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A quenched invariance principle for stationary processes

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Dedicated to Michael Lin on his seventieth birthday

Abstract. In this note, we prove a conditionally centered version of the quenched weak invariance principle under the Hannan condition, for stationary processes. In the course, we obtain a (new) construction of the fact that any stationary process may be seen as a functional of a Markov chain.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a probability space and θ be an invertible bi-measurable transformation of X, preserving μ, and assume that θ is ergodic. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{A}$ such that $\mathcal{F}_0 \subset \theta^{-1}(\mathcal{F}_0)$. Define a filtration $(\mathcal{F}_n)_{n\in\mathbf{Z}}$, by $\mathcal{F}_n = \theta^{-n}\mathcal{F}_0$ and denote $\mathcal{F}_{-\infty} = \cap_{n\in\mathbb{Z}}\mathcal{F}_n$. For every $n \in \overline{\mathbb{Z}}$, we denote by $\mathbb{E}_n$ the conditional expectation with respect to $\mathcal{F}_n$ and we define the projection $P_n := \mathbb{E}_n - \mathbb{E}_{n-1}$.

Let f be $\mathcal{F}_0$-measurable. We want to study the stationary process $(f\circ\theta^n)_{n\in\mathbb{N}}$ under the following condition introduced by Hannan (1973) and known as the *Hannan condition*:

$$\sum_{i=0}^{\infty} \|P_i f\|_2 = \sum_{i=0}^{\infty} \|P_0(f\circ\theta^i)\|_2 < \infty. \tag{1.1}$$

If $\mathbb{E}_{-\infty}(f) = 0$, the Hannan Condition guarantees the CLT see Hannan (1973)) and even the weak invariance principle (WIP) (see Dedecker et al. (2007)). The condition has been shown to be very useful in applications, see for instance Dedecker et al. (2007) or Wu (2007) (see also Cuny (2012)). In general, as shown in Durieu (2009), the Hannan Condition is independent of the so-called Dedecker-Rio and Maxwell-Woodroofe conditions, that are also sufficient for the WIP. The "natural"

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L^p versions of the Hannan Condition have been used in Wu (2007) and more recently in Cuny (2012).

Let us denote $S_n = S_n(f) = \sum_{i=1}^n f \circ \theta^i$. We are interested in the *quenched* versions of the CLT or of the WIP (see the next section for a proper definition). It is known that the Hannan condition is not sufficient in general to ensure the validity of quenched CLT. This fact has been proven by Volný and Woodroofe (2010a). However, by Cuny and Peligrad (2012) and Volný and Woodroofe (2010b) the quenched CLT holds under the Hannan condition for $(S_n - \mathbb{E}_0(S_n))_{n\ge 1}$. In this paper, we prove that under the Hannan condition $(S_n - \mathbb{E}_0(S_n))_{n\in\mathbb{N}}$ satisfies the quenched WIP as well. Then we deduce the quenched WIP for $(S_n)_{n\in\mathbb{N}}$ itself under the following stronger condition.

$$\sum_{n\ge 1} \frac{\|\mathbb{E}_0(f\circ\theta^n)\|_2}{\sqrt{n}} < \infty. \tag{1.2}$$

In the proof, we make use of the operator Q defined by $Qg := \mathbb{E}_0(g\circ\theta)$ for every $g \in L^1(X, \mathcal{F}_0, \mu)$. It turns out that this operator is a Markov operator, and allows to see the process $(f\circ\theta^n)_{n\in\mathbb{N}}$ as a functional of a Markov chain. This is explained in full details in Section 4. An important difference with previous constructions (Wu and Woodroofe (2004)) is that the state space is the original X and the functional is f itself.

2. Results

Let $\mu(\cdot,\cdot)$ denote a regular conditional probability on $\mathcal{A}$ given $\mathcal{F}_0$. For every $x \in X$, write $\mu_x := \mu(x,\cdot)$. Thus, for every $x \in X$, μ_x is a probability measure on $\mathcal{A}$, and for every $A \in \mathcal{A}$, $\mu(\cdot, A)$ is a version of $\mu(A|\mathcal{F}_0)$.

