilde{q}}$. Hence, by Theorem 4.2, $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}^*(\Psi_\alpha)$.

Conversely, let $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}^*(\Psi_\alpha)$ with $1 < \alpha < 2$. Then, by Theorem 4.2, there is $\widetilde{q} \in \mathbb{R}^d$ such that $\widetilde{\mu} * \delta_{-\widetilde{q}} \in \mathfrak{R}(\Psi_\alpha)$. Thus there is $\mu \in \mathfrak{D}(\Psi_\alpha)$ such that $\int_0^t f_\alpha(s) dX_s^{(\mu)}$ is convergent in probability as $t \to \infty$ to a random variable with distribution $\widetilde{\mu} * \delta_{-\widetilde{q}}$. Choose q_t such that $q_t \to -\widetilde{q}$. Then $\int_0^t f_\alpha(s) dX_s^{(\mu)} - q_t$ is convergent in probability as $t \to \infty$ to a random variable with distribution $\widetilde{\mu}$. Hence $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$.

Next we consider the case $\alpha = 1$. Let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \mathfrak{R}_{\mathrm{e}}(\Psi_1)$. Then $\widetilde{\mu}$ is the distribution of $\text{p-lim}_{t\to\infty} \left(\int_0^t f_1(s) dX_s^{(\mu)} - q_t \right)$ for some q_t and some $\mu = \mu_{(A,\nu,\gamma)}$ with $\int_{|x|>1} |x| \nu(dx) < \infty$ (Theorem 2.8). Suppose that $\widetilde{\nu} \neq 0$. Then, using Lemma 5.4 of Sato (2005), we see that (4.2) holds with $\alpha = 1$. Hence, by Lemma 4.4, $\widetilde{\nu}$ satisfies (4.4) and (4.5) with $\alpha = 1$. That is, $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}^*(\Psi_1)$.

Let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \mathfrak{R}_{\mathrm{e}}^*(\Psi_1)$. If $\widetilde{\nu} = 0$, then let $\nu = 0$. If $\widetilde{\nu} \neq 0$, then Lemma 4.4 guarantees the existence of ν satisfying $\nu(\{0\}) = 0$, $\int (|x|^2 \wedge |x|) \nu(dx) < \infty$, and condition (4.2) with $\alpha = 1$. Let $\mu = \mu_{(A,\nu\gamma)}$ with $A = \widetilde{A}$ and γ arbitrary. By Theorem 2.8, $\mu \in \mathfrak{D}_{\mathrm{e}}(\Psi_1)$. Thus, for some q_t, $\int_0^t f_1(s) dX_s^{(\mu)} - q_t$ is convergent in probability as $t \to \infty$. Denote the limit law by ρ. Then we have $\rho = \mu_{(\widetilde{A},\widetilde{\nu},\gamma_\rho)}$ with some $\gamma_\rho \in \mathbb{R}^d$ by Lemma 5.4 of Sato (2005). Therefore $\int_0^t f_1(s) dX_s^{(\mu)} - q_t - \gamma_\rho + \widetilde{\gamma}$ has limit law $\mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$. That is, $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}(\Psi_1)$.

(iii) The assertion is clear from Theorem 2.4 (iv). $\square$

Proposition 4.15. *We have*

$$\mathfrak{R}_{\mathrm{e}}(\Psi_\beta) \supsetneqq \mathfrak{R}_{\mathrm{e}}(\Psi_\alpha) \qquad \text{for } -\infty < \beta < \alpha \leqslant 2, \tag{4.26}$$

$$\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha) \supsetneqq \bigcup_{\alpha' > \alpha} \mathfrak{R}_{\mathrm{e}}(\Psi_{\alpha'}) \qquad \text{for } -\infty < \alpha < 2, \tag{4.27}$$

$$\bigcap_{\beta < \alpha} \mathfrak{R}_{\mathrm{e}}(\Psi_\beta) \supsetneqq \mathfrak{R}_{\mathrm{e}}(\Psi_\alpha) \qquad \text{for } 0 < \alpha \leqslant 2. \tag{4.28}$$

Proof. First let us prove non-strict inclusion in (4.26). Let $-\infty < \beta < \alpha \leqslant 2$. Let $\widetilde{\mu} = \mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$. If $\widetilde{\nu} = 0$, then evidently $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}(\Psi_\beta)$. If $\widetilde{\nu} \neq 0$, then $\alpha < 2$ and $\widetilde{\nu}$ has expression (4.4) together with (4.5), which implies $\widetilde{\mu} \in \mathfrak{R}_{\mathrm{e}}(\Psi_\beta)$, since $u^{-\alpha-1} = u^{-\beta-1} u^{\beta-\alpha}$.

