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Convergence of the Tóth lattice filling curve to the Tóth-Werner plane filling curve

C. M. Newman and K. Ravishankar

Courant Institute of Math. Sciences, New York University, New York, NY 10012, USA
E-mail address: newman@courant.nyu.edu
URL: http://math.nyu.edu/faculty/newman/index.html

Dept. of Mathematics, SUNY College at New Paltz, New Paltz, NY 12561, USA
E-mail address: ravishak@newpaltz.edu
URL: http://mcs.newpaltz.edu/faculty/ravi.html

Abstract. In this paper we consider the lattice filling curve, defined implicitly by Tóth (1995) and explicitly by Tóth and Werner (1998), which forms the boundary between forward and backward coalescing random walks starting from even and odd space-time sub-lattices of $\mathbb{Z}^2$ respectively. We show that this lattice filling curve converges in the diffusive scaling limit to a plane filling curve which is the boundary between the forward and backward Brownian webs. A one-dimensional projection of the two-dimensional result proves the convergence of Tóth's self-repelling walk to the Tóth-Werner continuum self repelling motion. Our main new result is the tightness of the rescaled lattice model distributions.

1. Introduction

This paper concerns the convergence of the scaling limit of a specific random lattice filling curve related to Tóth's "true" self-avoiding walk on $\mathbb{Z}$ (Tóth, 1995; Tóth and Werner, 1998) to its continuum limit, the Tóth-Werner random plane filling curve (Tóth and Werner, 1998). This is one of a number of natural situations in two dimensions where the lattice filling curve (and its continuum counterpart) represents the boundary between a spanning tree and its dual tree.

In this paper, the spanning tree is that obtained from the graphs of coalescing one-dimensional random walks starting from all lattice points of 1+1 dimensional space time (Tóth and Werner, 1998) and the continuum spanning tree is the corresponding continuum object obtained from the Brownian web (Tóth and Werner,

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1998; Fontes et al., 2002, 2004). Other important examples are the uniform spanning tree and the minimal spanning tree.

Some results, e.g. about tightness, for the scaling limits of these lattice trees were obtained in Aizenman et al. (1999), but without consideration of the corresponding lattice or plane filling curves. It was argued by Schramm (2000) that for the uniform spanning tree, the limiting plane filling curve should be given by SLE_8. This, along with the closely related result that scaled loop-erased random walk converges to SLE_2, has been proved in a recent paper of Lawler, Schramm and Werner (2004). Although analogous results for minimal spanning trees and their corresponding plane filling curves remain open problems (Camia et al., 2006), related results concerning critical two-dimensional percolation and SLE_6 have been obtained (Schramm, 2000; Smirnov, 2001; Camia and Newman, 2004, 2006a,b). As we shall see, the situation is simplified in the case treated in this paper, compared to either the uniform or minimal spanning tree situation, because one of our two coordinates is directed (i.e., the time) and because graphs of one dimensional Brownian motions are simpler than traces of SLE_κ.

Returning to our case of coalescing walks, we proceed as in Tóth and Werner (1998) and consider the odd and even sub-lattices of $\mathbb{Z}^2$ (thought of as space-time) denoted respectively by $\mathbb{Z}^2_o$ and $\mathbb{Z}^2_e$. Suppose there are simple symmetric coalescing random walks starting from every point in $\mathbb{Z}^2$ with those starting from $\mathbb{Z}^2_e$ moving forward in time and those starting from $\mathbb{Z}^2_o$ moving backward in time. This can be defined easily by first introducing a countable family of independent Bernoulli random variables $\{\xi^+_{i,j} | (i,j) \in \mathbb{Z}^2_e\}$ where $\xi^+_{i,j} = \pm 1$ with probability $1/2$. If $Z^+(i)$ denotes the position of a forward coalescing random walk on $\mathbb{Z}^2_e$ at time i, then $Z^+(i+1) = Z^+(i) + \xi^+_{i,Z^+(i)}$. This defines a family of coalescing forward random walks starting from every point in $\mathbb{Z}^2_e$.

We now define a countable collection of mutually independent random variables $\{\xi^-_{i,j} | (i,j) \in \mathbb{Z}^2_o\}$ as follows. $\xi^-_{i+1,j} = -\xi^+_{i,j}$. If $Z^-(i+1)$ denotes the position of a backward coalescing random walk at time $i+1$, then $Z^-(i) = Z^-(i+1) + \xi^-_{i+1,Z^-(i+1)}$. This defines a family of coalescing backward random walks starting from every point in $\mathbb{Z}^2_o$. Moreover from the definition it is clear that the paths of the forward and backward random walks never cross. Such families of forward and dual backward coalescing walks were introduced by Arratia (1981a) — see also Arratia (1979, 1981b). They were also introduced in the continuous time context for the study of voter models by Harris (1977) (for the case of infinitely many colors, see for example Fontes et al., 2001); there the forward and backward walks represent color boundaries and color ancestry respectively.

Almost every realization of forward and backward random walks uniquely defines a $\mathbb{Z}^2 + (1/2, 0)$ lattice filling curve which is the boundary between the collections of forward and backward paths (see Section 11 of Tóth and Werner (1998) for a detailed construction). This boundary (which we call the Tóth lattice filling curve) was implicit in Tóth's study of the "true" self-avoiding walk with bond repulsion on $\mathbb{Z}$ (Tóth, 1995). It and its continuum analogue (which we call the Tóth-Werner plane filling curve) were defined explicitly by Tóth and Werner (1998) in their analysis of the true self repelling motion which is the natural candidate for the scaling limit of the lattice self-avoiding walk. They showed that in both the discrete and the continuum models the projection on the time axis of the lattice (respectively, plane) filling curve represents the the "true" self-avoiding walk (respectively, true self repelling motion). Moreover the spatial projection gives the local time of the walk (respectively, motion) at that point.

A limit theorem for the random walk local times proved in Tóth (1995) (Theorem 1 there) implies that the finite dimensional distributions of the lattice filling curve should converge to those of the plane filling curve. Therefore, if one could prove tightness for the family of lattice filling curves obtained by diffusive scaling of the curve in $\mathbb{Z}^2 + (1/2, 0)$, the convergence of Tóth's lattice filling curve to the Tóth-Werner plane filling curve would also be proved.