Let $f \in L^2(X, \mathcal{F}_0, \mu)$. Recall that $S_n = S_n(f) = \sum_{i=1}^n f\circ\theta^i$ and write, for every $t \in [0,1]$, $S_n(t) = S_{[nt]} + (nt - [nt]) f\circ\theta^{[nt]+1}$ and $\bar{S}_n(t) = S_n(t) - \mathbb{E}_0(S_n(t))$.

Definition 2.1. We say that $(S_n)_{n\ge 1}$ satisfies the quenched CLT if $\sigma^2 := \lim_n \frac{\mathbb{E}(S_n^2)}{n}$ exists and for μ-almost every $x \in X$, for every bounded continuous function φ on $\mathbb{R}$,

$$\int_X \varphi(S_n/\sqrt{n})d\mu_x \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\varphi(\sigma W))\,, \tag{2.1}$$

where W stands for a standard normal variable.

We say that $(\bar{S}_n)_{n\ge 1}$ satisfies the quenched CLT if the above holds with S_n replaced with $\bar{S}_n$.

Definition 2.2. We say that $(S_n)_{n\ge 1}$ satisfies the quenched WIP if $\sigma^2 := \lim_n \frac{\mathbb{E}(S_n^2)}{n}$ exists and for μ-almost every $x \in X$, for every bounded continuous function φ on $C([0,1], \|\cdot\|_\infty)$, we have

$$\int_X \varphi(S_n(t)/\sqrt{n})d\mu_x \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\varphi(\sigma W_t)),$$

where $(W_t)_{0\le t\le 1}$ stands for a standard Brownian motion.

We say that $(\bar{S}_n)_{n\ge 1}$ satisfies the quenched WIP if the above holds with S_n replaced with $\bar{S}_n$.

Theorem 2.3. *Let $f \in L^2(X, \mathcal{A}, \mu)$ satisfy the Hannan condition. Then there exists a martingale $(M_n)_n$ with stationary ergodic increments such that*

$$\mathbb{E}_0(\max_{1 \le n \le N} (\bar{S}_n - M_n)^2) = o(N) \qquad \mu\text{-a.s.} \tag{2.2}$$

In particular, $(\bar{S}_n)_{n \ge 1}$ satisfies the quenched WIP.

The fact that (2.2) implies the quenched WIP for $(\bar{S}_n)_{n\ge 1}$ follows from the fact that the quenched WIP holds for stationary martingale differences. This has been carefully proved by Derriennic and Lin (2001). Their proof was done in the setting of functionals of Markov chains. To see that the result holds in our setting one may see that their arguments apply in this situation or one may just use section 4.

Corollary 2.4. *Let $f \in L^2(X, \mathcal{A}, \mu)$ be such that (1.2) holds. Then, (1.1) holds and $\mathbb{E}_0(S_n) = o(\sqrt{n})$ μ-a.s. In particular, $(S_n)_{n\ge 1}$ satisfies the quenched WIP.*

The fact that (1.2) implies that (1.1) holds and that $\mathbb{E}_0(S_n) = o(\sqrt{n})$ μ-a.s. has been observed in Cuny and Peligrad (2012).

The quenched WIP for $(S_n)_{n\ge 1}$ has been obtained by Cuny and Merlevède (2012) (after this work has been finished) under the Maxwell-Woodroofe condition, namely when

$$\sum_{n \ge 1} \|\mathbb{E}_0(S_n)\|_2 / n^{3/2} < \infty \, .$$

As noticed in Maxwell and Woodroofe (2000), condition (1.2) implies also the Maxwell-Woodroofe condition, hence Corollary 2.4 also follows from Cuny and Merlevède (2012).

3. Proof of the results

Our method of proof is somehow classical, and follows a line that has been particularly well illustrated by Gordin and Peligrad (2011), in the study of the (usual) weak invariance principle.

