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Diffusive variance for a tagged particle in $d \le 2$ asymmetric simple exclusion

Sunder Sethuraman

Address: 396 Carver Hall, Department of Mathematics, Iowa State University, Ames, IA 50011, USA
E-mail address: sethuram@iastate.edu

Abstract. The study of equilibrium fluctuations of a tagged particle in finite-range simple exclusion processes has a long history. The belief is that the scaled centered tagged particle motion behaves as some sort of homogenized random walk. In fact, invariance principles have been proved in all dimensions $d \ge 1$ when the single particle jump rate is unbiased, in $d \ge 3$ when the jump rate is biased, and in $d = 1$ when the jump rate is in addition nearest-neighbor.

The purpose of this article is to give some partial results in the open cases in $d \le 2$. Namely, we show the tagged particle motion is "diffusive" in the sense that upper and lower bounds are given for the tagged particle variance at time t on order $O(t)$ in $d = 2$ when the jump rate is biased, and also in $d = 1$ when in addition the jump rate is not nearest-neighbor. Also, a characterization of the tagged particle variance is given. The main methods are in analyzing H_{-1} norm variational inequalities.

1. Introduction and Results

One of the interesting questions in Spitzer's seminal paper on particle systems Spitzer (1970) asks for the asymptotics of a distinguished or "tagged," particle as it interacts with others. Although the tagged particle is not in general Markovian, due to the particle interactions, the understanding is that it behaves in some sense as a "homogenized" random walk. In the context of finite-range translation-invariant simple exclusion processes, this belief has been substantiated in large part through a quilt of results sometimes depending on the specific form of the single particle jump rate p, and the dimension d of the underlying lattice $\mathbb{Z}^d$.

For instance, laws of large numbers, both in equilibrium Saada (1987) and non-equilibrium Rezakhanlou (1994b) have been shown. Also, equilibrium central limit

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theorems and invariance principles when p is mean-zero Arratia (1983),Rost and Vares (1985), Kipnis and Varadhan (1986), Varadhan (1995), and when p has a drift in $d \geq 3$ Sethuraman, Varadhan, and Yau (2000) and in $d = 1$ when p is in addition nearest-neighbor Kipnis (1986) have been proved. See also Landim, Olla and Volchan (1998), Landim and Volchan (2000) for fluctuations in $d = 1$ with respect to a non-translation invariant p. Non-trivial non-equilibrium fluctuation results have even been derived in $d \geq 1$ when p is symmetric (excluding the $d = 1$ nearest-neighbor case) Rezakhanlou (1994a), and recently in the exceptional case in $d = 1$ when p is symmetric and nearest-neighbor Jara and Landim (2006). In addition, large deviations results have been proved in some cases Quastel, Rezakhanlou and Varadhan (1999), Seppäläinen (1998). Some of these results and others are reviewed in Ferrari (1996), section 4.VIII Liggett (1985), chapter 4.III Liggett (1999), chapter 6 Spohn (1991), and sections 4.3, 8.4 and 11.5 Kipnis and Landim (1999).

In terms of equilibrium fluctuations, however, open are the behaviors in $d = 2$ when p has a drift, and also in $d = 1$ when in addition p is not nearest-neighbor. The difficulty in their solution is roughly that in low dimensions with asymmetry one has to deal with more involved particle interactions than in high dimensions, where transience estimates can be used, and under symmetry, when reversibility helps. The main goal of this article is to shed light on the open low dimensional cases by giving some upper and lower bounds on the variance of the tagged particle at time t which are "diffusive," that is on order $O(t)$ (Theorems 1.2 and 1.3). In addition, a characterization of the variance, which recasts an expression in the literature (cf. equation (1.18) De Masi and Ferrari (1985)) in terms of certain "dynamical" and "static" contributions, is given (Theorem 1.1).

The method of the upper bounds is to bound above the variance of a "drift" additive functional as $O(t)$ by estimating certain H_{-1} variational formulas with the help of integral estimates in the spirit of Bernardin's work for occupation times Bernardin (2004). In particular, one of the main contributions of this article is to give a framework for tagged particle H_{-1} norms in which "environment" and "tagged-shift" dynamics are understood. The variance characterization, and lower bounds follow from explicit computations, and comparisons with "symmetrized" variances as in Loulakis (2005).

Loosely speaking, the simple exclusion process follows the motion of a collection of random walks on the lattice $\mathbb{Z}^d$ in which jumps to already occupied vertices are suppressed. More precisely, let $\Sigma = \{0,1\}^{\mathbb{Z}^d}$ and let $\eta(t) \in \Sigma$ represent the state of the process at time t. That is, the configuration at time t is given in terms of occupation variables $\eta(t) = \{\eta_i(t) : i \in \mathbb{Z}^d\}$ where $\eta_i(t) = 0$ or 1 according to whether the vertex $i \in \mathbb{Z}^d$ is empty or full at time t. Let $p = \{p(i,j) : i,j \in \mathbb{Z}^d\}$ be the single particle transition rates. Throughout this article we concentrate on the translation-invariant finite-range case: $p(i,j) = p(0,j-i) = p(j-i)$ and $p(x) = 0$ for $|x| > R$ and an integer $R < \infty$. In addition, to avoid technicalities, we concentrate on the situation when $(p(i) + p(-i))/2$ is irreducible, and $p(0) = 0$. We will say p is nearest-neighbor when the range $R = 1$.

The system $\eta(t)$ is a Markov process on $D(\mathbb{R}_+, \Sigma)$ with semi-group T_t and generator, well defined on functions ϕ supported on a finite number of vertices, namely "local" functions,

$$(L\phi)(\eta) \;=\; \sum_{i,j\in\mathbb{Z}^d} \eta_i(1-\eta_j)p(j-i)(\phi(\eta^{i,j}) - \phi(\eta)) \tag{1.1}$$

where $\eta^{i,j}$ is the "exchanged" configuration, $(\eta^{i,j})_i = \eta_j$, $(\eta^{i,j})_j = \eta_i$ and $(\eta^{i,j})_k = \eta_k$ for $k \neq i,j$. We note the transition rate $\eta_i(1-\eta_j)p(j-i)$ for $\eta \to \eta^{i,j}$ represents the exclusion property.

With respect to a configuration η, distinguish now one of the particles and call it the tagged particle. Let $x(t) \in \mathbb{Z}^d$ be its position at time t. To compensate for the non-Markovian character of the tagged motion, we form the larger process $(x(t), \eta(t))$ which is Markovian. In fact, as is standard practice, we will consider the system in the reference frame of the tagged particle, $(x(t), \zeta(t))$ where $\zeta(t) = \pi_{x(t)}\eta(t)$. Here, for a configuration $\eta \in \Sigma$, the k-shifted state is $\pi_k\eta$ where $(\pi_k\eta)_l = \eta_{k+l}$ for $l \in \mathbb{Z}^d$. The "reference frame" process $\zeta(t)$ is also Markovian with semi-group $\mathcal{T}_t$, and generator $\mathcal{L}$ well defined on local functions,

$$
\begin{aligned}
(\mathcal{L}\phi)(\zeta) \quad = \quad & \sum_{i,j \in \mathbb{Z}^d \setminus \{0\}} \zeta_i(1-\zeta_j)p(j-i)(\phi(\zeta^{i,j}) - \phi(\zeta)) \\
& + \sum_{j \in \mathbb{Z}^d \setminus \{0\}} (1-\zeta_j)p(j)(\phi(\tau_j\zeta)) - \phi(\zeta))
\end{aligned}
$$

where $\tau_j\zeta = \pi_j(\zeta^{0,j})$ accounts for the reference frame shift when the tagged particle displaces by j.

Naturally, $\mathcal{L}$ splits as $\mathcal{L} = \mathcal{L}^e + \mathcal{L}^t$ where $(\mathcal{L}^e\phi)(\zeta) = \sum_{i,j \in \mathbb{Z}^d \setminus \{0\}} \zeta_i(1-\zeta_j)p(j-i)(\phi(\zeta^{i,j}) - \phi(\zeta))$ and $(\mathcal{L}^t\phi)(\zeta) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}}(1-\zeta_j)p(j)(\phi(\tau_j\zeta)) - \phi(\zeta))$ correspond to movement around, and by the tagged particle, e.g. "environment" and "tagged-shift" motions, respectively. The main idea of the reference process is that, although the tagged particle is always at the origin ($\zeta_0(t) \equiv 1$), one can keep track of the position of the tagged particle by counting the various reference "j-shifts" (cf. (1.2)). We refer to Liggett (1985) for details of the construction of these processes.

We now discuss the equilibria for these systems. Let P_ρ, for $\rho \in [0,1]$, be the infinite Bernoulli product measure over $\mathbb{Z}^d$ with coin-tossing marginal $P_\rho\{\eta_i = 1\} = 1 - P_\rho\{\eta_i = 0\} = \rho$. It is known that P_ρ and $Q_\rho = P_\rho(\cdot|\zeta_0 = 1)$ are invariant extremal measures for L and $\mathcal{L}$ respectively Saada (1987). We remark with respect to P_ρ, the semi-group T_t and generator L can be extended to $L^2(P_\rho)$ (cf. section IV.4 Liggett (1985)); similarly, with respect to Q_ρ, $\mathcal{T}_t$ and $\mathcal{L}$ can be extended to $L^2(Q_\rho)$. We note the adjoints L^* and $\mathcal{L}^*$ with respect to P_ρ and Q_ρ, corresponding to time-reversal, are straightforwardly computed and identified as generators corresponding to reversed jump rates $p(-\cdot)$. It will sometimes be convenient to write $\mathcal{L}$ into symmetric and anti-symmetric parts, $\mathcal{L} = \mathcal{S} + \mathcal{A}$ where $\mathcal{S} = (\mathcal{L} + \mathcal{L}^*)/2$ and $\mathcal{A} = (\mathcal{L} - \mathcal{L}^*)/2$. We note the operator $\mathcal{S}$ is the generator of a reference frame process with symmetric jump rates $(p(\cdot)+p(-\cdot))/2$. Also, as before, $\mathcal{S}$ and $\mathcal{A}$ can be split into "environment" and "tagged-shift" parts, $\mathcal{S} = \mathcal{S}^e + \mathcal{S}^t$ and $\mathcal{A} = \mathcal{A}^e + \mathcal{A}^t$.

We denote E_ρ for expectation with respect to the reference process measure starting from Q_ρ. Denote also, for vector-valued functions $f, g : \Sigma \to \mathbb{R}^m$ and $m \ge 1$, the innerproduct $\langle f, g \rangle_\rho = E_\rho[f \cdot g]$, and L^2 norm $\|f\|_0 = \sqrt{\langle f, f \rangle_\rho}$ with respect to Q_ρ.

We now specify a family of martingales associated with the exclusion process. For $j \in \mathbb{Z}^d$, let $N_j(t)$ denote the counting processes which count the number of j-shifts made by the reference process, e.g. j-displacements of the tagged particle, up to time $t \ge 0$. By subtracting appropriate compensators, we can then form the martingale $M_j(t) = N_j(t) - A_j(t)$ where $A_j(t) = \int_0^t p(j)(1-\zeta_j(s))ds$. These martingales, as jumps are not simulateneous, are orthogonal for $j \in \mathbb{Z}^d$. Then, the tagged particle position $x(t)$ may be written into the sum of a martingale and an additive functional term,

$$x(t) \; = \; \sum_j jN_j(t) \; = \; \sum_j jM_j(t) + \sum_j jA_j(t).$$

These relations, by stationarity of the process measure, give the quadratic variation $E_\rho[M_j^2(t)] = (1-\rho)p(j)t$ and mean position, $E_\rho[x(t)] = (\sum_j jp(j))(1-\rho)t$. Then, after centering,

$$x(t) - E_\rho[x(t)] \; = \; M(t) + A(t) \tag{1.2}$$

with martingale $M(t) = \sum jM_j(t)$ and "drift" $A(t) = \int_0^t \mathfrak{F}(\zeta(s))ds$ with $\mathfrak{F}(\zeta) = \sum jp(j)(\rho - \zeta_j)$.

Let now

$$V(t) \; = \; E_\rho\bigg[|x(t) - E_\rho[x(t)]|^2\bigg].$$

Define also the measure $d\mu_{k,\rho} = (\zeta_k/\rho)dQ_\rho$ and its expectation $E_{k,\rho}$ for $k \in \mathbb{Z}^d \backslash \{0\}$. The first result is a characterization of the variance. In a different form, it was first derived by De Masi and Ferrari (cf. equation (1.18) De Masi and Ferrari (1985)), however, the interpretation below seems new. See also Sethuraman (2006) for analogous expressions in zero-range processes.

Theorem 1.1. *In $d \geq 1$,*

$$V(t) \; = \; (1-\rho)\sum_j |j|^2 p(j)t + 2\rho \sum_j jp(j) \cdot \int_0^t \bigg\{E_\rho[x(s)] - E_{-j,\rho}[x(s)]\bigg\} ds.$$

The first term above, $(1-\rho)\sum |j|^2 p(j)t$, is the mean quadratic variation of the martingale $M(t)$ and can be thought of as a "dynamical" part of the variation. The second term, however, as a difference in expected tagged particle positions from different initial measures, is in a sense variation due to initial conditions.

We note in $d = 1$ when p is totally asymmetric and nearest-neighbor, say $p(1) > 0$ and $p(i) = 0$ for $i \neq 1$, the second term in the decomposition vanishes as the "extra" particle at -1, being behind, cannot interfere with the tagged particle position; in this case, $V(t) = p(1)(1-\rho)t$ and moreover it is known the tagged motion is actually a Poisson process with rate $p(1)(1-\rho)$ (cf. Corollary VIII.4.9 Liggett (1985)). Also, in $d = 1$ when p is nearest-neighbor, the formula can be evaluated to some extent, and the limit $\lim_{t\to\infty} V(t)/t = (1-\rho)|p(1) - p(-1)|$ has been proved De Masi and Ferrari (1985).

However, for the next upper bounds, other methods are used.

Theorem 1.2. *When p has a drift, $\sum jp(j) \neq 0$, in $d = 2$, and in $d = 1$ when additionally p is not nearest-neighbor, we have a constant $C = C(d,p,\rho)$ such that*

$$V(t) \; \leq \; Ct.$$

For a general lower bound, we only give an estimate on a "Tauberian" quantity which resembles $V(t)$.

Theorem 1.3. *In $d \geq 1$ and for $0 \leq \rho < 1$, excluding the nearest-neighbor symmetric case in $d = 1$ when $p(1) = p(-1)$, we have a constant $C = C(d, p, \rho) > 0$ such that*

$$\liminf_{\lambda \downarrow 0} \ \lambda^2 \int_0^\infty e^{-\lambda t} V(t) dt \ \geq \ C.$$

The lower bound, by formal (non-rigorous) analogies, suggests

$$\frac{1}{T} \int_0^\infty e^{-t/T} V(t) dt \ \sim \ \frac{1}{T} \int_0^T V(t) dt \ \sim \ V(T) \ \geq \ CT.$$

We note also our proofs of Theorems 1.2 and 1.3 only give gross estimates on the constants $C(d, p, \rho)$.