The central object in the definition of the Tóth-Werner plane filling curve is a process which is a natural candidate for the diffusive limit of the forward and backward coalescing walks. It consists of forward and backward coalescing non-crossing Brownian motions stating from "every" space-time point. Such an object was defined by Arratia (1981a) and was an essential part of the paper of Tóth and Werner (1998). One version of this object (Fontes et al., 2002), which is defined as the closure of a countable collection of coalescing Brownian motions starting from a dense countable set of space-time points, has more than one path starting from non-generic random points. Arratia, as well as Tóth and Werner, introduced certain criteria (flow continuity and right continuity respectively) to pick out a unique path. On the other hand the scaling limit of the forward and backward coalescing walks will have multiple paths starting from random points. This seems to make the tightness argument for coalescing walks difficult in the formulations of Arratia and of Tóth and Werner. Fontes, Isopi, Newman and Ravishankar (2004; 2006) introduced a space and metric for the walks as well as the continuum object which allowed them to keep *all* the paths passing through all points and thus were able to prove first tightness for coalescing walks and then convergence to the continuum object which they called the double Brownian web (DBW).

In this paper we use the ideas introduced in Fontes et al. (2004, 2006) to obtain tightness for the family of measures describing the rescaled lattice filling curves and prove the convergence of the distribution of the rescaled Tóth lattice filling curve to the distribution of the Tóth-Werner plane filling curve. This is stated as Theorem 4.2 below and is the major result of this paper. While the main contribution is the tightness argument, we have also included a short section on the proof of convergence of finite dimensional distributions for completeness.

We will first give a brief description and definition of the Tóth-Werner plane filling curve (ω_{BW}) discussed above and also a characterization for it. In the remainder of the paper we will prove that the (rescaled) random lattice filling curve ω_n (the boundary between the forward and backward coalescing random walk paths) converges in distribution to the random plane filling curve ω_{BW}. The appendix gives a brief description of the metric space where the Brownian web and double Brownian web are defined and relevant characterization and convergence results.

2. Characterization of the Tóth-Werner plane filling curve

When taking the scaling limit, one can focus on fixed finite regions of $\mathbb{R}^2$, or consider the whole $\mathbb{R}^2$ at once. The second option avoids dealing with boundary issues, but requires an appropriate choice of metric.

A convenient way of dealing with the whole $\mathbb{R}^2$ is to replace the Euclidean metric with a distance function $\Delta(\cdot,\cdot)$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$\Delta(u,v) = \inf_{\varphi} \int (1+|\varphi|^2)^{-1}\, ds, \tag{2.1}$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parametrized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere. We denote by $\mathcal{I}$ the point at infinity in $\dot{\mathbb{R}}^2$.

Let $\mathbb{C}$ be the space of continuous curves $\omega : [-\infty,\infty] \to \dot{\mathbb{R}}^2$, such that $\omega(-\infty) = \omega(\infty) = \mathcal{I}$. We now define a metric D on $\mathbb{C}$, by

$$D(\omega_1,\omega_2) = \sup_{-\infty\le t\le\infty} \Delta(\omega_1(t),\omega_2(t)).$$

Let $\mathcal{F}_{\mathbb{C}}$ denote the Borel sigma algebra of subsets of $(\mathbb{C},D)$. If ν_n is a sequence of probability measures on $(\mathbb{C},\mathcal{F}_{\mathbb{C}})$, then by the Arzela-Ascoli theorem, a sufficient condition for tightness can be stated as follows:

Theorem 2.1. *A sequence of probability measures ν_n on $(\mathbb{C},\mathcal{F}_{\mathbb{C}})$ is tight if it satisfies the following: for any $\rho > 0$,*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \nu_n[\sup_{s,t\in[-\infty,\infty],\,|t-s|<\delta} |\Delta(\omega(t),\omega(s))| \ge \rho] = 0.$$

We define the first time that a continuous curve $\omega \in \mathbb{C}$ visits y as $\tau(y) = \inf\{t|\omega(t) = y\}$. In this and subsequent sections, we will use the notations for the Brownian web and double Brownian web from the Appendix (Section 5).

Definition 2.2. *Let ω' denote an element of the sample space of the DBW. We call an ω'-dependent point y in the plane "generic" if there is a unique forward and unique backward path from y.*

Remark 2.3. *We point out that from Proposition 5.2 in the Appendix it follows that any deterministic point is generic (μ_{DBW})-almost surely. By an elementary argument using Fubini's theorem it follows that non-generic points have zero Lebesgue measure (μ_{DBW})-almost surely.*

Definition 2.4. *For an ω'-dependent generic point $y \in \mathbb{R}^2$ let f_y^{BW}, b_y^{BW} denote the forward and backward paths starting from y. Define the random variable $\tau_{BW}(y)$ on $((\mathcal{H},\mathcal{F}_{\mathcal{H}}) \times (\mathcal{H}^b,\mathcal{F}^b_{\mathcal{H}}),\mu_{DBW})$ where μ_{DBW} is the double Brownian web measure as $\tau_{BW}(y) =$ planar area enclosed by the four paths $f_y^{BW}, f_0^{BW}, b_y^{BW}, b_0^{BW}$.*

Henceforth, we let $y_1, y_2, \cdots$ be a deterministic dense countable set of points in $\mathbb{R}^2$ ($y_j = \mathcal{I}$ is not allowed). We note that Tóth and Werner (1998) defined a measure μ_{BW} on $(\mathbb{C},\mathcal{F}_{\mathbb{C}})$ with the property that the finite dimensional distributions of $\{\tau(y_i), i = 1,2,\cdots\}$ under μ_{BW} are those of $\{\tau_{BW}(y_i), i = 1,2,\cdots\}$.

Proposition 2.5. *If there exists a measure μ'_{BW} on $(\mathbb{C},\mathcal{F}_{\mathbb{C}})$ such that the finite dimensional distributions of $\{\tau(y_i), i = 1,2,\cdots\}$ under μ'_{BW} are equal to the finite dimensional distributions of $\{\tau_{BW}(y_i), i = 1,2,\cdots\}$ then $\mu'_{BW} = \mu_{BW}$. We denote a corresponding $\mathbb{C}$-valued random variable by ω_{BW}.*

Proposition 2.5 follows from the next two lemmas.

Lemma 2.6. a) $\{\tau_{BW}(y) : y \text{ is generic }\}$ *is* μ_{DBW}*-almost surely dense in* $\mathbb{R}$.

b) *For* μ_{DBW}*-almost every* ω', *if* $z_1(\omega')$ *and* $z_2(\omega')$ *are generic and also*

$$\tau_{BW}(z_1(\omega')) < \tau_{BW}(z_2(\omega')),$$

then there exists at least one (actually infinitely many) $\tau_{BW}(y_i)$*'s in*

$$(\tau_{BW}(z_1), \tau_{BW}(z_2))$$

c) $\{\tau_{BW}(y_i), i = 1, 2, \cdots\}$ *is* μ_{DBW}*-almost surely dense in* $\mathbb{R}$.