We first establish a maximal inequality (under μ_x, for μ-a.e. $x \in X$). Then, we combine this inequality with an approximation argument to prove (2.2), which in turn allows to deduce the quenched WIP, from the case of stationary martingale differences.

To illustrate this approach we first give a sketch of the proof of the usual WIP, under the Hannan condition. For another (more recent) use of this approach, we refer to Cuny (2012), where the almost sure invariance principle is established under the Hannan condition.

Let U be the unitary operator defined by $Ug = g \circ \theta$. Then $UP_i = P_{i+1}U$. Let $f \in L^2(X, \mathcal{F}_0, \mu)$ be such that $\mathbb{E}_{-\infty}(f) = 0$. Let us denote $f_i = P_0 U^i f$, $i \in \mathbb{N}$. Then, $f = \sum_{i\in\mathbb{N}} U^{-i} f_i$ and $S_n(f) = \sum_{i\in\mathbb{Z}} \sum_{j=0}^{n-1} U^{j-i} f_i$. Hence, as in Wu (2007), we see that for every $1 \le n \le N$,

$$|S_n(f)| \le \sum_{i\in\mathbb{N}} U^{-i} \max_{1\le k\le N} \Big| \sum_{j=0}^{k-1} U^j f_i \Big| \, ,$$

while for every $i \in \mathbb{N}$, the process $\big(\sum_{j=0}^{k-1} U^j f_i\big)_{k\geq 1}$ is a martingale. Hence, by Doob's Maximal Inequality and orthogonality of $(U^j f_i)_{0\leq j\leq N-1}$,

$$\big(E(\max_{n\leq N} S_n^2(f)\big)^{1/2} \leq 2\sum_{i\in\mathbb{N}} \big(E(S_N^2(f_i))\big)^{1/2} = 2\sqrt{N}\sum_{i\in\mathbb{N}} \|f_i\|_2. \tag{3.1}$$

Write $M_n = \sum_{i=1}^n U^i m$ where $m = \sum_{i\in\mathbb{N}} P_0 U^i f$ is well-defined by (1.1). The WIP will follow, if we can prove that

$$|| \max_{n\leq N} (S_n(f) - M_n)||_2 = o(\sqrt{N}). \tag{3.2}$$

Let $r \geq 1$. Define $m^{(r)} = \sum_{k=0}^r f_k$, $f^{(r)} = \sum_{k=0}^r P_{-k} f$. Then, we have $S_n(f-m) = S_n(f-f^{(r)}) + S_n(f^{(r)} - m^{(r)}) + S_n(m^{(r)} - m)$. For the first term we use (3.1). For the second term we notice that $f^{(r)} - m^{(r)}$ is a coboundary (i.e. $f^{(r)} - m^{(r)} = g - Ug$ with $g \in L^2$). Finally, the third term may be estimated using Doob's maximal inequality. For further details, see the proof of Theorem 2.3.

The proof extends easily to the non-adapted case, i.e. $f \in L^2(X, \mathcal{F}_\infty, \mu)$, where $\mathcal{F}_\infty := \vee_{n\in\mathbb{N}} \mathcal{F}_n$ and to Hilbert valued variables. A different proof of (3.2) for non-adapted Hilbert valued variables under Hannan's condition has been given in Dedecker et al. (2012), where also L^p versions of (3.2) may be found.

We now turn to the proof of Theorem 2.3.

To avoid technical difficulties (and since it is also convenient for the next section) we assume that X is a Polish space and that $\mathcal{A}$ is the σ-algebra of its Borel sets. It is known (see for instance Neveu (1970, Proposition V.4.4)) that in this case there exists a regular version of the conditional probability given $\mathcal{F}_0$ on $\mathcal{A}$. In the general case we can (like in Volný (1989)) transport the situation from X to a Polish space Y with a measure preserving transformation S by a mapping $\psi : X \to Y$ which preserves the measure and for which $\psi \circ T = S \circ \psi$.
We use the notations of the introduction.