However, well-known when p is mean-zero and not nearest-neighbor in $d = 1$, biased in $d \geq 3$, or biased and nearest-neighbor in $d = 1$, the variance is on order $V(t) = O(t)$ Kipnis and Varadhan (1986),Varadhan (1995), Sethuraman, Varadhan, and Yau (2000), Kipnis (1986); in the excluded $d = 1$ nearest-neighbor symmetric case, due to "trapping" phenomena, $V(t) = O(\sqrt{t})$ Arratia (1983). Also, when $\rho = 1$, there is no motion and $V(t) \equiv 0$.

We remark now, in terms of remaining open questions, the limit

$$\lim_{t \to \infty} \frac{1}{t} V(t) \ = \ \sigma^2(d, p, \rho), \tag{1.3}$$

and full invariance principles should hold more generally in $d \leq 2$ when $\sum jp(j) \neq 0$.

We suspect more detailed H_{-1} norm estimation might allow martingale approximation of the tagged position $x(t)$ leading to limits (1.3) and invariance principles in this situation. Namely, one wants to show the "drift" $\mathfrak{F}$ (cf. (1.2)) can be approximated in terms of $\mathcal{L}u_\epsilon$ where u_ϵ is a local function satisfying $\|\mathfrak{F} - \mathcal{L}u_\epsilon\|_{H_{-1}} < \epsilon$. This type of program was done in Sethuraman, Varadhan, and Yau (2000) in $d \geq 3$ using "transience estimates" which unfortunately are not available in $d \leq 2$. We hope however the basic H_{-1} estimates given in this article will serve as building blocks for subsequent work.

The structure of the article is to prove first the variance characterization and lower bound in section 2. The upper bound is proved in section 4 with the aid of some preliminaries in section 3 and technical computations in section 5.

2. Proofs of Theorems 1.1 and 1.3

Let s and a be the symmetric and anti-symmetric parts of p, $s(i) = (p(i) + p(-i))/2$ and $a(i) = (p(i) - p(-i))/2$ for $i \in \mathbb{Z}^d$. Recall the "drift" function $\mathfrak{F} = \sum jp(j)(\rho - \zeta_j)$ in the introduction, and define analogous "drifts" $\mathfrak{F}_s(\zeta) = \sum js(j)(\rho - \zeta_j)$ and $\mathfrak{F}_{\leftarrow}(\zeta) = \sum jp(-j)(\rho - \zeta_j)$ corresponding to rates $s(\cdot)$ and $p(-\cdot) = s(\cdot) - a(\cdot)$ respectively.

Proof of Theorem 1.1. Following decomposition (1.2), write

$$\begin{aligned} V(t) &= (1-\rho)\sum_j |j|^2 p(j)t + 2E_\rho[M(t)\cdot A(t)] + E_\rho[|A(t)|^2] \\ &= (1-\rho)\sum_j |j|^2 p(j)t + 2\int_0^t E_\rho[M(s)\cdot \mathfrak{F}(\zeta(s))]ds + E_\rho[|A(t)|^2] \\ &= (1-\rho)\sum_j |j|^2 p(j)t + 2\int_0^t E_\rho[x(s)\cdot \mathfrak{F}(\zeta(s))]ds \qquad (2.1) \end{aligned}$$

where we note $E_\rho[|A(t)|^2] = 2\int_0^t E_\rho[A(s)\cdot\mathfrak{F}(\zeta(s))]ds$. We now reverse time at s, and note the time-reversed process $\zeta(s-\cdot)$ with respect to process measure started from Q_ρ has the same distribution as the process with reversed jump rates. In particular, $N_j(s)$ with respect to the process begun from Q_ρ has the same distribution as $N_{-j}(t)$ with respect to the reversed process. Hence, as $x(t) = \sum jN_j(t)$, we have $E_\rho[x(s)\cdot\mathfrak{F}(\zeta(s))] = E^*_\rho[-x(s)\cdot\mathfrak{F}(\zeta(0))]$ where E^*_ρ is expectation with respect to the reversed process begun with Q_ρ. Then, by spatial reflection, simple manipulations, and recalling the measure $d\mu_{k,\rho} = (\zeta_k/\rho)dQ_\rho$ with expectation $E_{k,\rho}$, we have

$$\begin{aligned} -E^*_\rho[x(s)\cdot\mathfrak{F}(\zeta(0))] &= -E_\rho\bigg[\sum_k kN_{-k}(s)\cdot\sum_j jp(j)(\rho-\zeta_{-j}(0))\bigg] \qquad (2.2) \\ &= \rho\sum_j jp(j)\cdot\Big\{E_\rho[x(s)] - E_{-j,\rho}[x(s)]\Big\}. \end{aligned}$$

□

Proof of Theorem 1.3. The proof follows straightforwardly from Propositions 2.1 and 2.2 below which allow comparisons with the tagged particle variance for the symmetrized process. □

Let $E^{\mathcal{S}}_\rho$ be expectation with respect to the symmetric reference process generated by $\mathcal{S}$ with initial distribution Q_ρ. Let also $V_s(t) = E^{\mathcal{S}}_\rho[|x(t) - E_\rho[x(t)]|^2]$ be the corresponding variance of the tagged particle at time t. Then, the following estimate is proved in Kipnis and Varadhan (1986).

Proposition 2.1. *In $d \geq 1$ and for $0 \leq \rho < 1$, except for the nearest-neighbor symmetric case in $d = 1$ when $p(1) = p(-1)$, we have a constant $C = C(d,p,\rho) > 0$ such that $V_s(t) \geq Ct$ for all $t \geq 0$.*

Form now, for $\lambda > 0$, two resolvent equations,

$$\lambda u_\lambda - \mathcal{L}u_\lambda = \mathfrak{F} \quad \text{and} \quad \lambda v_\lambda - \mathcal{S}v_\lambda = \mathfrak{F}_s$$

with respect to $u_\lambda = (\lambda-\mathcal{L})^{-1}\mathfrak{F} = \int_0^\infty e^{-\lambda t}(T_t\mathfrak{F})dt$ and $v_\lambda = (\lambda-\mathcal{S})^{-1}\mathfrak{F}_s$. We now state a comparison, in whose proof, the last part is Corollary 1 Loulakis (2005).

Proposition 2.2. *We have*

$$\begin{aligned} \int_0^\infty e^{-\lambda t}[V(t) - V_s(t)]dt &= \frac{2}{\lambda^2}\Big[\langle\mathfrak{F}_s, (\lambda-\mathcal{S})^{-1}\mathfrak{F}_s\rangle_\rho - \langle\mathfrak{F}_\leftarrow, (\lambda-\mathcal{L})^{-1}\mathfrak{F}\rangle_\rho\Big] \\ &= \frac{2}{\lambda^2}\Big[\lambda\|u_\lambda - v_\lambda\|_0^2 + \langle u_\lambda - v_\lambda, (-\mathcal{S})(u_\lambda - v_\lambda)\rangle_\rho\Big]. \end{aligned}$$

We note, as $-\mathcal{S}$ is a non-negative operator, the Dirichlet form $\langle u_\lambda - v_\lambda, (-\mathcal{S})(u_\lambda - v_\lambda)\rangle_\rho \geq 0$, and so as a consequence, $\int_0^\infty e^{-\lambda t}V(t)dt \geq \int_0^\infty e^{-\lambda t}V_s(t)dt$.

Proof. We first evaluate further (2.2) as

$$-E_\rho[x(s)\cdot \mathfrak{F}_{\leftarrow}(0)] \;=\; -E_\rho[A(s)\cdot \mathfrak{F}_{\leftarrow}(0)]$$

after the martingale part in $x(s) = M(s) + A(s)$ vanishes. Then, the last term of (2.1) equals

$$-2\int_0^t E_\rho[A(s)\cdot \mathfrak{F}_{\leftarrow}(\zeta(0))]ds \;=\; -2\int_0^t\int_0^s \langle \mathfrak{F}_{\leftarrow}, T_s\mathfrak{F}\rangle_\rho dr ds.$$

Hence, by two integration by parts,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}V(t)dt &= \lambda^{-2}(1-\rho)\sum_j |j|^2p(j) - 2\lambda^{-2}\int_0^\infty e^{-\lambda t}\langle \mathfrak{F}_{\leftarrow}, T_t\mathfrak{F}\rangle_\rho dt\\ &= \lambda^{-2}(1-\rho)\sum_j |j|^2p(j) - 2\lambda^{-2}\langle \mathfrak{F}_{\leftarrow}, (\lambda-\mathcal{L})^{-1}\mathfrak{F}\rangle_\rho.\end{aligned}$$

Since, $\sum |j|^2p(j) = \sum |j|^2s(j)$ and $\mathfrak{F} = \mathfrak{F}_{\leftarrow} = \mathfrak{F}_s$ when $p(\cdot) = s(\cdot)$, we obtain the first equality in the proposition directly.

For the second equality, we compute, using $\mathfrak{F} + \mathfrak{F}_{\leftarrow} = 2\mathfrak{F}_s$, the two resolvent equations and $\langle u_\lambda, \mathcal{A}u_\lambda\rangle_\rho = 0$, that

$$\begin{aligned}\langle \mathfrak{F}_s, v_\lambda\rangle_\rho - \langle \mathfrak{F}_{\leftarrow}, u_\lambda\rangle_\rho &= \langle \mathfrak{F}_s, v_\lambda\rangle_\rho + \langle \mathfrak{F}, u_\lambda\rangle_\rho - 2\langle \mathfrak{F}_s, u_\lambda\rangle_\rho\\ &= \langle v_\lambda, (-\mathcal{S})v_\lambda\rangle_\rho + \langle u_\lambda, (-\mathcal{S})u_\lambda\rangle_\rho\\ &\qquad + \lambda\|u_\lambda\|_0^2 + \lambda\|v_\lambda\|_0^2 - 2\langle\, \mathfrak{F}_s, u_\lambda\rangle_\rho\\ &= \langle v_\lambda, (-\mathcal{S})v_\lambda\rangle_\rho + \langle u_\lambda, (-\mathcal{S})u_\lambda\rangle_\rho\\ &\qquad + \lambda\|u_\lambda - v_\lambda\|_0^2 + 2\lambda\langle v_\lambda, u_\lambda\rangle_\rho - 2\langle \mathfrak{F}_s, u_\lambda\rangle_\rho.\end{aligned}$$

Since $2\langle \lambda v_\lambda, u_\lambda\rangle_\rho - 2\langle \mathfrak{F}_s, u_\lambda\rangle_\rho = -2\langle (-S)v_\lambda, u_\lambda\rangle_\rho$, we have the right-side equals $\lambda\|u_\lambda - v_\lambda\|_0^2 + \langle u_\lambda - v_\lambda, (-\mathcal{S})(u_\lambda - v_\lambda)\rangle_\rho$ as desired. □

3. Preliminaries for Upper Bound

We discuss here some definitions and results useful for the upperbound.

3.1. *Duality.* As the tagged particle is always at the origin with respect to the reference process, consider the underlying lattice $\mathbb{Z}^d \setminus \{0\}$. Let $\mathcal{E}_d$ denote the collection of finite subsets of $\mathbb{Z}^d \setminus \{0\}$, and let $\mathcal{E}_{d,n}$ be those subsets of cardinality $n \geq 0$. Let $\beta_\rho = \sqrt{\rho(1-\rho)}$ and, for non-empty $B \in \mathcal{E}_d$, let Ψ_B be the function

$$\Psi_B(\zeta) \;=\; \prod_{x\in B}\frac{\zeta_x - \rho}{\beta_\rho}$$

when $0 < \rho < 1$, and $\Psi_B \equiv 0$ when $\rho = 0$ or 1. By convention, we set $\Psi_\emptyset \equiv 1$. One can check that $\{\Psi_B : B \in \mathcal{E}_d\}$ is a Hilbert basis of $L^2(Q_\rho)$. In particular, any function $f \in L^2(Q_\rho)$ has decomposition

$$f \;=\; \sum_{n\geq 0}\sum_{B\in\mathcal{E}_{d,n}} \mathfrak{f}(B)\Psi_B$$

with coefficient $\mathfrak{f} : \mathcal{E}_d \to I\!R$ which in general depends on ρ. Then, for $f, g \in L^2(Q_\rho)$, we define innerproduct

$$\langle \mathfrak{f}, \mathfrak{g} \rangle \;=\; \langle f, g\rangle_\rho \;=\; \sum_{B\in\mathcal{E}_d} \mathfrak{f}(B)\mathfrak{g}(B),$$

and L^2 norm by $\|\mathfrak{f}\|_0^2 = \|f\|_0^2 = \langle f, f\rangle_\rho$.

Let also $\mathcal{C}_{d,n}$ be the subspace of coefficient functions on $\mathcal{E}_{d,n}$. When f is in the span of $\{\Psi_B : B \in \mathcal{E}_{d,n}\}$, we have $\mathfrak{f} \in \mathcal{C}_{d,n}$, and we say both f and its coefficient $\mathfrak{f}$ are of degree n. Note also, when f is local, then $\mathfrak{f}$ is also local on $\mathcal{E}_d$, that is with support on a finite number of subsets of $\mathbb{Z}^d \setminus \{0\}$.

The operators $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{A}$ have counterparts $\mathfrak{L} = \mathfrak{L}^e + \mathfrak{L}^t$, $\mathfrak{S} = \mathfrak{S}^e + \mathfrak{S}^t$ and $\mathfrak{A} = \mathfrak{A}^e + \mathfrak{A}^t$ which act on "coefficient" functions $\mathfrak{f}$:

$$\mathcal{L}^e f \;=\; \sum_{B\in\mathcal{E}} (\mathfrak{L}^e\mathfrak{f})(B)\Psi_B,\;\; \mathcal{S}^e f \;=\; \sum_{B\in\mathcal{E}} (\mathfrak{S}^e\mathfrak{f})(B)\Psi_B,\;\; \text{and}\;\; \mathcal{A}^e f \;=\; \sum_{B\in\mathcal{E}} (\mathfrak{A}^e\mathfrak{f})(B)\Psi_B$$

with analogous expressions for $\mathfrak{L}^t$, $\mathfrak{S}^t$ and $\mathfrak{A}^t$.

Recall the symmetric and anti-symmetric parts of p, $s(i) = (p(i) + p(-i))/2$ and $a(i) = (p(i) - p(-i))/2$ for $i \in \mathbb{Z}^d$; by assumption $s(0) = a(0) = 0$. For $B \subset \mathbb{Z}^d \setminus \{0\}$, denote

$$B_{x,y} = \begin{cases} B \setminus \{x\} \cup \{y\} & \text{when } x \in B, y \notin B \\ B \setminus \{y\} \cup \{x\} & \text{when } x \notin B, y \in B \\ B & \text{otherwise} \end{cases}$$

and

$$\tau_x B \;=\; \begin{cases} B + x & \text{when } -x \notin B \\ (B + x) \setminus \{0\} \cup \{x\} & \text{when } -x \in B \end{cases}$$

where as usual $B + x = \{i + x : i \in B\}$ for B nonempty, and $\emptyset + x = \emptyset$. As in Sethuraman, Varadhan, and Yau (2000), the symmetric parts $\mathfrak{S}^e$ and $\mathfrak{S}^t$ can be computed as

$$\begin{aligned}
(\mathfrak{S}^e\mathfrak{f})(B) \;&=\; \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d\setminus\{0\}} s(y-x)[\mathfrak{f}(B_{x,y}) - \mathfrak{f}(B)] \\
(\mathfrak{S}^t\mathfrak{f})(B) \;&=\; (1-\rho) \sum_{\substack{z\notin B\\ z\in\mathbb{Z}^d\setminus\{0\}}} s(z)[\mathfrak{f}(\tau_{-z}B) - f(B)] + \rho \sum_{z\in B} s(z)[f(\tau_{-z}B) - f(B)] \\
&\quad +\beta_\rho \sum_{\substack{z\notin B\\ z\in\mathbb{Z}^d\setminus\{0\}}} s(z)[\mathfrak{f}(B\cup\{z\}) - \mathfrak{f}(\tau_{-z}(B\cup\{z\}))] \\
&\quad +\beta_\rho \sum_{z\in B} s(z)[\mathfrak{f}(B\setminus\{z\}) - \mathfrak{f}(\tau_{-z}(B\setminus\{z\}))].
\end{aligned}$$

Note that $\mathfrak{S}^e\mathfrak{f} \in \mathcal{C}_{d,n}$ for $\mathfrak{f} \in \mathcal{C}_{d,n}$, and so $\mathfrak{S}^e$ "preserves" degrees. However, $\mathfrak{S}^t$ does not "preserve" degrees but, as will be seen, we will not need to deal directly with $\mathfrak{S}^t$ in our calculations.

