



Spatial birth and death processes as solutions of stochastic equations

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Abstract. Spatial birth and death processes are obtained as solutions of a system of stochastic equations. The processes are required to be locally finite, but may involve an infinite population over the full (noncompact) type space. Conditions are given for existence and uniqueness of such solutions, and for temporal and spatial ergodicity. For birth and death processes with constant death rate, a sub-criticality condition on the birth rate implies that the process is ergodic and converges exponentially fast to the stationary distribution.

1. Introduction

Spatial birth and death processes in which the birth and death rates depend on the configuration of the system were first studied by Preston (1975). His approach was to consider the solution of the backward Kolmogorov equation, and he worked under the restriction that there were only a finite number of individuals alive at any time. Under certain conditions, the processes exist and are temporally ergodic, that is, there exists a unique stationary distribution. The more general setting considered here requires only that the number of points alive in any compact set remains finite at all times.

Specifically, we assume that our population is represented as a countable subset of points in a complete, separable metric space S (typically, $S \subset \mathbb{R}^d$). We will identify the subset with the counting measure η given by assigning unit mass to each point, that is, $\eta(B)$ is the number of points in a set $B \in \mathcal{B}(S)$. ($\mathcal{B}(S)$ will denote the Borel subsets of S .) We will use the terms point process and random counting measure interchangeably. With this identification in mind, let $\mathcal{N}(S)$ be

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the collection of counting measures on the metric space S . The state space for our process will be some subset of $\mathcal{N}(S)$. All processes and random variables are defined on a complete probability space (Ω, \mathcal{F}, P) .

The spatial birth and death process is specified in terms of non-negative functions $\lambda : S \times \mathcal{N}(S) \rightarrow [0, \infty)$ and $\delta : S \times \mathcal{N}(S) \rightarrow [0, \infty)$ and a reference measure β on S (typically Lebesgue measure m_d , if $S \subset \mathbb{R}^d$). λ is the birth rate and δ the death rate. If the point configuration at time t is $\eta \in \mathcal{N}(S)$, then the probability that a point in a set $B \subset S$ is added to the configuration in the next time interval of length Δt is approximately $\int_B \lambda(x, \eta) \beta(dx) \Delta t$ and the probability that a point $x \in \eta$ is deleted from the configuration in the next time interval of length Δt is approximately $\delta(x, \eta) \Delta t$. Under these assumptions, the generator of the process should be of the form

$$AF(\eta) = \int (F(\eta + \delta_x) - F(\eta)) \lambda(x, \eta) \beta(dx) + \int (F(\eta - \delta_x) - F(\eta)) \delta(x, \eta) \eta(dx) \quad (1.1)$$

for F in an appropriate domain.

Following the work of Preston, spatial birth and death processes quickly found application in statistics when Ripley (1977) observed that spatial point patterns could be simulated by constructing a spatial birth and death process having the distribution of the desired pattern as its stationary distribution and then simulating the birth and death process for a long time, a procedure now known as Markov chain Monte Carlo.

The two best-known classes of spatial point processes are Poisson random measures and Gibbs distributions.

1.1. Poisson random measures. Let β be a σ -finite measure on S , (S, d_S) a complete, separable metric space. ξ is a Poisson random measure on S with mean measure β if for each $B \in \mathcal{B}(S)$, $\xi(B)$ has a Poisson distribution with expectation $\beta(B)$ and $\xi(B)$ and $\xi(C)$ are independent if $B \cap C = \emptyset$. Taking $\lambda = \delta \equiv 1$, then the Poisson random measure with mean measure β gives the unique stationary distribution for the birth and death process with generator

$$AF(\eta) = \int (F(\eta + \delta_x) - F(\eta)) \beta(dx) + \int (F(\eta - \delta_x) - F(\eta)) \eta(dx). \quad (1.2)$$

Letting μ_β^0 denote this distribution, the stationarity can be checked by verifying that

$$\int_{\mathcal{N}(S)} AF(\eta) \mu_\beta^0(d\eta) = 0.$$

This assertion follows from the standard identity

$$E\left[\int_S h(\xi - \delta_x, x) \xi(dx)\right] = E\left[\int_S h(\xi, x) \beta(dx)\right]. \quad (1.3)$$

See Daley and Vere-Jones (1988), p. 188, Equation (6.4.11).

1.2. Gibbs distributions. Assume that $\beta(S) < \infty$. Consider the class of spatial point processes specified through a density (Radon-Nikodym derivative) with respect to a Poisson point process with mean measure β , that is, the distribution of the point process is given by

$$\mu_{\beta, H}(d\eta) = \frac{1}{Z_{\beta, H}} e^{-H(\eta)} \mu_\beta^0(d\eta), \quad (1.4)$$

where $H(\eta)$ is referred to as the *energy function*, $Z_{\beta,H}$ is a normalizing constant, and μ_β^0 is the law of a Poisson process with mean measure β . Therefore, the state space for this process is $\mathcal{S} = \{\eta \in \mathcal{N}(S); H(\eta) < \infty\}$, the set of configurations with positive density. We assume that H is hereditary in the sense of Ripley (1977), that is $H(\eta) < \infty$ and $\tilde{\eta} \subset \eta$ implies $H(\tilde{\eta}) < \infty$. Ripley showed that such a measure $\mu_{\beta,H}$ is the stationary distribution of a spatial birth and death process. In fact, there is more than one birth and death process that has $\mu_{\beta,H}$ as a stationary distribution; we simply require that $\lambda(x, \eta) > 0$ if $H(\eta + \delta_x) < \infty$ and that λ and δ satisfy

$$\lambda(x, \eta)e^{-H(\eta)} = \delta(x, \eta + \delta_x)e^{-H(\eta + \delta_x)}. \tag{1.5}$$

This equation is a detailed balance condition which ensures that births from η to $\eta + \delta_x$ match deaths from $\eta + \delta_x$ to η and that the process is time-reversible with (1.4) as its stationary distribution. Again, this assertion can be verified by showing that

$$\int AF(\eta)\mu_{\beta,H}(d\eta) = \frac{1}{Z_{\beta,H}} \int AF(\eta)e^{-H(\eta)}\mu_\beta^0(d\eta) = 0.$$

This identity again follows from (1.3).

Notice that equation (1.5) says that any pair of birth and death rates such that

$$\frac{\lambda(x, \eta)}{\delta(x, \eta + \delta_x)} = \exp\{-H(\eta + \delta_x) + H(\eta)\}$$

will give rise to a process with stationary distribution given by (1.4). We can always take $\delta(x, \eta) = 1$, that is, whenever a point is added to the configuration, it lives an exponential length of time independently of the configuration of the process.

For example, consider a spatial point process on a compact set $S \subset \mathbb{R}^d$ given by a Gibbs distribution with pairwise interaction potential $\rho(x_1, x_2) \geq 0$, that is, for $\eta = \sum_{i=1}^m \delta_{x_i}$,

$$\begin{aligned} H_\rho(\eta) &= \sum_{i < j} \rho(x_i, x_j) \\ &= \frac{1}{2} \left[\int \int \rho(x, y)\eta(dx)\eta(dy) - \int \rho(x, x)\eta(dx) \right] \end{aligned} \tag{1.6}$$

and the distribution of the point process is absolutely continuous with respect to the spatially homogeneous Poisson process with constant intensity 1 (or equivalently with Lebesgue mean measure) on S . Taking $\delta(x, \eta) \equiv 1$ and $\lambda(x, \eta) = \exp\{-\int \rho(x, y)\eta(dy)\}$, the distribution determined by (1.6) is the stationary distribution for the birth and death process with infinitesimal generator

$$AF(\eta) = \int e^{-\int \rho(x, y)\eta(dy)}(F(\eta + \delta_x) - F(\eta))dx + \int (F(\eta - \delta_x) - F(\eta))\eta(dx). \tag{1.7}$$

Another example is the area-interaction point process introduced by Baddeley and van Lieshout (1995). This point process is absolutely continuous with respect to the spatial Poisson process with Lebesgue mean measure m_d on $S \subset \mathbb{R}^d$ and $H(\eta) = \eta(S) \log \rho - m_d(\eta \oplus G)$, so the Radon-Nikodym derivative is given by

$$L(\eta) = \frac{1}{Z} \rho^{\eta(S)} \gamma^{-m_d(\eta \oplus G)}. \tag{1.8}$$

Again, Z is the normalizing constant, ρ and γ are positive parameters, and G is a compact (typically convex) subset of \mathbb{R}^d referred to as the *grain*. The set $\eta \oplus G$ is

given by

$$\eta \oplus G = \cup\{x \oplus G; x \in \eta\}.$$

The parameter γ controls the area-interaction among the points of η : the process is *attractive* if $\gamma > 1$ and *repulsive* otherwise. (See Lemma 3.6.) If $\gamma = 1$ the point process is just the Poisson random measure with mean measure ρm_d . The case $\gamma > 1$ is related to the *Widow-Rowlinson model* introduced by Widow and Rowlinson (1970). The case of *area-exclusion* corresponds to a suitable limit $\gamma \rightarrow 0$. A birth and death process with stationary distribution given by the area-interaction distribution can be obtained by taking the unit death rate and the birth rate given by

$$\lambda(x, \eta) = \rho \gamma^{-m_d((x+G) \setminus (\eta \oplus G))}. \quad (1.9)$$

1.3. *Overview.* The spatial birth and death processes that correspond to the Gibbs distributions discussed above involve finite configurations and it is straightforward to see that they are uniquely characterized by their birth and death rates, for example, as solutions of the martingale problem associated with the generator A given in (1.1); however, if the configurations are infinite and the total birth and death rates are infinite, the existence and uniqueness of the processes are not so clear. In Section 2, we represent these processes as solutions of a system of stochastic equations and give conditions for existence and uniqueness of solutions for the equations as well as for the corresponding martingale problems. These equations are very useful in studying the asymptotic behavior of the birth and death processes, including temporal and/or spatial-ergodicity and the speed of convergence to the stationary distribution.