For an adapted (i.e. $\mathcal{F}_0$-measurable) function $f \in L^2$ we have

$$f = \sum_{i=0}^\infty P_{-i} f + \mathbb{E}_{-\infty}(f) = \sum_{i=0}^\infty U^{-i} P_0 U^i f + \mathbb{E}_{-\infty}(f) = \sum_{i=0}^\infty U^{-i} f_i + \mathbb{E}_{-\infty}(f)$$

where $f_i = P_0 U^i f$, $i = 0, 1, \ldots$. Therefore, since for every $n \geq 0$, $\mathbb{E}_0(\mathbb{E}_{-\infty}(f)) = \mathbb{E}_{-\infty}(f)$, we have

$$\bar{S}_n(f) = S_n(f) - \mathbb{E}_0(S_n(f)) = \sum_{i=0}^{n-1}\sum_{j=1}^{n-i} U^j f_i. \tag{3.3}$$

Denote, for $h \in L^1$, $Qh = \mathbb{E}_0(Uh)$. Then Q is a Dunford-Schwartz operator (it is a contraction in all L^p, $1 \leq p \leq \infty$). Notice that $Q^n h = \mathbb{E}_0(U^n f)$. The use of the operator Q is crucial in our proof. Its relevance to the problem is made more clear in the next section.

Let us recall several facts from ergodic theory that will be needed in the sequel.

By the Dunford-Schwartz (or Hopf) ergodic theorem (cf. Krengel (1985, Lemma 6.1)), for every $h \in L^1$, denoting $h^* = \sup_{n\geq 1} (1/n) \sum_{i=0}^{n-1} Q^i(|h|)$, we have

$$\sup_{\lambda > 0} \lambda \mu(h^* > \lambda) \leq \|h\|_1. \tag{3.4}$$

We will make use of the weak L^2-space

$$L^{2,w} := \{f \in L^1 \ : \ \sup_{\lambda>0} \lambda^2 \mu\{|f| \geq \lambda\} < \infty\}.$$

Recall (see for instance Ledoux and Talagrand (1991), section "notation") that there exists a norm $\|\cdot\|_{2,w}$ on $L^{2,w}$ that makes it a Banach space and which is equivalent to the pseudo-norm $(\sup_{\lambda>0} \lambda^2\mu\{|f| \geq \lambda\})^{1/2}$.

Then it follows from (3.4), that for every $h \in L^2$,

$$((h^2)^*)^{1/2} \in L^{2,w} \quad \text{and} \quad \|((h^2)^*)^{1/2}\|_{2,w} \leq \|h\|_2. \tag{3.5}$$

We obtain

Lemma 3.1. *Let $f \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$. For every $N \geq 1$ we have*

$$(\mathbb{E}_0(\max_{1\leq n\leq N} \bar{S}_n^2(f)))^{1/2} \leq \sqrt{N} \sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2} \qquad \mu\text{-}a.s. \tag{3.6}$$

In particular, if f satisfies the Hannan condition, then, by (3.5)

$$\sup_{N\geq 1} \frac{\mathbb{E}_0(\max_{1\leq n\leq N} \bar{S}_n^2(f))}{N} < \infty \qquad \mu\text{-}a.s.$$

Proof. Let $N \geq n \geq 1$. From (3.3) it follows that

$$|\bar{S}_n(f)| \leq \sum_{i=0}^{N-1} \max_{1\leq k\leq N} |\sum_{j=1}^{k} U^j f_i|.$$

Notice that for every $i \geq 0$, the process $(U^j f_i)_j$ is a sequence of martingale increments. We will use the Doob maximal inequality conditionally, in particular we will use

$$\big(E_0(\max_{n\leq N} |\bar{S}_n(f_i)|^2)\big)^{1/2} \leq 2\big[E_0(\bar{S}_N^2(f_i))\big]^{1/2}.$$

