ALEA, Lat. Am. J. Probab. Math. Stat. 10 (2), 609–624 (2013)

Functional Laws of Large Numbers in Hölder Spaces

Alfredas Račkauskas and Charles Suquet

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
E-mail address: alfredas.rackauskas@mif.vu.lt

Laboratoire P. Painlevé (UMR 8524 CNRS), Bât. M2 U.F.R. de Mathématiques, Université Lille 1 Sciences et Technologies, F-59655 Villeneuve d'Ascq Cedex France
E-mail address: Charles.Suquet@univ-lille1.fr
URL: http://math.univ-lille1.fr/~suquet/

Abstract. Let $S_n = X_1 + \cdots + X_n$, $n \geq 1$, where $(X_i)_{i\geq 1}$ are random variables. Let μ be a constant and I be the identity function on $[0, 1]$. We study the almost sure convergence to μI of the two polygonal line partial sums processes ζ_n and ζ_n^{ad} with respective vertices $(k/n, S_k)$ and (τ_k, S_k), $0 \leq k \leq n$, where $\tau_k = T_k/T_n$ and $T_k = |X_1| + \cdots + |X_k|$. These convergences are considered in the space $C[0, 1]$ or in the Hölder spaces $\mathcal{H}_\alpha^o[0, 1]$, $0 \leq \alpha < 1$. In $C[0, 1]$, any strong law of large numbers satisfied by S_n is inherited by ζ_n. In $\mathcal{H}_\alpha^o[0, 1]$, assuming moreover that the X_i's are i.i.d., $n^{-1}\zeta_n$ converges almost surely to μI if and only if $\mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty$ and $\mu = \mathbf{E}\,X_1$. In contrast, the same convergence for ζ_n^{ad} is equivalent to $\mathbf{E}\,|X_1| < \infty$ and $\mu = \mathbf{E}\,X_1$.

1. Introduction and main results

On the same probability space $(\Omega, \mathcal{F}, P)$, let us consider a sequence of real valued random variables $(X_i)_{i\geq 1}$ together with its partial sums $(S_n)_{n\geq 0}$

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad n \geq 1,$$

and its polygonal line partial sums processes $(\zeta_n)_{n\geq 1}$, where

$$\zeta_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1].$$

In the case when the X_i's are i.i.d., limit theorems establish a strong relationship between the degree of integrability of X_1 and the asymptotic behavior of S_n and of

Received by the editors September 3, 2012; accepted June 10, 2013.

2010 *Mathematics Subject Classification.* Primary 60F17; Secondary 60B12.

Key words and phrases. adaptive partial sums process, Hölder space, Marcinkiewicz-Zygmund strong law of large numbers, partial sums process, uniform law of large numbers, weighted increments.

The research of the first author was partially supported by the Research Council of Lithuania, grant No. MIP-053/2012.

This work was supported by cooperation agreement Lille-Vilnius PHC Gilibert 25448RC.

ζ_n. To mention the two most famous examples, it is well known that the almost sure convergence of $n^{-1}S_n$ to some constant μ is equivalent to $\mathbf{E}\,|X_1| < \infty$ and $\mathbf{E}\,X_1 = \mu$, by Kolmogorov's strong law of large numbers (SLLN) and its converse; similarly, the convergence in distribution of $n^{-1/2}(S_n - n\mu)$ is equivalent to $\mathbf{E}\,X_1^2 < \infty$ and $\mathbf{E}\,X_1 = \mu$, by central limit theorem (CLT) and its converse. One may look at the central limit theorem as a convergence rate result for the strong law of large numbers, showing that a convergence rate better or equal than $n^{-1/2}$ cannot be obtained in the SLLN. Intermediate convergence rates of the form $n^{1/p-1}$, $0 < p < 2$ are obtained from Marcinkiewicz-Zygmund SLLN under the existence of p-th moment.

As for ζ_n, by Donsker-Prohorov theorem or functional central limit theorem (FCLT), the process $n^{1/2}(n^{-1}\zeta_n - \mu I)$ converges in distribution to the Brownian motion in the classical space of continuous functions $C[0,1]$ if and only if $\mathbf{E}\,X_1^2 < \infty$ and $\mu = \mathbf{E}\,X_1$, where I is the identity function,

$$I : [0,1] \to [0,1], \quad t \mapsto I(t) = t.$$

When $\mathbf{E}\,|X_1|^p < \infty$ for some $p > 2$, the convergence of $n^{1/2}(n^{-1}\zeta_n - \mu I)$ can be strengthened in a convergence in some Hölder space $\mathcal{H}_\alpha^o[0,1]$, giving a FCLT in $\mathcal{H}_\alpha^o$, see Račkauskas and Suquet (2004) for the precise connection between the degree of integrability of X_1 and the strength of the relevant Hölder topology. Alternatively, one can also modify the construction of ζ_n in an adaptive way to obtain Hölder convergence under mild integrability assumptions, see Račkauskas and Suquet (2001).

It is a natural question then, to ask whether all these functional central limit theorems in $C[0,1]$ or in Hölder spaces may be viewed as convergence rate results for some corresponding *functional* strong law of large numbers (FSLLN).

Our aim in this contribution is to discuss various functional laws of large numbers for ζ_n or for some adaptive modification, in terms of the degree of integrability of X_1.

Throughout the paper, $\xrightarrow[n\to\infty]{\text{a.s.}}$ denotes almost sure convergence and $C[0,1]$ is the Banach space of continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the so-called supremum or uniform norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, $f \in C[0,1]$.

The simplest law of large numbers for ζ_n reads as follows.

Theorem 1.1. *Assume that the X_i's are i.i.d. Then the convergence*

$$\frac{1}{n}\zeta_n \xrightarrow[n\to\infty]{\text{a.s.}} \mu I \quad \text{in the space} \quad C[0,1] \tag{1.1}$$

holds if and only if $\mathbf{E}\,|X_1| < \infty$ and $\mu = \mathbf{E}\,X_1$.

Since the supremum norm of a polygonal line is reached at some vertex, the above functional strong law of large numbers for ζ_n can be viewed as an uniform law of large numbers for the partial sums as follows.

Theorem 1.2. *Assume that the X_i's are i.i.d. Then the convergence*

$$\frac{1}{n}\max_{1\le k\le n} |S_k - k\mu| \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{1.2}$$

holds if and only if $\mathbf{E}\,|X_1| < \infty$ and $\mu = \mathbf{E}\,X_1$.

As a matter of fact, functional strong law of large numbers in $C[0,1]$ are easily inherited from the corresponding strong law for S_n, according to the following result whose proof is a simple exercise in analysis. It is worth noticing that no direct assumption on the dependence structure of the X_i's is made here.

Theorem 1.3. *Let X_i be random variables with an arbitrary dependence structure. Let μ be a real number and $(b_n)_{n\geq 1}$ be a non-decreasing sequence of positive numbers going to infinity. Then the following three convergences are equivalent.*

i) $b_n^{-1}(S_n - n\mu) \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

ii) $nb_n^{-1}\|n^{-1}\zeta_n - \mu I\|_\infty \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

iii) $b_n^{-1}\max_{1\leq k\leq n}|S_k - k\mu| \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

Combining classical Marcinkiewicz-Zygmund SLLN with Theorem 1.3 gives the following functional Marcinkiewicz-Zygmund strong law of large numbers for ζ_n (whose special case $p = 1$ is equivalent to Theorem 1.1).