The uniqueness conditions given here are direct analogs of Liggett's conditions for existence and uniqueness for lattice indexed interacting particle systems (Liggett, 1972). Stochastic equations for lattice indexed systems were formulated in Kurtz (1980) using time-changed Poisson processes and existence and uniqueness given under Liggett's conditions. Stochastic equations for spatial birth and death processes of the type considered here were formulated in Garcia (1995) using a spatial version of the time-change approach. Existence and uniqueness were again given under analogs of Liggett's conditions.

One disadvantage to the time-change approach taken in Kurtz (1980) and Garcia (1995) is that the filtration to which the process is adapted depends on the solution. A stochastic equation that avoids this difficulty can be formulated by representing the birth process as a thinning of a Poisson random measure. Intuitively, the approach is analogous to the rejection method for simulating random variables. The fact that counting processes and more general marked counting processes can be obtained by thinning Poisson random measures is well known, particularly in the context of simulation. (See, for example, Daley and Vere-Jones (2003), Section 7.5.) Stochastic equations exploiting this approach were formulated for lattice systems in Kurtz and Protter (1996) and for general spatial birth processes by Massoulié (1998). In both cases, uniqueness was obtained under conditions analogous to Liggett's.

Section 3 considers temporal ergodicity for birth and death processes in non-compact S (more precisely, S and β with $\beta(S) = \infty$) and spatial ergodicity for $S = \mathbb{R}^d$ and translation invariant birth rates. We give conditions for ergodicity and exponential convergence to the stationary distribution. It is well known that these

processes are temporally and spatially ergodic if the birth and death rates are constant (the stationary measure being Poisson). More generally, in Theorem 3.10, we show that if the birth rate satisfies the conditions of Theorem 2.13 with $M < 1$ and the death rate is constant ($\delta \equiv 1$), the system is temporally ergodic and for every initial distribution, the distribution of the solution converges at an exponential rate to the stationary distribution. For $S = \mathbb{R}^d$ and λ translation invariant, from the stochastic equation, we see that spatial ergodicity of the initial distribution implies spatial ergodicity of the solution at each time $0 < t < \infty$. Unfortunately, it is not clear, in general, how to carry this conclusion over to $t = \infty$, that is, to the limiting distribution of the solution, although in the case $M < 1$, spatial ergodicity holds for the unique stationary distribution as well. We give some additional conditions under which spatial ergodicity of the limiting distribution can be obtained.

Fernández, Ferrari and Garcia (2002) study ergodicity of spatial birth and death processes using a graphical representation to construct the stationary distribution that is closely related to the stochastic equations we consider here. They give conditions for an exponential rate of convergence to the stationary distribution and for spatial ergodicity of the stationary distribution similar to those given here, but for a more restricted class of models.

Throughout, $\overline{\mathcal{C}}(S)$ will denote the space of bounded continuous functions on S and $\mathcal{B}(S)$ the Borel subsets of S .

The stochastic equations we consider will be driven by a Poisson random measure N on $U \times [0, \infty)$ for an appropriate space U , having mean measure of the form $\nu \times m_1$, where m_1 is Lebesgue measure on $[0, \infty)$. Then for $B \in \mathcal{B}(U)$ with $\nu(B) < \infty$, $N(B, t) \equiv N(B \times [0, t])$ is just an ordinary Poisson process with intensity $\nu(B)$. For a filtration $\{\mathcal{F}_t\}$, we say that N is *compatible* with $\{\mathcal{F}_t\}$ if and only if for each $B \in \mathcal{B}(U)$ with $\nu(B) < \infty$, $N(B, \cdot)$ is $\{\mathcal{F}_t\}$ -adapted and $N(B, t + s) - N(B, t)$ is independent of \mathcal{F}_t for $s, t \geq 0$.

2. Spatial birth and death processes as solutions of stochastic equations

A birth and death process as described in the previous section can be represented as the solution of a system of stochastic equations. The approach is similar to Garcia (1995) where such processes were obtained as solutions of time-change equations. We assume that the individuals in the birth and death process are represented by points in a Polish space S . Typically, S will be $\mathbb{R}^d, \mathbb{Z}^d$, or a subset of one of these, but we do not rule out more general spaces. Let $K_1 \subset K_2 \subset \dots$ satisfy $\cup_k K_k = S$, and let $c_k \in \overline{\mathcal{C}}(S)$ satisfy $c_k \geq 0$ and $\inf_{x \in K_k} c_k(x) > 0$. $\mathcal{N}(S)$ will denote the collection of counting measures on S and \mathcal{S} will denote $\{\zeta \in \mathcal{N}(S) : \int_S c_k(x) \zeta(dx) < \infty, k = 1, 2, \dots\}$. Without loss of generality, we can assume that $c_1 \leq c_2 \leq \dots$. Let $\mathcal{C} = \{f \in \overline{\mathcal{C}}(S) : |f| \leq ac_k \text{ for some } k \text{ and } a > 0\}$, and topologize \mathcal{S} by the weak* topology generated by \mathcal{C} , that is, $\zeta_n \rightarrow \zeta$ if and only if $\int_S f d\zeta_n \rightarrow \int_S f d\zeta$ for all $f \in \mathcal{C}$. (Note that \mathcal{C} is linear and that, with this topology, \mathcal{S} is Polish.) $D_{\mathcal{S}}[0, \infty)$ will denote the space of cadlag \mathcal{S} -valued functions with the Skorohod (J_1) topology. We assume that λ and δ are nonnegative, Borel measurable functions on $S \times \mathcal{C}$.

Let β be a σ -finite, Borel measure on S . We assume

Condition 2.1. For each compact $\mathcal{K} \subset \mathcal{S}$, the birth rate λ satisfies

$$\sup_{\zeta \in \mathcal{K}} \int_S c_k(x) \lambda(x, \zeta) \beta(dx) < \infty, \quad t > 0, \quad k = 1, 2, \dots, \tag{2.1}$$

and

$$\delta(x, \zeta) < \infty, \quad \zeta \in \mathcal{S}, \quad x \in \zeta. \tag{2.2}$$

We also assume that λ and δ satisfy the following continuity condition.

Condition 2.2. *If*

$$\lim_{n \rightarrow \infty} \int_S c_k(x) |\zeta_n - \zeta|(dx) = 0, \tag{2.3}$$

for each $k = 1, 2, \dots$, then

$$\lambda(x, \zeta) = \lim_{n \rightarrow \infty} \lambda(x, \zeta_n), \quad \delta(x, \zeta) = \lim_{n \rightarrow \infty} \delta(x, \zeta_n). \tag{2.4}$$

Note that since (2.3) implies ζ_n converges to ζ in \mathcal{S} , the continuity condition (2.4) is weaker than continuity in \mathcal{S} ; however, we have the following condition under which convergence in \mathcal{S} implies (2.3).

Lemma 2.3. *Suppose $\zeta_0, \zeta_1, \zeta_2, \dots \in \mathcal{S}$ and $\zeta_n \leq \zeta_0, n = 1, 2, \dots$. If $\zeta_n \rightarrow \zeta$ in \mathcal{S} , then (2.3) holds.*

Proof. $\zeta_n \leq \zeta_0$ implies that, considered as a measure, $\zeta_n \ll \zeta_0$ and $\frac{d\zeta_n}{d\zeta_0} = \frac{\zeta_n(\{x\})}{\zeta_0(\{x\})} \leq 1$, almost everywhere ζ_0 . Furthermore, $\zeta_n \rightarrow \zeta$ in \mathcal{S} implies $\zeta_n(\{x\}) \rightarrow \zeta(\{x\})$ for each $x \in \zeta_0$, since the support of ζ_0 consists of a countable collection of isolated points. Consequently,

$$c_k(x) \geq c_k(x) \left| \frac{\zeta_n(\{x\})}{\zeta_0(\{x\})} - \frac{\zeta(\{x\})}{\zeta_0(\{x\})} \right| \rightarrow 0,$$

and since $\int_S c_k(x) \zeta_0(dx) < \infty$, the dominated convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_S c_k(x) |\zeta_n - \zeta|(dx) = \lim_{n \rightarrow \infty} \int_S c_k(x) \left| \frac{\zeta_n(\{x\})}{\zeta_0(\{x\})} - \frac{\zeta(\{x\})}{\zeta_0(\{x\})} \right| \zeta_0(dx) = 0.$$

□

Lemma 2.4. *Suppose $\mathcal{H} \subset \mathcal{S}$ and $\zeta_0 \in \mathcal{S}$ satisfy $\zeta \leq \zeta_0, \zeta \in \mathcal{H}$. If δ satisfies (2.2) and Condition 2.2, then*

$$\sup_{\zeta \in \mathcal{H}} \delta(x, \zeta) < \infty.$$

Proof. \mathcal{H} is relatively compact in \mathcal{S} , so any sequence $\{\zeta_n\} \subset \mathcal{H}$ has a subsequence that converges in \mathcal{S} and, by Lemma 2.3, satisfies (2.3). Fix $x \in S$, and let $\{\zeta_n\}$ satisfy $\lim_{n \rightarrow \infty} \delta(x, \zeta_n) = \sup_{\zeta \in \mathcal{H}} \delta(x, \zeta)$. Then there is a subsequence that converges to some $\widehat{\zeta} \in \mathcal{S}$ and hence, $\sup_{\zeta \in \mathcal{H}} \delta(x, \zeta) = \delta(x, \widehat{\zeta}) < \infty$. □