For μ-a.e. $x \in X$ and every $i \geq 0$, $(U^j f_i)_j$ remains a sequence of martingale increments under μ_x. Denoting by $\|\cdot\|_{1,\mu_x}$ the norm in $L^2(\mu_x)$, it follows from the Doob maximal inequality that

$$\|\max_{n\leq N} |\bar{S}_n(f)|\|_{2,\mu_x} \leq \sum_{i=0}^{N-1} \|\max_{1\leq n\leq N} |\bar{S}_n(f_i)|\|_{2,\mu_x} \leq 2\sum_{i=0}^{N-1} \|\bar{S}_N(f_i)\|_{2,\mu_x}$$

hence

$$\big(\mathbb{E}_0(\max_{n\leq N} |\bar{S}_n(f)|^2)\big)^{1/2} \leq 2\sum_{i=0}^{N-1} \big[\mathbb{E}(\bar{S}_N^2(f_i))\big]^{1/2} = 2\sum_{i=0}^{N-1} \big[\mathbb{E}(\sum_{j=1}^{N} U^j f_i^2)\big]^{1/2} =$$

$$= 2\sum_{i=0}^{N-1} \big(\sum_{j=1}^{N} Q^j f_i^2\big)^{1/2} \leq 2\sqrt{N} \sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2}.$$

Now, using (3.4) and (3.5), we see that $\sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2}$ is in $L^{2,w}$, which finishes the proof. □

Proof of Theorem 2.3. By Hannan's condition $m = \sum_{k\geq 0} P_0(U^k f)$ is well defined and $M_n = \sum_{k=1}^{n} U^k m$ is a martingale with stationary and ergodic increments.

Let $r \geq 1$. We have

$$f = \sum_{k=0}^{r} P_0(U^k f) - \sum_{k=1}^{r} \big(\mathbb{E}_0(U^k f) - \mathbb{E}_{-1}(U^{k-1} f)\big) + \mathbb{E}_{-1}(U^r f).$$

Hence, denoting $m^{(r)} = \sum_{k=0}^{r} P_0(U^k f)$ and $M_n^{(r)} = \sum_{l=1}^{n} U^l m^{(r)}$, we obtain

$$S_n - M_n = M_n^{(r)} - M_n - U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f)) + \sum_{k=1}^{r} \mathbb{E}_0(U^k f) + \sum_{l=1}^{n} U^l(\mathbb{E}_{-1}(U^r f))$$

and

$$\begin{aligned} S_n - M_n - \mathbb{E}_0(S_n) = \quad & \quad (3.7)\\ = M_n^{(r)} - M_n - [U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f)) - \mathbb{E}_0(U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f))] + \\ + \sum_{l=1}^{n} U^l(\mathbb{E}_{-1}(U^r f)) - \mathbb{E}_0(\sum_{l=1}^{n} U^l(\mathbb{E}_{-1}(U^r f))) \end{aligned}$$

By Doob maximal inequality, denoting $h^{(r)} := (m - m^{(r)})^2$, we have

$$\mathbb{E}_0(\max_{1\leq n\leq N}(M_n^{(r)} - M_n)^2) \leq 4 \sum_{1\leq k\leq N} Q^k h^{(r)} \leq CN(h^{(r)})^* \tag{3.8}$$

(recall that $h^* = \sup_{n\geq 1}(1/n)\sum_{i=0}^{n-1} Q^i(|h|)$).

Let $K > 0$. Denote $Z^{(r)} = \sum_{k=1}^{r} \mathbb{E}_0(U^k f)$ and $Z_K^{(r)} = Z^{(r)} \mathbf{1}_{|Z^{(r)}|>K}$

$$\begin{aligned} &\mathbb{E}_0\Big(\max_{1\leq n\leq N} \big| U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f)) - \mathbb{E}_0(U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f))\big|^2\Big) \\ &\leq 4\mathbb{E}_0(\max_{1\leq n\leq N} |U^n Z^{(r)}|^2) \leq 4K^2 + 4\mathbb{E}_0(\sum_{n=1}^{N} |U^n Z_K^{(r)}|^2) \\ &\leq 4K^2 + 4\sum_{n=1}^{N} Q^n((Z_K^{(r)})^2) \leq 4(K^2 + N((Z_K^{(r)})^2)^*) \end{aligned} \tag{3.9}$$