Theorem 1.4. *Assume that the X_i's are i.i.d. Then the following statements hold true.*

a) If $\mathbf{E}\,|X_1|^p < \infty$ for some $p \in (0, 1)$, then $n^{-1/p}\|\zeta_n\|_\infty$ goes to zero almost surely.

b) If $\mathbf{E}\,|X_1|^p < \infty$ for some $p \in [1, 2)$, then $n^{1-1/p}\|n^{-1}\zeta_n - (\mathbf{E}\,X_1)I\|_\infty$ goes to zero almost surely.

c) If $n^{-1/p}\|\zeta_n - c_n I\|_\infty$ goes to zero almost surely for some $p \in (0, 2)$ and some sequence $(c_n)_{n\geq 1}$ of real numbers, then $\mathbf{E}\,|X_1|^p < \infty$.

Remark 1.5. There is a large literature on the Marcinkiewicz-Zygmund strong law of large numbers for dependent variables, see e.g. Rio (1995), Fazekas and Klesov (2000), Fazekas (2006) and the references therein. From these results, FSLLN in $C[0, 1]$ for dependent variables are easily inherited via Theorem 1.3.

Remark 1.6. When $\mu = 0$, any FSLLN of the form *ii)* in Theorem 1.3 verified by ζ_n in $C[0, 1]$ is satisfied also by the polygonal line ξ_n with vertices $(\tau_{n,k}, S_k)$, $0 \leq k \leq n$ where the $\tau_{n,k}$ are deterministic or random in $[0, 1]$, with $\min_{0\leq k\leq n}\tau_{n,k} = 0$, $\max_{0\leq k\leq n}\tau_{n,k} = 1$ and $\tau_{n,k} \neq \tau_{n,j}$ every time $S_k \neq S_j$. This results from *iii)* in Theorem 1.3. When ζ_n satisfies *ii)* with $\mu \neq 0$, ξ_n satisfies the same functional law of large numbers if and only if $b_n^{-1}\max_{0\leq k\leq n}|k - n\tau_{n,k}|$ converges almost surely to zero.

Intuitively, the more concentrated is the distribution of X_1 (in the i.i.d. case), the closer to μI should be the paths of $n^{-1}\zeta_n$. In the functional framework of $C[0, 1]$ this closeness is expressed by a convergence rate in the uniform norm, and there is no gain in assuming that $\mathbf{E}\,|X_1|^p < \infty$ for $p > 2$. This can lead us to look for a different closeness "in shape", more sensitive to the degree of integrability of X_1, by considering stronger norms than the uniform one. This question is natural since both functions ζ_n and μI have a much stronger global regularity than the continuity.

For $\alpha \in [0, 1)$ we consider the Hölderian modulus of smoothness of a function $f : [0, 1] \to \mathbb{R}$ defined by

$$\omega_\alpha(f, \delta) := \sup_{\substack{s,t\in[0,1]\\0<t-s<\delta}} \frac{|f(t) - f(s)|}{|t - s|^\alpha}, \quad \delta \in (0, 1).$$

The Hölder space $\mathcal{H}_\alpha^o[0, 1]$ is then the set of functions $f \in C[0, 1]$ such that $\omega_\alpha(f, \delta)$ converges to zero as δ goes to zero, endowed with the norm $\|f\|_\alpha = |f(0)| + \omega_\alpha(f, 1)$.

The following result gives a characterization of the functional strong law of large numbers for ζ_n in the ladder of spaces $\mathcal{H}^o_\alpha[0,1]$, $0 \le \alpha < 1$. As the space $\mathcal{H}^o_0[0,1]$ is isomorphic to $C[0,1]$, the special case $\alpha = 0$ is equivalent to Theorem 1.1.

Theorem 1.7. *Let $0 \le \alpha < 1$. When the X_i's are i.i.d., the following statements are equivalent:*

a) $\mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty$ *and* $\mu = \mathbf{E}\,X_1$*;*

b) $n^{-1}\zeta_n \xrightarrow[n\to\infty]{\text{a.s.}} \mu I$ *in the space* $\mathcal{H}^o_\alpha[0,1]$*;*

c) for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n - n\mu I\|_\alpha > \varepsilon n) < \infty.$$

In terms of increments of partial sums, Theorem 1.7 can be stated in the following form.

Theorem 1.8. *Let $0 \le \alpha < 1$. Then (a) of Theorem 1.7 is also equivalent with each of the following statements:*

b') $n^{-1+\alpha} \max\limits_{0\le j<k\le n} \dfrac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} \xrightarrow[n\to\infty]{\text{a.s.}} 0$*;*

c') for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\sup_{0\le j<k\le n} \frac{|S_k - S_j - \mu(k-j)|}{(k-j)^\alpha} > \varepsilon n^{1-\alpha}\Big) < \infty.$$

In the special case where $\alpha = 0$, condition b') is equivalent to (1.2), while c') is equivalent to

c") for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\sup_{0\le k\le n} |S_k - k\mu| > \varepsilon n\Big) < \infty.$$

Remark 1.9. Assuming $\mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty$ for some $0 \le \alpha < 1$ and $\mathbf{E}\,X_1 = 0$ we have particularly

$$\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n\to\infty]{\text{a.s.}} 0 \tag{1.3}$$

and one can ask what is the best possible rate of this convergence. From the classical Donsker-Prohorov invariance principle, we have that if $\mathbf{E}\,X_1^2 < \infty$, then

$$\sqrt{n}||n^{-1}\zeta_n||_\infty \xrightarrow[n\to\infty]{\mathcal{D}} \|W\|_\infty,$$

where $W = (W_t, t \in [0,1])$ is a standard Wiener process and $\xrightarrow[n\to\infty]{\mathcal{D}}$ stands for the convergence in distribution. For $0 < \alpha < 1$ we observe different rates of convergence. For $0 < \alpha < 1/2$ we have from Račkauskas and Suquet (2004) that

$$\sqrt{n}\|n^{-1}\zeta_n\|_\alpha \xrightarrow[n\to\infty]{\mathcal{D}} \|W\|_\alpha$$

if and only if $\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|X_1| > t) = 0$, a condition which is stronger than $\mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty$ which in turn is stronger than $\mathbf{E}\,X_1^2 < \infty$. If $1/2 \le \alpha < 1$ and the random variables X_k's are regularly varying with exponent $a > 2$, then we have from Mikosch and Račkauskas (2010) that

$$n^{1-\alpha} a_n^{-1} \|n^{-1}\zeta_n\|_\alpha \xrightarrow[n\to\infty]{\mathcal{D}} Y, \tag{1.4}$$

where Y has Fréchet distribution with parameter a: $P(Y \le x) = \exp\{-x^{-a}\}$, $x \ge 0$, and $a_n = \inf\{x \in \mathbb{R} : P(|X_1| \le x) \ge 1 - 1/n\}$. Since $a_n = n^{1/a}\ell_n$ with slowly varying ℓ_n, the normalization in (1.4) is $n^{1-1/a-\alpha}\ell_n^{-1}$. Let us observe that $1-1/a-\alpha < 1/2$, so the rate in (1.3) is slower for $1/2 \le \alpha < 1$ than for $0 \le \alpha < 1/2$.