Lemma 2.5. *Suppose that for each $x \in S$, there exists $k(x)$ such that $\lambda(x, \zeta + \delta_y) = \lambda(x, \zeta)$ for $y \notin K_{k(x)}$. Then λ satisfies Condition 2.2 and similarly for δ .*

Proof. Note that $\zeta \in \mathcal{S}$ implies $\zeta(K_{k(x)}) < \infty$ and that $\zeta_n \rightarrow \zeta$ implies that for n sufficiently large, ζ_n restricted to $K_{k(x)}$ coincides with ζ restricted to $K_{k(x)}$, so $\lambda(x, \zeta_n) = \lambda(x, \zeta)$. □

Let N be a Poisson random measure on $S \times [0, \infty)^3$ with mean measure $\beta(dx) \times ds \times e^{-r} dr \times du$. Let η_0 be an \mathcal{S} -valued random variable independent of N , and let

$\widehat{\eta}_0$ be the point process on $S \times [0, \infty)$ obtained by associating to each “count” in η_0 an independent, unit exponential random variable, that is, for $\eta_0 = \sum_{i=1}^{\infty} \delta_{x_i}$, set

$$\widehat{\eta}_0 = \sum_{i=1}^{\infty} \delta_{(x_i, \tau_i)}, \tag{2.5}$$

where the $\{\tau_i\}$ are independent unit exponentials, independent of η_0 and N . The birth and death process η should satisfy a stochastic equation of the form

$$\begin{aligned} \eta_t(B) = & \int_{B \times [0, t] \times [0, \infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) \mathbf{1}_{(\int_s^t \delta(x, \eta_v) dv, \infty)}(r) N(dx, ds, dr, du) \\ & + \int_{B \times [0, \infty)} \mathbf{1}_{(\int_0^t \delta(x, \eta_s) ds, \infty)}(r) \widehat{\eta}_0(dx, dr). \end{aligned} \tag{2.6}$$

To be precise, let η be a process with sample paths in $D_S[0, \infty)$ that is adapted to a filtration $\{\mathcal{F}_t\}$ with respect to which N is compatible. (Note that (2.1) ensures that the integral with respect to N on the right exists and determines an \mathcal{S} -valued random variable, and the continuity condition (2.4) and the finiteness of $\delta(x, \zeta)$ ensure that $\delta(s, \eta_t)$ is a cadlag function of t , so that the integrals $\int_s^t \delta(x, \eta_v) dv$ exist.) Then η is a solution of (2.6) if and only if the identity (2.6) holds almost surely for all $B \in \mathcal{B}(S)$ and $t \geq 0$ (allowing $\infty = \infty$).

Lemma 2.6. *Suppose Condition 2.1 holds. If η is a solution of (2.6), then for each $T > 0$,*

$$\int_0^T \int_S c_k(x) \lambda(x, \eta_s) \beta(dx) ds < \infty \quad a.s., \tag{2.7}$$

η_T^* defined by

$$\eta_T^*(B) = \int_{B \times [0, T] \times [0, \infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) N(dx, ds, dr, du)$$

is an element of \mathcal{S} ,

$$\eta_t \leq \eta_T^* + \eta_0, \quad 0 \leq t \leq T,$$

and

$$\lim_{s \rightarrow t+} \int_S c_k(x) |\eta_s - \eta_t|(dx) = 0, \quad t \geq 0.$$

Proof. Since for almost every $\omega \in \Omega$, the closure of $\{\eta_s : 0 \leq s \leq T\}$ is compact, Condition 2.1 implies

$$\sup_{s \leq T} \int_S c_k(x) \lambda(x, \eta_s) \beta(dx) < \infty \quad a.s.,$$

and hence (2.7). Letting

$$\tau_c = \inf \left\{ t : \int_0^t \int_S c_k(x) \lambda(x, \eta_s) \beta(dx) ds > c \right\},$$

we have

$$E \left[\int_S c_k(x) \eta_{T \wedge \tau_c}^*(dx) \right] = E \left[\int_0^{T \wedge \tau_c} \int_S c_k(x) \lambda(x, \eta_s) \beta(dx) ds \right] \leq c,$$

and since $\lim_{c \rightarrow \infty} \tau_c = \infty$ a.s., it follows that $\int_S c_k(x) \eta_T^*(dx) < \infty$, a.s. implying $\eta_T^* \in \mathcal{S}$ a.s. The last statement then follows by Lemma 2.3. \square

If η is a solution of (2.6) and a point at x was born at time $s \leq t$, then the “residual clock time” $r - \int_s^t \delta(x, \eta_v) dv$ is an \mathcal{F}_t -measurable random variable. In particular, the counting-measure-valued process given by

$$\begin{aligned} \widehat{\eta}_t(B \times D) &= \int_{B \times [0, t] \times [0, \infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) \mathbf{1}_D \left(r - \int_s^t \delta(x, \eta_v) dv \right) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_D(r - \int_0^t \delta(x, \eta_{s-}) ds) \widehat{\eta}_0(dx, dr) \end{aligned} \tag{2.8}$$

is $\{\mathcal{F}_t\}$ -adapted.

Let $\widehat{\mathcal{S}}$ denote the collection of counting measures ζ on $S \times [0, \infty)$ such that $\zeta(\cdot \times [0, \infty)) \in \mathcal{S}$. We can formulate an alternative equation for the $\widehat{\mathcal{S}}$ -valued process $\widehat{\eta}$ by requiring that

$$\begin{aligned} &\int_{S \times [0, \infty)} f(x, r) \widehat{\eta}_t(dx, dr) \\ &= \int_{S \times [0, \infty)} f(x, r) \widehat{\eta}_0(dx, dr) \\ &\quad + \int_{S \times [0, t] \times [0, \infty)^2} f(x, r) \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) N(dx, ds, dr, du) \\ &\quad - \int_0^t \int_{S \times [0, \infty)} \delta(x, \eta_s) f_r(x, r) \widehat{\eta}_s(dx, dr) ds, \end{aligned} \tag{2.9}$$

for all $f \in \widehat{\mathcal{C}}$, where $\widehat{\mathcal{C}}$ is the collection of $f \in \overline{\mathcal{C}}(S \times [0, \infty))$ such that $f_r \equiv \frac{\partial}{\partial r} f \in \overline{\mathcal{C}}(S \times [0, \infty))$, $f(x, 0) = 0$, $\sup_r |f(\cdot, r)|, \sup_r |f_r(\cdot, r)| \in \mathcal{C}$, and there exists $r_f > 0$ such that $f_r(x, r) = 0$ for $r > r_f$. Note that if $f \in \widehat{\mathcal{C}}$ and

$$f^*(x, r) = \int_0^r |f_r(x, u)| du, \tag{2.10}$$

then $f^* \in \widehat{\mathcal{C}}$. In (2.9), $\widehat{\eta}_0$ can be any $\widehat{\mathcal{S}}$ -valued random variable that is independent of N .

2.1. *Martingale problems.* Let $\mathcal{D}(\widehat{A})$ be the collection of functions F of the form $F(\widehat{\zeta}) = e^{-\int_{S \times [0, \infty)} f(x, r) \widehat{\zeta}(dx, dr)}$, for non-negative $f \in \widehat{\mathcal{C}}$. Suppose that $\widehat{\eta}$ is a solution of (2.9) with sample paths in $D_{\widehat{\mathcal{S}}}[0, \infty)$. Assuming Condition 2.1,

$$\int_0^t \int_S c_k(x) \lambda(x, \eta_s) \beta(dx) ds < \infty, \quad k = 1, 2, \dots \tag{2.11}$$

By Itô's formula

$$\begin{aligned}
 F(\hat{\eta}_t) &= F(\hat{\eta}_0) \\
 &+ \int_{S \times [0, t] \times [0, \infty)^2} F(\hat{\eta}_{s-}) (e^{-f(x, r)} - 1) \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) N(dx, ds, dr, du) \\
 &+ \int_0^t F(\hat{\eta}_s) \int_{S \times [0, \infty)} \delta(x, \eta_s) f_r(x, r) \hat{\eta}_s(dx, dr) ds, \\
 &= F(\hat{\eta}_0) \\
 &+ \int_{S \times [0, t] \times [0, \infty)^2} F(\hat{\eta}_{s-}) (e^{-f(x, r)} - 1) \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) \tilde{N}(dx, ds, dr, du) \\
 &+ \int_0^t F(\hat{\eta}_s) \left(\int_{S \times [0, \infty)} \lambda(x, \eta_s) (e^{-f(x, r)} - 1) e^{-r} \beta(dx) dr \right. \\
 &\quad \left. + \int_{S \times [0, \infty)} \delta(x, \eta_s) f_r(x, r) \hat{\eta}_s(dx, dr) \right) ds,
 \end{aligned} \tag{2.12}$$

where $\tilde{N}(dx, ds, dr, du) = N(dx, ds, dr, du) - \beta(dx) \times ds \times e^{-r} dr \times du$. It follows from (2.11) that the stochastic integral term on the right is a local martingale, and since f^* given by (2.10) is in $\hat{\mathcal{C}}$, it follows that

$$\int_0^t \int_{S \times [0, \infty)} \delta(x, \eta_s) |f_r(x, r)| \hat{\eta}_s(dx, dr) ds < \infty, \quad t > 0. \tag{2.13}$$

Consequently, defining

$$\begin{aligned}
 \hat{A}F(\hat{\zeta}) &= F(\hat{\zeta}) \left(\int_{S \times [0, \infty)} \lambda(x, \zeta) (e^{-f(x, r)} - 1) e^{-r} \beta(dx) dr \right. \\
 &\quad \left. + \int_{S \times [0, \infty)} \delta(x, \zeta) f_r(x, r) \hat{\zeta}(dx, dr) \right),
 \end{aligned} \tag{2.14}$$

any solution of (2.9) must be a solution of the local martingale problem for \hat{A} . We say that $\hat{\eta}$ is a solution of the *local martingale problem* for \hat{A} if there exists a filtration $\{\mathcal{F}_t\}$ such that $\hat{\eta}$ is $\{\mathcal{F}_t\}$ -adapted and

$$M_F(t) = F(\hat{\eta}_t) - F(\hat{\eta}_0) - \int_0^t \hat{A}F(\hat{\eta}_s) ds \tag{2.15}$$

is a $\{\mathcal{F}_t\}$ -local martingale for each $F \in \mathcal{D}(\hat{A})$, that is, for each F of the form $F(\hat{\zeta}) = e^{-\int f d\hat{\zeta}}$, $f \in \hat{\mathcal{C}}$, $f \geq 0$. In particular, let $\bar{f}(x) = \sup_r f(x, r)$ and

$$\tau_{f, c} = \inf \left\{ t : \int_0^t \int_S \bar{f}(x) \lambda(x, \eta_s) \beta(dx) ds > c \right\}.$$

Then $M_F(\cdot \wedge \tau_{f, c})$ is a martingale. Note that $\tau_{f, c}$ is a $\{\mathcal{F}_t^n\}$ -stopping time.