To deal with the last term in (3.7), we apply Lemma 3.1 to $\mathbb{E}_{-1}(U^r f)$, noticing that in this case f_i is replaced with $P_0(U^i \mathbb{E}_{-1}(U^r f)) = P_0(U^{i+r} f) = f_{i+r}$ when $i \geq 1$ and for $i = 0$, $P_0(\mathbb{E}_{-1}(U^r f)) = 0$). Hence

$$\begin{aligned} \mathbb{E}_0(\max_{1\leq n\leq N} |\sum_{l=0}^{n-1} U^l(\mathbb{E}_{-1}(U^r f)) - \mathbb{E}_0(\sum_{l=0}^{n-1} U^l(\mathbb{E}_{-1}(U^r f)))|^2) \\ \leq N \sum_{i\geq r} ((f_i^2)^*)^{1/2}. \end{aligned} \tag{3.10}$$

Combining (3.8), (3.9) and (3), we obtain that for every $K > 0$ and every $r \in \mathbb{N}$,

$$\limsup_{N\to\infty} \frac{\mathbb{E}_0(\max_{1\leq n\leq N}(\bar{S}_n - M_n)^2)}{N} \leq C(h^{(r)})^* + ((Z_K^{(r)})^2)^* + \sum_{i\geq r} ((f_i^2)^*)^{1/2} \ \mu\text{-a.s.}$$

Now, $\|(((Z_K^{(r)})^2)^*)^{1/2}\|_{2,w} \leq C\|Z_K^{(r)}\|_2 \underset{K\to\infty}{\longrightarrow} 0$. Hence there exists a sequence (K_l) going to infinity such that

$$((Z_{K_l}^{(r)})^2)^* \underset{l\to\infty}{\longrightarrow} 0 \qquad \mu\text{-a.s.}$$

Hence

$$\limsup_{N\to\infty} \frac{\mathbb{E}_0(\max_{1\leq n\leq N}(\bar{S}_n - M_n)^2)}{N} \leq C(h^{(r)})^* + \sum_{i\geq r}((f_i^2)^*)^{1/2} < \infty \qquad \mu\text{-a.s.}$$

The second term clearly goes to 0 μ-a.s., when $r \to \infty$ (by Lemma 3.1), and the first one goes to 0 μ-a.s. (along a subsequence) by (3.5). □

4. Markov Chains

In most of the literature, quenched limit theorems for stationary sequences use a Markov Chain setting: the process is represented as a functional $(f(W_n))_n$ of a stationary and homogeneous Markov Chain (W_n); the limit theorem is said "quenched" it it remains true for almost every starting point.

Every (strictly) stationary sequence of random variables admits a Markov Chain representation. This has been observed by Wu and Woodroofe (2004), using an idea from Rosenblatt (1971). A remark-survey on equivalent representations of stationary processes can be found in Volný (2010). Here we show that the operator Q introduced above leads to another Markov Chain representation of stationary processes.

Before going to the proof, we want to emphasize that the construction below makes use of a σ-algebra $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{F} \subset \theta^{-1}(\mathcal{F})$. By taking $\mathcal{F} = \mathcal{A}$ itself, one obtains a Markov operator Q given by $Qg = g \circ \theta$, which leads to a trivial representation of $(f \circ \theta^n)_{n\in\mathbb{Z}}$ as a functional of a Markov chain, which is useless here. Hence, the point is to use the construction below with a suitable $\mathcal{F}$. Notice that Proposition 4.1 applies to $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, in particular the process $(f\circ\theta^n)_{n\in\mathbb{Z}}$ is *adapted*.

Let $(X, \mathcal{A}, \mu)$ be a probability space and θ be an invertible bi-measurable transformation of X preserving the measure μ.