In the Gaussian case the limiting behavior of $||\zeta_n||_{1/2}$ has been investigated by Siegmund and Venkatraman (1995) and Kabluchko (2008). It is proved that centered and normed sequence $b_n(||n^{-1}\zeta_n||_{1/2} - a_n)$ with appropriately chosen sequences (b_n) and (a_n) converges in distribution to a double exponential random variable. In the non-Gaussian light-tailed case, the limiting distribution of $||\zeta_n||_{1/2}$ has been obtained in Kabluchko and Wang (2012).

Theorem 1.7 indicates that the construction of polygonal line process ζ_n is not adapted to the structure of Hölder topology with respect to the law of large numbers. Next we introduce another construction which we call *adaptive*. For this we use the random partition of the interval $[0,1]$ generated by the points τ_k constructed as follows. For $n \ge 1$, put

$$T_n := \sum_{i=1}^{n} |X_i|.$$

Then we define the triangular array $\{\tau_{n,k}, 0 \le k \le n\}$ by setting $\tau_{n,0} := 0$, $\tau_{n,n} := 1$ and for $1 \le k < n$,

$$\tau_{n,k} := \begin{cases} \dfrac{T_k}{T_n} & \text{on the event } \{T_n > 0\} \\ 0 & \text{on the event } \{T_n = 0\}. \end{cases}$$

For notational convenience, we write τ_k for $\tau_{n,k}$ every time there is no ambiguity on which n is involved by τ_k. Now we denote by ζ_n^{ad} the random polygonal line process with vertices the points $V_k := (\tau_k, S_k)$. To express this in explicit formulas, we can start with:

$$\zeta_n^{\mathrm{ad}}(\tau_k) = S_k, \quad 0 \le k \le n.$$

We note in passing that this writing is consistent, since on the event $\{\tau_k = \tau_{k-1}\}$, $X_k = 0$ and $S_{k-1} = S_k$. Next we remark that for every $t \in [0,1] \setminus \{\tau_k,\ 0 \le k \le n\}$, there is a unique (random) integer $1 \le j \le n$ such that $\tau_{j-1} < t < \tau_j$ and then

$$\zeta_n^{\mathrm{ad}}(t) = \frac{\tau_j - t}{\tau_j - \tau_{j-1}} S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} S_j = S_{j-1} + \frac{t - \tau_{j-1}}{\tau_j - \tau_{j-1}} X_j.$$

It is worth noticing here that, due to the direct definition of $\tau_0 = 0$ and $\tau_n = 1$, the above formulas are still valid on the event $\{T_n = 0\}$, with $j = n$.

Before stating the functional law of large numbers for ζ_n^{ad}, one can get an intuition by a rough comparison of ζ_n and ζ_n^{ad}. For the first process, the increment $S_k - S_{k-1}$ $(1 \le k \le n)$ is realized in the deterministic interval $[(k-1)/n, k/n]$ with lenght $1/n$. For ζ_n^{ad}, the same increment is realized in the random interval $I_k = [\tau_{k-1}, \tau_k]$. In mean, the lenght $|I_k| = \tau_k - \tau_{k-1}$ is asymptotically equivalent to $1/n$ (or equal if $P(X_1 = 0) = 0$). Indeed it is easily seen that $\mathbf{E}\,|I_k| = P(T_n > 0)/n$ for $1 \le k < n$ while $\mathbf{E}\,|I_n| = P(T_n > 0)/n + P(T_n = 0)$. As the X_k's are i.i.d., $P(T_n = 0) = P(X_1 = 0)^n$ so, discarding the degenerated case where $P(X_1 = 0) = 1$, each $|I_k|$ is equivalent to $1/n$.

This apparent similarity between ζ_n and ζ_n^{ad} may be misleading when working with Hölder topologies. In this context, what matters is the slope of the polygonal line between two consecutive vertices. Contrary to a fixed interval of length $1/n$, I_k

reacts to a value of X_k which can be big with big probability if $\mathbf{E}\,|X_1|^{1/(1-\alpha)} = \infty$. Roughly speaking, for the same big increment $S_k - S_{k-1}$, the slope is weaker for ζ_n^{ad} than for ζ_n. This is the key of the better Hölderian behavior of ζ_n^{ad} as illustrated by the following result.

Theorem 1.10. *Let $0 \le \alpha < 1$. The following statements are equivalent*

a) $\mathbf{E}\,|X_1| < \infty$ *and* $\mathbf{E}\,X_1 = \mu$*;*

b) $n^{-1}\zeta_n^{\mathrm{ad}} \xrightarrow[n\to\infty]{\text{a.s.}} \mu I$ *in the space* $\mathcal{H}_\alpha^o[0,1]$*;*

c) for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(\|\zeta_n^{\mathrm{ad}} - n\mu I\|_\alpha > \varepsilon n) < \infty.$$

2. FSLLN in $C[0,1]$

Since Theorems 1.1 and 1.2 are contained in Theorem 1.4, we need only to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3: Observing that the supremum norm of a polygonal line is reached at some vertex gives immediately the equivalence between conditions *ii*) and *iii*) in Theorem 1.3. Obviously *iii*) implies *i*). The proof of the implication *i*) ⇒ *ii*) is purely analytical and is provided by the following lemma. □

Lemma 2.1. *Let μ be a real number and $(b_n)_{n\ge1}$ be a non-decreasing sequence of positive numbers going to infinity. Let $(s_n)_{n\ge1}$ be a sequence of real numbers such that $b_n^{-1}(s_n - n\mu)$ converges to zero. Then*

$$\max_{1\le k\le n} \frac{|s_k - k\mu|}{b_n} \xrightarrow[n\to\infty]{} 0. \tag{2.1}$$

Proof: Let ε be an arbitrary positive number. By the convergence to zero of $b_n^{-1}(s_n - n\mu)$, there is some integer $k_0 = k_0(\varepsilon)$ such that for every $k \ge k_0$, $b_k^{-1}|s_k - k\mu| < \varepsilon$. Together with the non-decreasingness of the sequence $(b_n)_{n\ge1}$, this leads to the upper bound

$$\begin{aligned}\max_{1\le k\le n} \frac{|s_k - k\mu|}{b_n} &\le \frac{1}{b_n}\max_{k<k_0}|s_k - k\mu| + \max_{k_0\le k\le n}\frac{|s_k - k\mu|}{b_k}\frac{b_k}{b_n}\\ &\le \frac{1}{b_n}\max_{k<k_0}|s_k - k\mu| + \varepsilon.\end{aligned}$$

Since b_n goes to infinity, it follows that

$$\limsup_{n\to\infty}\max_{1\le k\le n}\frac{|s_k - k\mu|}{b_n} \le \varepsilon,$$

which gives (2.1) by arbitraryness of ε. □

Proof of Theorem 1.4: Let $p \in (0,2)$. From the classical Marcinkiewicz-Zygmund SLLN, we know that if $\mathbf{E}\,|X_1|^p < \infty$, then $n^{-1/p}S_n$ goes a.s. to zero in the case $0 < p < 1$, while $n^{-1/p}(S_n - n\mathbf{E}\,X_1)$ goes a.s. to zero in the case $1 \le p < 2$. Accounting the equivalence of i) and ii) in Theorem 1.3, this immediately gives the statements *a*) by choosing $\mu = 0$ and *b*) with $\mu = \mathbf{E}\,X_1$.