Conversely, if $\hat{\eta}$ is a solution of the local martingale problem for \hat{A} with sample paths in $D_{\hat{S}}[0, \infty)$, then under Condition 2.1, (2.11) and (2.13) hold. If $\gamma \in C_{\mathbb{R}}[0, \infty)$ has compact support and $f(x, r) = \int_0^r \gamma(u) duc_k(x)$, then $f \in \hat{\mathcal{C}}$ and it follows that

$$\int_0^t \int_S c_k(x) \delta(x, \eta_s) \hat{\eta}_s(dx, dr) ds < \infty, \quad t > 0.$$

To formulate the main theorem of this section, we need to introduce the notion of a *weak solution* of a stochastic equation.

Definition 2.7. A stochastic process $\tilde{\eta}$ with sample paths in $D_{\mathcal{S}}[0, \infty)$ is a *weak solution* of (2.8) if there exists a probability space (Ω, \mathcal{F}, P) , a Poisson random measure N on $S \times [0, \infty)^3$ with mean measure $\beta(dx) \times ds \times e^{-r} dr \times du$ and a stochastic process $\hat{\eta}$ defined on (Ω, \mathcal{F}, P) , such that $\tilde{\eta}$ and $\hat{\eta}$ have the same distribution on $D_{\mathcal{S}}[0, \infty)$, $\hat{\eta}$ is adapted to a filtration with respect to which N is compatible, and N and $\hat{\eta}$ satisfy (2.8).

Theorem 2.8. *Suppose that λ and δ satisfy Conditions 2.1 and 2.2. Then each solution of the stochastic equation (2.8) (or equivalently, (2.9)) is a solution of the local martingale problem for \hat{A} defined by (2.14), and each solution of the local martingale problem for \hat{A} is a weak solution of the stochastic equation.*

Proof. The first part of the theorem follows from the discussion above.

To prove the second part, we apply a Markov mapping result of Kurtz (1998). Let $\{D_i\} \subset \mathcal{B}(S \times [0, \infty)^2)$ be countable, closed under intersections, generate $\mathcal{B}(S \times [0, \infty)^2)$, and satisfy $\int_{D_i} \beta(dx) e^{-r} dr ds < \infty$. Then N is completely determined by $N(D_i, t)$. Define

$$Z_i(t) = Z_i(0)(-1)^{N(D_i, t)},$$

where $Z_i(0)$ is ± 1 . Note that

$$N(D_i, t) = -\frac{1}{2} \int_0^t Z_i(s-) dZ_i(s), \tag{2.16}$$

and if the $Z_i(0)$ are iid with $P\{Z_i(0) = 1\} = P\{Z_i(0) = -1\} = \frac{1}{2}$ and independent of N , then for each $t \geq 0$, the $Z_i(t)$ are iid and independent of N . For $z \in \{-1, 1\}^\infty$, we will let $(-1)^{\mathbf{1}_{D(x,r,u)} z}$ denote

$$(-1)^{\mathbf{1}_{D(x,r,u)} z} = ((-1)^{\mathbf{1}_{D_1(x,r,u)} z_1}, (-1)^{\mathbf{1}_{D_2(x,r,u)} z_2}, \dots).$$

Then $Z = (Z_1, Z_2, \dots)$ is a solution of the martingale problems for

$$CG(z) = \int_{S \times [0, \infty)^2} (G((-1)^{\mathbf{1}_{D(x,r,u)} z}) - G(z)) e^{-r} dr du \beta(dx).$$

We can take the domain for C to be the collection of functions that depend on only finitely many coordinates of z . With this domain, the martingale problem for C is well-posed.

If $\hat{\eta}$ is a solution of (2.9), then $(\hat{\eta}, Z)$ is a solution of the local martingale problem for

$$\begin{aligned} & \hat{A}(FG)(\hat{\zeta}, z) \\ &= F(\hat{\zeta}) \left(\int_{S \times [0, \infty)^2} \left((\mathbf{1}_{[0, \lambda(x, \zeta)]}(u) e^{-f(x,r)} + \mathbf{1}_{(\lambda(x, \zeta), \infty)}(u)) G((-1)^{\mathbf{1}_{D(x,r,u)} z}) \right. \right. \\ & \qquad \qquad \qquad \left. \left. - G(z) \right) e^{-r} \beta(dx) dr du \right. \\ & \qquad \left. - G(z) \int_{S \times [0, \infty)} \delta(x, \zeta) f_r(x, r) \hat{\zeta}(dx, dr) \right). \end{aligned} \tag{2.17}$$

Let $\tilde{\eta}$ be a solution of the local martingale problem for \hat{A} . For $a = (a_1, a_2, \dots)$ with $a_k > 0$, $k = 1, 2, \dots$, define

$$\begin{aligned} \tau_a(t) = \\ \inf\left\{u : \int_0^u 1 \vee \sum_{k=1}^{\infty} a_k \left[\int_S c_k(x) \lambda(x, \tilde{\eta}_s) \beta(dx) + \int_{S \times S} c_k(x) \delta(x, \tilde{\eta}_s) \tilde{\eta}_s(dx) \right] ds \geq t\right\}, \\ H_a(\zeta) = 1 \vee \sum_{k=1}^{\infty} a_k \left[\int_S c_k(x) \lambda(x, \zeta) \beta(dx) + \int_{S \times S} c_k(x) \delta(x, \zeta) \zeta(dx) \right], \end{aligned}$$

and $\tilde{\eta}_t^a = \tilde{\eta}_{\tau_a(t)}$. Then $\tilde{\eta}^a$ is a solution of the martingale problem for

$$\hat{A}^a \equiv \frac{1}{H_a} \hat{A}. \quad (2.18)$$

For $F \in \mathcal{D}(\hat{A})$, $\hat{A}^a F$ is bounded, and we can select $a^n = (a_1^n, a_2^n, \dots)$ so that $a^n \geq a^{n+1}$ and $\tau_{a^n}(t) \rightarrow t$ a.s.

Let $\mu(dz) = \prod_{k=1}^{\infty} (\frac{1}{2} \delta_{\{-1\}}(dz_k) + \frac{1}{2} \delta_{\{1\}}(dz_k))$, and set $c_G = \int G d\mu$. Then

$$\int \hat{\mathbb{A}}(FG)(\hat{\zeta}, z) \mu(dz) = c_G \hat{A}F(\hat{\zeta}),$$

and more generally,

$$\int \frac{1}{H_a} \hat{\mathbb{A}}A(FG)(\hat{\zeta}, z) \mu(dz) = c_G \frac{1}{H_a} \hat{A}F(\hat{\zeta}).$$

Applying Corollary 3.5 of Kurtz (1998) to $H_a^{-1} \hat{\mathbb{A}}$ for each a , we conclude that if $\tilde{\eta}$ is a solution of the local martingale problem for \hat{A} , then there exists a solution $(\hat{\eta}, Z)$ of the local martingale problem for $\hat{\mathbb{A}}$ such that $\hat{\eta}$ and $\tilde{\eta}$ have the same distribution. Finally, applying (2.16), we can construct the corresponding Poisson random measure N and show that $\hat{\eta}$ and N satisfy (2.9). \square

The natural (local) martingale problem for η is really the martingale problem for A given by (1.1); however, there will be solutions $\hat{\eta}$ of the local martingale problem for \hat{A} (and hence of the stochastic equation) such that the corresponding η is not a solution of the local martingale problem for A . Intuitively, conditioned on $\mathcal{F}_t^\eta = \sigma(\eta_s : s \leq t)$, the residual clock times should be independent unit exponentials, independent of \mathcal{F}_t^η . That need not be the case, since we are free to pick the residual clock times at time zero in any way we please. It also need not be the case if the solution of the martingale problem fails to be unique. The following results clarify the relationship between the martingale problems for A and \hat{A} .