Also, the anti-symmetric parts $\mathfrak{A}^e$ and $\mathfrak{A}^t$ are decomposed into sums of three operators which preserve, increase, and decrease the degree of the function acted upon: $\mathfrak{A}^e = \mathfrak{A}_0^e + \mathfrak{A}_+^e + \mathfrak{A}_-^e$ and $\mathfrak{A}^t = \mathfrak{A}_0^t + \mathfrak{A}_+^t + \mathfrak{A}_-^t$ where

$$\begin{aligned}
(\mathfrak{A}_0^e\mathfrak{f})(B) &= (1-2\rho)\sum_{\substack{x\in B\\ y\not\in B, y\in\mathbb{Z}^d\setminus\{0\}}} a(y-x)[\mathfrak{f}(B_{x,y})-\mathfrak{f}(B)]\\
(\mathfrak{A}_+^e\mathfrak{f})(B) &= 2\beta_\rho\sum_{x,y\in B} a(y-x)\mathfrak{f}(B\setminus\{y\})\\
(\mathfrak{A}_-^e\mathfrak{f})(B) &= -2\beta_\rho\sum_{\substack{x,y\not\in B\\ x,y\in\mathbb{Z}^d\setminus\{0\}}} a(y-x)\mathfrak{f}(B\cup\{x\})\\
(\mathfrak{A}_0^t\mathfrak{f})(B) &= (1-\rho)\sum_{\substack{z\not\in B\\ z\in\mathbb{Z}^d\setminus\{0\}}} a(z)[\mathfrak{f}(\tau_{-z}B)-\mathfrak{f}(B)] + \rho\sum_{z\in B} a(z)[\mathfrak{f}(\tau_{-z}B)-\mathfrak{f}(B)]\\
(\mathfrak{A}_+^t\mathfrak{f})(B) &= \beta_\rho\sum_{z\in B} a(z)[\mathfrak{f}(B\setminus\{z\})-\mathfrak{f}(\tau_{-z}(B\setminus\{z\}))]\\
(\mathfrak{A}_-^t\mathfrak{f})(B) &= \beta_\rho\sum_{\substack{z\not\in B\\ z\in\mathbb{Z}^d\setminus\{0\}}} a(z)[\mathfrak{f}(B\cup\{z\})-\mathfrak{f}(\tau_{-z}(B\cup\{z\}))].
\end{aligned}$$

It will also be helpful to write $\mathfrak{A}$ in terms of its explicit "degree" actions,

$$\mathfrak{A} = \sum_{n\geq 0}\Big(\mathfrak{A}_{n,n-1}+\mathfrak{A}_{n,n}+\mathfrak{A}_{n,n+1}\Big)$$

where $\mathfrak{A}_{m,n}$ is the part which takes a degree m function to a degree n function. Here, by convention $\mathfrak{A}_{0,-1}\equiv 0$ is the zero operator; one also sees $\mathfrak{A}_{0,0}=\mathfrak{A}_{1,0}=\mathfrak{A}_{0,1}\equiv 0$. Similarly, $\mathfrak{A}^e$ and $\mathfrak{A}^t$ can be decomposed in terms of degree actions $\mathfrak{A}^e_{m,n}$ and $\mathfrak{A}^t_{m,n}$ so that $\mathfrak{A}_{m,n}=\mathfrak{A}^e_{m,n}+\mathfrak{A}^t_{m,n}$ for $m,n\geq 0$. We later evaluate in Proposition 4.1, and its proof in section 5, some of the relevant actions.

3.2. *Variational Formulas.* Define, for $\lambda>0$ and local ϕ, the $H_{1,\lambda,\mathcal{L}}$ norm $\|\cdot\|_{1,\lambda,\mathcal{L}}$ by

$$\|\phi\|^2_{1,\lambda,\mathcal{L}} = \langle\phi,(\lambda-\mathcal{S})\phi\rangle_\rho + \langle\mathcal{A}\phi,(\lambda-\mathcal{S})^{-1}\mathcal{A}\phi\rangle_\rho$$

where we note $\langle\phi,(-\mathcal{S})\phi\rangle_\rho, \langle\mathcal{A}\phi,(\lambda-\mathcal{S})^{-1}\mathcal{A}\phi\rangle_\rho\geq 0$ as $-\mathcal{S}$ is a non-negative operator. The $H_{1,\lambda,\mathcal{L}}$ Hilbert space is then the completion over local functions with respect to this norm.

To define a dual norm, consider for $f\in L^2(Q_\rho)$ and local ϕ that

$$\langle f,\phi\rangle_\rho \leq \|f\|_0\|\phi\|_0 \leq \lambda^{-1/2}\|f\|_0\|\phi\|_{1,\lambda,\mathcal{L}}\ .$$

Then, the dual norm of $\|\cdot\|_{1,\lambda,\mathcal{L}}$, given by

$$\|f\|_{-1,\lambda,\mathcal{L}} = \sup_{\substack{\phi \text{ local}\\ \|\phi\|_{1,\lambda,\mathcal{L}}=1}}\langle f,\phi\rangle_\rho,$$

is always finite with bound $\|f\|^2_{-1,\lambda,\mathcal{L}}\leq\lambda^{-1}\|f\|_0^2$. Let $H_{-1,\lambda,\mathcal{L}}$ be the corresponding Hilbert space with respect to $\|\cdot\|_{-1,\lambda,\mathcal{L}}$. An equivalent expression for $\|f\|_{-1,\lambda,\mathcal{L}}$, given in the next result, is proved in p. 46-47 Olla (1994).

Proposition 3.1. *For $f \in L^2(Q_\rho)$ and $\lambda > 0$, we have*

$$\begin{aligned}\|f\|^2_{-1,\lambda,L} &= \langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho \\ &= \sup_{g \text{ local}} \Big\{ 2\langle f, g\rangle_\rho - \langle g, (\lambda - \mathcal{S})g\rangle_\rho - \langle \mathcal{A}g, (\lambda - \mathcal{S})^{-1}\mathcal{A}g\rangle_\rho \Big\} \\ &= \inf_{g \text{ local}} \Big\{ \langle f - \mathcal{A}g, (\lambda - \mathcal{S})^{-1}(f - \mathcal{A}g)\rangle_\rho + \langle g, (\lambda - \mathcal{S})g\rangle_\rho \Big\}.\end{aligned}$$

Hence, when $\mathcal{L} = \mathcal{S}$ is symmetric, we have for f local, $\|f\|^2_{1,\lambda,\mathcal{S}} = \langle f, (\lambda - \mathcal{S})f\rangle_\rho$ and $\|f\|^2_{-1,\lambda,\mathcal{S}} = \langle f, (\lambda - \mathcal{S})^{-1} f\rangle_\rho$. In this context, it will be useful to define corresponding H_1 and H_{-1} "coefficient" norms, that is, $\|\mathfrak{f}\|^2_{1,\lambda,\mathfrak{S}} = \langle \mathfrak{f}, (\lambda - \mathfrak{S})\mathfrak{f}\rangle = \|f\|^2_{1,\lambda,\mathcal{S}}$, and $\|\mathfrak{f}\|^2_{-1,\lambda,\mathfrak{S}} = \sup_{\mathfrak{g} \text{ local}}\{2\langle \mathfrak{f}, \mathfrak{g}\rangle - \|\mathfrak{g}\|^2_{1,\lambda,\mathfrak{S}}\} = \|f\|^2_{-1,\lambda,\mathcal{S}}$.

Also, in the following, it will be convenient to denote, when B and its coefficient $\mathcal{B}$ are symmetric exclusion-type operators, that $\|f\|^2_{1,\lambda,B} = \langle f, (\lambda - B)f\rangle_\rho = \langle \mathfrak{f}, (\lambda - \mathcal{B})\mathfrak{f}\rangle = \|\mathfrak{f}\|^2_{1,\lambda,\mathcal{B}}$ and $\|f\|_{-1,\lambda,B} = \sup_{g \text{ local}}\{2\langle f, g\rangle_\rho - \|g\|^2_{1,\lambda,B}\} = \sup_{\mathfrak{g} \text{ local}}\{2\langle \mathfrak{f}, \mathfrak{g}\rangle - \|\mathfrak{g}\|^2_{1,\lambda,\mathcal{B}}\} = \|\mathfrak{f}\|^2_{-1,\lambda,\mathcal{B}}$.

3.3. *Some Variance Bounds and Comparisons.* For a real local mean-zero function f, $E_\rho[f] = 0$, denote the variance

$$\sigma_t^2(f) = E_\rho\left[\left(\int_0^t f(\zeta(s))ds\right)^2\right].$$

A well known upperbound on $\sigma_t^2(f)$, which connects with H_{-1} norms, and proved say in Proposition 6.1, appendix 1 Kipnis and Landim (1999), is given in the next statement.

Proposition 3.2. *There is a universal constant C_1 such that for $t \geq 0$,*

$$\sigma_t^2(f) \leq C_1 t \langle f, (t^{-1} - \mathcal{L})^{-1} f\rangle_\rho .$$

We now compare $\langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho$ with other quadratic forms depending on the dimension d. Let $\mathcal{L}_{\mathrm{nn}}$ be the reference process generator corresponding to nearest-neighbor jump rates p_{nn} supported on standard vectors $\{\pm e_l\}$ of $\mathbb{Z}^d$ where

$$p_{\mathrm{nn}}(\pm e_l) = \begin{cases} \max[\pm e_l \cdot \sum jp(j), 0] & \text{when } \pm e_l \cdot \sum jp(j) \neq 0 \\ 1 & \text{when } \pm e_l \cdot \sum jp(j) = 0 \end{cases}$$

for $1 \leq l \leq d$, and $p_{\mathrm{nn}}(z) = 0$ for $|z| \neq 1$. Note that $s_{\mathrm{nn}}(z) = (p_{\mathrm{nn}}(z) + p_{\mathrm{nn}}(-z))/2 > 0$ for $|z| = 1$.

When $d = 1$, define also operator $\mathcal{N}$ on local functions f by

$$(\mathcal{N}f)(\zeta) = f(\zeta^{-1,1}) - \mathfrak{f}(\zeta), \tag{3.1}$$

that is, the symmetric exchange operator on bond connecting -1 and 1. Its coefficient operator $\mathfrak{N}$ defined on local functions $\mathfrak{f}$ is then $(\mathfrak{N}\mathfrak{f})(B) = \mathfrak{f}(B_{-1,1}) - \mathfrak{f}(B)$.

The next proposition, which indicates the H_{-1} norm with respect to $\mathcal{L}$ is on the same order as that for a nearest-neighbor dynamics with the same drift, is Theorem 2.1 Sethuraman (2003) for $d \geq 2$ and proved by the proof of Theorem 2.2 Sethuraman (2003) for $d = 1$ (cf. Lemma 3.5 and p. 50 Sethuraman (2003)).

Proposition 3.3. *We have a constant $C = C(d,p)$, such that for $\lambda > 0$ and local f in $d \geq 2$,*

$$C^{-1}\langle f, (\lambda - \mathcal{L}_{\mathrm{nn}})^{-1} f\rangle_\rho \leq \langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho \leq C\langle f, (\lambda - \mathcal{L}_{\mathrm{nn}})^{-1} f\rangle_\rho,$$

and in $d = 1$,

$$C^{-1}\langle f, (\lambda - \mathcal{L}_{\rm nn} - \mathcal{N})^{-1} f\rangle_\rho \ \leq \ \langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho \ \leq \ C\langle f, (\lambda - \mathcal{L}_{\rm nn} - \mathcal{N})^{-1} f\rangle_\rho.$$

Let $\mathcal{S}_{\rm nn}$ and $\mathcal{A}_{\rm nn}$ be the symmetric and anti-symmetric parts of $\mathcal{L}_{\rm nn} = \mathcal{S}_{\rm nn} + \mathcal{A}_{\rm nn}$. Let also $\mathcal{S}^e_{\rm nn}$ and $\mathcal{S}^t_{\rm nn}$ be the "environment" and "tagged-shift" parts of $\mathcal{S}_{\rm nn} = \mathcal{S}^e_{\rm nn} + \mathcal{S}^t_{\rm nn}$. We denote also by $\mathfrak{A}_{\rm nn}$ and $\mathfrak{S}^e_{\rm nn}$ the respective coefficients of $\mathcal{A}_{\rm nn}$ and $\mathcal{S}^e_{\rm nn}$.

Recall the H_1 and H_{-1} norm expressions $\|\cdot\|_{\pm,\lambda,\mathcal{B}}$ for symmetric operators $\mathcal{B}$ at the end of subsection 3.2. The following bound allows us to bound H_1 and H_{-1} norms of the non-local "tagged-shift" operator $\mathcal{S}^t_{\rm nn}$ in terms of the more manageable "environment" operator $\mathcal{S}^e_{\rm nn}$. The proof is postponed to the last subsection of this section.

Proposition 3.4. *We have a constant* $C = C(n, p)$ *such that for* $\lambda > 0$ *and local* f *with degree* n *in* $d \geq 2$,

$$\|f\|_{1,\lambda,\mathcal{S}^e_{\rm nn}} \ \leq \ \|f\|_{1,\lambda,\mathcal{S}_{\rm nn}} \ \leq \ C\|f\|_{\mathcal{S}^e_{\rm nn}},$$

and so consequently,

$$C^{-1}\|f\|_{-1,\lambda,\mathcal{S}^e_{\rm nn}} \ \leq \ \|f\|_{-1,\lambda,\mathcal{S}_{\rm nn}} \ \leq \|f\|_{-1,\lambda,\mathcal{S}^e_{\rm nn}}.$$

In $d = 1$, *the inequalities hold with* $\mathcal{S}^e_{\rm nn}$ *replaced by* $\mathcal{S}^e_{\rm nn} + \mathcal{N}$.

3.4. *"Extended" Coefficient Functions.* To aid later computations, we now extend the underlying space $\mathbb{Z}^d \setminus \{0\}$ to $\mathbb{Z}^d$. We concentrate on dimension $d \leq 2$ for simplicity. Let $\bar{\mathcal{E}}_d$ be the set of finite subsets of $\mathbb{Z}^d$, and let $\bar{\mathcal{E}}_{d,n}$ be those subsets of $\mathbb{Z}^d$ with cardinality n. Let also $\bar{\mathcal{C}}_{d,n}$ denote the collection of functions on $\bar{\mathcal{E}}_{d,n}$.