By the converse part in classical Marcinkiewicz-Zygmund SLLN, if $n^{-1/p}(S_n - c_n)$ goes to zero almost surely for some $p \in (0,2)$ and some sequence $(c_n)_{n\ge1}$ of real numbers, then $\mathbf{E}\,|X_1|^p < \infty$. Together with Theorem 1.3, this gives the statement c), noting that $\|\zeta_n - c_n I\|_\infty \geq |\zeta_n(1) - c_n I(1)| = |S_n - c_n|$. $\square$

3. Some Hölderian tools

The Hölder norm of a polygonal line function is very easy to compute according to the following lemma for which we refer e.g. to Markevičiūtė et al. (2012) Lemma 3, where it is proved in a more general setting.

Lemma 3.1. *Let $t_0 = 0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ and f be a real-valued polygonal line function on $[0,1]$ with vertices at the t_i's, i.e. f is continuous on $[0,1]$ and its restriction to each interval $[t_i, t_{i+1}]$ is an affine function. Then for any $0 \leq \alpha < 1$,*

$$\sup_{0\leq s<t\leq 1} \frac{|f(t) - f(s)|}{(t-s)^\alpha} = \max_{0\leq i<j\leq n} \frac{|f(t_j) - f(t_i)|}{(t_j - t_i)^\alpha}.$$

Let D_j denotes the set of dyadic numbers of level j in $[0,1]$, that is $D_0 := \{0,1\}$ and for $j \geq 1$, $D_j := \{(2l-1)2^{-j};\ 1 \leq l \leq 2^{j-1}\}$. For $r \in D_j$ set $r^- := r - 2^{-j}$, $r^+ := r + 2^{-j}$, $j \geq 0$. For $f : [0,1] \to \mathbb{R}$ and $r \in D_j$ let us define

$$\lambda_r(f) := \begin{cases} f(r^+) + f(r^-) - 2f(r) & \text{if } j \geq 1, \\ f(r) & \text{if } j = 0. \end{cases}$$

The following sequential norm defined on $\mathcal{H}^o_\alpha[0,1]$ by

$$\|f\|^{\mathrm{seq}}_\alpha := \sup_{j\geq 0} 2^{\alpha j} \max_{r\in D_j} |\lambda_r(f)|,$$

is equivalent to the natural norm $\|f\|_\alpha$, see Ciesielski (1960). Let us define also $\mathbb{D}_j := \{k2^{-j},\ 0 \leq k < 2^j\}$, so that $\mathbb{D}_j = \{0\} \cup \bigcup_{1\leq i\leq j} D_i$. In what follows, we denote by log the logarithm with basis 2 ($\log 2 = 1$).

Lemma 3.2. *For $0 \leq \alpha < 1$,*

$$\|\zeta_n - \mu I\|^{\mathrm{seq}}_\alpha \leq 2 \max_{0\leq j\leq \log n} 2^{\alpha j} \max_{r\in\mathbb{D}_j} \Bigg| \sum_{nr<i\leq n(r+2^{-j})} (X_i - \mu) \Bigg| + 4n^\alpha \max_{1\leq i\leq n} |X_i - \mu|.$$

Proof: First we remark that for $j \geq 1$,

$$\max_{r\in D_j} |\lambda_r(f)| \leq \max_{r\in D_j} |f(r^+) - f(r)| + \max_{r\in D_j} |f(r) - f(r^-)|.$$

As r^+ and r^- belong to $\mathbb{D}_j$, this gives:

$$\sup_{j\geq 1} 2^{\alpha j} \max_{r\in D_j} |\lambda_r(f)| \leq 2 \sup_{j\geq 1} 2^{\alpha j} \max_{r\in \mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

It follows that if $f \in \mathcal{H}^o_\alpha[0,1]$ and $f(0) = 0$,

$$\|f\|^{\mathrm{seq}}_\alpha \leq 2 \sup_{j\geq 0} 2^{\alpha j} \max_{r\in\mathbb{D}_j} |f(r + 2^{-j}) - f(r)|.$$

This inequality can be applied to the random polygonal line $\zeta_n - \mu I$. Moreover, it is clear that there is no loss of generality in assuming $\mu = 0$ for notational simplicity. We claim that for $r \in \mathbb{D}_j$,

$$\big|\zeta_n(r + 2^{-j}) - \zeta_n(r)\big| \leq \begin{cases} \big|S_{[nr+n2^{-j}]} - S_{[nr]}\big| + 2\max_{1\leq i\leq n} |X_i| & \text{if } \ j \leq \log n, \\ 2n2^{-j} \max_{1\leq i\leq n} |X_i| & \text{if } \ j > \log n. \end{cases}$$

Indeed, if $j \leq \log n$, this follows immediately by triangular inequality. If $j > \log n$, $2^{-j} < 1/n$ and then with r in say $[i/n, (i+1)/n)$, either $r + 2^{-j}$ is in $(i/n, (i+1)/n]$ or belongs to $\bigl((i+1)/n, (i+2)/n\bigr]$. In the first case, noting that the slope of ζ_n on $[i/n, (i+1)/n)$ is exactly nX_{i+1}, we have

$$|\zeta_n(r + 2^{-j}) - \zeta_n(r)| = n|X_{i+1}|2^{-j} \leq 2^{-j} n \max_{1 \leq i \leq n} |X_i|.$$

If r and $r + 2^{-j}$ are in consecutive intervals, the same argument applies after chaining, so

$$\begin{aligned}|\zeta_n(r + 2^{-j}) - \zeta_n(r)| &\leq |\zeta_n(r) - \zeta_n((i+1)/n)| + |\zeta_n((i+1)/n) - \zeta_n(r + 2^{-j})| \\ &\leq 2^{-j+1} n \max_{1 \leq i \leq n} |X_i|.\end{aligned}$$

To complete the proof of the lemma, it remains to note that if $j \leq \log n$, $2^{j\alpha} \leq n^\alpha$ and if $j > \log n$, $2^{j\alpha} 2^{-j} n = (2^{-j})^{1-\alpha} n \leq (n^{-1})^{1-\alpha} n = n^\alpha$. $\square$

4. Proof of Theorems 1.7 and 1.8

Theorem 1.8 follows immediately from Theorem 1.7 because the Hölder norm of a polygonal line is reached at two vertices (Lemma 3.1). We shall prove Theorem 1.7 following the scheme:

$$(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

Proof of $(a) \Rightarrow (c)$. Since (a) yields $\mathbf{E}\,|X_1| < \infty$, we can assume without loss of generality that $\mathbf{E}\,X_1 = 0$. Define for every positive ε,

$$P_1(n, \varepsilon) := P\left(\max_{0 \leq j \leq \log n} 2^{\alpha j} \max_{r \in \mathbb{D}_j} \Bigl| \sum_{nr < k \leq n(r + 2^{-j})} X_k \Bigr| > \varepsilon n \right)$$

and

$$P_2(n, \varepsilon) := P\left(n^{-1+\alpha} \max_{1 \leq k \leq n} |X_k| > \varepsilon \right).$$