Remark 2.7. *We point out that the proof of Lemma 2.6 follows the arguments used to prove Lemma 3.2 of Tóth and Werner (1998) except for modifications to take care of the fact that there can be more than one path starting from some random points.*

Proof: Proof of claim a): Suppose a) is not true. Then with positive μ_{DBW}-probability there exist generic points z_1', z_2', with $\tau_{BW}(z_1') < \tau_{BW}(z_2')$ such that if $t \in (\tau_{BW}(z_1'), \tau_{BW}(z_2'))$, then no generic y exists in $\mathbb{R}^2$ such that $\tau_{BW}(y) = t$. From properties of Brownian motion we know that the four paths $f_{z_1'}^{BW}, f_{z_2'}^{BW}, b_{z_1'}^{BW}, b_{z_2'}^{BW}$ μ_{DBW}-almost surely enclose a region $A^{BW}(z_1', z_2')$ of positive Lebesgue measure (in fact with nonempty interior, as we shall see shortly). Moreover if $z_3' \in A^{BW}(z_1', z_2')$ is generic (and z_3' is distinct from z_1', z_2') then $\tau_{BW}(z_3') \in (\tau_{BW}(z_1'), \tau_{BW}(z_2'))$. This implies that all points in the region $A^{BW}(z_1', z_2')$ are non-generic. This in turn implies that with positive μ_{DBW}-probability non-generic points have positive Lebesgue measure contradicting Remark 2.3. This proves claim a)

Proof of claim b): If z_1' and z_2' are generic, then z_1' can not lie on $f_{z_2'}^{BW}$ or $b_{z_2'}^{BW}$ and z_2' can not lie on $f_{z_1'}^{BW}$ or $b_{z_1'}^{BW}$. Then the planar region $A^{BW}(z_1', z_2')$ strictly enclosed by the four paths above is a nonempty open set (of area $= \tau_{BW}(z_2') - \tau_{BW}(z_1')$) and hence contains at least one (in fact infinitely many) y_i since the y_i's are dense in $\mathbb{R}^2$. By the definition of τ_{BW}, there must be at least one $\tau_{BW}(y_i)$ in the time interval $(\tau_{BW}(z_1'), \tau_{BW}(z_2'))$. This proves claim b).

Claim c) follows from claims a) and b).

Lemma 2.8. *Let* μ_1' *and* μ_2' *be two measures on* $(\mathbb{C}, \mathcal{F}_{\mathbb{C}})$ *(and let the corresponding random variables be denoted by* ω_1 *and* ω_2*) such that* $\{\tau_{\omega_1}(y_i), i = 1, 2, \cdots\}$ *and* $\{\tau_{\omega_2}(y_i), i = 1, 2, \cdots\}$ *are both dense (*μ_1' *and* μ_2' *respectively) almost surely. If the finite dimensional distribution of* $\{\tau_{\omega_1}(y_i), i = 1, 2, \cdots\}$ *and* $\{\tau_{\omega_2}(y_i), i = 1, 2, \cdots\}$ *are equal then* $\mu_1' = \mu_2'$.

Proof: For a given $t \in (-\infty, \infty)$, let $I_n(t) = max\{\tau(y_i) : i \leq n \text{ and } \tau_i \leq t\}$ and $Z_n^1(t) = \omega_1(I_n(t)), Z_n^2(t) = \omega_2(I_n(t))$. Since $\{\tau(y_i)\}$ is dense and $\omega_1(.)$, $\omega_2(\cdot)$ are continuous we have $Z_n^1(t) \to \omega_1(t)$and $Z_n^2(t) \to \omega_2(t)$ almost surely. Since the finite dimensional distributions of $\{\tau_{\omega_1}(y_i), i = 1, 2\cdots\}$ and $\{\tau_{\omega_2}(y_i), i = 1, 2, \cdots\}$ are identical, the finite dimensional distributions of $Z_n^1(t)$ and $Z_n^2(t)$ are also identical. This proves that $\mu_1' = \mu_2'$.

Proof of Proposition 2.5: Since the finite dimensional distributions of $\{\tau(y_i), i = 1, 2, \cdots\}$ under μ_{BW}' are equal to the finite dimensional distributions of $\{\tau_{BW}(y_i), i = 1, 2, \cdots\}$, it follows from Lemma 2.6 that $\{\tau(y_i), i = 1, 2, \cdots\}$ is almost surely dense. Lemma 2.8 then implies that $\mu_{BW}' = \mu_{BW}$.

3. Tightness

Let $L, T > 0$ and define $\Lambda_{T,L} = [-T,T] \times [-L,L]$. Let $a_n > 0$ be a sequence of positive numbers converging to zero and let μ_n be the measure on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}}) \times (\mathcal{H}^b, \mathcal{F}^b_{\mathcal{H}})$ induced by forward coalescing random walks on the rescaled lattice $\mathcal{L}_e(a_n) = \{(a_n u, \sqrt{a_n} v) | (u,v) \in \mathbb{Z}^2_e\}$ and backward coalescing random walks on the rescaled lattice $\mathcal{L}_o(a_n) = \{(a_n u, \sqrt{a_n} v) | (u,v) \in \mathbb{Z}^2_o\}$. Almost every realization of forward random walks on $\mathcal{L}_e(a_n)$ and backward random walks on $\mathcal{L}_o(a_n)$ uniquely defines a curve $\omega_n = (\omega^n_t, \omega^n_x) \in \mathbb{C}$ that fills the lattice $\mathcal{L}(a_n) = \{(a_n u, \sqrt{a_n} v) | (u,v) \in \mathbb{Z}^2 + (1/2, 0)\}$,which is the boundary between the collections of forward and backward paths.