For $n \leq 2$, let $\mathfrak{f} \in \mathcal{C}_{d,n}$ be a coefficient function. We now give extensions $\mathfrak{f}_{\rm ext}$ and $\mathfrak{f}_\odot$ belonging to $\bar{\mathcal{C}}_{d,n}$; we also give an "inverse" of the $\odot$ extension, namely $\mathfrak{g}_{\rm res}$, which restricts $\mathfrak{g} \in \bar{\mathcal{C}}_{d,n}$ to $\mathcal{C}_{d,n}$. In addition, we define some related operators, an innerproduct, and norms, acting on these functions.

Extension $\mathfrak{f}_{\rm ext}$. This extension assigns to sets $B \ni 0$ the "local" average of "nearest-neighbor" sets and is well suited for later comparisons of Dirichlet forms over $\mathbb{Z}^d \setminus \{0\}$ and $\mathbb{Z}^d$ (cf. Proposition 3.6). More precisely, when $n = 1$, let

$$\mathfrak{f}_{\rm ext}(\{x\}) \ = \ \begin{cases} \mathfrak{f}(\{x\}) & \text{for } x \in \mathbb{Z}^d \setminus \{0\} \\ \frac{1}{2d}\sum_{|z|=1} \mathfrak{f}(\{z\}) & \text{for } x = 0. \end{cases}$$

When $n = 2$, for distinct $x, y \in \mathbb{Z}^d \setminus \{0\}$, let $\mathfrak{f}_{\rm ext}(\{x, y\}) = \mathfrak{f}(\{x, y\})$, and

$$\mathfrak{f}_{\rm ext}(\{0, y\}) \ = \ \begin{cases} \frac{1}{2d-1}\sum_{\substack{z \neq y \\ |z|=1}} f(\{z, y\}) & \text{when } |y| = 1 \\ \frac{1}{2d}\sum_{|z|=1} \mathfrak{f}(\{z, y\}) & \text{when } |y| \geq 2. \end{cases}$$

Extension $\mathfrak{f}_\odot$. This type of extension vanishes on sets involving the origin and allows H_{-1} norm comparisons over $\mathbb{Z}^d \setminus \{0\}$ and $\mathbb{Z}^d$ (cf. Proposition 3.6). Let

$$\mathfrak{f}_\odot(B) = \begin{cases} \mathfrak{f}(B) & \text{when } B \in \mathcal{E}_{d,n} \\ 0 & \text{otherwise.} \end{cases}$$

Restriction $\mathfrak{g}_{\rm res}$. For $\mathfrak{g} \in \bar{\mathcal{C}}_{d,n}$, let $\mathfrak{g}_{\rm res} \in \mathcal{C}_{d,n}$ be the restriction of $\mathfrak{g}$ to subsets $B \in \mathcal{E}_{d,n}$. This restriction is useful in extending operators with respect to $\mathbb{Z}^d \setminus \{0\}$ to underlying space $\mathbb{Z}^d$ (cf. definition of $\bar{\mathfrak{A}}_{{\rm nn};n,m}$ below).

Operator $\mathfrak{S}_{\rm ext}$. Recall operator $\mathcal{S}^e_{\rm nn}$ and its coefficient form $\mathfrak{S}^e_{\rm nn}$ from subsection 3.3. We now extend $\mathfrak{S}^e_{\rm nn}$ on local $\mathcal{C}_{d,n}$ functions to $\mathfrak{S}_{\rm ext}$ acting on local $\bar{\mathcal{C}}_{d,n}$ functions in the usual way, namely transitions are now allowed into the origin. Define the nearest-neighbor operator, acting on local $\mathfrak{g} \in \bar{\mathcal{C}}_{d,n}$, by

$$(\mathfrak{S}_{\rm ext}\mathfrak{g})(B) \;=\; \sum_{\substack{|i-j|=1\\ i,j\in\mathbb{Z}^d}} \Big(\mathfrak{g}(B_{i,j}) - \mathfrak{g}(B)\Big).$$

Operator $\bar{\mathfrak{A}}_{{\rm nn};n,m}$. Recall operator $\mathcal{A}_{\rm nn}$ and its coefficient form $\mathfrak{A}_{\rm nn}$ in subsection 3.3. With respect to $\mathfrak{A}_{{\rm nn};n,m}$, the part of $\mathfrak{A}_{\rm nn}$ which takes degree n functions to degree m, define on local $\mathfrak{g} \in \bar{\mathcal{C}}_{d,n}$ that

$$(\bar{\mathfrak{A}}_{{\rm nn};n,m}\mathfrak{g})(B) \;=\; \begin{cases} (\mathfrak{A}_{{\rm nn};n,m}\mathfrak{g}_{\rm res})(B) & \text{when } B \subset \mathbb{Z}^d \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Extended Innerproduct and Norms. The innerproduct naturally extends to L^2 functions in $\bar{\mathcal{C}}_{d,n}$:

$$\langle \mathfrak{f}, \mathfrak{g}\rangle_{\rm ext} \;=\; \sum_{\substack{|B|=n\\ B\subset\mathbb{Z}^d}} \mathfrak{f}(B)\mathfrak{g}(B).$$

Also, H_1 and H_{-1} norms of $\mathfrak{f} \in \bar{\mathcal{C}}_{d,n}$, with respect to $\mathfrak{S}_{\rm ext}$, are defined for $\lambda > 0$:

$$\begin{aligned} \|\mathfrak{f}\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} &= \langle \mathfrak{f}, (\lambda - \mathfrak{S}_{\rm ext})\mathfrak{f}\rangle_{\rm ext} \\ &= \lambda\langle \mathfrak{f},\mathfrak{f}\rangle_{\rm ext} + \frac{1}{2}\sum_{\substack{|B|=n\\ B\subset\mathbb{Z}^d}}\sum_{\substack{|i-j|=1\\ i,j\in\mathbb{Z}^d}} (\mathfrak{f}(B_{i,j}) - \mathfrak{f}(B))^2 \qquad (3.2)\\ \|\mathfrak{f}\|^2_{-1,\lambda,\mathfrak{S}_{\rm ext}} &= \sup_{\mathfrak{g} \text{ local on } \bar{\mathcal{E}}_d} \big\{2\langle \mathfrak{f},\mathfrak{g}\rangle_{\rm ext} - \langle \mathfrak{g}, (\lambda - \mathfrak{S}_{\rm ext})\mathfrak{g}\rangle_{\rm ext}\big\}. \end{aligned}$$

In addition, we have the following useful bounds which relate further the various extensions.

Lemma 3.5. *For $0 < \lambda \leq 1$, we have a constant $\mathcal{C}(d)$ such that for $\mathfrak{g} \in \mathcal{C}_{d,1}$ and any extension $\mathfrak{g}' \in \bar{\mathcal{C}}_{d,1}$,*

$$\|\mathfrak{g}_{\rm ext}\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} \;\leq\; C\Big[\|\mathfrak{g}'\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} + |\mathfrak{g}'(\{0\}) - \sum_{|z|=1}\mathfrak{g}'(\{z\})|^2\Big].$$

Proof. Note first $\mathfrak{g}_{\rm ext} = \mathfrak{g}' + [\sum_{|z|=1}\mathfrak{g}'(\{z\}) - \mathfrak{g}'(\{0\})]\omega_0$ where $\omega_0 \in \bar{\mathcal{C}}_{d,1}$ and $\omega_0(\{x\}) = 1$ for $x = 0$ and vanishes otherwise. Then,

$$\|\mathfrak{g}_{\rm ext}\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} \leq 2\|\mathfrak{g}'\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} + 2\big[\sum_{|z|=1}\mathfrak{g}'(\{z\}) - \mathfrak{g}'(\{0\})\big]^2\|\omega_0\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}}.$$

By calculation, using (3.2), $\|\omega_0\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}} \leq \lambda + C$ and so the result follows. □

Recall symmetric operators $\mathcal{S}^e_{\rm nn}$ and $\mathcal{N}$, and their coefficients $\mathfrak{S}^e_{\rm nn}$ and $\mathfrak{N}$ from subsection 3.3, and H_1 and H_{-1} norm expressions $\|\cdot\|_{\pm,\lambda,\mathcal{B}}$ for symmetric operators $\mathcal{B}$ at the end of subsection 3.2.

Proposition 3.6. *For $n \leq 2$ and $\lambda > 0$, we have a constant $C = C(d,n,p)$ such that for $\mathfrak{f} \in \mathcal{C}_{d,n}$ in $d = 2$,*

$$C^{-1}\|\mathfrak{f}\|_{1,\lambda,\mathfrak{S}^e_{\rm nn}} \;\leq\; \|\mathfrak{f}_{\rm ext}\|_{1,\lambda,\mathfrak{S}_{\rm ext}} \;\leq\; C\|\mathfrak{f}\|_{1,\lambda,\mathfrak{S}^e_{\rm nn}} \qquad (3.3)$$

and

$$\|\mathfrak{f}\|_{-1,\lambda,\mathfrak{S}^e_{\rm nn}} \ \leq \ C\|\mathfrak{f}_\odot\|_{-1,\lambda,\mathfrak{S}_{\rm ext}} \ .$$

In $d = 1$, the inequalities hold with operator $\mathfrak{S}^e_{\rm nn}$ replaced by $\mathfrak{S}^e_{\rm nn} + \mathfrak{N}$.

We postpone the proof to the last subsection of this section.

3.5. *"Free Particle" Bounds.* For later detailed analysis, it will be helpful to "remove the hard-core exclusion." In other words, we want to get equivalent bounds in terms of operators which govern completely independent or "free" motions. We follow the treatment of Bernardin (2004) with respect to occupation times.

"Free Particle" Generator $\mathfrak{S}_{\rm free}$. Let $\upsilon_{d,n} = (\mathbb{Z}^d)^n$ and consider n independent random walks with symmetric nearest-neighbor symmetric jump rates on $\mathbb{Z}^d$ for $d \geq 1$. The process $x_t = (x^1_t, \ldots, x^n_t)$ evolves on $\upsilon_{d,n}$ and has generator $\mathfrak{S}_{\rm free}$ acting on local, namely finitely supported, functions on $\upsilon_{n,d}$,

$$(\mathfrak{S}_{\rm free}\mathfrak{f})(x) \ = \ \frac{1}{2d}\sum_{\substack{1\leq j\leq n\\ |z|=1}}\Big(\phi(x+z\omega_j) - \phi(x)\Big)$$

where $z\omega_j = (0,\ldots,0,z,0,\ldots,0)$ is the state with z in the jth place.

Free Innerproduct and Norms. With respect to local functions on $\upsilon_{d,n}$, define

$$\langle\phi,\psi\rangle_{\rm free} = \frac{1}{n!}\sum_{x\in\upsilon_{d,n}}\phi(x)\psi(x).$$

Define also, for $\lambda > 0$, $H_{1,\lambda}$ and $H_{-1,\lambda}$ norms $\|\phi\|^2_{1,\lambda,{\rm free}} = \langle\phi,(\lambda - \mathfrak{S}_{\rm free})\phi\rangle_{\rm free}$ and

$$\|\phi\|^2_{-1,\lambda,{\rm free}} \ = \ \sup_{\psi \text{ local on } \upsilon_{d,n}}\big\{2\langle\phi,\psi\rangle_{\rm free} - \|\psi\|^2_{1,\lambda,{\rm free}}\big\}.$$

Extension $\mathfrak{f}_{\rm free}$. Let $\mathcal{G}_n \subset \upsilon_{d,n}$ be those points whose coordinates are distinct. For a function $\mathfrak{f} \in \bar{\mathcal{C}}_{d,n}$, define the natural extension to $\upsilon_{d,n}$ by

$$\mathfrak{f}_{\rm free}(x) \ = \ \mathfrak{f}(U)$$

where U is the set formed from coordinates of $x \in \upsilon_{d,n}$. Note $\mathfrak{f}_{\rm free}$ is supported on $\mathcal{G}_n$.

Extension $\tilde{\mathfrak{f}}$. We now give an extension $\tilde{\mathfrak{f}}$ on $\upsilon_{d,n}$ which allows some H_1 and H_{-1} norm comparsions (cf. Proposition 3.7). Let τ be the arrival time into $\mathcal{G}_n$,

$$\tau \ = \ \inf\big\{t \geq 0 : x_t \in \mathcal{G}_n\big\}.$$

Then, for $\mathfrak{f} \in \bar{\mathcal{C}}_{d,n}$, define for $x \in \upsilon_{n,d}$ that

$$\tilde{\mathfrak{f}}(x) \ = \ E_x[\mathfrak{f}_{\rm free}(x_\tau)].$$

Free Bounds and Relations. The next result relates $\mathfrak{S}_{\rm ext}$ and $\mathfrak{S}_{\rm free}$ with respect to H_1 and H_{-1} norms of $\mathfrak{f}$ and $\tilde{\mathfrak{f}}$, and is a part of Theorems 3.1 and 3.2 Bernardin (2004).

Proposition 3.7. *We have, for a constant $C = C(d,n,p)$, $\lambda > 0$, and $\mathfrak{f} \in \bar{\mathcal{C}}_{d,n}$, that*

$$C^{-1}\|\tilde{\mathfrak{f}}\|_{1,\lambda,{\rm free}} \ \leq \ \|\mathfrak{f}\|_{1,\lambda,\mathfrak{S}_{\rm ext}} \ \leq C\|\tilde{\mathfrak{f}}\|_{1,\lambda,{\rm free}}.$$

Also,

$$\|\mathfrak{f}\|_{-1,\lambda,\mathfrak{S}_{\rm ext}} \ \leq\ C\|1_{\mathcal{G}_n}\tilde{\mathfrak{f}}\|_{-1,\lambda,{\rm free}}.$$

The following relations, which follow from straightforward manipulations, will also be useful.

Lemma 3.8. *Let $\mathfrak{g}\in\mathcal{C}_{d,1}$ be a local function, and let $\mathfrak{g}'\in\bar{\mathcal{C}}_{d,1}$ be any extension. Then, for $x\in\mathbb{Z}^d$*

$$\widetilde{\mathfrak{g}'}(x) \ =\ \mathfrak{g}'_{\rm free}(x), \quad \text{and } 1_{\mathcal{G}_1}(\widetilde{\mathfrak{A}_{nn;1,1}\mathfrak{g}})_\odot(x) \ =\ (\bar{\mathfrak{A}}_{nn;1,1}\mathfrak{g}')_{\rm free}(x).$$

Also, for $x,y\in\mathbb{Z}^d$,

$$1_{\mathcal{G}_2}(\widetilde{\mathfrak{A}_{nn;1,2}\mathfrak{g}})_\odot(x,y) \ =\ (\bar{\mathfrak{A}}_{nn;1,2}\mathfrak{g}')_{\rm free}(x,y).$$

Fourier Transform Expressions. It will be convenient to express "free" $H_{1,\lambda}$ and $H_{-1,\lambda}$ norms in terms of Fourier transforms. Let ψ be a local function on $\upsilon_{d,n}$ and let $\widehat{\psi}$ be its Fourier transform

$$\widehat{\psi}(s_1,\ldots,s_n) = \frac{1}{\sqrt{n!}}\sum_{x\in\upsilon_{d,n}} e^{2\pi i(x_1\cdot s_1+\cdots+x_n\cdot s_n)}\psi(x)$$

where $s_1,\ldots,s_n\in[0,1]^d$. Compute

$$\widehat{\mathfrak{S}_{\rm free}\psi}(s_1,\ldots,s_n) \ =\ -\Big[\sum_{j=1}^n\theta_d(s_j)\Big]\widehat{\psi}(s_1,\ldots,s_n)$$

where $\theta_d(u) = (2/2d)\sum_{\substack{z\in\mathbb{Z}^d\\|z|=1}}\sin^2(\pi(u\cdot z)) = (2/d)\sum_{j=1}^n\sin^2(\pi u_j)$. Hence, we have

$$\|\psi\|^2_{1,\lambda,{\rm free}} \ =\ \int_{\substack{s\in([0,1]^d)^n\\ s=(s_1,\ldots,s_n)}}\Big(\lambda+\sum_{j=1}^n\theta_d(s_j)\Big)|\widehat{\psi}(s_1,\ldots,s_n)|^2ds$$

and

$$\|\psi\|^2_{-1,\lambda,{\rm free}} \ =\ \int_{\substack{s\in([0,1]^d)^n\\ s=(s_1,\ldots,s_n)}}\frac{|\widehat{\psi}(s_1,\ldots,s_n)|^2}{\lambda+\sum_{j=1}^n\theta_d(s_j)}ds.$$

3.6. *Putting Bounds Together.* We now incorporate the previous bounds into a single statement.