Proposition 2.9. *Suppose that λ and δ satisfy Conditions 2.1 and 2.2. If $\hat{\eta}$ is a solution of the local martingale problem for \hat{A} and at each time t , the residual clock times are independent of \mathcal{F}_t^η and are independent unit exponentials, then η is a solution of the local martingale problem for A .*

Proof. By assumption, we can write $\hat{\eta}_t = \sum_i \delta_{(X_i(t), R_i(t))}$, where the $R_i(t)$ are independent unit exponentials, independent of \mathcal{F}_t^η , and in particular, independent of η_t . For $f \in \hat{\mathcal{C}}$ and $F(\hat{\zeta}) = e^{-\int_{S \times [0, \infty)} f(x, r) \hat{\zeta}(dx, dr)}$, since (2.15) can be localized by $\{\mathcal{F}_t^\eta\}$ -stopping times, it follows that

$$E[F(\hat{\eta}_t) | \mathcal{F}_t^\eta] - E[F(\hat{\eta}_0) | \mathcal{F}_0^\eta] - \int_0^t E[\hat{A}F(\eta_s) | \mathcal{F}_s^\eta] ds$$

is a $\{\mathcal{F}_t^\eta\}$ -local martingale. By the independence of the $R_i(t)$,

$$E[F(\widehat{\eta}_t)|\mathcal{F}_t^\eta] = \prod_i \int_0^\infty e^{-f(X_i(t),r)} e^{-r} dr = e^{-\int_S g(x)\eta_t(dx)} \equiv G(\eta_t),$$

where g is defined so that $e^{-g(x)} = \int_0^\infty e^{-f(x,r)} e^{-r} dr$. Integrating by parts gives

$$\int_0^\infty e^{-f(x,r)} f_r(x,r) e^{-r} dr = 1 - \int_0^\infty e^{-f(x,r)} e^{-r} dr = 1 - e^{-g(x)},$$

and hence

$$\begin{aligned} E[\widehat{A}F(\eta_s)|\mathcal{F}_s^\eta] &= G(\eta_s) \int_{S \times [0, \infty)} \lambda(x, \eta_s) (e^{-f(x,r)} - 1) e^{-r} \beta(dx) dr \\ &\quad + \sum_j \left(\prod_{i \neq j} \int_0^\infty e^{-f(X_i(s),r)} e^{-r} dr \right) \\ &\quad \int_0^\infty e^{-f(X_j(s),r)} \delta(X_j(s), \eta_s) f_r(X_j(s), r) e^{-r} dr \\ &= G(\eta_s) \int_S \lambda(x, \eta_s) (e^{-g(x)} - 1) \beta(dx) \\ &\quad + \sum_j \left(\prod_{i \neq j} e^{-g(X_i(s))} \right) \delta(X_j(s), \eta_s) (1 - e^{-g(X_j(s))}) \\ &= AG(\eta_s), \end{aligned}$$

and the proposition follows. □

We have the following converse for the previous proposition.

Theorem 2.10. *Suppose that λ and δ satisfy Conditions 2.1 and 2.2. If η is a solution of the local martingale problem for A , then there exists a solution $\widehat{\eta}$ of the local martingale problem for \widehat{A} such that η and $\widehat{\eta}(\cdot \times [0, \infty))$ have the same distribution on $D_S[0, \infty)$ and at each time $t \geq 0$, the residual clock times are independent, unit exponentials that are independent of \mathcal{F}_t^η .*

Proof. For $\zeta = \sum_i \delta_{x_i} \in \mathcal{S}$, let $\alpha(\zeta, d\widehat{\zeta})$ denote the distribution on $\widehat{\mathcal{S}}$ of $\sum_i \delta_{(x_i, \tau_i)}$, where the τ_i are independent, unit exponential random variables. Then, by the calculation in the proof of Proposition 2.9,

$$G(\zeta) = \int_{\widehat{\mathcal{S}}} F(\widehat{\zeta}) \alpha(\zeta, d\widehat{\zeta}) \quad AG(\zeta) = \int_{\widehat{\mathcal{S}}} \widehat{A}F(\widehat{\zeta}) \alpha(\zeta, d\widehat{\zeta}),$$

for $F \in \mathcal{D}(\widehat{A})$. More generally, $A^a G(\zeta) = \int_{\widehat{\mathcal{S}}} \widehat{A}^a F(\widehat{\zeta}) \alpha(\zeta, d\widehat{\zeta})$, where \widehat{A}^a is defined as in (2.18). The theorem then follows by Corollary 3.5 of Kurtz (1998). □

Corollary 2.11. *Let $\nu \in \mathcal{P}(\mathcal{S})$, and define $\widehat{\nu} \in \mathcal{P}(\widehat{\mathcal{S}})$ by*

$$\int_{\widehat{\mathcal{S}}} h d\widehat{\nu} = \int_S \int_{\widehat{\mathcal{S}}} h(\widehat{\zeta}) \alpha(\zeta, d\widehat{\zeta}) \nu(d\zeta).$$

If uniqueness holds for the martingale problem for $(\widehat{A}, \widehat{\nu})$, or equivalently, weak uniqueness holds for the stochastic equation (2.9), then uniqueness holds for the martingale problem for (A, ν) .

Proof. If η is a solution of the martingale problem for (A, ν) , then Theorem 2.10 gives a corresponding solution of the martingale problem for $(\widehat{A}, \widehat{\nu})$. Uniqueness for the latter then implies uniqueness of the former. \square

2.2. *Existence.* We now turn to the question of existence of solutions of (2.6). We assume that Conditions 2.1 and 2.2 hold. The pair (λ, δ) will be called *attractive* if $\zeta_1 \subset \zeta_2$ implies $\lambda(x, \zeta_1) \leq \lambda(x, \zeta_2)$ and $\delta(x, \zeta_1) \geq \delta(x, \zeta_2)$. If (λ, δ) is attractive and we set $\eta^0 \equiv 0$, then η^n defined by

$$\begin{aligned} \eta_t^{n+1}(B) &= \int_{B \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_s^n)]}(u) \mathbf{1}_{(\int_s^t \delta(x,\eta_v^n) dv, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\eta_s^n) ds, \infty)}(r) \widehat{\eta}_0(dx, dr) \end{aligned} \quad (2.19)$$

is monotone increasing and either η^n converges to a process with values in S , or

$$\int_0^T \int_S c_k(x) \lambda(x, \eta_s^n) \beta(dx) ds \rightarrow \infty, \quad (2.20)$$

for some T and k . To see this, let

$$\tau_c^n = \inf\{t : \int_0^t \int_S c_k(x) \lambda(x, \eta_s^n) \beta(dx) ds > c\}.$$

Then

$$\begin{aligned} E &\left[\sup_{t \leq T \wedge \tau_c^n} \left(\int_S c_k(x) \eta_s^{n+1}(dx) - \int_{A \times [0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\eta_s^n) ds, \infty)}(r) \widehat{\eta}_0(dx, dr) \right) \right] \\ &\leq E \left[\int_0^{T \wedge \tau_c^n} \int_S c_k(x) \lambda(x, \eta_s^n) \beta(dx) ds \right] \\ &\leq c, \end{aligned}$$

and $\tau_c^1 \geq \tau_c^2 \geq \dots$. Either

$$\lim_{c \rightarrow \infty} \lim_{n \rightarrow \infty} \tau_c^n = \infty \quad (2.21)$$

or (2.20) holds for some T .

If (2.21) holds almost surely, the limit η^∞ is the minimal solution of (2.6) in the sense that any other solution η will satisfy $\eta_t^\infty(B) \leq \eta_t(B)$ for all $B \in \mathcal{B}(S)$ and $t \geq 0$.

For an arbitrary pair (λ, δ) satisfying Conditions 2.1 and 2.2, we define an attractive pair by setting

$$\overline{\lambda}(x, \zeta) = \sup_{\zeta' \subset \zeta} \lambda(x, \zeta') \quad \underline{\delta}(x, \zeta) = \inf_{\zeta' \subset \zeta} \delta(x, \zeta').$$

Let η_0 be an S -valued random variable independent of N , and let $\widehat{\eta}_0$ be defined as in (2.5). We assume that $\overline{\lambda}$ satisfies (2.1), which implies

$$\int c_k(x) \overline{\lambda}(x, \zeta) \beta(dx) < \infty, \quad \zeta \in S, k = 1, 2, \dots, \quad (2.22)$$

and that there exists a solution $\overline{\eta}$ for the pair $(\overline{\lambda}, \underline{\delta})$.

We consider a different sequence of approximate equations. Let $\{K_n\}$ be the sets in the definition of \mathcal{C} , and let η^n satisfy

$$\begin{aligned} \eta_t^n(B) = & \int_{B \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\bar{\eta}_{s-} \cap K_n \cap \eta_{s-}^n)]}(u) \mathbf{1}_{(\int_s^t \delta(x,\bar{\eta}_v \cap K_n \cap \eta_v^n) dv, \infty)}(r) N(dx, ds, dr, du) \\ & + \int_{B \times [0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\bar{\eta}_v \cap K_n \cap \eta_v^n) dv, \infty)}(r) \widehat{\eta}_0(dx, dr). \end{aligned} \tag{2.23}$$

Existence and uniqueness for this equation follow from the fact that only finitely many births can occur in a bounded time interval in K_n . Consequently, the equation can be solved from one such birth to the next. Since $\lambda(x, \bar{\eta}_{s-} \cap K_n \cap \eta_{s-}^n) \leq \bar{\lambda}(x, \bar{\eta}_{s-})$ and $\delta(x, \bar{\eta}_s \cap K_n \cap \eta_s^n) \geq \underline{\delta}(x, \bar{\eta}_s)$, it follows that $\eta_t^n \subset \bar{\eta}_t$ and hence that

$$\begin{aligned} \eta_t^n(B) = & \int_{B \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0,\lambda(x, K_n \cap \eta_{s-}^n)]}(u) \mathbf{1}_{(\int_s^t \delta(x, K_n \cap \eta_v^n) dv, \infty)}(r) N(dx, ds, dr, du) \\ & + \int_{B \times [0,\infty)} \mathbf{1}_{(\int_0^t \delta(x, K_n \cap \eta_v^n) dv, \infty)}(r) \widehat{\eta}_0(dx, dr). \end{aligned} \tag{2.24}$$

Also, note that for $g \in \mathcal{C}$,

$$\int_0^t \int_S g(x) \delta(x, K_n \cap \eta_s^n) \eta_s^n(dx) ds \leq \int_0^t \int_S g(x) r \mathbf{1}_{[0,\bar{\lambda}(x,\bar{\eta}_{s-})]}(u) N(dx, ds, dr, du) < \infty. \tag{2.25}$$

Define $F(\widehat{\zeta}) = e^{-\int_{S \times [0,\infty)} f(x,r) \widehat{\zeta}(dx,dr)}$, $f \in \widehat{\mathcal{C}}$ nonnegative. Setting

$$\begin{aligned} \widehat{A}_n F(\widehat{\zeta}) = & F(\widehat{\zeta}) \left(\int_{S \times [0,\infty)} \lambda(x, K_n \cap \zeta) (e^{-f(x,r)} - 1) e^{-r} \beta(dx) dr \right. \\ & \left. + \int_{S \times [0,\infty)} \delta(x, K_n \cap \zeta) f_r(x, r) \widehat{\zeta}(dx, dr) \right), \end{aligned} \tag{2.26}$$

as in (2.12),

$$F(\widehat{\eta}_t^n) - F(\widehat{\eta}_0) - \int_0^t \widehat{A}_n F(\widehat{\eta}_s^n) ds$$

is a local martingale.