It was shown by Toth (1995) that ω_n can be parametrized in a natural way using the forward and backward paths. For all $(u,v) \in \mathbb{Z}^2, n \in \mathbb{N}$ let $f^n_{(u,v)}$ be the forward path starting from the $\mathcal{L}_e(a_n)$ site adjacent to $(a_n(u+1/2), \sqrt{a_n} v)$ and similarly define a backward path $b^n_{(u,v)}$. The epoch s when ω_n visits $(a_n(u+1/2), \sqrt{a_n} v)$ is given by the area $A^n((0,0),(u,v))$ in $\mathbb{R}^2$ enclosed by $f^n_{(0,0)}, f^n_{(u,v)}, b^n_{(u,v)}$ and $b^n_{(0,0)}$ and the line segments joining $(0,0), (a_n, 0)$ and $(a_n u, \sqrt{a_n} v), (a_n(u+1), \sqrt{a_n} v)$. Let μ'_n denote the distribution of the corresponding $\mathbb{C}$-valued random variable $\omega_n(.)$ on $((\mathcal{H}, \mathcal{F}_{\mathcal{H}}) \times (\mathcal{H}^b, \mathcal{F}^b_{\mathcal{H}}), \mu_n)$ which identifies μ_n- almost every element of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}}) \times (\mathcal{H}^b, \mathcal{F}^b_{\mathcal{H}})$ with its unique boundary curve (separating backward and forward paths). We gave a characterization of the distribution of ω_{BW} in Proposition 2.5. In the rest of this paper we prove that $\omega_n \Rightarrow \omega_{BW}$.

Let $A'_{T,L}(\epsilon, \delta)$ denote the event in $(\mathbb{C}, \mathcal{F}_{\mathbb{C}})$ that $\exists\ s_1, s_2$ with $|s_1 - s_2| \leq \delta$ and with $\omega(s_1), \omega(s_2) \in \Lambda_{T,L}$ such that $|\omega(s_1) - \omega(s_2)| \geq \epsilon$. Using standard properties of random walks (or Brownian motion) it is not difficult to show that for all $C > 0$, for all $\epsilon' > 0$ there exists $L(\epsilon'), T(\epsilon')$ such that $\inf_n(\mu'_n(\omega(s) \in \Lambda_{T,L}$ for all $s \in [-C, C])) \geq 1 - \epsilon'$. (For example, consider the rectangle $\Lambda_{L^{2+\alpha}, L^{1+\alpha}}$ where $0 < \alpha < 1$ and the forward random walk paths in $\mathcal{L}_e(a_n)$ which start from 0 and L. These paths will stay within $\Lambda_{L^{2+\alpha}, L^{1+\alpha}}$, meet before time $L^{2+\alpha}$ and enclose an area greater than $CL^{3-\alpha}$ with probability approaching one as $L \to \infty$. Of course, the same analysis applies to the backward paths.) Then it follows from Theorem 2.1 that the family of measures $\{\mu'_n\}$ is tight if

$$\forall L, T > 0, \forall \epsilon > 0, \limsup_{\delta \downarrow 0} \limsup_{n \to \infty} \mu'_n(A'_{T,L}(\epsilon, \delta)) = 0.$$

Theorem 3.1. *The family of measures $\{\mu'_n\}$ is tight.*

Proof: We prove the theorem by contradiction. Suppose that $\{\mu'_n\}$ is not tight. Then since $A'_{T,L}(\epsilon, \delta)$ decreases as δ decreases, $\exists$ some $L, T > 0$, $\epsilon > 0$ and subsequence $\{n_j\}$ of $\{n\}$ such that for all $\delta > 0$,

$$\mu'_{n_j}(A'_{T,L}(\epsilon, \delta)) \geq \beta > 0, \forall j \in \mathbb{N}\,, \tag{3.2}$$

for some $\beta > 0$.

To describe the strategy of the proof, we first recall that μ'_n, a measure on scaled lattice filling curves, was induced by the measure μ_n on the collections of scaled backward and forward random walk paths, and also that μ_n converges to the Double Brownian web measure μ_{DBW} as $n \to \infty$. Our strategy is then to show that (3.2) would lead to a contradiction concerning the DBW — namely, that the DBW would, with strictly positive probability, contain forward and backward paths passing through each of two distinct points (t_1, x_1) and (t_2, x_2), which altogether would enclose *zero* area. As we shall see, this would contradict standard properties of the forward and backward Brownian motions from which the DBW is constructed.

Let $0 < \gamma < \epsilon/2$ be given and $\bar{B}((t,x),a)$ be the closed ball of radius a centered at (t,x). Define $A_{T,L}(\epsilon,\delta,\gamma)$ as the event that $(K, K^b) \in \mathcal{H} \times \mathcal{H}^b$ is such that there exist $(t_1,x_1),(t_2,x_2) \in \Lambda_{L,T}$ with $|(t_1,x_1)-(t_2,x_2)| \geq \epsilon$ and K contains forward paths $f^{\gamma}_{(t_1,x_1)}, f^{\gamma}_{(t_2,x_2)}$ which start or pass through $(t'_1,x'_1) \in \bar{B}((t_1,x_1),\gamma),(t'_2,x'_2) \in \bar{B}((t_2,x_2),\gamma)$ respectively with $t'_1 \geq t_1, t'_2 \geq t_2$ and K^b contains backward paths $b^{\gamma}_{(t_1,x_1)}, b^{\gamma}_{(t_2,x_2)}$ which start or pass through $(t''_1,x''_1) \in \bar{B}((t_1,x_1),\gamma),(t''_2,x''_2) \in \bar{B}((t_2,x_2),\gamma)$ respectively with $t''_1 \leq t_1, t''_2 \leq t_2$ and the area enclosed by $f^{\gamma}_{(t_1,x_1)}, b^{\gamma}_{(t_1,x_1)}$, the line segment joining (t'_1,x'_1) and (t''_1,x''_1), $f^{\gamma}_{(t_2,x_2)}, b^{\gamma}_{(t_2,x_2)}$, and the line segment joining (t'_2,x'_2) and (t''_2,x''_2) is less than or equal to δ.