Proposition 3.9. *In $d\leq 2$, for local degree one functions $f\in\mathcal{C}_{d,1}$, we have a constant $C=C(d,p,\rho)$ such that for $t\geq 1$,*

$$\sigma_t^2(f)/t \ \leq\ C\inf_{\substack{\mathfrak{g}\in\bar{\mathcal{C}}_{d,1},\\ \rm local}}\Big\{\|(\mathfrak{f}_\odot)_{\rm free}-(\bar{\mathfrak{A}}_{\rm nn;1,1}\mathfrak{g})_{\rm free}\|^2_{-1,t^{-1},{\rm free}}+\|\mathfrak{g}_{\rm free}\|^2_{1,t^{-1},{\rm free}}$$
$$+\|(\bar{\mathfrak{A}}_{\rm nn;1,2}\mathfrak{g})_{\rm free}\|^2_{-1,t^{-1},{\rm free}}+|\mathfrak{g}_{\rm free}(0)-\sum_{|z|=1}\mathfrak{g}_{\rm free}(z)|^2\Big\}.$$

Proof. In the following, the constant $C = C(d,p,\rho)$ can change from line to line. We have, in sequence, from Propositions 3.2, 3.3, 3.1, 3.4 and 3.6, when $d=2$ that

$$\begin{aligned}
\sigma_t^2(f)/t &\le C\langle f, (t^{-1}-\mathcal{L})^{-1} f\rangle_\rho \\
&\le C\langle f, (t^{-1}-\mathcal{L}_{\rm nn})^{-1} f\rangle_\rho \\
&= C \inf_{g \text{ local}} \{\|f-\mathcal{A}_{\rm nn} g\|^2_{-1,t^{-1},\mathcal{S}_{\rm nn}} + \|g\|^2_{1,t^{-1},\mathcal{S}_{\rm nn}}\} \\
&\le C \inf_{g \text{ local}} \left\{\|f-\mathcal{A}_{\rm nn} g\|^2_{-1,t^{-1},\mathcal{S}^e_{\rm nn}} + \|g\|^2_{1,t^{-1},\mathcal{S}^e_{\rm nn}}\right\} \\
&\le C \inf_{\mathfrak{g} \text{ local}} \left\{\|\mathfrak{f}_\odot - (\mathfrak{A}_{\rm nn}\mathfrak{g})_\odot\|^2_{-1,t^{-1},\mathfrak{S}_{\rm ext}} + \|\mathfrak{g}_{\rm ext}\|^2_{1,t^{-1},\mathfrak{S}_{\rm ext}}\right\}.
\end{aligned}$$

When $d=1$, in the fourth line of the sequence above, $\mathcal{S}^e_{\rm nn}$ is replaced by $\mathcal{S}^e_{\rm nn}+\mathcal{N}$.

The last infimum, by first restricting to $\mathfrak{g}\in\mathcal{C}_{d,1}$ and Schwarz inequality, second using Lemma 3.5 to estimate $\|\mathfrak{g}_{\rm ext}\|^2_{1,t^{-1},\mathfrak{S}_{\rm ext}}$ in terms of $\|\mathfrak{g}'\|^2_{1,t^{-1},\mathfrak{S}_{\rm ext}}$ for $\mathfrak{g}'\in\bar{\mathcal{C}}_{d,1}$, and then third applying Proposition 3.7 and Lemma 3.8 to estimate in terms of "free" norms on local functions in $\bar{\mathcal{C}}_{d,1}$, is further bounded by twice

$$\begin{aligned}
&\inf_{\substack{\mathfrak{g}\in\mathcal{C}_{d,1},\\ \text{local}}} \Big\{\|\mathfrak{f}_\odot - (\mathfrak{A}_{{\rm nn};1,1}\mathfrak{g})_\odot\|^2_{-1,t^{-1},\mathfrak{S}_{\rm ext}} \\
&\qquad\qquad + \|(\mathfrak{A}_{{\rm nn};1,2}\mathfrak{g})_\odot\|^2_{-1,t^{-1},\mathfrak{S}_{\rm ext}} + \|\mathfrak{g}_{\rm ext}\|^2_{1,t^{-1},\mathfrak{S}_{\rm ext}}\Big\} \\
&\le C \inf_{\substack{\mathfrak{g}'\in\bar{\mathcal{C}}_{d,1},\\ \text{local}}} \Big\{\|(\mathfrak{f}_\odot)_{\rm free} - (\bar{\mathfrak{A}}_{{\rm nn};1,1}\mathfrak{g}')_{\rm free}\|^2_{-1,t^{-1},{\rm free}} \\
&\qquad + \|(\bar{\mathfrak{A}}_{{\rm nn};1,2}\mathfrak{g}')_{\rm free}\|^2_{-1,t^{-1},{\rm free}} \\
&\qquad + \|\mathfrak{g}'_{\rm free}\|^2_{1,t^{-1},{\rm free}} + |\mathfrak{g}'_{\rm free}(0) - \sum_{|z|=1}\mathfrak{g}'_{\rm free}(z)|^2\Big\}.
\end{aligned}$$

□

3.7. *Proofs of Propositions 3.4 and 3.6.*

Proof of Proposition 3.4. The H_1 lower bound follows as

$$\langle f, (-\mathcal{S}_{\rm nn})f\rangle_\rho = \langle f, (-\mathcal{S}^e_{\rm nn})f\rangle_\rho + \langle f, (-\mathcal{S}^t_{\rm nn})f\rangle_\rho$$

and

$$\langle f, (-\mathcal{S}^t_{\rm nn})f\rangle_\rho = \frac{1}{2}\sum_{|z|=1} s_{\rm nn}(z) E_\rho[(1-\zeta_z)(f(\tau_z\zeta)-f(\zeta))^2] \ge 0.$$

For the H_1 upper bound, note

$$\begin{aligned}
E_\rho[(1-\zeta_z)(f(\tau_z\zeta)-f(\zeta))^2] &\le E_\rho[(f(\tau_z\zeta)-f(\zeta))^2] \\
&= \sum_{\substack{B\subset\mathbb{Z}^d\setminus\{0\}\\ |B|=n}} (\mathfrak{f}(\tau_{-z}B)-\mathfrak{f}(B))^2,
\end{aligned}$$

and by the proof of Lemma 5.1 Landim, Olla and Varadhan (2002),

$$\sum_{\substack{B\subset\mathbb{Z}^d\setminus\{0\}\\ |B|=n}} (\mathfrak{f}(\tau_{-z}B)-\mathfrak{f}(B))^2 \le C_z n \sum_{\substack{B\subset\mathbb{Z}^d\setminus\{0\}\\ |B|=n}} \sum_{i\sim j} (\mathfrak{f}(B_{i,j})-\mathfrak{f}(B))^2 \tag{3.4}$$

where C_z is a constant depending on z, and $i \sim j$ means a "neighboring" pair $i, j \in \mathbb{Z}^d \setminus \{0\}$ with $|i-j| = 1$, or also $(i,j) = (1,-1)$ and $(-1,1)$ when $d = 1$. Also

$$\sum_{\substack{B \subset \mathbb{Z}^d \setminus \{0\} \\ |B|=n}} \sum_{i \sim j} (\mathfrak{f}(B_{i,j}) - \mathfrak{f}(B))^2 \;\leq\; \begin{cases} C' \langle f, (-\mathcal{S}^e_{\rm nn}) f\rangle_\rho & \text{when } d \geq 2 \\ C' \langle f, (-\mathcal{S}^e_{\rm nn} - \mathcal{N}) f)\rangle_\rho & \text{when } d = 1 \end{cases}$$

where $C' = C'(s_{\rm nn})$. The H_1 estimates in the proposition follow now by adding over $|z| = 1$. Also, the H_{-1} bounds are deduced from the H_1 bounds through simple estimates with the definition of $\|f\|^2_{-1,\lambda,\mathcal{B}}$ (cf. subsection 3.2). □

Proof of Proposition 3.6. We prove the statement for $d = 2$, and mention at the end modifications for $d = 1$. In the following, $C = C(n,p)$ denotes a constant which can change from line to line. The lowerbound inequality in (3.3) follows from overcounting:

$$\begin{aligned} \langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle &= \frac{1}{2} \sum_{\substack{B \subset \mathbb{Z}^2 \setminus \{0\} \\ |B|=n}} \sum_{\substack{|i-j|=1 \\ i,j \neq 0}} (\mathfrak{f}(B_{i,j}) - \mathfrak{f}(B))^2 s_{\rm nn}(j-i) \\ &\leq \frac{C}{2} \sum_{\substack{B \subset \mathbb{Z}^2 \\ |B|=n}} \sum_{\substack{|i-j|=1 \\ i,j \in \mathbb{Z}^2}} (\mathfrak{f}_{\rm ext}(B_{i,j}) - \mathfrak{f}_{\rm ext}(B))^2 \;=\; C\langle f_{\rm ext}, (-\mathfrak{S}_{\rm ext})\mathfrak{f}_{\rm ext}\rangle_{\rm ext}; \end{aligned}$$

also, we have $\|\mathfrak{f}\|_0^2 \leq \langle \mathfrak{f}_{\rm ext}, \mathfrak{f}_{\rm ext}\rangle_{\rm ext}$.

For the upperbound in (3.3), as $s_{\rm nn}(z) > 0$ for $|z| = 1$, we have

$$\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle \;\geq\; C \sum_{\substack{B \subset \mathbb{Z}^2 \setminus \{0\} \\ |B|=n}} \sum_{\substack{|i-j|=1 \\ i,j \in \mathbb{Z}^2 \setminus \{0\}}} (\mathfrak{f}(B_{i,j}) - \mathfrak{f}(B))^2$$

and so

$$\begin{aligned} \langle f_{\rm ext}, (-\mathfrak{S}_{\rm ext})\mathfrak{f}_{\rm ext}\rangle_{\rm ext} \;\leq\; & C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle \\ & + C \sum_{\substack{|i-j|=1 \\ i,j \in \mathbb{Z}^2}} \sum_{\substack{B \text{ or } B_{i,j} \ni 0 \\ |B|=n}} (\mathfrak{f}_{\rm ext}(B_{i,j}) - \mathfrak{f}_{\rm ext}(B))^2. \end{aligned} \tag{3.5}$$

When $n = 1$, the last term of (3.5) is on order

$$\sum_{|z|=1} (\mathfrak{f}_{\rm ext}(\{0\}) - \mathfrak{f}_{\rm ext}(\{z\}))^2 \;=\; \frac{1}{16} \sum_{|w|,|z|=1} (\mathfrak{f}(\{w\}) - \mathfrak{f}(\{z\}))^2 \;\leq\; C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle.$$

Here, for the last inequality, we build a path from $w_0 = e_1$ to $w_1 = e_1 + e_2$ to $w_2 = e_2$ and so on to $w_7 = e_1 - e_2$ back to $w_8 = e_1$, and bound each of the finite number of terms $(\mathfrak{f}(\{w\}) - \mathfrak{f}(\{z\}))^2 \leq 8 \sum_{i=0}^7 (\mathfrak{f}(\{w_i\}) - \mathfrak{f}(\{w_{i+1}\}))^2 \leq C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle$.

When $n = 2$, the last sum in (3.5) is on order

$$\begin{aligned} & \sum_{y \neq 0} \Bigg[\sum_{\substack{|z|=1 \\ y+z \neq 0}} (\mathfrak{f}_{\rm ext}(\{0, y+z\}) - \mathfrak{f}_{\rm ext}(\{0,y\}))^2 + \sum_{\substack{|z|=1 \\ z \neq y}} (\mathfrak{f}_{\rm ext}(\{z,y\}) - \mathfrak{f}_{\rm ext}(\{0,y\}))^2 \Bigg] \\ &= \sum_{|y| \geq 2} \sum_{|z|=1} \Big[(\mathfrak{f}_{\rm ext}(\{0, y+z\}) - \mathfrak{f}_{\rm ext}(\{0,y\}))^2 + (\mathfrak{f}_{\rm ext}(\{z,y\}) - \mathfrak{f}_{\rm ext}(\{0,y\}))^2 \Big] \\ &\qquad + \text{finite number of remaining terms.} \end{aligned}$$

The first line is straightforwardly bounded by $C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle$. The remaining finite number of terms are handled as follows: For $|y| = |z| = 1$, the terms with $y + z \neq 0$

are bounded

$$\begin{aligned}\Big(\mathfrak{f}_{\rm ext}(\{0,y+z\}) - \mathfrak{f}_{\rm ext}(\{0,y\})\Big)^2 &= \Big(\frac{1}{4}\sum_{|x|=1}\mathfrak{f}(\{x,y+z\}) - \frac{1}{3}\sum_{\substack{|w|=1\\ w\neq y}}\mathfrak{f}(\{w,y\})\Big)^2 \\ &\leq C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle\end{aligned}$$

and the terms with $|y| = |z| = 1$ and $z \neq y$ are bounded

$$\Big(\mathfrak{f}_{\rm ext}(\{z,y\}) - \mathfrak{f}_{\rm ext}(\{0,y\})\Big)^2 = \Big(\mathfrak{f}(\{z,y\}) - \frac{1}{3}\sum_{\substack{|w|=1\\ w\neq y}}\mathfrak{f}(\{w,y\})\Big)^2 \leq C\langle \mathfrak{f}, (-\mathfrak{S}^e_{\rm nn})\mathfrak{f}\rangle$$

through similar arguments using the path built in the $n = 1$ case.

Also, more directly, $\langle \mathfrak{f}_{\rm ext}, \mathfrak{f}_{\rm ext}\rangle_{\rm ext} \leq C\|\mathfrak{f}\|_0^2$ to finish the upperbounds in the first statement of the proposition.