Next, we can show (4.27) similarly to Step 2 of the proof of Proposition 4.5. The strict inclusion in (4.26) follows from this.

To show (4.28), notice that α-stable distributions belong to the left-hand side but do not to the right-hand side. $\square$

Proposition 4.16. *If* $\alpha \in (-\infty, 1) \cup (1, \infty)$, *then* $\mathfrak{R}^0(\Psi_\alpha) = \mathfrak{R}(\Psi_\alpha)$. *If* $\alpha = 1$, *then* $\mathfrak{R}^0(\Psi_1) \subsetneqq \mathfrak{R}(\Psi_1)$.

Proof. This is obvious from Theorems 2.4, 2.8, and Proposition 4.1. □

Proposition 4.17. *We have*

$$\mathfrak{R}^0(\Psi_\beta) \supsetneqq \mathfrak{R}^0(\Psi_\alpha) \qquad \text{for } -\infty < \beta < \alpha \leqslant 2, \tag{4.29}$$

$$\mathfrak{R}^0(\Psi_\alpha) \supsetneqq \bigcup_{\alpha' > \alpha} \mathfrak{R}^0(\Psi_{\alpha'}) \qquad \text{for } -\infty < \alpha < 2, \tag{4.30}$$

$$\bigcap_{\beta < \alpha} \mathfrak{R}^0(\Psi_\beta) \supsetneqq \mathfrak{R}^0(\Psi_\alpha) \qquad \text{for } 0 < \alpha \leqslant 2. \tag{4.31}$$

Proof. If $\alpha \neq 1$ and $\beta \neq 1$, or if $\beta < 1 = \alpha$, then (4.29) is a consequence of Corollary 4.6 and Proposition 4.16. If $\beta = 1 < \alpha < 2$, then (4.29) is shown from Theorems 2.4, 2.8, and the remark to Theorem 2.8. We also have (4.30) for $\alpha \neq 1$ by the same reason. To show (4.30) for $\alpha = 1$, Step 2 of the proof of Proposition 4.5 works. Use Propositions 4.8 and 4.16, to prove (4.31). □

5. Comments

We mention some remaining problems directly related to the preceding three sections.

1. The proof of the relation (3.2) between Ψ_α and $\Phi_{\beta,\alpha}$ in Theorem 3.1 is long and complicated in the case where $\alpha = 1$. A simpler proof is desirable.

2. Theorem 3.1 suggests the relation

$$\Phi_{\beta',\alpha} = \Phi_{\beta,\alpha}\Phi_{\beta',\beta} = \Phi_{\beta',\beta}\Phi_{\beta,\alpha} \quad \text{for } -\infty < \beta' < \beta < \alpha < \infty.$$

This seems to be verifiable similarly to the theorem.

3. In the description of $\mathfrak{R}(\Psi_1)$ in Theorem 4.2 (ii), it is desirable to give to the condition (4.6) an expression directly related to $\widetilde{\nu}$.

4. It is not known whether there exists a non-Gaussian distribution in the class $\bigcap_{\alpha<2} \mathfrak{R}(\Psi_\alpha)$.

5. In connection with Proposition 4.16, a description of $\mathfrak{R}^0(\Psi_1)$ should be given.

6. It is not known whether the strict inclusions in (4.24), (4.28), and (4.31) concerning $\mathfrak{R}(\Psi_\alpha)$, $\mathfrak{R}_{\mathrm{e}}(\Psi_\alpha)$, and $\mathfrak{R}^0(\Psi_\alpha)$ hold not only for $0 < \alpha \leqslant 2$ but also for $-\infty < \alpha \leqslant 0$.

7. Let $-\infty < \beta < \alpha < 2$. In connection with Lemma 3.2, it is not known whether $\mu \in \mathfrak{D}^0(\Psi_\alpha)$ whenever $\mu \in \mathfrak{D}^0(\Psi_\beta)$ and $\Psi_\beta\mu \in \mathfrak{D}^0(\Phi_{\beta,\alpha})$.

8. It is an open problem whether there is a function f for which

$$\mathfrak{D}^0(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}_{\mathrm{c}}(\Phi_f) \subsetneqq \mathfrak{D}_{\mathrm{e}}(\Phi_f)$$

but which has asymptotic behavior different from that of Theorem 2.8.

9. Description of $\mathfrak{R}(\Phi_{\beta,\alpha})$ as well as of $\mathfrak{R}_{\mathrm{c}}$, $\mathfrak{R}_{\mathrm{e}}$, and $\mathfrak{R}^0$ for $\Phi_{\beta,\alpha}$ is to be made. They are known for $\Phi_{-1,0}$, since $\Phi_{-1,0} = \Phi$.

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