According to Lemma 3.2 it is enough to prove for $i = 1, 2$ and for each $\varepsilon > 0$ that

$$\sum_{n=1}^{\infty} n^{-1} P_i(n, \varepsilon) < \infty. \tag{4.1}$$

First let us check (4.1) with $i = 2$. Since $P_2(n, \varepsilon) \leq \sum_{k=1}^n P(|X_k| > \varepsilon n^{1-\alpha})$ and the X_k's are identically distributed, we get

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1} P_2(n, \varepsilon) &\leq \sum_{n=1}^{\infty} P(|X_1| > \varepsilon n^{1-\alpha}) \leq \sum_{n=1}^{\infty} \int_{n-1}^{n} P(|X_1| > \varepsilon t^{1-\alpha})\,\mathrm{d}t \\ &= \int_0^\infty P(|X_1| > \varepsilon t^{1-\alpha})\,\mathrm{d}t \\ &= \int_0^\infty P\bigl(|X_1|^{1/(1-\alpha)} > \varepsilon^{1/(1-\alpha)} t\bigr)\,\mathrm{d}t \\ &= \varepsilon^{-1/(1-\alpha)} \mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty,\end{aligned}$$

by (a).

Next we prove (4.1) for $i = 1$. Introducing

$$X_i' = X_i \mathbf{1}_{\{|X_i| \le n^{1-\alpha}\}} - \mathbf{E}\, X_i \mathbf{1}_{\{|X_i| \le n^{1-\alpha}\}},$$
$$X_i'' = X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}} - \mathbf{E}\, X_i \mathbf{1}_{\{|X_i| > n^{1-\alpha}\}},$$

and recalling that $\mathbf{E}\, X_k = 0$, we decompose X_i in the sum $X_i' + X_i''$ of two truncated and centered random variables. We split $P_1(n,\varepsilon) \le P_1'(n,\varepsilon/2) + P_1''(n,\varepsilon/2)$ where in the definition of $P_1(n,\varepsilon)$ one has to substitute respectively X_i by X_j' to define $P_1'(n,\varepsilon)$ and by X_i'' to define $P_1''(n,\varepsilon)$.

First we estimate $P_1''(n,\varepsilon)$. By Markov inequality we have

$$\begin{aligned} P_1''(n,\varepsilon) &\le \varepsilon^{-1} n^{-1} \sum_{1 \le j \le \log n} 2^{\alpha j} \mathbf{E} \max_{r \in \mathbb{D}_j} \Big| \sum_{nr < k \le n(r+2^{-j})} X_k'' \Big| \\ &\le \varepsilon^{-1} \mathbf{E}\, |X_1''| \sum_{j=0}^{\log n} 2^{\alpha j} \\ &\le \frac{2^\alpha n^\alpha}{\varepsilon(2^\alpha - 1)} \mathbf{E}\, |X_1''|. \end{aligned} \tag{4.2}$$

Next we note that $\mathbf{E}\, |X_1''| \le 2\mathbf{E}\, |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}}$ and

$$\begin{aligned} \mathbf{E}\, |X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} &= \int_0^\infty P\left(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s \right) \mathrm{d}s \\ &= \int_0^{n^{1-\alpha}} P\left(|X_1| \mathbf{1}_{\{|X_1| > n^{1-\alpha}\}} > s \right) \mathrm{d}s + \int_{n^{1-\alpha}}^\infty P\left(|X_1| > s \right) \mathrm{d}s \\ &= n^{1-\alpha} P(|X_1| > n^{1-\alpha}) + \int_{n^{1-\alpha}}^\infty P\left(|X_1| > s \right) \mathrm{d}s. \end{aligned} \tag{4.3}$$

Now from (4.2) and (4.3), we obtain

$$\sum_{n=1}^\infty n^{-1} P_1''(n,\varepsilon) \le \frac{c(\alpha)}{\varepsilon} \left(\sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) + \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s)\, \mathrm{d}s \right).$$

The first series in the right hand side converges since

$$\sum_{n=1}^\infty P(|X_1| > n^{1-\alpha}) \le \mathbf{E}\, |X_1|^{1/(1-\alpha)},$$

as already seen above when bounding $\sum_{n=1}^\infty n^{-1} P_2(n,\varepsilon)$. For the second series, we can write

$$\begin{aligned} \sum_{n=1}^\infty n^{-1+\alpha} \int_{n^{1-\alpha}}^\infty P(|X_1| > s)\, \mathrm{d}s &= \sum_{n=1}^\infty n^{-1+\alpha} \sum_{k=n}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s)\, \mathrm{d}s \\ &= \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} P(|X_1| > s)\, \mathrm{d}s \sum_{1 \le n \le k} n^{-1+\alpha} \\ &\le c'(\alpha) \sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} k^\alpha P(|X_1| > s)\, \mathrm{d}s. \end{aligned}$$

Noting that $k^\alpha = (k^{1-\alpha})^{\alpha/(1-\alpha)} \le s^{\alpha/(1-\alpha)}$ for every $s \ge k^{1-\alpha}$, we get

$$\begin{aligned}\sum_{n=1}^\infty n^{-1+\alpha}\int_{n^{1-\alpha}}^\infty P(|X_1|>s)\,\mathrm{d}s &\le c'(\alpha)\sum_{k=1}^\infty \int_{k^{1-\alpha}}^{(k+1)^{1-\alpha}} s^{\alpha/(1-\alpha)}P(|X_1|>s)\,\mathrm{d}s\\ &\le c'(\alpha)\int_0^\infty s^{\alpha/(1-\alpha)}P(|X_1|>s)\,\mathrm{d}s\\ &= \frac{c'(\alpha)(1-\alpha)}{\alpha}\mathbf{E}\,|X_1|^{1/(1-\alpha)}<\infty.\end{aligned}$$

This achieves the verification of the convergence of $\sum_{n=1}^\infty n^{-1}P_1''(n,\varepsilon)$.

Now consider the series $\sum_{n=1}^\infty n^{-1}P_1'(n,\varepsilon)$. We claim that one can find some real $q>1/(1-\alpha)$ and some constant $c(\alpha,q)$ such that

$$P_1'(n,\varepsilon)\le \frac{c(\alpha,q)}{\varepsilon^q}n^{1-q(1-\alpha)}\mathbf{E}\,|X_1'|^q. \tag{4.4}$$

Postponing the verification of (4.4), let us see how it gives the convergence of $\sum_{n=1}^\infty n^{-1}P_1'(n,\varepsilon)$.