For the second statement after (3.3), write

$$\begin{aligned}\|\mathfrak{f}\|^2_{-1,\lambda,\mathfrak{S}^e_{\rm nn}} &= \sup_{\phi \text{ local}}\big\{2\langle \mathfrak{f},\phi\rangle - \|\phi\|^2_{1,\lambda,\mathfrak{S}^e_{\rm nn}}\big\} \\ &= \sup_{\phi\in\mathcal{C}_{2,n} \text{ local}}\big\{2\langle \mathfrak{f},\phi\rangle - \|\phi\|^2_{1,\lambda,\mathfrak{S}^e_{\rm nn}}\big\}.\end{aligned}$$

The last step follows as for $\mathfrak{f} \in \mathcal{C}_{2,n}$ with $\phi = \sum_m \phi_m$ decomposed in degrees, $\langle \mathfrak{f}, \phi\rangle = \langle \mathfrak{f}, \phi_n\rangle$ and as $\mathfrak{S}^e_{\rm nn}$ preserves degrees, $\|\phi\|^2_{1,\lambda,\mathfrak{S}^e_{\rm nn}} = \sum_m \|\phi_m\|^2_{1,\lambda,\mathfrak{S}^e_{\rm nn}}$; so one does best by choosing $\phi = \phi_n$.

Continuing, as $\langle \mathfrak{f}, \phi\rangle = \langle \mathfrak{f}_\odot, \phi_{\rm ext}\rangle_{\rm ext}$ and using the proved lowerbound in (3.3), $\|\mathfrak{f}\|^2_{-1,\lambda,\mathfrak{S}^e_{\rm nn}}$ is bounded above by

$$\sup_{\phi\in\mathcal{C}_{2,n} \text{ local}}\big\{2\langle \mathfrak{f}_\odot, \phi_{\rm ext}\rangle_{\rm ext} - C^{-1}\|\phi_{\rm ext}\|^2_{1,\lambda,\mathfrak{S}_{\rm ext}}\big\} \leq C\|\mathfrak{f}_\odot\|^2_{-1,\lambda,\mathfrak{S}_{\rm ext}}.$$

The modifications for $d = 1$ take advantage of inequalities

$$\begin{aligned}\langle \mathfrak{f}, (-\mathfrak{N})\mathfrak{f}\rangle &= \frac{1}{2}\sum_{\substack{B\subset\mathbb{Z}\setminus\{0\}\\ |B|=n}}(\mathfrak{f}(B_{1,-1}) - \mathfrak{f}(B))^2 \\ &= \frac{1}{2}\sum_{\substack{B\subset\mathbb{Z}\setminus\{0\}\\ |B|=n}}(\mathfrak{f}_{\rm ext}(B_{1,-1}) - \mathfrak{f}_{\rm ext}(B))^2 \\ &\leq \frac{1}{2}\sum_{\substack{B\subset\mathbb{Z}\\ |B|=n}}(\mathfrak{f}_{\rm ext}(B_{1,-1}) - \mathfrak{f}_{\rm ext}(B))^2 \\ &\leq C\sum_{\substack{B\subset\mathbb{Z}\\ |B|=n}}\Big[(\mathfrak{f}_{\rm ext}(B_{1,0}) - \mathfrak{f}_{\rm ext}(B))^2 + (\mathfrak{f}_{\rm ext}(B_{0,-1}) - \mathfrak{f}_{\rm ext}(B))^2\Big]\end{aligned}$$

which hold as $B_{-1,1} = ((B_{1,0})_{0,-1})_{1,0}$ and by applying Schwarz inequality. The arguments are now similar to those in $d = 2$. □

4. Proof of Theorem 1.2

First, by (1.2) and that quadratic variation $E_\rho[|M(t)|^2] = (1-\rho)t\sum_j |j|^2 p(j) = O(t)$, we need only bound

$$E_\rho[|A(t)|^2] \leq \sum |j|^2\sigma_t^2(\rho - \zeta_j)p(j) = O(t).$$

Clearly, it is sufficient to show that $\sigma_t^2(\rho-\zeta_{j_0}) = O(t)$ for $j_0 \in \mathbb{Z}^2 \setminus \{0\}$ and $t \geq 1$.

To accomplish this, through Proposition 3.9, it will be useful to compute, for a local function $\mathfrak{g} \in \bar{\mathcal{C}}_{d,1}$, Fourier transforms $\widehat{(\bar{\mathfrak{A}}_{\mathrm{nn};1,1}\mathfrak{g})}_{\mathrm{free}}$ and $\widehat{(\bar{\mathfrak{A}}_{\mathrm{nn};1,2}\mathfrak{g})}_{\mathrm{free}}$ where $\bar{\mathfrak{A}}_{\mathrm{nn};n,m}$ are the nearest-neighbor operators defined in subsection 3.4. When $d=2$, let $a_1 = a_{\mathrm{nn}}(e_1)$ and $a_2 = a_{\mathrm{nn}}(e_2)$, and when $d=1$ let $a_1 = a(1)$. Note, by the assumption $\sum jp(j) \neq 0$, that $a_1^2 + a_2^2 > 0$ in $d=2$ and $|a_1| > 0$ in $d=1$.

Let $\gamma(r) = e^{2\pi i r} - e^{-2\pi i r} = 2i\sin(2\pi r)$ for $r \in [0,1]$. The following proposition is proved in section 5.

Proposition 4.1. *In $d \leq 2$, for local $\mathfrak{g} \in \bar{\mathcal{C}}_{d,1}$ and a constant $C = C(d,p,\rho)$,*

$$\widehat{(\bar{\mathfrak{A}}_{\mathrm{nn};1,1}\mathfrak{g})}_{\mathrm{free}}(v) \quad = \quad \rho\Bigg[\sum_{i=1}^d a_i\gamma(v_i)\Bigg]\widehat{\mathfrak{g}_{\mathrm{free}}}(v) + \delta_0(v)$$

where $|\delta_0(v)| \leq \kappa(v)\sum_{|z|\leq 1}|\mathfrak{g}_{\mathrm{free}}(z)|$ and $\kappa(v)$ is a bounded function such that

$$\kappa(v)^2 \;\leq\; C|v-z|^2$$

as $v \to z$ for $z = (0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ in $d=2$, and $z=0$ and 1 in $d=1$. Also,

$$\begin{aligned}\sqrt{2}\widehat{(\bar{\mathfrak{A}}_{\mathrm{nn};1,2}\mathfrak{g})}_{\mathrm{free}}(v,w) \quad &= \quad 2\beta_\rho\Bigg[\sum_{i=1}^d a_i\gamma(v_i+w_i) + \alpha_d(v,w)\Bigg]\widehat{\mathfrak{g}_{\mathrm{free}}}(v+w)\\ &\qquad + \beta_\rho\Bigg[-\sum_{i=1}^d a_i\gamma(v_i) + \alpha_d(v,w)\Bigg]\widehat{\mathfrak{g}_{\mathrm{free}}}(v)\\ &\qquad + \beta_\rho\Bigg[-\sum_{i=1}^d a_i\gamma(w_i) + \alpha_d(w,v)\Bigg]\widehat{\mathfrak{g}_{\mathrm{free}}}(w) + \delta_1(v,w)\end{aligned}$$

where, for $r,s \in [0,1]^d$,

$$\alpha_d(r,s) \quad = \quad \sum_{i=1}^d a_i\Bigg[\gamma(r_i) + \gamma(s_i) - \gamma(r_i+s_i)\Bigg]$$

and $|\delta_1(v,w)| \leq \kappa(v,w)\sum_{|z|\leq 1}|\mathfrak{g}_{\mathrm{free}}(z)|$ and $\kappa(v,w)$ is a bounded function such that

$$\kappa(v,w)^2 \;\leq\; C[|v-z_1|^2 + |w-z_2|^2]$$

as $(v,w) \to (z_1,z_2)$ for $z_1, z_2 = (0,0)$,$(0,1)$, $(1,0)$ and $(0,1)$ in $d=2$, and $z_1,z_2 = 0$ and 1 in $d=1$.

Let now $f(\zeta) = \rho - \zeta_{j_0}$. As

$$(\mathfrak{f}_\odot)_{\mathrm{free}}(z) \;=\; \begin{cases} -\beta_\rho & z = j_0\\ 0 & \text{otherwise,}\end{cases}$$

we calculate

$$\widehat{(\mathfrak{f}_\odot)}_{\mathrm{free}}(v) \;=\; -\beta_\rho e^{2\pi i(j_0\cdot v)} \;=\; -\beta_\rho + \delta_2(v)$$

where $\delta_2(v) = -\beta_\rho(e^{2\pi i(j_0\cdot v)} - 1)$ and so $|\delta_2(v)|^2 \leq C|v-z|^2$ as $v \to z$ for $z = (0,0), (0,1)$, $(1,0)$, and $(1,1)$ in $d=2$, and $z=0$ and 1 in $d=1$.

We now apply Propositions 3.9 and 4.1. Write, for local $\mathfrak{g} \in \bar{\mathcal{C}}_{d,1}$ and $\lambda = t^{-1}$, in Fourier expression (cf. subsection 3.5), that $\sigma_t^2(f)/t$ is less than

$$\begin{aligned} &2\int_{[0,1]^d} \frac{|-\beta_\rho - \rho\big[\sum_{i=1}^d a_i\gamma(v_i)\big]\widehat{\mathfrak{g}_{\rm free}}(v)|^2}{\lambda+\theta_d(v)} + (\lambda+\theta_d(v))|\widehat{\mathfrak{g}_{\rm free}}(v)|^2\ dv \qquad (4.1)\\ &+2\int_{[0,1]^d} \frac{|\delta_0(v)+\delta_2(v)|^2}{\lambda+\theta_d(v)}dv \ + \ \Big|\mathfrak{g}_{\rm free}(0) - \sum_{|z|=1}\mathfrak{g}_{\rm free}(z)\Big|^2\\ &+\frac{3}{2}\beta_\rho^2\int_{([0,1]^d)^2}\frac{dvdw}{\lambda+\theta_d(v)+\theta_d(w)}\\ &\quad\times\Big|2\widehat{\mathfrak{g}_{\rm free}}(v+w)\sum_{i=1}^d a_i\gamma(v_i+w_i) - \widehat{\mathfrak{g}_{\rm free}}(v)\sum_{i=1}^d a_i\gamma(v_i) - \widehat{\mathfrak{g}_{\rm free}}(w)\sum_{i=1}^d a_i\gamma(w_i)\Big|^2\\ &+\frac{3}{2}\beta_\rho^2\int_{([0,1]^d)^2}\frac{|\alpha_d(v,w)\widehat{\mathfrak{g}_{\rm free}}(v,w)+\alpha_d(v,w)\widehat{\mathfrak{g}_{\rm free}}(v)+\alpha_d(w,v)\widehat{\mathfrak{g}_{\rm free}}(w)|^2}{\lambda+\theta_d(v)+\theta_d(w)}dvdw\\ &+\frac{3}{2}\int_{([0,1]^d)^2}\frac{|\delta_1(v,w)|^2}{\lambda+\theta_d(v)+\theta_d(w)}dvdw. \end{aligned}$$

Note that the infimum on the six lines of (4.1) over local $\mathfrak{g} \in \bar{\mathcal{C}}_{d,1}$ is the same as if over L^2 functions in $\bar{\mathcal{C}}_{d,1}$.

The strategy now follows three steps. In Step 1, we bound uniformly in $\lambda > 0$,

$$\inf_{\mathfrak{g}}\int_{[0,1]^d} \frac{|-\beta_\rho - \rho\big[\sum_{i=1}^d a_i\gamma(v_i)\big]\widehat{\mathfrak{g}_{\rm free}}(v)|^2}{\lambda+\theta_d(v)} + (\lambda+\theta_d(v))|\widehat{\mathfrak{g}_{\rm free}}(v)|^2\ dv, \qquad (4.2)$$

and find the L^2 minimizer function $\mathfrak{g}_\lambda$.

In Step 2 we show $\mathfrak{g}_\lambda$ is a real function and $(\mathfrak{g}_\lambda)_{\rm free}(0) = \sum_{|z|=1}(\mathfrak{g}_\lambda)_{\rm free}(z) = 0$. Also, we show for $x \in \mathbb{Z}^d$ that $\sup_{\lambda>0}|(\mathfrak{g}_\lambda)_{\rm free}(x)| < \infty$. Then, as

$$\sup_{\lambda>0}\sup_{v\in[0,1]^d}\frac{|\delta_0(v)+\delta_2(v)|^2}{\lambda+\theta_d(v)} \ < \ \infty \ \text{ and } \ \sup_{\lambda>0}\sup_{v,w\in[0,1]^d}\frac{|\delta_1(v,w)|^2}{\lambda+\theta_d(v)+\theta_d(w)} \ < \ \infty,$$

the integrals in the second and sixth lines of (4.1) are uniformly bounded. Also, the other term in absolute value in the second line of (4.1), with $\mathfrak{g} = \mathfrak{g}_\lambda$, vanishes.

Finally, in Step 3 we show that the two integrals, with $\mathfrak{g} = \mathfrak{g}_\lambda$, in the third through fifth lines of (4.1) are uniformly bounded in $\lambda > 0$. Hence, $\sigma_t^2(f)/t$ is uniformly bounded over $t \geq 1$, completing the proof of Theorem 1.2. □

We now argue these steps.

Step 1. By straightforward optimizations on the quadratic expression in the integrand, observe infimum (4.2) evaluates to

$$\beta_\rho^2\int_{[0,1]^d}\frac{\lambda+\theta_d(v)}{\rho^2\big|\sum_{i=1}^d a_i\gamma(v_i)\big|^2+(\lambda+\theta_d(v))^2}dv \qquad (4.3)$$

with minimizer

$$\widehat{(\mathfrak{g}_\lambda)_{\rm free}}(v) \ = \ \frac{\beta_\rho\rho\sum_{i=1}^d a_i\gamma(v_i)}{\rho^2\big|\sum_{i=1}^d a_i\gamma(v_i)\big|^2+(\lambda+\theta_d(v))^2}\ .$$

We now check (4.3) is uniformly finite in $\lambda > 0$: As noted near equation (5.6) Bernardin (2004), which considers almost the same integral, problems arise when $v = (0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ in $d = 2$; and in $d = 1$, when $v = 0$ and 1.

In $d = 2$, by using a possible sign change, the uniform bound of (4.3) is equivalent to bounding

$$\int_V \frac{\lambda + v_1^2 + v_2^2}{(c_1 v_1 + c_2 v_2)^2 + (\lambda + v_1^2 + v_2^2)^2} dv_1 dv_2$$

where $V \in I\!R_+ \times I\!R_+$ is a neighborhood of the origin and c_1, c_2 are arbitrary constants with $c_1^2 + c_2^2 > 0$. As the difficulty is when $c_1 v_1 + c_2 v_2 = 0$, bounding the above integral is the same as bounding, with $c_1/\sqrt{c_1^2 + c_2^2} = \sin(\phi_0)$ and $c_2/\sqrt{c_1^2 + c_2^2} = \cos(\phi_0)$,

$$\int_0^1 \int_0^{\pi/2} \frac{(\lambda + r^2) r}{(c_1^2 + c_2^2) r^2 \sin^2(\phi + \phi_0) + (\lambda + r^2)^2} d\phi dr$$

or more simply on order

$$\int_0^1 \int_0^{\pi/2} \frac{(\lambda + r^2) r}{(c_1^2 + c_2^2) r^2 \sin^2(\phi) + (\lambda + r^2)^2} d\phi dr$$

which is finite uniformly in $\lambda > 0$ (cf. Lemma 5.2 Bernardin (2004) for similar calculations).

In $d = 1$, (4.3) is on order

$$\int_0^1 \frac{\lambda + v^2}{v^2 + (\lambda + v^2)^2} dv$$

which also, by straightforward computation, is finite uniformly in $\lambda > 0$.