Suppose (3.2) is true and let δ_i and γ_k be sequences converging to zero. Then we have

$$\liminf_{i\to\infty} \liminf_{k\to\infty} \liminf_{j\to\infty} \mu_{n_j}(A_{T,L}(\epsilon,\delta_i,\gamma_k)) \geq \beta > 0.$$

Since the measures $\{\mu_{n_j}\}$ converge to the Double Brownian Web (DBW) (Soucaliuc et al., 2000; Fontes et al., 2004) and $A_{T,L}(\epsilon,\delta_i,\gamma_k)$ is a closed subset of $(\mathcal{H} \times \mathcal{H}^b, d_{\mathcal{H}^D})$ [see Section 5.2], we have,

$$0 < \beta \leq \liminf_{i\to\infty} \liminf_{k\to\infty} \liminf_{j\to\infty} \mu_{n_j}(A_{T,L}(\epsilon,\delta_i,\gamma_k)) \tag{3.3}$$

$$\leq \liminf_{i\to\infty} \liminf_{k\to\infty} \limsup_{j\to\infty} \mu_{n_j}(A_{T,L}(\epsilon,\delta_i,\gamma_k)) \tag{3.4}$$

$$\leq \liminf_{i\to\infty} \liminf_{k\to\infty} \mu_{DBW}(A_{T,L}(\epsilon,\delta_i,\gamma_k)) . \tag{3.5}$$

Since $A_{T,L}(\epsilon,\delta_i,\gamma)$ decreases when γ decreases

$$\liminf_{k\to\infty} \mu_{DBW}(A_{T,L}(\epsilon,\delta_i,\gamma_k)) = \mu_{DBW}(\cap_{k=1}^{\infty} A_{T,L}(\epsilon,\delta_i,\gamma_k))$$

and since $A_{T,L}(\epsilon,\delta,\gamma_k)$ decreases when δ decreases

$$\liminf_{i\to\infty} \mu_{DBW}(\cap_{k=1}^{\infty} A_{T,L}(\epsilon,\delta_i,\gamma_k)) = \mu_{DBW}(\cap_{i=1}^{\infty} \cap_{k=1}^{\infty} A_{T,L}(\epsilon,\delta_i,\gamma_k)) .$$

If $K \times K^b \in \cap_{i=1}^{\infty} \cap_{k=1}^{\infty} A_{L,T}(\epsilon,\delta_i,\gamma_k)$ then, since K and K^b are compact subsets of Π and Π^b respectively, $K \times K^b$ contains pairs of forward and backward paths passing through two points $(t_1,x_1),(t_2,x_2) \in \Lambda_{T,L}$ with $|(t_i,x_1)-(t_2,x_2)| \geq \epsilon$,which enclose zero area.

Therefore we conclude that with positive probability the DBW contains forward and backward paths passing through two points $(t_1,x_1),(t_2,x_2) \in \Lambda_{L,T}$ which enclose zero area. Suppose $t_1 = t_2$. Then this implies that the DBW contains either a forward or a backward path which traverses a distance $\epsilon/2$ in zero time contradicting the fact that realizations of the DBW are compact sets of continuous functions of time. Suppose $t_1 \neq t_2$ (we assume $t_1 < t_2$). Then it follows that there exist real numbers $a < b$ and forward (respectively, backward) paths in the DBW passing through points $(\tilde{t}_1,\tilde{x}_1),\tilde{t}_1 > t_1$ (respectively, $(\tilde{t}_2,\tilde{x}_2),\tilde{t}_2 < t_2$) which enclose zero area during the time interval $[a,b]$. That is, $K \in \mathcal{H}$ contains a path $f_{(\tilde{t}_1,\tilde{x}_1)}(t)$ passing through $(\tilde{t}_1,\tilde{x}_1)$ and $K^b \in \mathcal{H}^b$ contains a path $b_{(\tilde{t}_2,\tilde{x}_2)}(t)$ such that

$f^{BW}_{(\tilde{t}_1,\tilde{x}_1)}(t) = b^{BW}_{(\tilde{t}_2,\tilde{x}_2)}(t)$ for a.e. $t \in [a,b]$. But from the construction of the DBW (see Soucaliuc et al. (2000); Fontes et al. (2006) and also Prop. 4.2 of Fontes et al. (2004)) we know that the points where a forward and a backward path meet have the same distribution as the record points of Brownian motion which in turn have the same distribution as the zeros of Brownian motion. This would lead to the conclusion that with positive probability the set of zeros of Brownian motion has positive Lebesgue measure, but this contradicts the well known property of Brownian motion that the set of zeroes of Brownian motion has zero Lebesgue measure almost surely. Therefore we conclude that the family of measures $\{\mu'_n\}$ is tight.

4. Finite Dimensional Distributions

Let $y_i, i = 1, 2, \cdots$ be a dense countable set of points in $\mathbb{R}^2$ and define a $[-\infty, \infty]$-valued random variable τ_i, the first hitting time of y_i, as $\tau_i = \inf\{t : \omega(t) = y_i\}$. Recall that the random lattice filling curve in the lattice $\mathcal{L}(a_n)$ is denoted by ω_n. Let τ_i^ϵ be the approximation to τ_i defined as

$$\tau_i^\epsilon = \int_0^1 g_\epsilon(r)\tau'_{B(y_i,r)}dr$$

where $\tau'_{B(y_i,r)}$ is the first hitting time of an open disk of radius r centered at y_i and the function g_ϵ is defined (for example) as follows:

$$g_\epsilon(r) = \begin{cases} 0 & \text{for } r \leq \epsilon, \\ \frac{4}{\epsilon^2}(r-\epsilon) & \text{for } \epsilon < r < 3\epsilon/2, \\ \frac{-4}{\epsilon^2}(r - 3\epsilon/2) + \frac{2}{\epsilon} & \text{for } 3\epsilon/2 \leq r < 2\epsilon, \\ 0 & \text{for } r \geq 2\epsilon. \end{cases} \tag{4.6}$$

Note that g_ϵ satisfies $\int_\epsilon^{2\epsilon} g_\epsilon(r)dr = 1$. It is easy to see that τ_i^ϵ as a function on $(\mathbb{C}, D)$, is continuous and $\lim_{\epsilon\to 0} \tau_i^\epsilon = \tau_i$.

Theorem 4.1. *Let μ'' denote a sub-sequential limit of the sequence $\{\mu'_n\}$ along a sub-sequence μ'_{n_k} and denote the corresponding random variable by ω''. Then the finite dimensional distributions of the random variables $\{\tau_i(\omega'')\}$ are identical to the finite dimensional distributions of $\{\tau_{BW}(y_i)\}$.*

Proof: Since for all ϵ and i, τ_i^ϵ is a continuous function on $(\mathbb{C}, D)$, it follows that the finite dimensional distributions of $\{\tau_i^\epsilon(\omega_{n_k})\}$ converge to those of $\{\tau_i^\epsilon(\omega'')\}$. We want to show that the finite dimensional distributions of $\{\tau_i^\epsilon(\omega'')\}$ converge to those of $\{\tau_{BW}(y_i)\}$ as $\epsilon \to 0$. For each $i = 1, 2, \cdots$ let $y_i(n) = (y_i^t(n), y_i^x(n)) \in \mathcal{L}(a_n)$ be a sequence of points converging to y_i as $n \to \infty$. Let $\tau_i^n(\omega_n) = \inf\{t | \omega_n(t) = y_i(n)\}$. Then it is easy to see that for each ϵ and i, $\tau_i^\epsilon(\omega_n) \leq \tau_i^n(\omega_n)$ for large enough $n = n(\epsilon), \mu'_n - a.s.$