Uniqueness for (2.24) implies that the residual clock times at time t are conditionally independent, unit exponentials given \mathcal{F}_t^n . Consequently, as in Proposition 2.9, for $G(\zeta) = e^{-\int_S g(x) \zeta(dx)}$, $g \in \mathcal{C}$ nonnegative, and polli

$$\begin{aligned} A_n G(\zeta) = & \int (G(\zeta + \delta_x) - G(\zeta)) \lambda(x, K_n \cap \zeta) \beta(dx) \\ & + \int (G(\zeta - \delta_x) - G(\zeta)) \delta(x, K_n \cap \zeta) \zeta(dx), \\ G(\eta_t^n) - G(\eta_0^t) - & \int_0^t A_n G(\eta_s^n) ds \end{aligned} \tag{2.27}$$

is a local martingale. Exploiting the fact that $\eta_t^n \subset \bar{\eta}_t$, the relative compactness of $\{\eta^n\}$, in the sense of convergence in distribution in $D_{\mathcal{C}}[0, \infty)$ follows.

Proposition 2.12. *Suppose that Conditions 2.1 and 2.2 hold. If $(x, \zeta) \rightarrow \lambda(x, \zeta)$ and $(x, \zeta) \rightarrow \delta(x, \zeta)$ are continuous on $S \times \mathcal{C}$, then $\zeta \rightarrow AG(\zeta)$ is continuous, and any limit point of $\{\eta^n\}$ is a solution of the local martingale problem for A , and hence a weak solution of (2.8).*

Proof. By (2.22), we can select a_k so that

$$\Gamma(t) \equiv \int_0^t 1 \vee \sum_k a_k \int_S c_k(x) \bar{\lambda}(x, \bar{\eta}_s) \beta(dx) ds < \infty, \quad \forall t > 0 \quad a.s.$$

and by (2.25), it follows that

$$\tau_m = \inf\{t : \Gamma(t) \geq m\}$$

is a localizing sequence for (2.27) for all g and n . The estimates also give the necessary uniform integrability to ensure that limit points of (2.27) are local martingales. \square

2.3. Existence and Uniqueness. If $\sup_{\zeta \in \mathcal{S}} \int_S \lambda(x, \zeta) \beta(dx) < \infty$, then a solution of (2.6) has only finitely many births per unit time and it is easy to see that (2.6) has a unique solution. Condition 2.1, however, only ensures that there are finitely many births per unit time in each K_k , and uniqueness requires additional conditions. The conditions we use are essentially the same as those used for existence and uniqueness of the solution of the time change system in Garcia (1995). From now on, we are going to assume that $\delta(x, \eta) = 1$, for all $x \in S$ and $\eta \in \mathcal{S}$.

Let N be a Poisson random measure on $S \times [0, \infty)^3$ with mean measure $\beta(dx) \times ds \times e^{-r} dr \times du$. Let η_0 be an \mathcal{S} -valued random variable independent of N , and let $\hat{\eta}_0$ be defined as in (2.5). Suppose $\{\mathcal{F}_t\}$ is a filtration such that $\hat{\eta}_0$ is \mathcal{F}_0 -measurable and N is $\{\mathcal{F}_t\}$ -compatible. We consider the equation

$$\begin{aligned} \eta_t(B) = & \int_{B \times [0, t] \times [0, \infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ & + \int_{B \times [0, \infty)} \mathbf{1}_{(t, \infty)}(r) \hat{\eta}_0(dx, dr). \end{aligned} \quad (2.28)$$

Theorem 2.13. *Assume Conditions 2.1 and 2.2. Suppose that*

$$a(x, y) \geq \sup_{\eta} |\lambda(x, \eta + \delta_y) - \lambda(x, \eta)|$$

and that there exists a positive function c such that

$$M = \sup_x \int_S \frac{c(x)a(x, y)}{c(y)} \beta(dy) < \infty.$$

Then, there exists a unique solution of (2.28).

Example 2.14. Let $d(x, \eta) = \inf\{d_S(x, y) : y \in \eta\}$, where d_S is a distance in S such that (S, d_S) is complete separable metric space. Suppose $\lambda(x, \eta) = h(d(x, \eta))$. Then $a(x, y) = \sup_{r > d_S(x, y)} |h(r) - h(d_S(x, y))|$. If h is increasing, then $a(x, y) = h(\infty) - h(d_S(x, y))$ and

$$|\lambda(x, \eta^1) - \lambda(x, \eta^2)| \leq \int (h(\infty) - h(d_S(x, y))) |\eta^1 - \eta^2|(dy).$$

If h is decreasing, then $a(x, y) = h(d_S(x, y)) - h(\infty)$ and

$$|\lambda(x, \eta^1) - \lambda(x, \eta^2)| \leq \int (h(d_S(x, y)) - h(\infty)) |\eta^1 - \eta^2|(dy).$$

Theorem 2.13 is a consequence of the following lemmas that hold under the conditions of the theorem.

Lemma 2.15. For any $\eta^1, \eta^2 \in \mathcal{S}$ we have

$$|\lambda(x, \eta^1) - \lambda(x, \eta^2)| \leq \int_S a(x, y) |\eta^1 - \eta^2|(dy). \tag{2.29}$$

Proof. Since η^1 and η^2 contain countably many points, there exist $\{y_1, y_2, \dots\}$ and $\{z_1, z_2, \dots\}$ such that

$$\eta^2 = \eta^1 + \sum_{i=1}^I \delta_{y_i} - \sum_{j=1}^J \delta_{z_j}$$

(where I and J may be infinity) and hence

$$|\eta^1 - \eta^2|(B) = \sum_{i=1}^I \delta_{y_i}(B) + \sum_{j=1}^J \delta_{z_j}(B).$$

By the definition of a and Condition 2.2

$$\begin{aligned} |\lambda(x, \eta^1) - \lambda(x, \eta^2)| &= \lim_{n \rightarrow \infty} |\lambda(x, \eta^1) - \lambda(x, \eta^1 + \sum_{i=1}^{I \wedge n} \delta_{y_i} - \sum_{j=1}^{J \wedge n} \delta_{z_j})| \tag{2.30} \\ &\leq \lim_{n \rightarrow \infty} \left(\sum_{i=1}^{I \wedge n} a(x, y_i) + \sum_{j=1}^{J \wedge n} a(x, z_j) \right) \\ &\leq \int_S a(x, y) |\eta^1 - \eta^2|(dy). \end{aligned}$$

□

Define

$$\eta_0(B, t) = \int_{B \times [0, \infty)} \mathbf{1}_{(t, \infty)}(r) \widehat{\eta}_0(dx, dr).$$

Let η be $\{\mathcal{F}_t\}$ -adapted with sample paths in $D_{\mathcal{S}}[0, \infty)$. Then by Condition 2.1

$$\Phi \eta_t(B) = \eta_0(B, t) + \int_{B \times [0, t] \times [0, \infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_s)]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \tag{2.31}$$

defines a process adapted to $\{\mathcal{F}_t\}$ with sample paths in $D_{\mathcal{S}}[0, \infty)$.