Step 2. Noting $\overline{\gamma(r)} = -\gamma(r)$, we now show $\widehat{(\mathfrak{g}_\lambda)_{\rm free}}$ is the transform of a real function:

$$\begin{aligned}
\overline{\int_{[0,1]^d} e^{2\pi i v\cdot x} \widehat{(\mathfrak{g}_\lambda)_{\rm free}}(v) dv} &= -\int_{[0,1]^d} e^{-2\pi i v\cdot x} \widehat{(\mathfrak{g}_\lambda)_{\rm free}}(v) dv \\
&= \int_{[0,1]^d} e^{-2\pi i v\cdot x} \widehat{(\mathfrak{g}_\lambda)_{\rm free}}(\vec{1} - v) dv \\
&= \int_{[0,1]^d} e^{2\pi i v\cdot x} \widehat{(\mathfrak{g}_\lambda)_{\rm free}}(v) dv
\end{aligned}$$

where $\vec{1}$ is the vector with components all 1. The last sequence also shows $(\mathfrak{g}_\lambda)_{\rm free}$ is odd, that is $(\mathfrak{g}_\lambda)_{\rm free}(x) = -(\mathfrak{g}_\lambda)_{\rm free}(-x)$ for $x \in \mathbb{Z}^d$. Then, $\sum_{|z|=1} (\mathfrak{g}_\lambda)_{\rm free}(z) = (\mathfrak{g}_\lambda)_{\rm free}(0) = 0$. Also, for $x \in \mathbb{Z}^d$, again as $(\mathfrak{g}_\lambda)_{\rm free}$ is odd,

$$\begin{aligned}
&\sup_{\lambda > 0} |(\mathfrak{g}_\lambda)_{\rm free}(x)| \\
&\quad = \sup_{\lambda > 0} \left| \int_{[0,1]^d} i \sin(2\pi v \cdot x) \widehat{(\mathfrak{g}_\lambda)_{\rm free}}(v) dv \right| \qquad (4.4) \\
&\quad \leq C \sup_{\lambda > 0} \int_{[0,1]^d} \frac{|\sin(2\pi v \cdot x)| |\sum_{i=1}^d a_i \gamma(v_i)| dv}{\rho^2 |\sum_{i=1}^d a_i \gamma(v_i)|^2 + (\lambda + \theta_d(v))^2}
\end{aligned}$$

where $C = C(\rho)$. As with (4.3) above, the only problem with the denominator in $d = 2$ comes at points $v = (0,0), (0,1), (1,0)$ and $(1,1)$, and in $d = 1$ at $v = 0$ and 1.

The bound on (4.4) in $d = 2$, similar to the calculation in Step 1, is the same as bounding

$$\int_0^1 \int_0^{\pi/2} \frac{r d\phi dr}{\sin^2(\phi) + r^2}$$

which is finite. The bound on (4.4) in $d = 1$ is also finite and simpler.

Step 3. The two integrals in the third through fifth lines of (4.1), after adding and subtracting $2\beta_\rho b$ with $b = -\beta_\rho/\rho$, are bounded up to a constant $C = C(p,\rho)$ by

$$C\int_{([0,1]^d)^2} \frac{\big|b - \big[\sum_{i=1}^d a_i\gamma(v_i+w_i)\big]\widehat{(g_\lambda)_{\rm free}}(v+w)\big|^2}{\lambda + \theta_d(v)+\theta_d(w)} dvdw \tag{4.5}$$

$$+C\int_{([0,1]^d)^2} \frac{dvdw}{\lambda+\theta_d(v)+\theta_d(w)} \tag{4.6}$$

$$\times\Big\{\big|b - \big[\sum_{i=1}^d a_i\gamma(v_i)\big]\widehat{(g_\lambda)_{\rm free}}(v)\big|^2 + \big|b - \big[\sum_{i=1}^d a_i\gamma(w_i)\big]\widehat{(g_\lambda)_{\rm free}}(w)\big|^2\Big\}$$

$$+C\int_{([0,1]^d)^2} \frac{|\alpha_d(v,w)\widehat{(g_\lambda)_{\rm free}}(v+w)|^2}{\lambda+\theta_d(v)+\theta_d(w)} dvdw \tag{4.7}$$

$$+C\int_{([0,1]^d)^2} \frac{|\alpha_d(v,w)\widehat{(g_\lambda)_{\rm free}}(v)|^2 + |\alpha_d(w,v)\widehat{(g_\lambda)_{\rm free}}(w)|^2}{\lambda+\theta_d(v)+\theta_d(w)} dvdw \tag{4.8}$$

The first integral (4.5), noting $\big[\sum_{i=1}^d a_i\gamma(r_i)\big]^2 = -\big|\sum_{i=1}^d a_i\gamma(r_i)\big|^2$, is on order

$$\int_{([0,1]^d)^2} \frac{(\lambda+\theta_d(v+w))^2}{\big|\sum_{i=1}^d a_i\gamma(v_i+w_i)\big|^2 + (\lambda+\theta_d(v+w))^2}\frac{dvdw}{\lambda+\theta_d(v)+\theta_d(w)}$$

which in $d = 2$ is bounded simply and uniformly in $\lambda > 0$ by

$$\int_{([0,1]^2)^2} \frac{dvdw}{\theta_2(v)+\theta_2(w)} < \infty.$$

In $d = 1$, as $\sup_{\lambda>0}\sup_{v,w\in[0,1]}(\lambda+\theta_1(v+w))/(\lambda+\theta_1(v)+\theta_1(w)) < \infty$, we bound on order by

$$\int_{[0,1]^2} \frac{\lambda+\theta_1(v+w)}{|\gamma(v+w)|^2 + (\lambda+\theta_1(v+w))^2} dvdw.$$

Then, as

$$\sup_{\lambda>0}\sup_{v,w\in[0,1]} \frac{\theta_1(v+w)}{|\gamma(v+w)|^2+(\lambda+\theta_1(v+w))^2} < \infty,$$

we need only bound

$$\int_{[0,1]^2} \frac{\lambda dvdw}{|\gamma(v+w)|^2+(\lambda+\theta_1(v+w))^2} \le \int_0^2 \frac{\lambda ds}{\sin^2(2\pi s) + (\lambda+\sin^2(\pi s))^2}$$

which is uniformly finite in $\lambda > 0$.

The second integral (4.6) is analogously, and more simply, bounded in $d = 1, 2$.

For the third integral (4.7), on order we need to bound

$$\int_{([0,1]^d)^2} \frac{|\alpha_d(v,w)|^2|\sum_{i=1}^d a_i\gamma(v_i+w_i)|^2}{[|\sum_{i=1}^d a_i\gamma(v_i+w_i)|^2 + (\lambda+\theta_d(v+w))^2]^2}\frac{dvdw}{\lambda+\theta_d(v)+\theta_d(w)}.$$

In $d=2$, noting the form of $\alpha_d(v,w)$, the integral is bounded on order by

$$\int_{([0,1]^2)^2} \frac{|\sum_{i=1}^2 a_i(\gamma(v_i)+\gamma(w_i))|^2|\sum_{i=1}^2 a_i\gamma(v_i+w_i)|^2 dvdw}{[|\sum_{i=1}^2 a_i\gamma(v_i+w_i)|^2+\theta_2^2(v+w)]^2[\theta_2(v)+\theta_2(w)]} + \int_{([0,1]^2)^2} \frac{dvdw}{\theta_2(v)+\theta_2(w)}.$$

The first term is considered and bounded, modulo constants, in Bernardin (2004, Lemma 5.3) through an analysis of singularities of the denominator. The second term is clearly bounded. In $d=1$, write, for $v,w\in[0,1]$,

$$\begin{aligned}\alpha_1(v,w) &= 2ia_1\Big[\sin(2\pi v)[1-\cos(2\pi w)]+\sin(2\pi w)[1-\cos(2\pi v)]\Big]\\ &= 8ia_1\sin(\pi v)\sin(\pi w)\sin(\pi(v+w)).\end{aligned}$$

Then, the uniform bound on (4.7) follows from the bound on the integrand

$$\sup_{\lambda>0}\sup_{v,w\in[0,1]} \frac{|\alpha_1(v,w)|^2|a_1\gamma(v+w)|^2}{[|a_1\gamma(v+w)|^2+(\lambda+\theta_1(v+w))^2]^2[\lambda+\theta_1(v)+\theta_1(w)]} < \infty.$$

The last integral (4.8) is bounded on order by

$$\int_{([0,1]^d)^2} \frac{|\alpha_d(v,w)|^2}{|\sum_{i=1}^d a_i\gamma(v_i)|^2+\theta_d(v)^2}\frac{dvdw}{\theta_d(v)+\theta_d(w)}.$$

As

$$\alpha_d(v,w) = \sum_{j=1}^d a_j\Big[\gamma(v_j)(1-e^{2\pi i w_j})+\gamma(w_j)(1-e^{-2\pi i v_j})\Big]$$

and $\sup_{r\in(0,1)^d}(\sum_{j=1}^d|1-e^{\pm 2\pi i r_j}|^2)/\theta_d(r)<\infty$, the last integral is on order

$$\int_{[0,1]^d} \frac{\sum_{j=1}^d|1-e^{\pm 2\pi i v_j}|^2}{|\sum_{i=1}^d a_i\gamma(v_i)|^2+\theta_d(v)^2}dv. \tag{4.9}$$

In $d=2$, the singularities are at $v=(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, and as in Steps $1,2$ the bound on (4.9) is the same as

$$\int_0^1\int_0^{\pi/2} \frac{r}{\sin^2(\phi)+r^2}d\phi dr$$

which is finite. In $d=1$, as $\sup_{r\in(0,1)}|1-e^{\pm 2\pi i r}|^2/|\gamma(r)|^2<\infty$, the integrand in (4.9) is itself finite.

5. Proof of Proposition 4.1

We prove the proposition in $d=2$. The argument in $d=1$ is analogous, and follows in particular by choosing $a_2=0$.

To make notation simple, in the following, we will omit the brackets for singletons $\{x\}$ and two-tuple sets $\{x,y\}$ and denote them as x and x,y. Also, we will drop the suffix "nn" with respect to operators $\mathfrak{A}_{n,m}=\mathfrak{A}_{\text{nn};n,m}$. Recall $\{e_1,e_2\}$ denotes the standard basis of $\mathbb{Z}^2$.

First, from the formulas in subsection 3.1, we compute the actions of $\mathfrak{A}_{1,1}$ and $\mathfrak{A}_{1,2}$ on local one-degree functions, $\mathfrak{g} \in \mathcal{C}_{2,1}$. For $x \in \mathbb{Z}^2 \setminus \{0\}$,

$$\begin{aligned}
(\mathfrak{A}^e_{1,1}\mathfrak{g})(x) &= (1-2\rho)\sum_{y\neq x,0}[\mathfrak{g}(y)-\mathfrak{g}(x)]a_{\rm nn}(y-x) \qquad \text{and}\\
(\mathfrak{A}^t_{1,1}\mathfrak{g})(x) &= -(1-\rho)\sum_{y\neq x,0}[\mathfrak{g}(y)-\mathfrak{g}(x)]a_{\rm nn}(y-x)-\rho[\mathfrak{g}(x)-\mathfrak{g}(-x)]a(x)
\end{aligned}$$

which together give

$$(\mathfrak{A}_{1,1}\mathfrak{g})(x) = -\rho\sum_{y\neq x,0}[\mathfrak{g}(y)-\mathfrak{g}(x)]a_{\rm nn}(y-x)-\rho[\mathfrak{g}(x)-\mathfrak{g}(-x)]a_{\rm nn}(x).$$

Also, for distinct $x, y \in \mathbb{Z}^d \setminus \{0\}$,

$$\begin{aligned}
(\mathfrak{A}^e_{1,2}\mathfrak{g})(x,y) &= 2\beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(y)]a_{\rm nn}(y-x)\\
(\mathfrak{A}^t_{1,2}\mathfrak{g})(x,y) &= \beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(x-y)]a_{\rm nn}(y)+\beta_\rho[\mathfrak{g}(y)-\mathfrak{g}(y-x)]a_{\rm nn}(x).
\end{aligned}$$

Then, we may write for $x \in \mathbb{Z}^2$ and local $\mathfrak{g} \in \bar{\mathcal{C}}_{2,1}$ that $(\bar{\mathfrak{A}}_{1,1}\mathfrak{g})(x)$ (cf. subsection 3.4) equals

$$\begin{cases}
-\rho[\mathfrak{g}(x+e_1)-\mathfrak{g}(x-e_1)]a_1-\rho[\mathfrak{g}(x+e_2)-\mathfrak{g}(x-e_2)]a_2 & \text{for } x\neq \pm e_1,\pm e_2,0\\
\mp\rho[\mathfrak{g}(\pm 2e_1)-\mathfrak{g}(\mp e_1)]a_1-\rho[\mathfrak{g}(e_2\pm e_1)-\mathfrak{g}(-e_2\pm e_1)]a_2 & \text{for } x=\pm e_1\\
\mp\rho[\mathfrak{g}(\pm 2e_2)-\mathfrak{g}(\mp e_2)]a_2-\rho[\mathfrak{g}(e_1\pm e_2)-\mathfrak{g}(-e_1\pm e_2)]a_1 & \text{for } x=\pm e_2\\
0 & \text{otherwise.}
\end{cases}$$

Also, for $x, y \in \mathbb{Z}^2$, we write (noting $(\bar{\mathfrak{A}}^{\cdot}_{1,2}\mathfrak{g})(x,y) = (\bar{\mathfrak{A}}^{\cdot}_{1,2}\mathfrak{g})(\{x,y\}) = (\bar{\mathfrak{A}}^{\cdot}_{1,2}\mathfrak{g})(y,x)$),

$$(\bar{\mathfrak{A}}^e_{1,2}\mathfrak{g})(x,y) = \begin{cases}
2\beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(x+e_1)]a_1 & \text{for } y=x+e_1,\ x\neq 0,-e_1\\
2\beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(x+e_2)]a_2 & \text{for } y=x+e_2,\ x\neq 0,-e_2\\
0 & \text{otherwise.}
\end{cases}$$

and $(\bar{\mathfrak{A}}^t_{1,2}\mathfrak{g})(x,y)$ equals

$$\begin{cases}
\pm\beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(x\mp e_1)]a_1 & \text{for } x\neq\pm e_1,\pm e_2,0,\ y=\pm e_1\\
\pm\beta_\rho[\mathfrak{g}(x)-\mathfrak{g}(x\mp e_2)]a_2 & \text{for } x\neq\pm e_1,\pm e_2,0,\ y=\pm e_2\\
\beta_\rho[\pm(\mathfrak{g}(e_1)-\mathfrak{g}(e_1\mp e_2))a_2 & \\
\qquad +(\mathfrak{g}(\pm e_2)-\mathfrak{g}(\pm e_2-e_1))a_1] & \text{for } x=e_1, y=\pm e_2\\
\beta_\rho[\pm(\mathfrak{g}(-e_1)-\mathfrak{g}(-e_1\mp e_2))a_2 & \\
\qquad -(\mathfrak{g}(\pm e_2)-\mathfrak{g}(\pm e_2+e_1))a_1] & \text{for } x=-e_1, y=\pm e_2\\
\beta_\rho[-(\mathfrak{g}(e_1)-\mathfrak{g}(2e_1))a_1+(\mathfrak{g}(-e_1)-\mathfrak{g}(-2e_1))a_1] & \text{for } x=e_1, y=-e_1\\
\beta_\rho[-(\mathfrak{g}(e_2)-\mathfrak{g}(2e_2))a_2+(\mathfrak{g}(-e_2)-\mathfrak{g}(-2e_2))a_2] & \text{for } x=e_2, y=-e_2\\
0 & \text{otherwise.}
\end{cases}$$