Therefore for any $m \in \mathbb{N}$ and $t_j \in \mathbb{R}$, $j = 1, 2, \cdots, m$ and large enough $n(\epsilon)$,

$$\mu'_n(\tau_1^\epsilon > t_1, \tau_2^\epsilon > t_2, \cdots \tau_m^\epsilon > t_m) \leq \mu'_n(\tau_1^n > t_1, \tau_2^n > t_2, \cdots \tau_m^n > t_k). \tag{4.7}$$

By definition, $\tau_i^n(\omega_n)$ is the area $A^n((0,0), (y_i^t(n) - a_n/2, y_i^x(n)))$. From a convergence property of the double Brownian web (Fontes et al., 2006) we know that $\{f^n_{y_1}, f^n_{y_2} \cdots f^n_{y_m}, b^n_{y_1}, b^n_{y_2}, \cdots b^n_{y_m}\}$ converge jointly in distribution to coalescing forward and backward Brownian motions $\{W_1, W_2, \cdots W_m, W_1^b, W_2^b, \cdots W_m^b\}$ starting from $y_1, y_2, \cdots y_m$. By Skorohod's theorem there exist random variables

$$\{f'^n_{y_1}, f'^n_{y_2} \cdots f'^n_{y_m}, b'^n_{y_1}, b'^n_{y_2} \cdots b'^n_{y_m}\} \text{ and } \{W'_1, W'_2, \cdots W'_m, W'^b_1, W'^b_2, \cdots W'^b_m\}$$

on some probability space which have the same joint distribution as

$$\{f^n_{y_1}, f^n_{y_2} \cdots f^n_{y_m}, b^n_{y_1}, b^n_{y_2}, \cdots b^n_{y_m}\} \text{ and } \{W_1, W_2, \cdots W_m, W^b_1, W^b_2, \cdots W^b_m\}$$

respectively such that $f'^n_{y_i}$ converges almost surely to W'_i and $b^n_{y_i}$ converges to W'^b_i almost surely for all $1 \le i \le m$. Let us denote the area enclosed by $f'^n_{y_i}, b'^n_{y_i}, f'^n_0, b'^n_0$ (and two line segments: one connecting the starting points of $f'^n_{y_i}$ and $b'^n_{y_i}$, another connecting those of f'^n_0 and b'^n_0) by $A'^n((0-a_n/2, 0), (y^t_i(n) - a_n/2, y^x_i(n)))$. Then using the property of Brownian motion that two coalescing Brownian motions starting distance d apart enclose an area before coalescing which goes to zero almost surely as $d \to 0$ we can conclude that $A'^n((0-a_n/2, 0), (y^t_i(n) - a_n/2, y^x_i(n)))$ converges almost surely to $A'^{BW}((0,0), (y^t_i, y^x_i))$ where $A'^{BW}((0,0), (y^t_i, y^x_i))$ is the area enclosed by $W'_i, W'^b_i, W'_0, W'^b_0$. From this it follows that for all $m \in \mathbb{N}$,the joint distribution of the areas
$A^n((0-a_n/2, 0), (y^t_1(n) - a_n/2, y^x_1(n))), A^n((0-a_n/2, 0), (y^t_2(n) - a_n/2, y^x_2(n)),$
$\cdots A^n((0-a_n/2, 0), (y^t_m(n) - a_n/2, y^x_m(n))$ converges to the joint distribution of $A^{BW}((0,0), (y^t_1, y^x_1)), A^{BW}((0,0), (y^t_2, y^x_2), \cdots A^{BW}((0,0), (y^t_m, y^x_m))$. Thus we conclude that the finite dimensional distributions of $\tau^n_i(\omega_n)$ converge as $n \to \infty$ to those of $\{\tau_i(\omega_{BW})\}$. Therefore taking a subsequential limit along the sequence $\{n_k\}$ in Equation (4.7), we have

$$\mu''(\tau^\epsilon_1 > t_1, \tau^\epsilon_2 > t_2, \cdots \tau^\epsilon_m > t_m) \le \mu_{DBW}(\tau_1 > t_1, \tau_2 > t_2, \cdots \tau_m > t_m). \tag{4.8}$$

To complete the proof, we need to get an inequality in the opposite direction when $\epsilon \to 0$. Let $(t,x) = y \in \mathbb{R}^2$ and for all $\epsilon > 0$ again let $B(y,\epsilon)$ denote the open ball of radius ϵ centered at y. Let $\tilde{\epsilon} = \frac{|y|}{8}\hat{\epsilon}$. Let us denote a forward path in $K \in \mathcal{H}$ starting or passing through y' by $\tilde{f}^K_{y'}$ and similarly a backward path in $K^b \in \mathcal{H}^b$ starting from or passing through y'' by $\tilde{b}^{K^b}_{y''}$. Let $(K, K^b) = K^d \in \mathcal{H} \times \mathcal{H}^b$. Let $z, z' \in B(0, 4\tilde{\epsilon})$ and $w, w' \in B(y, 4\tilde{\epsilon})$, where $(t^z, x^z) = z, (t'^z, x'^z) = z'$, with $t^z \ge t'^z$, $(t^w, x^w) = w, (t'^w, x'^w) = w'$, with $t^w \ge t'^w$. Also we denote by $A^{K^d}(z, z', w, w')$ the area enclosed by $\tilde{f}^K_z, \tilde{f}^K_w, \tilde{b}^{K^b}_{z'}, \tilde{b}^{K^b}_{w'}$ and the line segment joining z, z' and the line segment joining w, w'.

Define for all $y_1, y_2, \cdots$,

$$\tau_{B(y_i,4\epsilon)} = \tau_{B(y_i,4\epsilon)}(K^d) = \inf_{w,w',z,z'}(A^{K^d}(z,z',w,w')),$$

where the infimum is over z, z', w, w' as above but with y replaced by y_i and $\tilde{\epsilon}$ by ϵ. We know that $\mu_n \Rightarrow \mu_{DBW}$ and that the family of measures $\{\mu_n\}$ is tight (Fontes et al., 2004). Thus for all $\epsilon' > 0$, there exists a compact set $M_{\epsilon'} \in \mathcal{H} \times \mathcal{H}^b$ such that $\mu_n(M_{\epsilon'}) \ge 1 - \epsilon'$ for all $n \in \mathbb{N}$. Using the compactness of $M_{\epsilon'}$ it is easy to see that $\tau_{B(y_i,4\epsilon)}(K^d) 1_{\{M_{\epsilon'}\}}(K^d)$ is a lower semi-continuous function on $\mathcal{H} \times \mathcal{H}^b$.