Lemma 2.16. Let η^1 and η^2 be adapted to $\{\mathcal{F}_t\}$ and have sample paths in $D_{\mathcal{S}}[0, \infty)$. Then

$$\begin{aligned} \sup_x c(x) E \left[\int_S a(x, y) |\Phi \eta^1(t) - \Phi \eta^2(t)|(dy) \right] &\tag{2.32} \\ \leq M \int_0^t \sup_x c(x) E \left[\int_S a(x, y) |\eta_s^1 - \eta_s^2|(dy) \right] e^{-(t-s)} ds. \end{aligned}$$

Proof. Let $\xi^i = \Phi\eta^i$. Then

$$\begin{aligned}
& \sup_z c(z) E\left[\int_S a(z, x) |\xi_t^1 - \xi_t^2|(dx)\right] \\
& \leq \sup_z c(z) E\left[\int_{S \times [0, t] \times [0, \infty)^2} a(z, x) |\mathbf{1}_{[0, \lambda(x, \eta_s^1)]}(u) - \mathbf{1}_{[0, \lambda(x, \eta_s^2)]}(u)| \right. \\
& \quad \left. \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du)\right] \\
& \leq \sup_z c(z) E\left[\int_{S \times [0, t]} a(z, x) |\lambda(x, \eta_s^1) - \lambda(x, \eta_s^2)| e^{-(t-s)} \beta(dx) ds\right] \\
& \leq \sup_z c(z) \int_{S \times [0, t]} a(z, x) E\left[\int_S a(x, y) |\eta_s^1 - \eta_s^2|(dy)\right] e^{-(t-s)} \beta(dx) ds \\
& \leq \sup_z c(z) \int_S \frac{a(z, x)}{c(x)} \beta(dx) \int_0^t \sup_x c(x) E\left[\int_S a(x, y) |\eta_s^1 - \eta_s^2|(dy)\right] e^{-(t-s)} ds \\
& \leq M \int_0^t \sup_x c(x) E\left[\int_S a(x, y) |\eta_s^1 - \eta_s^2|(dy)\right] e^{-(t-s)} ds. \tag{2.33}
\end{aligned}$$

□

Proof. (Theorem 2.13) Uniqueness follows by (2.32) and Gronwall's inequality. To prove existence, we proceed by iteration. Let $\eta_t^0 = \eta_0(\cdot, t)$, and for $n \geq 1$, define $\eta^{n+1} = \Phi\eta^n$. Then

$$\begin{aligned}
& \sup_x c(x) E\left[\int_S a(x, y) |\eta_t^{n+1} - \eta_t^n|(dy)\right] \\
& \leq M \int_0^t \sup_x c(x) E\left[\int_S a(x, y) |\eta_s^n - \eta_s^{n-1}|(dy)\right] e^{-(t-s)} ds \\
& \leq M^2 \int_0^t \int_0^{s_1} \sup_x c(x) E\left[\int_S a(x, y) |\eta_{s_2}^{n-1} - \eta_{s_2}^{n-2}|(dy)\right] e^{-(s_1-s_2)} ds_2 e^{-(t-s_1)} ds_1 \\
& \leq M^n \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} \sup_x c(x) E\left[\int_S a(x, y) |\eta_{s_{n-1}}^1 - \eta_{s_{n-1}}^0|(dy)\right] \\
& \quad e^{-(s_{n-1}-s_n)} ds_n \dots e^{-(t-s_1)} ds_1.
\end{aligned}$$

Therefore, there exists $C > 0$ such that

$$\sup_x c(x) E\left[\int_S a(x, y) |\eta_t^{n+1} - \eta_t^n|(dy)\right] \leq \frac{C^n t^n}{n!} \sup_{s \leq t} \sup_x c(x) E\left[\int_S a(x, y) |\eta_s^1 - \eta_s^0|(dy)\right],$$

and the convergence of η^n to a solution of (2.28) follows. □

3. Ergodicity for spatial birth and death processes

3.1. *Temporal ergodicity.* The statement that a Markov process is ergodic can carry several meanings. At a minimum, it means that there exists an unique stationary distribution for the process. Under this condition, the corresponding stationary process is ergodic in the sense of triviality of its tail σ -algebra. A second, stronger meaning of ergodicity for Markov processes is that for all initial distributions, the distribution of the process at time t converges to the (unique) stationary distribution as $t \rightarrow \infty$.

One approach to the first kind of ergodicity involves using the stochastic equation to construct a “coupling from the past.” Following an idea of Kendall and Møller (2000), for $\eta^1 \subset \eta^2$, define

$$\bar{\lambda}(x, \eta^1, \eta^2) = \sup_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta) \quad \underline{\lambda}(x, \eta^1, \eta^2) = \inf_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta).$$

Note that for $\eta^1 \subset \eta^2$

$$|\bar{\lambda}(x, \eta^1, \eta^2) - \underline{\lambda}(x, \eta^1, \eta^2)| \leq \int_S a(x, y) |\eta^1 - \eta^2|(dy).$$

We assume that N is defined on $S \times (-\infty, \infty) \times [0, \infty)^2$, that is, for all positive and negative time, and consider a system starting from time $-T$, that is, for $t \geq -T$

$$\begin{aligned} \eta_t^{1,T}(B) &= \int_{B \times [-T, t] \times [0, \infty)^2} \mathbf{1}_{[0, \underline{\lambda}(x, \eta_{s-}^{1,T}, \eta_{s-}^{2,T})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_{(t+T, \infty)}(r) \widehat{\eta}_{-T}^{1,T}(dx, dr) \\ \eta_t^{2,T}(B) &= \int_{B \times [0, t] \times [0, \infty)^2} \mathbf{1}_{[0, \bar{\lambda}(x, \eta_{s-}^{1,T}, \eta_{s-}^{2,T})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_{(t+T, \infty)}(r) \widehat{\eta}_{-T}^{2,T}(dx, dr), \end{aligned} \tag{3.1}$$

where we require $\eta_{-T}^{1,T} \subset \eta_{-T}^{2,T}$. Suppose $\lambda(x, \eta) \leq \Lambda(x)$ for all η and

$$\int_S c_k(x) \Lambda(x) \beta(dx) < \infty, \quad k = 1, 2, \dots, \tag{3.2}$$

which implies Condition 2.1, and suppose Condition 2.2 holds. Then we can obtain a solution of (3.1) by iterating

$$\begin{aligned} \eta_t^{1,T,n+1}(B) &= \int_{B \times [-T, t] \times [0, \infty)^2} \mathbf{1}_{[0, \underline{\lambda}(x, \eta_{s-}^{1,T,n}, \eta_{s-}^{2,T,n})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_{(t+T, \infty)}(r) \widehat{\eta}_{-T}^{1,T}(dx, dr) \\ \eta_t^{2,T,n+1}(B) &= \int_{B \times [-T, t] \times [0, \infty)^2} \mathbf{1}_{[0, \bar{\lambda}(x, \eta_{s-}^{1,T,n}, \eta_{s-}^{2,T,n})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_{(t+T, \infty)}(r) \widehat{\eta}_{-T}^{2,T}(dx, dr), \end{aligned} \tag{3.3}$$

where we take $\eta_t^{1,T,1} \equiv \emptyset$ and

$$\begin{aligned} \eta_t^{2,T,1}(B) &= \int_{B \times [-T, t] \times [0, \infty)^2} \mathbf{1}_{[0, \Lambda(x)]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \\ &\quad + \int_{B \times [0, \infty)} \mathbf{1}_{(t+T, \infty)}(r) \widehat{\eta}_{-T}^{2,T}(dx, dr). \end{aligned}$$

Note that $\eta^{1,T,n} \subset \eta^{2,T,n}$, $\{\eta^{1,T,n}\}$ is monotone increasing, and $\{\eta^{2,T,n}\}$ is monotone decreasing, and the limit, which must exist, will be a solution of (3.1).

For $C \subset \mathbb{R}$, define $(C + t) = \{(s + t) : s \in C\}$, and define the time-shift of N by $R_t N(B \times C \times D \times E) = N(B \times (C + t) \times D \times E)$. Taking $T = \infty$ in (3.3), the

iterates

$$\eta_t^{1,\infty,n+1}(B) = \int_{B \times (-\infty, t] \times [0, \infty)^2} \mathbf{1}_{[0, \Delta(x, \eta_s^{1,\infty,n}, \eta_s^{2,\infty,n})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du) \quad (3.4)$$

$$\eta_t^{2,\infty,n+1}(B) = \int_{B \times (-\infty, t] \times [0, \infty)^2} \mathbf{1}_{[0, \bar{\lambda}(x, \eta_s^{1,\infty,n}, \eta_s^{2,\infty,n})]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du),$$

satisfy $\eta_t^{m,\infty,n} = H^{m,n}(R_t N)$, $m = 1, 2$, for deterministic mappings $H^{m,n} : \mathcal{N}(S \times (-\infty, \infty) \times [0, \infty)^2) \rightarrow \mathcal{N}(S)$ and the limits $\eta_t^{m,\infty}$ satisfy

$$\eta_t^{m,\infty} = H^m(R_t N), \quad (3.5)$$

where $H^m = \lim_{n \rightarrow \infty} H^{m,n}$. It follows that $(\eta_t^{1,\infty}, \eta_t^{2,\infty})$ is stationary and ergodic.

Lemma 3.1. *Suppose that λ satisfies (3.2) and Condition 2.2. Then*

$$\eta_t^{1,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{1,\infty,n} \text{ and } \eta_t^{2,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{2,\infty,n}$$

exist and are stationary.

Applying Theorem 2.8, any stationary solution of the martingale problem can be represented as a weak solution η of the stochastic equation on the doubly infinite time interval and hence coupled to versions of $\eta^{1,\infty,n}$ and $\eta^{2,\infty,n}$ so that $\eta_t^{1,\infty,n} \subset \eta_t \subset \eta_t^{2,\infty,n}$, $-\infty < t < \infty$. Consequently, we have the following.

Lemma 3.2. *Suppose that λ satisfies (3.2) and Condition 2.2. If*

$$\lim_{n \rightarrow \infty} \int_S c_k(x) |\eta_t^{2,\infty,n} - \eta_t^{1,\infty,n}|(dx) = 0 \quad \text{a.s.}$$

for $k = 1, 2, \dots$, then $\eta \equiv \eta^{2,\infty} = \eta^{1,\infty}$ a.s. is a stationary solution of (2.28) and the distribution of $\eta_t^{2,\infty}$ is the unique stationary distribution for A .