Let $\mathcal{F} \subset \mathcal{A}$ be a σ-algebra such that $\mathcal{F} \subset \theta^{-1}\mathcal{F}$. Denote $\mathbb{E}(\cdot|\mathcal{F})$ the conditional expectation with respect to $\mathcal{F}$ and define an operator Q on $L^\infty(X, \mathcal{F}, \mu)$ by

$$Qh = \mathbb{E}(h \circ \theta|\mathcal{F}). \tag{4.1}$$

Then Q is a positive contraction satisfying $Q1 = 1$ and it is the dual of a positive contraction T of $L^1(X, \mathcal{F}, \mu)$, namely $Tg = (\mathbb{E}(g|\mathcal{F}))\circ\theta^{-1}$. By Neveu (1970, Proposition V.4.3), if X is a Polish space and $\mathcal{A}$ the σ-algebra of its Borel sets, there exists a transition probability $Q(x, dy)$ on $X \times \mathcal{F}$ such that for every $h \in L^\infty(X, \mathcal{F}, \mu)$,

$$Qh = \int_X h(y)Q(\cdot, dy).$$

Clearly, the transition probability Q preserves the measure μ, hence the canonical Markov chain induced by Q, with initial distribution μ, may be extended to $\mathbb{Z}$.

Now define a sequence of random variables $(W_n)_{n\in\mathbb{Z}}$ defined from $(X, \mathcal{A})$ to $(X, \mathcal{F})$ by $W_n(x) = \theta^n(x)$. Then we have

Proposition 4.1. *Let $(X, \mathcal{A})$ be a Polish space with its Borel σ-algebra. Let μ be a probability on $\mathcal{A}$ and θ be an invertible bi-measurable transformation of X preserving the measure μ. Let $\mathcal{F} \subset \mathcal{A}$ be a σ-algebra such that $\mathcal{F} \subset \theta^{-1}\mathcal{F}$. Then $(W_n)_{n\in\mathbb{Z}}$ is a Markov chain with state space $(X, \mathcal{F})$, transition probability Q (given by (4.1)) and stationary distribution μ. In particular, for every $f \in L^2(X, \mathcal{F}, \mu)$, the process $(f \circ \theta^n)$ is a functional of a stationary Markov chain.*

Proof.
It suffices to show that for every $n \geq 1$, and any $\varphi_0, \ldots, \varphi_n$ bounded measurable functions from $\mathbb{R}$ to $\mathbb{R}$, we have

$$\int_X \varphi_0(W_0)\ldots\varphi_n(W_n)d\mu = \int_X \varphi_0(W_0)\ldots\varphi_{n-1}(W_{n-1})Q\varphi_n(W_{n-1})d\mu, \qquad (4.2)$$

the result for general blocks with possibly negative indices follows by stationarity. By definition of (W_n), (4.2) holds for $n = 1$.

Assume that (4.2) holds for a given $n \geq 1$. Let $\varphi_0, \ldots, \varphi_{n+1}$ be bounded $\mathcal{F}$-measurable functions from X to $\mathbb{R}$. Using the definition of Q, (4.2) for our given n, and stationarity, we obtain

$$\int_X \varphi_0(W_0)\ldots\varphi_n(W_{n+1})d\mu = \int_X \varphi_0\varphi_1\circ\theta\ldots\varphi_{n+1}\circ\theta^{n+1}d\mu$$
$$= \int_X \varphi_0(\theta^{-n})\ldots\varphi_{n-1}\circ\theta^{-1}\varphi_n\varphi_{n+1}\circ\theta d\mu = \int_X \varphi_0(\theta^{-n})\ldots\varphi_n Q\varphi_{n+1}d\mu$$
$$= \int_X \varphi_0\ldots\varphi_n\circ\theta^n Q\varphi_n\circ\theta^n d\mu = \int_X \varphi_0(W_0)\ldots\varphi_n(W_n)Q\varphi_n(W_n)d\mu$$

which proves our result by induction. □

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