First we note that $\mathbf{E}\,|X_1'|^q \le 2^q\mathbf{E}\,|X_1|^q\mathbf{1}_{\{|X_i|\le n^{1-\alpha}\}}$ and

$$\begin{aligned}\mathbf{E}\,|X_1|^q\mathbf{1}_{\{|X_i|\le n^{1-\alpha}\}} &= \int_0^\infty P\left(|X_1|^q\mathbf{1}_{\{|X_i|\le n^{1-\alpha}\}}>t\right)\,\mathrm{d}t\\ &= \int_0^{n^{q(1-\alpha)}} P\left(|X_1|^q\mathbf{1}_{\{|X_i|\le n^{1-\alpha}\}}>t\right)\,\mathrm{d}t\\ &= \int_0^{n^{q(1-\alpha)}} P\left(n^{q(1-\alpha)}\ge|X_1|^q>t\right)\,\mathrm{d}t\\ &\le \int_0^{n^{q(1-\alpha)}} P(|X_1|^q>t)\,\mathrm{d}t = \int_0^{n^{1-\alpha}} qs^{q-1}P(|X_1|>s)\,\mathrm{d}s.\end{aligned}$$

Combining this estimate with (4.4) reduces the problem to the convergence of the series

$$\Sigma(q,\alpha) := \sum_{n=1}^\infty n^{-q(1-\alpha)}\int_0^{n^{1-\alpha}} s^{q-1}P(|X_1|>s)\,\mathrm{d}s. \tag{4.5}$$

For this aim, it is convenient to exchange the summations as follows:

$$\begin{aligned}\Sigma(q,\alpha) &= \sum_{n=1}^\infty n^{-q(1-\alpha)}\sum_{k=1}^n\int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1}P(|X_1|>s)\,\mathrm{d}s\\ &= \sum_{k=1}^\infty \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} s^{q-1}P(|X_1|>s)\,\mathrm{d}s\sum_{n=k}^\infty n^{-q(1-\alpha)}\\ &\le c'(q,\alpha)\sum_{k=1}^\infty \int_{(k-1)^{(1-\alpha)}}^{k^{(1-\alpha)}} k^{1-q(1-\alpha)}s^{q-1}P(|X_1|>s)\,\mathrm{d}s.\end{aligned}$$

Now we note that in the last integral above, $s\le k^{(1-\alpha)}$, so $k\ge s^{1/(1-\alpha)}$ and as $1-q(1-\alpha)<0$, $k^{1-q(1-\alpha)}\le (s^{1/(1-\alpha)})^{1-q(1-\alpha)} = s^{1/(1-\alpha)-q}$ and finally $k^{1-q(1-\alpha)}s^{q-1}\le s^{1/(1-\alpha)-1}$. This leads to

$$\Sigma(q,\alpha)\le c'(q,\alpha)\int_0^\infty s^{1/(1-\alpha)-1}P(|X_1|>s)\,\mathrm{d}s = c''(q,\alpha)\mathbf{E}\,|X_1|^{1/(1-\alpha)}<\infty.$$

To complete the proof, it only remains to check (4.4). In the case where $0 \le \alpha < 1/2$, we can choose $q = 2$ and then (4.4) is obtained by Chebyshev inequality:

$$\begin{aligned} P_1'(n,\varepsilon) &\le \sum_{j=0}^{\log n} \sum_{r\in\mathbb{D}_j} P\left(\Big|\sum_{nr<k\le n(r+2^{-j})} X_k'\Big| > \varepsilon n 2^{-j\alpha}\right) \\ &\le \frac{1}{n^2\varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} \sum_{r\in\mathbb{D}_j} \mathbf{E}\Big|\sum_{nr<k\le n(r+2^{-j})} X_k'\Big|^2 \\ &= \frac{1}{n^2\varepsilon^2} \sum_{j=0}^{\log n} 2^{2j\alpha} 2^j \mathbf{E}\, {X_1'}^2 \\ &\le \frac{2^{1+2\alpha}}{\varepsilon^2(2^{1+2\alpha}-1)} n^{-1+2\alpha} \mathbf{E}\, {X_1'}^2. \end{aligned}$$

If $\alpha \ge 1/2$, then by Markov and Rosenthal (1970) inequalities we obtain for any $q > 1/(1-\alpha) \ge 2$,

$$\begin{aligned} P_1'(n,\varepsilon) &\le \frac{1}{n^q\varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} \sum_{r\in\mathbb{D}_j} \mathbf{E}\Big|\sum_{nr<k\le n(r+2^{-j})} X_k'\Big|^q \\ &\le \frac{C_q}{n^q\varepsilon^q} \sum_{j=0}^{\log n} 2^{jq\alpha} 2^j \Big(\big(n2^{-j}\mathbf{E}\,(X_1')^2\big)^{q/2} + n2^{-j}\mathbf{E}\,|X_1'|^q\Big), \end{aligned}$$

denoting by C_q the universal constant in Rosenthal inequality. Since $q > 2$, $(\mathbf{E}\,(X_1')^2)^{q/2} = \big((\mathbf{E}\,|X_1'|^2)^{1/2}\big)^q \le \big((\mathbf{E}\,|X_1'|^q)^{1/q}\big)^q = \mathbf{E}\,|X_1'|^q$. Moreover, in the range of summation, $n2^{-j} \ge 1$ and as $q/2 > 1$, $(n2^{-j})^{q/2} \ge n2^{-j}$. This gives

$$P_1'(n,\varepsilon) \le \frac{2C_q\mathbf{E}\,|X_1'|^q}{\varepsilon^q n^{q/2}} \sum_{j=0}^{\log n} 2^{j(q(\alpha-1/2)+1)} \le \frac{4C_q}{(2^{q(\alpha-1/2)+1}-1)\varepsilon^q} n^{1-q(1-\alpha)} \mathbf{E}\,|X_1'|^q,$$

so (4.4) is verified. □

Proof of (c) ⇒ (b): Since $\zeta_n(0) = 0$, we have

$$\|\zeta_n - n\mu I\|_\alpha = \omega_\alpha(\zeta_n - n\mu I, 1) = \sup_{0\le s<t\le 1} \frac{|\zeta_n(t) - \zeta_n(s) - n\mu(t-s)|}{|t-s|^\alpha}.$$

Applying Lemma 3.1 to the polygonal line $\zeta_n - \mu I$ gives

$$\|\zeta_n - \mu I\|_\alpha = n^\alpha \max_{1\le i<j\le n} \frac{|S_j - S_i - (j-i)\mu|}{|j-i|^\alpha}. \tag{4.6}$$

From (4.6), it is clear that the sequence $(Z_n)_{n\ge 1}$ defined by

$$Z_n = n^{-\alpha}\|\zeta_n - n\mu I\|_\alpha, \quad n \ge 1,$$

is non-decreasing. Now by (c) we have for every positive ε,

$$\sum_{n=1}^{\infty} n^{-1} P(Z_n > \varepsilon n^{1-\alpha}) < \infty.$$

Exploiting the monotonicity of $(Z_n)_{n\geq 1}$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1}P(Z_n > \varepsilon n^{1-\alpha}) &= \sum_{i=0}^{\infty} \sum_{2^i\leq n<2^{i+1}} n^{-1}P(Z_n > \varepsilon n^{1-\alpha}) \\ &\geq \sum_{i=0}^{\infty} \sum_{2^i\leq n<2^{i+1}} 2^{-i-1}P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}) \\ &= \sum_{i=0}^{\infty} 2^{-1}P(Z_{2^i} > \varepsilon 2^{(i+1)(1-\alpha)}).\end{aligned}$$