Theorem 3.3. *Let $\lambda : S \times \mathcal{N}(S) \rightarrow [0, \infty)$ satisfy the conditions of Theorem 2.13 with $M < 1$. Then $\eta \equiv \eta^{2,\infty} = \eta^{1,\infty}$ a.s. is a stationary solution of (2.28) and the distribution of $\eta_t^{2,\infty}$ is the unique stationary distribution for A .*

Proof. As in the proof of Theorem 2.13,

$$\begin{aligned} & \sup_x c(x) E \left[\int_S a(x, y) |\eta_t^{2,\infty,n+1} - \eta_t^{1,\infty,n+1}|(dy) \right] \\ & \leq M \int_{-\infty}^t \sup_x c(x) E \left[\int_S a(x, y) |\eta_s^{2,\infty,n} - \eta_s^{1,\infty,n}|(dy) \right] e^{-(t-s)} ds \\ & = M \sup_x c(x) E \left[\int_S a(x, y) |\eta_t^{2,\infty,n} - \eta_t^{1,\infty,n}|(dy) \right], \end{aligned}$$

where the equality follows by the stationarity of $\eta^{2,\infty,n}$ and $\eta^{1,\infty,n}$. Since the expression on the left is nonincreasing, its limit ρ exists, and we have $0 \leq \rho \leq M\rho$. But $M < 1$, so $\rho = 0$. \square

Definition 3.4. $\lambda(x, \cdot)$ is *nondecreasing*, if $\eta_1 \subset \eta_2$ implies $\lambda(x, \eta_1) \leq \lambda(x, \eta_2)$.

Note that if λ is nondecreasing, then for $\eta_1 \subset \eta_2$, $\bar{\lambda}(x, \eta_1, \eta_2) = \lambda(x, \eta_2)$ and $\underline{\lambda}(x, \eta_1, \eta_2) = \lambda(x, \eta_1)$. The following lemma is immediate.

Lemma 3.5. *Let λ be nondecreasing and satisfy (3.2) and Condition 2.2. Then $\eta_t^{1,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{1,\infty,n}$ and $\eta_t^{2,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{2,\infty,n}$ are, respectively, the minimal and maximal stationary solutions of the martingale problem for A .*

For λ nondecreasing, the minimal stationary distribution can also easily be obtained as a temporal limit.

Lemma 3.6. *If uniqueness holds for (2.28) and $\lambda(x, \cdot)$ is nondecreasing, then the process η_t is attractive, that is*

$$\eta_0^1 \subset \eta_0^2 \quad \text{implies} \quad \eta_t^1 \subset \eta_t^2 \quad (3.6)$$

for all $t \geq 0$.

Proof. The conclusion is immediate from coupling the two processes using the same underlying Poisson random measure. \square

Theorem 3.7. *Suppose λ satisfies (3.2) and Condition 2.2. If $\lambda(x, \cdot)$ is nondecreasing and $\eta_0 = \emptyset$, then the distribution of η_t converges to the minimal stationary distribution.*

Proof. Note that, if we set $\eta_{-t}^t = \emptyset$, then η_t has the same distribution as η_0^t , and by Lemma 3.6, $\eta_s^t \subset \eta_s^{1,\infty}$ for $s \geq -t$. Since for each $s \geq -t$, η_s^t is monotone increasing in t , $\tilde{\eta}_s = \lim_{t \rightarrow \infty} \eta_s^t$ exists and must be a stationary process. Since $\eta^{1,\infty}$ is the minimal stationary process, we must have $\tilde{\eta}_s = \eta_s^{1,\infty}$. \square

The same argument gives the following result on the maximal stationary distribution.

Theorem 3.8. *Suppose λ satisfies (3.2) and Condition 2.2. If $\lambda(x, \cdot)$ is nondecreasing and*

$$\eta_0(B) = \int_{B \times (-\infty, 0] \times [0, \infty)^2} \mathbf{1}_{[0, \Lambda(x)]}(u) \mathbf{1}_{(t-s, \infty)}(r) N(dx, ds, dr, du),$$

then the distribution of η_t converges to the maximal stationary distribution.

Remark 3.9. Note that η_0 is a Poisson random measure with mean measure $\mu(B) = \int_B \Lambda(x) \beta(dx)$.

We can also use the stochastic equation and estimates similar to those used in the proof of uniqueness to give conditions for ergodicity in the sense of convergence as $t \rightarrow \infty$ for all initial distributions.

Theorem 3.10. *Let $\lambda : S \times \mathcal{N}(S) \rightarrow [0, \infty)$ satisfy the conditions of Theorem 2.13 with $M < 1$. Then the process obtained as a solution of the system of stochastic equations (2.28) is temporally ergodic and the rate of convergence is exponential.*

Proof. Suppose η^1 and η^2 are solutions of the system (2.28) with distinct initial configurations η_0^1 and η_0^2 (equivalently, $\hat{\eta}_0^1$ and $\hat{\eta}_0^2$). Then, by exactly the same

argument as used for the proof of Lemma 2.16 we obtain

$$\begin{aligned} & \sup_x c(x) E\left[\int_S a(x, y) |\eta_t^1 - \eta_t^2|(dy)\right] \\ & \leq e^{-t} \sup_x c(x) E\left[\int_S a(x, y) |\eta_0^1 - \eta_0^2|(dy)\right] \\ & \quad + M \int_0^t \sup_x c(x) E\left[\int_S a(x, y) |\eta_s^1 - \eta_s^2|(dy)\right] e^{-(t-s)} ds. \end{aligned} \quad (3.7)$$

Multiply (3.7) by e^t and apply Gronwall's inequality to obtain the exponential rate of convergence. \square

3.2. Spatial ergodicity. In this section, we take $S = \mathbb{R}^d$ and assume that λ is translation invariant in the following sense. For arbitrary $x, y \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$, write

$$T_x y = x + y \text{ and } T_x B = B + x = \{y + x; y \in B\}.$$

Then, T_x induces a transformation S_x on $\mathcal{N}(\mathbb{R}^d)$ through the equation

$$(S_x \eta)(B) = \eta(T_x B), \quad \eta \in \mathcal{N}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}^d). \quad (3.8)$$

Note that if $\eta = \sum \delta_{x_i}$, then $S_x \eta = \sum \delta_{x_i - x}$.

Definition 3.11. We say that λ is *translation invariant* if $\lambda(x + y, \eta) = \lambda(x, S_y \eta)$ for $x, y \in \mathbb{R}^d, \eta \in \mathcal{N}(\mathbb{R}^d)$.

Definition 3.12. An $\mathcal{N}(\mathbb{R}^d)$ -valued random variable η is *translation invariant* if the distribution of $S_y \eta$ does not depend on y . A probability distribution $\mu \in \mathcal{P}(\mathcal{N}(\mathbb{R}^d))$ is *translation invariant* if $\int f(\eta) \mu(d\eta) = \int f(S_y \eta) \mu(d\eta)$, for all $y \in \mathbb{R}^d$ and all bounded, measurable functions f .

Definition 3.13. Let η be a translation invariant, $\mathcal{N}(\mathbb{R}^d)$ -valued random variable. A measurable subset $G \subset \mathcal{N}(\mathbb{R}^d)$ is *almost surely translation invariant* for η , if

$$\mathbf{1}_G(\eta) = \mathbf{1}_G(S_x \eta) \quad a.s.$$

for every $x \in \mathbb{R}^d$. η is *spatially ergodic* if $P\{\eta \in G\}$ is 0 or 1 for each almost surely translation invariant $G \subset \mathcal{N}(\mathbb{R}^d)$.

Similarly, for $x \in \mathbb{R}^d$, we define $S_x N$ so that the spatial coordinate of each point is shifted by $-x$. Almost sure translation invariance of a set $G \subset \mathcal{N}(\mathbb{R}^d \times [0, \infty)^3)$ and spatial ergodicity are defined analogously to Definition 3.13. Spatial ergodicity for N follows from its independence properties.

Lemma 3.14. Suppose λ is translation invariant. If η_0 is translation invariant and spatially ergodic and the solution of (2.28) is unique, then for each $t > 0$, η_t is translation invariant and spatially ergodic.

Proof. $\{S_y \eta_t, t \geq 0\}$ is the solution of (2.28) with η_0 replaced by $S_y \eta_0$ and N replaced by $S_y N$. By uniqueness, $S_y \eta_t$ must have the same distribution as η_t giving the stationarity. Also, by uniqueness, for measurable $G \subset \mathcal{N}(\mathbb{R}^d)$ there exists a measurable $\widehat{G} \subset \mathcal{N}(\mathbb{R}^d) \times \mathcal{N}(\mathbb{R}^d \times [0, \infty)^3)$ such that

$$\mathbf{1}_{\{S_x \eta_t \in G\}} = \mathbf{1}_{\{(S_x \eta_0, S_x N) \in \widehat{G}\}} \quad a.s.$$

for all $x \in \mathbb{R}^d$. Consequently, spatial ergodicity for η_t follows from the spatial ergodicity of (η_0, N) . \square

Remark 3.15. If η is temporally ergodic and π is the unique stationary distribution, then it must be translation invariant since $\{\eta_t\}$ stationary (in time) implies $\{S_x\eta_t\}$ is stationary.

Lemma 3.16. *Suppose that λ is translation invariant and satisfies (3.2) and Condition 2.2. Then for each t , $\eta_t^{1,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{1,\infty,n}$ and $\eta_t^{2,\infty} \equiv \lim_{n \rightarrow \infty} \eta_t^{2,\infty,n}$ are spatially ergodic.*

Proof. As in (3.5), $\eta_t^{1,\infty} = H^1(R_t N)$ can be written as a deterministic transformation $F(t, N)$ of N and that $S_y \eta_t^{1,\infty} = F(t, S_y N)$. The spatial ergodicity of $\eta_t^{1,\infty}$ then follows from the spatial ergodicity of N . \square

Corollary 3.17. *If in addition to the conditions of Lemma 3.16, λ satisfies the conditions of Lemma 3.2, then the unique stationary distribution is spatially ergodic. In particular, if λ satisfies the conditions of Theorem 2.13 with $M < 1$, then the unique stationary distribution is spatially ergodic.*

Corollary 3.18. *If in addition to the conditions of Lemma 3.16, λ is nondecreasing, then the minimal and maximal stationary distributions are spatially ergodic.*

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