Therefore for small enough ϵ,

$$\begin{aligned}
& \mu_{DBW}(\tau_{B(y_1,4\epsilon)} > t_1, \tau_{B(y_2,4\epsilon)} > t_2, \cdots, \tau_{B(y_m,4\epsilon)} > t_m) \\
\leq\ & \mu_{DBW}(\tau_{B(y_1,4\epsilon)} 1_{\{M_{\epsilon'}\}} > t_1, \cdots, \tau_{B(y_m,4\epsilon)} 1_{\{M_{\epsilon'}\}} > t_m) + \epsilon' \\
\leq\ & \liminf_n \mu_n(\tau_{B(y_1,4\epsilon)} 1_{\{M_{\epsilon'}\}} > t_1, \cdots, \tau_{B(y_m,4\epsilon)} 1_{\{M_{\epsilon'}\}} > t_m) + \epsilon' \\
\leq\ & \liminf_n \mu_n(\tau_{B(y_1,4\epsilon)} > t_1, \cdots, \tau_{B(y_m,4\epsilon)} > t_m) + \epsilon' \\
\leq\ & \limsup_k \mu_{n_k}(\tau'_{B(y_1,4\epsilon)} > t_1, \cdots, \tau'_{B(y_m,4\epsilon)} > t_m) + \epsilon' \\
\leq\ & \limsup_k \mu'_{n_k}(\tau_1^{2\epsilon} \geq t_1, \cdots, \tau_m^{2\epsilon} \geq t_m) + \epsilon' \\
\leq\ & \mu''(\tau_1^{\epsilon} > t_1, \cdots, \tau_m^{\epsilon} > t_m) + \epsilon'. \qquad (4.9)
\end{aligned}$$

In the third line we have used lower semi-continuity and in the fifth line we have used the fact that for large enough n, $\tau'_{B(y_i,4\epsilon)}(\omega_n) \geq \tau_{B(y_i,4\epsilon)}(\omega_n)$. In the last line we have used the fact that $\{\tau_i^{2\epsilon} \geq t\}$ is a closed subset of $(\mathbb{C}, D)$ and that $\tau_i^{\epsilon} \geq \tau_i^{2\epsilon}$, μ''-almost surely.

Since the above equation is true for all $\epsilon' > 0$ we conclude that

$$\mu_{DBW}(\tau_{B(y_1,4\epsilon)} > t_1, \tau_{B(y_2,4\epsilon)} > t_2, \cdots, \tau_{B(y_m,4\epsilon)} > t_m) \qquad (4.10)$$
$$\leq \mu''(\tau_1^{\epsilon} > t_1, \tau_2^{\epsilon} > t_2, \cdots \tau_m^{\epsilon} > t_m). \qquad (4.11)$$

Now from (4.8) and (4.10) we have

$$\begin{aligned}
& \mu_{DBW}(\tau_{B(y_1,4\epsilon)} > t_1, \tau_{B(y_2,4\epsilon)} > t_2, \cdots, \tau_{B(y_m,4\epsilon)} > t_m) \\
\leq\ & \mu''(\tau_1^{\epsilon} > t_1, \tau_2^{\epsilon} > t_2, \cdots, \tau_m^{\epsilon} > t_m) \qquad (4.12) \\
\leq\ & \mu_{DBW}(\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_m > t_m). \qquad (4.13)
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ we conclude that the finite dimensional distributions of $\{\tau_i(\omega'')\}$ are identical to those of $\{\tau_i(\omega_{BW})\}$. Therefore it follows from Proposition 2.5 that $\mu'' = \mu_{BW}$. We have proved:

Theorem 4.2. $\omega_n \Rightarrow \omega_{BW}$*; i.e., the rescaled Tóth lattice filling curve converges in distribution to the Tóth-Werner plane filling curve in the diffusive scaling limit.*

5. Appendix

5.1. *Brownian Web:* We recall first Fontes, Isopi, Newman and Ravishankar's (2002; 2004) choice of the metric space in which the Brownian Web takes its values.

Let $(\bar{\mathbb{R}}^2, \rho)$ be the completion (or compactification) of $\mathbb{R}^2$ under the metric ρ, where

$$\rho((t_1, x_1), (t_2, x_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \qquad (5.14)$$

$\bar{\mathbb{R}}^2$ can be thought of as the image of $[-\infty, \infty] \times [-\infty, \infty]$ under the mapping

$$(t, x) \rightsquigarrow (\Psi(t), \Phi(t, x)) \equiv \left(\tanh(t), \frac{\tanh(x)}{1 + |t|} \right). \qquad (5.15)$$

This compactification of $\mathbb{R}^2$ is rather different than the one-point compactification introduced in Section 2 above; here the points at infinity consist of two intervals at $x = \pm\infty$ (and $-\infty < t < \infty$) together with two points at $t = \pm\infty$.

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions f from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $\Phi(t, f(t))$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} \{t_0\} \times C[t_0], \tag{5.16}$$

where $(t_0, f) \in \Pi$ represents a path in $\bar{\mathbb{R}}^2$ starting at $(t_0, f(t_0))$. For(t_0, f) in Π, we denote by $\hat{f}$ the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for $t < t_0$. Then we take

$$d((t_1, f_1), (t_2, f_2)) = (\sup_t |\Phi(t, \hat{f}_1(t)) - \Phi(t, \hat{f}_2(t))|) \vee |\Psi(t_1) - \Psi(t_2)|. \tag{5.17}$$

(Π, d) is a complete separable metric space.

Let now $\mathcal{H}$ denote the set of compact subsets of (Π, d), with $d_{\mathcal{H}}$ the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2). \tag{5.18}$$

$(\mathcal{H}, d_{\mathcal{H}})$ is also a complete separable metric space. Let $\mathcal{F}_{\mathcal{H}}$ denote the Borel σ-algebra generated by $d_{\mathcal{H}}$.

5.2. *Double Brownian Web:* Our description of the DBW and some of its properties will rely on a paper of Soucaliuc, Tóth and Werner (2000).