It follows that for every positive ε,

$$\sum_{i=0}^{\infty} P(Z_{2^i} > \varepsilon 2^{i(1-\alpha)}) < \infty. \tag{4.7}$$

For $N > 1$ let $J \geq 0$ be such that $2^{J-1} \leq N < 2^J$. Then

$$\begin{aligned}P\left(\sup_{n\geq N} \|n^{-1}\zeta_n - \mu I\|_\alpha > \varepsilon\right) &\leq P\left(\sup_{n\geq 2^{J-1}} n^{-1+\alpha}Z_n > \varepsilon\right) \\ &= P\left(\sup_{i\geq J}\max_{2^{i-1}\leq n<2^i} n^{-1+\alpha}Z_n > \varepsilon\right) \\ &\leq P\left(\sup_{i\geq J} 2^{(-1+\alpha)(i-1)}Z_{2^i} > \varepsilon\right) \\ &\leq \sum_{i=J}^{\infty} P\big(Z_{2^i} > \varepsilon 2^{(1-\alpha)(i-1)}\big).\end{aligned}$$

This upper bound goes to zero when $J\to\infty$ by (4.7). Hence $\sup_{n\geq N}\|n^{-1}\zeta_n - \mu I\|_\alpha$ converges in probability to zero as $N\to\infty$. From this it is easily deduced that $\|n^{-1}\zeta_n - \mu I\|_\alpha$ converges almost surely to zero as $n\to\infty$, which gives (b). □

Proof of (b) ⇒ (a): Putting $\zeta_n' := \zeta_n - n\mu I$, we note that

$$n^\alpha|X_n - \mu| = \frac{|\zeta_n'(1) - \zeta_n'(1-1/n)|}{(\frac{1}{n})^\alpha} \leq \|\zeta_n'\|_\alpha,$$

so (b) implies that

$$n^{-1+\alpha}|X_n - \mu| \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Then, by independence of the X_n's, the second Borel-Cantelli lemma yields

$$\sum_{n=1}^{\infty} P(|X_n - \mu| \geq n^{1-\alpha}) < \infty,$$

which by identical distribution of the X_n's can be recast as

$$\sum_{n=1}^{\infty} P(|X_1 - \mu| \geq n^{1-\alpha}) < \infty,$$

what gives $\mathbf{E}\,|X_1 - \mu|^{1/(1-\alpha)} < \infty$. It follows that $\mathbf{E}\,|X_1|^{1/(1-\alpha)} < \infty$ and as $(1-\alpha)^{-1} \geq 1$, $\mathbf{E}\,|X_1| < \infty$. Finally $n^{-1}S_n = n^{-1}\zeta_n(1)$ converges almost surely to μ by (b) and to $\mathbf{E}\,X_1$ by the classical strong law of large numbers. Hence $\mu = \mathbf{E}\,X_1$ and the proof of Theorem 1.7 is complete. □

5. Proof of Theorem 1.10

As a preliminary remark, it seems worth noticing that the proof of Theorem 1.10 cannot be easily reduced to the case where $\mu = 0$. Indeed the centering which substitutes X_i by $X_i' = X_i - \mu$ changes also the random partition of $[0,1]$ in another one, built on the X_i', and it seems difficult to find a simple relationship between the two corresponding polygonal lines.

In what follows, we put

$$\nu := \mathbf{E}\,|X_1|$$

and discard the trivial case where $\nu = 0$ since then all the X_i's would be almost surely null.

We shall prove Theorem 1.10 following the scheme:

$$(a) \Rightarrow (b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (a).$$

Proof of $(a) \Rightarrow (b)$*:* The Hölder norm of the polygonal line ζ_n^{ad} is reached at two vertices. Accounting the possibility of several consecutive equal τ_i's, this property can be translated here by

$$\begin{aligned}\Delta_n &:= \|n^{-1}\zeta_n^{\mathrm{ad}} - \mu I\|_\alpha \\ &= \frac{1}{n}\max_{0\le\tau_j<\tau_k\le 1}\frac{|S_k - S_j - n\mu(\tau_k-\tau_j)|}{(\tau_k-\tau_j)^\alpha}\mathbf{1}_{\{T_n>0\}} + \mu\mathbf{1}_{\{T_n=0\}} \\ &= \frac{T_n^\alpha}{n}\max_{0\le\tau_j<\tau_k\le 1}\frac{|S_k - S_j - \frac{n\mu}{T_n}(T_k-T_j)|}{(T_k-T_j)^\alpha}\mathbf{1}_{\{T_n>0\}} + \mu\mathbf{1}_{\{T_n=0\}}. \end{aligned} \tag{5.1}$$

By triangle inequality, on the event $\{T_n > 0\}$,

$$\left|\sum_{j<i\le k}\left(X_i - \frac{n\mu}{T_n}|X_i|\right)\right| \le \left(1 + \frac{n|\mu|}{T_n}\right)\sum_{j<i\le k}|X_i|$$

whence

$$(T_k - T_j)^\alpha \ge \left(1 + \frac{n|\mu|}{T_n}\right)^{-\alpha}\left|S_k - S_j - \frac{n\mu}{T_n}(T_k-T_j)\right|^\alpha.$$

With this lower bound for the denominator in (5.1), we obtain

$$\begin{aligned}\Delta_n &\le \left(\frac{T_n}{n} + |\mu|\right)^\alpha\left(\frac{1}{n}\max_{0\le\tau_j<\tau_k\le 1}\left|S_k - S_j - \frac{n\mu}{T_n}(T_k-T_j)\right|\right)^{1-\alpha}\mathbf{1}_{\{T_n>0\}} \\ &\quad + \mu\mathbf{1}_{\{T_n=0\}} \\ &\le \left(\frac{T_n}{n} + |\mu|\right)^\alpha\left(\frac{2}{n}\max_{1\le k\le n}\left|S_k - \frac{n\mu}{T_n}T_k\right|\right)^{1-\alpha}\mathbf{1}_{\{T_n>0\}} + \mu\mathbf{1}_{\{T_n=0\}}.\end{aligned}$$

We introduce some centering by writing, on the event $\{T_n > 0\}$,

$$S_k - \frac{n\mu}{T_n}T_k = S_k - k\mu - \frac{n\mu}{T_n}(T_k - k\nu) + \frac{\mu k}{T_n}(T_n - n\nu),$$

whence

$$\frac{1}{n}\max_{1\le k\le n}\left|S_k - \frac{n\mu}{T_n}T_k\right| \le \frac{1}{n}\max_{1\le k\le n}|S_k - k\mu| + \frac{2|\mu|}{T_n}\max_{1\le k\le n}|T_k - k\nu|.$$

Finally

$$\Delta_n \le \left(\frac{T_n}{n} + |\mu|\right)^{\alpha} \left(\frac{2}{n} \max_{1\le k\le n} |S_k - k\mu| + \frac{4|\mu|}{T_n} \max_{1\le k\le n} |T_k - k\nu|\right)^{1-\alpha} \mathbf{1}_{\{T_n>0\}} \\ + \mu \mathbf{1}_{\{T_n=0\}} \tag{5.2}$$

Now we recall that $P(T_n = 0) = P(X_1 = 0)^n$ which goes to zero, since we discarded the trivial case where $X_1 = 0$ almost surely. As the sequence T_n is non-decreasing, this implies that

$$\mathbf{1}_{\{T_n>0\}} \xrightarrow[n\to\infty]{\text{a.s.}} 1. \tag{5.3}$$

As $\mathbf{E}\,|X_1| < \infty$, we have the following convergences

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} \nu > 0, \tag{5.4}$$

by classical SLLN applied to the $|X_i|$'s,

$$\frac{1}{n} \max_{1\le k\le n} |S_k - k\mu| \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{5.5}$$

by Theorem 1.2,

$$\frac{1}{T_n} \max_{1\le k\le n} |T_k - k\nu| \mathbf{1}_{\{T_n>0\}} \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{5.6}$$

by (5.3), (5.4) and Theorem 1.2 applied to the $|X_i|$'s.