We now compute corresponding Fourier transforms. To simplify notation, we drop the subscript "free" and call $\mathfrak{g}_{\rm free} = \mathfrak{g}$. First, we have $\widehat{(\bar{\mathfrak{A}}_{1,1}\mathfrak{g})}_{\rm free}(v)$ (cf.

subsection 3.5) equals

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^2} e^{2\pi i x\cdot v}(\bar{\mathfrak{A}}_{1,1}\mathfrak{g})_{\rm free}(x)\\
&= \sum_{x\neq \pm e_1,\pm e_2,0} -\rho e^{2\pi i x\cdot v}[(\mathfrak{g}(x+e_1)-\mathfrak{g}(x-e_1))a_1 + (\mathfrak{g}(x+e_2)-\mathfrak{g}(x-e_2))a_2]\\
&\quad -\rho e^{2\pi i v_1}[(\mathfrak{g}(2e_1)-\mathfrak{g}(-e_1))a_1 + (\mathfrak{g}(e_2+e_1)-\mathfrak{g}(-e_2+e_1))a_2]\\
&\quad -\rho e^{-2\pi i v_1}[-(\mathfrak{g}(-2e_1)-\mathfrak{g}(e_1))a_1 + (\mathfrak{g}(e_2-e_1)-\mathfrak{g}(-e_2-e_1))a_2]\\
&\quad -\rho e^{2\pi i v_2}[(\mathfrak{g}(2e_2)-\mathfrak{g}(-e_2))a_2 + (\mathfrak{g}(e_1+e_2)-\mathfrak{g}(-e_1+e_2))a_1]\\
&\quad -\rho e^{-2\pi i v_2}[-(\mathfrak{g}(-2e_2)-\mathfrak{g}(e_2))a_2 + (\mathfrak{g}(e_1-e_2)-\mathfrak{g}(-e_1-e_2))a_1].
\end{aligned}$$

The sum further equals

$$\begin{aligned}
&- \sum_{\substack{x\neq 0,2e_1,\\ e_1\pm e_2, e_1}} \rho e^{2\pi i x\cdot v} e^{-2\pi i v_1}\mathfrak{g}(x)a_1 \ + \sum_{\substack{x\neq 0,-2e_1,\\ -e_1\pm e_2,-e_1}} \rho e^{2\pi i x\cdot v} e^{2\pi i v_1}\mathfrak{g}(x)a_1\\
&\quad - \sum_{\substack{x\neq 0,2e_2,\\ e_2\pm e_1, e_2}} \rho e^{2\pi i x\cdot v} e^{-2\pi i v_2}\mathfrak{g}(x)a_2 \ + \sum_{\substack{x\neq 0,-2e_2,\\ -e_2\pm e_1,-e_2}} \rho e^{2\pi i x\cdot v} e^{2\pi i v_2}\mathfrak{g}(x)a_2.
\end{aligned}$$

Recall now that $\gamma(r) = e^{2\pi i r} - e^{-2\pi i r} = 2i\sin(2\pi r)$. Combining and canceling terms gives that

$$\begin{aligned}
\widehat{(\bar{\mathfrak{A}}_{1,1}\mathfrak{g})_{\rm free}}(v) &= \rho[a_1\gamma(v_1)+a_2\gamma(v_2)]\widehat{\mathfrak{g}}(v)\\
&\qquad -\rho a_1(e^{-2\pi i v_1}-1)\mathfrak{g}(e_1) + \rho a_1(e^{2\pi i v_1}-1)\mathfrak{g}(-e_1)\\
&\qquad -\rho a_2(e^{-2\pi i v_2}-1)\mathfrak{g}(e_2) + \rho a_2(e^{2\pi i v_2}-1)\mathfrak{g}(-e_2)\\
&\qquad -\rho[a_1\gamma(v_1)+a_2\gamma(v_2)]\mathfrak{g}(0)\\
&= \rho[a_1\gamma(v_1)+a_2\gamma(v_2)]\widehat{\mathfrak{g}}(v) + \delta_0(v)
\end{aligned}$$

where $|\delta_0(v)| \leq \kappa(v)\sum_{|z|\leq 1}|\mathfrak{g}(z)|$ and $\kappa(v)$ is a bounded function on order $\kappa(v) = O(|v-z|)$ when $v\to z$ for $z = (0,0),(0,1),(1,0)$, and $(1,1)$.

We also compute that $\sqrt{2}\widehat{(\bar{\mathfrak{A}}^e_{1,2}\mathfrak{g})_{\text{free}}}(v,w)$ equals

$$\begin{aligned}
&\sum_{x,y\in\mathbb{Z}^2} e^{2\pi i(x\cdot v+y\cdot w)}(\bar{\mathfrak{A}}^e_{1,2}\mathfrak{g})_{\text{free}}(x,y)\\
&\quad = 2\beta_\rho a_1 \sum_{z\neq 0,-e_1} e^{2\pi i z\cdot(v+w)}(e^{2\pi i w_1}+e^{2\pi i v_1})[\mathfrak{g}(z)-\mathfrak{g}(z+e_1)]\\
&\qquad +2\beta_\rho a_2 \sum_{z\neq 0,-e_2} e^{2\pi i z\cdot(v+w)}(e^{2\pi i w_2}+e^{2\pi i v_2})[\mathfrak{g}(z)-\mathfrak{g}(z+e_2)]\\
&\quad = 2\beta_\rho a_1 \sum_{z\neq 0,\pm e_1} e^{2\pi i z\cdot(v+w)}(\gamma(w_1)+\gamma(v_1))\mathfrak{g}(z)\\
&\qquad +2\beta_\rho a_2 \sum_{z\neq 0,\pm e_2} e^{2\pi i z\cdot(v+w)}(\gamma(w_2)+\gamma(v_2))\mathfrak{g}(z)\\
&\qquad +2\beta_\rho a_1 e^{2\pi i(v_1+w_1)}(e^{2\pi i w_1}+e^{2\pi i v_1})\mathfrak{g}(e_1)\\
&\qquad -2\beta_\rho a_1 e^{-2\pi i(v_1+w_1)}(e^{-2\pi i w_1}+e^{-2\pi i v_1})\mathfrak{g}(-e_1)\\
&\qquad +2\beta_\rho a_2 e^{2\pi i(v_2+w_2)}(e^{2\pi i w_2}+e^{2\pi i v_2})\mathfrak{g}(e_2)\\
&\qquad -2\beta_\rho a_2 e^{-2\pi i(v_2+w_2)}(e^{-2\pi i w_2}+e^{-2\pi i v_2})\mathfrak{g}(-e_2)\\
&\quad = 2\beta_\rho[a_1(\gamma(w_1)+\gamma(v_1))+a_2(\gamma(w_2)+\gamma(v_2))]\widehat{\mathfrak{g}}(v+w)\\
&\qquad -2\beta_\rho[a_1(\gamma(w_1)+\gamma(v_1))+a_2(\gamma(w_2)+\gamma(v_2))]\mathfrak{g}(0)\\
&\qquad +2\beta_\rho a_1[(e^{2\pi i w_1}+e^{2\pi i v_1})\mathfrak{g}(e_1)-(e^{-2\pi i w_1}+e^{-2\pi i v_1})\mathfrak{g}(-e_1)]\\
&\qquad +2\beta_\rho a_2[(e^{2\pi i w_2}+e^{2\pi i v_2})\mathfrak{g}(e_2)-(e^{-2\pi i w_2}+e^{-2\pi i v_2})\mathfrak{g}(-e_2)].
\end{aligned}$$

Also, we have $\sqrt{2}\widehat{(\bar{\mathfrak{A}}^t_{1,2}\mathfrak{g})_{\text{free}}}(v,w)$ equals

$$\begin{aligned}
&\sum_{x,y\in\mathbb{Z}^2} e^{2\pi i(x\cdot v+y\cdot w)}(\bar{\mathfrak{A}}^t_{1,2}\mathfrak{g})_{\text{free}}(x,y)\\
&\quad = \beta_\rho a_1 \sum_{z\neq 0,\pm e_1,\pm e_2} (e^{2\pi i z\cdot v}e^{2\pi i w_1}+e^{2\pi i z\cdot w}e^{2\pi i v_1})[\mathfrak{g}(z)-\mathfrak{g}(z-e_1)]\\
&\qquad -\beta_\rho a_1 \sum_{z\neq 0,\pm e_1,\pm e_2} (e^{2\pi i z\cdot v}e^{-2\pi i w_1}+e^{2\pi i z\cdot w}e^{-2\pi i v_1})[\mathfrak{g}(z)-\mathfrak{g}(z+e_1)]\\
&\qquad +\beta_\rho a_2 \sum_{z\neq 0,\pm e_1,\pm e_2} (e^{2\pi i z\cdot v}e^{2\pi i w_2}+e^{2\pi i z\cdot w}e^{2\pi i v_2})[\mathfrak{g}(z)-\mathfrak{g}(z-e_2)]\\
&\qquad -\beta_\rho a_2 \sum_{z\neq 0,\pm e_1,\pm e_2} (e^{2\pi i z\cdot v}e^{-2\pi i w_2}+e^{2\pi i z\cdot w}e^{-2\pi i v_2})[\mathfrak{g}(z)-\mathfrak{g}(z+e_2)]\\
&\qquad +\beta_\rho(e^{2\pi i v_1}e^{2\pi i w_2}+e^{2\pi i v_2}e^{2\pi i w_1})\\
&\qquad\qquad \times[a_2(\mathfrak{g}(e_1)-\mathfrak{g}(e_1-e_2))+a_1(\mathfrak{g}(e_2)-\mathfrak{g}(e_2-e_1))]\\
&\qquad +\beta_\rho(e^{2\pi i v_1}e^{-2\pi i w_2}+e^{-2\pi i v_2}e^{2\pi i w_1})\\
&\qquad\qquad \times[-a_2(\mathfrak{g}(e_1)-\mathfrak{g}(e_1+e_2))+a_1(\mathfrak{g}(-e_2)-\mathfrak{g}(-e_2-e_1))]
\end{aligned}$$

$$
\begin{aligned}
&+\beta_\rho(e^{-2\pi i v_1}e^{2\pi i w_2}+e^{2\pi i v_2}e^{-2\pi i w_1})\\
&\qquad\times[a_2(\mathfrak{g}(-e_1)-\mathfrak{g}(-e_1-e_2))-a_1(\mathfrak{g}(e_2)-\mathfrak{g}(e_2+e_1))]\\
&+\beta_\rho(e^{-2\pi i v_1}e^{-2\pi i w_2}+e^{-2\pi i v_2}e^{-2\pi i w_1})\\
&\qquad\times[-a_2(\mathfrak{g}(-e_1)-\mathfrak{g}(-e_1+e_2))-a_1(\mathfrak{g}(-e_2)-\mathfrak{g}(-e_2+e_1))]\\
&+\beta_\rho a_1(e^{2\pi i v_1}e^{-2\pi i w_1}+e^{-2\pi i v_1}e^{2\pi i w_1})\\
&\qquad\times[-\mathfrak{g}(e_1)+\mathfrak{g}(2e_1)+\mathfrak{g}(-e_1)-\mathfrak{g}(-2e_1)]\\
&+\beta_\rho a_2(e^{2\pi i v_2}e^{-2\pi i w_2}+e^{-2\pi i v_2}e^{2\pi i w_2})\\
&\qquad\times[-\mathfrak{g}(e_2)+\mathfrak{g}(2e_2)+\mathfrak{g}(-e_2)-\mathfrak{g}(-2e_2)]\\
=\ &\beta_\rho\widehat{\mathfrak{g}}(v)[a_1(\gamma(w_1)-\gamma(v_1+w_1))+a_2(\gamma(w_2)-\gamma(v_2+w_2))]\\
&+\beta_\rho\widehat{\mathfrak{g}}(w)[a_1(\gamma(v_1)-\gamma(v_1+w_1))+a_2(\gamma(v_2)-\gamma(v_2+w_2))]\\
&+\beta_\rho\mathfrak{g}(0)[a_1(-\gamma(v_1)-\gamma(w_1)+2\gamma(v_1+w_1))\\
&\qquad\qquad+a_2(-\gamma(v_2)-\gamma(w_2)+2\gamma(v_2+w_2))]\\
&-\beta_\rho a_1\mathfrak{g}(e_1)[e^{-2\pi i w_1}+e^{-2\pi i v_1}+2e^{2\pi i(v_1+w_1)}]\\
&+\beta_\rho a_1\mathfrak{g}(-e_1)[e^{2\pi i w_1}+e^{2\pi i v_1}+2e^{-2\pi i(v_1+w_1)}]\\
&-\beta_\rho a_2\mathfrak{g}(e_2)[e^{-2\pi i w_2}+e^{-2\pi i v_2}+2e^{2\pi i(v_2+w_2)}]\\
&+\beta_\rho a_2\mathfrak{g}(-e_2)[e^{2\pi i w_2}+e^{2\pi i v_2}+2e^{-2\pi i(v_2+w_2)}]
\end{aligned}
$$

Putting terms together, we compute the Fourier transform for $(\bar{\mathfrak{A}}_{1,2}\mathfrak{g})_{\rm free}$. In particular, we note the last four lines of the computation for $\widehat{(\bar{\mathfrak{A}}^t_{1,2}\mathfrak{g})}_{\rm free}$ and the last two lines of the computation for $\widehat{(\bar{\mathfrak{A}}^e_{1,2}\mathfrak{g})}_{\rm free}$ are $O(1)$ as $(v,w)\to(z_1,z_2)$ for $z_1,z_2=(0,0),(0,1),(1,0)$ and $(1,1)$. However, they match in the sense, when they are added to each other, the sum is small on the desired order. We have

$$
\begin{aligned}
&\sqrt{2}\widehat{(\bar{\mathfrak{A}}_{1,2}\mathfrak{g})}_{\rm free}(v,w)\\
&\quad=2\beta_\rho[a_1(\gamma(w_1)+\gamma(v_1))+a_2(\gamma(w_2)+\gamma(v_2))]\widehat{\mathfrak{g}}(v+w)\\
&\quad\ +\beta_\rho\widehat{\mathfrak{g}}(v)[a_1(\gamma(w_1)-\gamma(v_1+w_1))+a_2(\gamma(w_2)-\gamma(v_2+w_2))]\\
&\quad\ +\beta_\rho\widehat{\mathfrak{g}}(w)[a_1(\gamma(v_1)-\gamma(v_1+w_1))+a_2(\gamma(v_2)-\gamma(v_2+w_2))]+\delta_1(v,w)
\end{aligned}
$$

where $|\delta_1(v,w)|\le\kappa(v,w)\sum_{|z|\le 1}|\mathfrak{g}(z)|$ and $\kappa(v,w)$ is a bounded function on order $\kappa^2(v,w)=O(|v-z_1|^2+|w-z_2|^2)$ when $(v,w)\to(z_1,z_2)$ for points $z_1,z_2=(0,0),(0,1),(1,0)$ and $(1,1)$.

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