We begin with an (ordered) dense countable set $\mathcal{D} \subset \mathbb{R}^2$, and a family of i.i.d. standard B.M.'s $B_1, B_1^b, B_2, B_2^b, \ldots$ and construct forward and backward paths $W_1, W_1^b, W_2, W_2^b, \ldots$ starting from $(t_j, x_j) \in \mathcal{D}$:

$$W_j(t) = x_j + B_j(t - t_j),\ t \geq t_j \tag{5.19}$$

$$W_j^b(t) = x_j + B_j^b(t_j - t),\ t \leq t_j. \tag{5.20}$$

Then we construct coalescing and "reflecting" paths $\tilde{W}_1, \tilde{W}_1^b, \ldots$ inductively, as follows.

$$\tilde{W}_1 = W_1; \quad \tilde{W}_1^b = W_1^b; \tag{5.21}$$

$$\tilde{W}_n = CR(W_n; \tilde{W}_1, \tilde{W}_1^b, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b); \tag{5.22}$$

$$\tilde{W}_n^b = CR(W_n^b; \tilde{W}_1, \tilde{W}_1^b, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b), \tag{5.23}$$

where the operation CR is defined in Soucaliuc et al. (2000), Subsubsection 3.1.4. We proceed to explain CR for the simplest case, in the definition of $\tilde{W}_2$.

As pointed out in Soucaliuc et al. (2000), the nature of the reflection of a forward Brownian path $\tilde{W}$ off a backward Brownian path $\tilde{W}^b$ (or vice-versa) is special. It is actually better described as a push of $\tilde{W}$ off $\tilde{W}^b$ (see Subsection 2.1 in Soucaliuc et al. (2000)). It does not have a simple explicit formula in general, but in the case of one forward path and one backward path, the form is as follows. Following our notation and construction, we ignore $\tilde{W}_1$ and consider $\tilde{W}_1^b$ and $\tilde{W}_2$ in the time interval $[t_2, t_1]$ (we suppose $t_2 < t_1$; otherwise, $\tilde{W}_1^b$ and $\tilde{W}_2$ are independent). Given W_2 and $\tilde{W}_1^b$, for $t_2 \leq t \leq t_1$,

$$\tilde{W}_2(t) = \begin{cases} W_2(t) + \sup_{t_2 \leq s \leq t}(W_2(s) - \tilde{W}_1^b(s))^-, & \text{if } W_2(t_2) > \tilde{W}_1^b(t_2); \\ W_2(t) - \sup_{t_2 \leq s \leq t}(W_2(s) - \tilde{W}_1^b(s))^+, & \text{if } W_2(t_2) < \tilde{W}_1^b(t_2). \end{cases} \tag{5.24}$$

After t_1, $\tilde{W}_2$ interacts only with $\tilde{W}_1$, by coalescence. We call

$$\mathcal{W}_n^D := \{\tilde{W}_1, \tilde{W}_1^b, \dots, \tilde{W}_n, \tilde{W}_n^b\}$$

coalescing/reflecting forward and backward Brownian motions (starting at the point $\{(t_1, x_1), \dots, (t_n, x_n)\}$*).* We will also use the alternative notation $\mathcal{W}^D(\mathcal{D}_n)$ below, essentially in place of $\mathcal{W}_n^D$, where $\mathcal{D}_n := \{(t_1, x_1), \dots, (t_n, x_n)\}$.

Remark 5.1. *In Theorem 8 of Soucaliuc et al. (2000), it is proved that the above construction is a.s. well-defined, gives a perfectly coalescing/reflecting system (see Subsubsection 3.1.1 in Soucaliuc et al. (2000)), and for every* $n \geq 1$*, the distribution of* $\mathcal{W}_n^D$ *does not depend on the ordering of* $\mathcal{D}_n$*. It also follows from that result that* $\{\tilde{W}_1, \dots, \tilde{W}_n\}$ *and* $\{\tilde{W}_1^b, \dots, \tilde{W}_n^b\}$ *are separately forward and backward coalescing Brownian motions, respectively.*

We now define dual spaces of paths going backward in time (Π^b, d^b) and a corresponding $(\mathcal{H}^b, d_{\mathcal{H}^b})$ in an obvious way, so that they are the dual versions of (Π, d) and $(\mathcal{H}, d_{\mathcal{H}})$, and then define $\mathcal{H}^D = \mathcal{H} \times \mathcal{H}^b$ and

$$d_{\mathcal{H}^D}((K_1, K_1^b), (K_2, K_2^b)) = \max(d_{\mathcal{H}}(K_1, K_2), d_{\mathcal{H}^b}(K_1^b, K_2^b)).$$

We now define

$$\mathcal{W}_n^D(\mathcal{D}) = \{\tilde{W}_1, \dots, \tilde{W}_n\} \times \{\tilde{W}_1^b, \dots, \tilde{W}_n^b\}, \tag{5.25}$$

$$\mathcal{W}^D(\mathcal{D}) = \{\tilde{W}_1, \tilde{W}_2, \dots\} \times \{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}, \tag{5.26}$$

$$\bar{\mathcal{W}}^D(\mathcal{D}) = \overline{\{\tilde{W}_1, \tilde{W}_2, \dots\}} \times \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}}. \tag{5.27}$$

The latter closures are in Π for the first factor and in Π^b for the second one.

From Remark 5.1, we have that

$$\bar{\mathcal{W}} := \overline{\{\tilde{W}_1, \tilde{W}_2, \dots\}} \text{ and } \bar{\mathcal{W}}^b := \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}}$$

are forward and backward Brownian webs, respectively.

We now state a proposition which describes some of the properties of the DBW.

Proposition 5.2. $\bar{\mathcal{W}}^D(\mathcal{D})$ *satisfies*

- (o^D) *From any deterministic* (t, x) *there is almost surely a unique forward path and unique backward path.*
- (i^D) *For any deterministic* $\mathcal{D}_n' := \{(s_1, y_1), \dots, (s_n, y_n)\}$ *the forward and backward paths from* $\mathcal{D}_n'$*, denoted* $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}_n')$*, are distributed as coalescing/reflecting forward and backward Brownian motions starting at* $\mathcal{D}_n'$*. In other words,* $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}_n')$ *has the same distribution as* $\mathcal{W}^D(\mathcal{D}_n')$*.*

Definition 5.3. *Let* Y^n *and* $Y^{b,n}$ *denote the forward and backward coalescing random walks on* $\mathcal{L}_e(a_n)$ *and* $\mathcal{L}_o(a_n)$ *respectively.*

Theorem 5.4. $(Y^n, Y^{b,n})$ *converges in distribution as* $n \to \infty$ *to the double Brownian web.*

For a proof of this theorem we refer the reader to Fontes et al. (2003).

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