Then the almost sure convergence of $\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha$ to zero results from (5.2) to (5.6). □

Proof of (b) ⇒ (a): We note that

$$\|n^{-1}\zeta_n^{\text{ad}} - \mu I\|_\alpha = \omega_\alpha(n^{-1}\zeta^{\text{ad}} - \mu I, 1) \ge |n^{-1}(\zeta^{\text{ad}}(1) - \zeta^{\text{ad}}(0)) - \mu| = \left|\frac{S_n}{n} - \mu\right|.$$

Then (b) implies that $n^{-1}S_n$ converges almost surely to μ, which gives (a) by the converse part in the classical Kolmogorov-Khintchine strong law of large numbers for i.i.d. random variables. □

Proof of (a) ⇒ (c): Writing $E_n := \{\Delta_n > \varepsilon\}$, we have to prove that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P(E_n) < \infty. \tag{5.7}$$

Recalling that $\nu = \mathbf{E}\,|X_1|$ is assumed to be positive, we split

$$P(E_n) = P(E_n') + P(E_n''),$$

where

$$E_n' := E_n \cap \left\{\left|\frac{T_n}{n} - \nu\right| \le \frac{\nu}{2}\right\}, \qquad E_n'' := E_n \cap \left\{\left|\frac{T_n}{n} - \nu\right| > \frac{\nu}{2}\right\}.$$

On the event E'_n, as $0 < n\nu/2 \leq T_n \leq 3n\nu/2$, we deduce from (5.2) the following upper bound.

$$\begin{aligned}\Delta_n &\leq \left(\frac{5\nu}{2}\right)^{\alpha}\left(\frac{2}{n}\max_{1\leq k\leq n}|S_k - k\mu| + \frac{8}{n}\max_{1\leq k\leq n}|T_k - k\nu|\right)^{1-\alpha}\\ &\leq a\left(\frac{1}{n}\max_{1\leq k\leq n}|S_k - k\mu|\right)^{1-\alpha} + b\left(\frac{1}{n}\max_{1\leq k\leq n}|T_k - k\nu|\right)^{1-\alpha},\end{aligned}$$

with $a = 2(5\nu)^{\alpha}$ and $b = 2^{3-4\alpha}(5\nu)^{\alpha}$.

It follows that

$$\begin{aligned}P(E'_n) &\leq P\left(\left(\frac{1}{n}\max_{1\leq k\leq n}|S_k - k\mu|\right)^{1-\alpha} > \frac{\varepsilon}{2a}\right) + P\left(\left(\frac{1}{n}\max_{1\leq k\leq n}|T_k - k\nu|\right)^{1-\alpha} > \frac{\varepsilon}{2b}\right)\\ &= P\left(\max_{1\leq k\leq n}|S_k - k\mu| > n\varepsilon_a\right) + P\left(\max_{1\leq k\leq n}|T_k - k\nu| > n\varepsilon_b\right)\end{aligned}$$

where $\varepsilon_a = (\varepsilon/2a)^{1/(1-\alpha)}$ and $\varepsilon_b = (\varepsilon/2b)^{1/(1-\alpha)}$. Therefore

$$\sum_{n=1}^{\infty} n^{-1}P(E'_n) < \infty, \tag{5.8}$$

by Theorem 1.8 (c") applied to the sequences $(X_i)_{i\geq 1}$ and $(|X_i|)_{i\geq 1}$.

Next we note that

$$\sum_{n=1}^{\infty} n^{-1}P(E''_n) \leq \sum_{n=1}^{\infty} n^{-1}P\left(|T_n - n\nu| > \frac{\nu}{2}n\right) < \infty, \tag{5.9}$$

by Theorem 1.8 (c") applied to the random variables $|X_i|$.

Gathering (5.8) and (5.9) gives (5.7), establishing (c). □

Proof of (c) $\Rightarrow$ (a)*:* As already observed above,

$$\|n^{-1}\zeta_n^{\mathrm{ad}} - \mu I\|_{\alpha} \geq \left|\frac{S_n}{n} - \mu\right|.$$

Then (c) implies that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1}P\left(|S_n - n\nu| > n\varepsilon\right) < \infty,$$

whence (a) follows by the part $(c'') \Rightarrow (a)$ in Theorem 1.8. □

References

Z. Ciesielski. On the isomorphisms of the spaces H_{α} and m. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **8**, 217–222 (1960). MR0132389.

I. Fazekas. On a general approach to the strong law of large number. Technical report, University of Debrecen (2006).

I. Fazekas and O. Klesov. A general approach to the strong laws of large numbers. *Teor. Veroyatnost. i Primenen.* **45** (3), 568–583 (2000). MR1967791.

Z. Kabluchko. Extreme-value analysis of standardized Gaussian increments. *ArXiv Mathematics e-prints* (2008). arXiv: 0706.1849v3.

Z. Kabluchko and Y. Wang. Limiting distribution for the maximal standardized increment of a random walk. *ArXiv Mathematics e-prints* (2012). arXiv:1211.3301.

J. Markevičiūtė, Ch. Suquet and A. Račkauskas. Functional central limit theorems for sums of nearly nonstationary processes. *Lith. Math. J.* **52** (3), 282–296 (2012). MR3020943.

T. Mikosch and A. Račkauskas. The limit distribution of the maximum increment of a random walk with regularly varying jump size distribution. *Bernoulli* **16** (4), 1016–1038 (2010). MR2759167.

A. Račkauskas and Ch. Suquet. Invariance principles for adaptive self-normalized partial sums processes. *Stochastic Process. Appl.* **95** (1), 63–81 (2001). MR1847092.

A. Račkauskas and Ch. Suquet. Necessary and sufficient condition for the functional central limit theorem in Hölder spaces. *J. Theoret. Probab.* **17** (1), 221–243 (2004). MR2054586.

E. Rio. A maximal inequality and dependent Marcinkiewicz-Zygmund strong laws. *Ann. Probab.* **23** (2), 918–937 (1995). MR1334177.

H. Rosenthal. On the subspaces of l_p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.* **8**, 273–303 (1970). MR0271721.

D. Siegmund and E. S. Venkatraman. Using the generalized likelihood ratio statistic for sequential detection of a change-point. *Ann. Statist.* **23** (1), 255–271 (1995). MR1331667.