Strongly Vertex-Reinforced-Random-Walk on a complete graph

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Abstract. We study Vertex-Reinforced-Random-Walk (VRRW) on a complete graph with weights of the form $w(n) = n^\alpha$, with $\alpha > 1$. Unlike for the Edge-Reinforced-Random-Walk, which in this case localizes a.s. on 2 sites, here we observe various phase transitions, and in particular localization on arbitrary large sets is possible, provided α is close enough to 1. Our proof relies on stochastic approximation techniques. At the end of the paper, we also prove a general result ensuring that any strongly reinforced VRRW on any bounded degree graph localizes a.s. on a finite subgraph.

1. Introduction

This paper considers a Vertex-Reinforced Random Walk (VRRW) on a finite complete graph with weights $w_\alpha(n) := (n + 1)^\alpha$ in the strongly reinforced regime $\alpha > 1$.

Such a process is a discrete time random process $(X_n)_{n\geq 0}$ living in $E = \{1, \ldots, N\}$ and such that for all $n \geq 0$ and $j \in E$,

$$\mathsf{P}(X_{n+1} = j \mid \mathcal{F}_n) = \frac{w_\alpha(Z_n(j))}{\sum_{k\neq X_n} w_\alpha(Z_n(k))}\mathbf{1}_{\{X_n\neq j\}},$$

where $Z_n(j) := \sum_{\ell=0}^n 1_{\{X_\ell=j\}}$ is the number of jumps to site j before time n, and $\mathcal{F}_n = \sigma(X_k; \ k \leq n)$.

The linear regime (i.e $\alpha = 1$) has been initially introduced by Pemantle (1992) on a finite graph and then extensively studied for different type of graphs by several authors (Pemantle and Volkov (1999); Tarrès (2004); Volkov (2006); Limic and Volkov (2010); Benaïm and Tarrès (2011)).

The main result of the present paper is the following:

Theorem 1.1. *Let $N \geq 2$ and $\alpha > 1$ be given. Then the following properties hold.*

(i) *With probability one there exists $2 \leq \ell \leq N$, such that (X_n) visits exactly ℓ sites infinitely often, and the empirical occupation measure converges towards the uniform measure on these ℓ sites.*

(ii) *Let $3 \leq k \leq N$. If $\alpha > (k-1)/(k-2)$, then the probability to visit strictly more than $k-1$ sites infinitely often is zero.*
        If $\alpha < (k-1)/(k-2)$, then for any $2 \leq \ell \leq k$, the probability that exactly ℓ sites are visited infinitely often is positive.

This result has to be compared with the situation for Edge Reinforced Random Walks (ERRW). Limic (2003) (see also Limic and Tarrès (2007)), proved that for any $\alpha > 1$, and for any free loops graph with bounded degree, the ERRW with weight w_α visits only 2 sites infinitely often. It has also to be compared to the situation on the graph $\mathbb{Z}$, where for any $\alpha > 1$, the VRRW with weights $(w_\alpha(n))$ visits a.s. only 2 sites infinitely often.

It might be interesting to notice also that when we add one loop to each site, i.e. when at each step, independently of the actual position of the walk, the probability to jump to some site i is proportional to $w_\alpha(Z_n(i))$, then Rubin's construction (see Davis (1990)) immediately shows that the walk visits a.s. only one site infinitely often. In fact with our techniques we can study a whole family of processes which interpolate between these two examples: for $c \geq 0$, consider the process with transitions probabilities given by

$$\mathsf{P}(X_{n+1} = j \mid \mathcal{F}_n) = \frac{w_\alpha(Z_n(j))\mathbf{1}_{\{X_n \neq j\}} + c\,w_\alpha(Z_n(j))\mathbf{1}_{\{X_n = j\}}}{\sum_{k \neq X_n} w_\alpha(Z_n(k)) + c\,w_\alpha(Z_n(X_n))},$$

with the same notation as above. The case $c = 0$ corresponds to the VRRW on a free loop complete graph, and the case $c = 1$ corresponds to the VRRW on a complete graph with loops. Then for any $c \geq 1$, the process visits a.s. only 1 site infinitely often, and when $c \in (0, 1)$, various phase transitions occur, exactly as in Theorem 1.1, except that the critical values are this time equal to $[k - (1 - c)]/[k - 2(1 - c)]$, for $2 \leq k \leq N$ and localization on 1 site is always possible and occurs even a.s. when $\alpha > (1 + c)/(2c)$. Since the proofs of these results are exactly similar as those for Theorem 1.1, we will not give further details here.

Finally let us observe that similar phase transitions as in Theorem 1.1 have been observed in some random graphs models, see for instance Chung et al. (2003); Oliveira and Spencer (2005).

The paper is organised as follows. In the next two sections we prove some general results for VRRW with weight w_α, on arbitrary finite graphs, which show that in the good cases a.s. the empirical occupation measure of the walk converges a.s. to linearly stable equilibria of some ordinary differential equation. In section 4 we study in detail the case of a complete graph, we describe precisely the set of linearly stable equilibria in this case, and deduce our main result, Theorem 1.1. Finally in the last section we prove a general result ensuring that strongly reinforced VRRWs on arbitrary bounded degree graphs localize a.s. on a finite subset.

2. A general formalism for VRRW

We present here a general and natural framework for studying VRRW based on the formalism and results introduced in Benaïm (1997) and Benaïm and Raimond (2010). Such a formalism heavily relies on stochastic approximation technics and specifically the dynamical system approach developed in Benaïm (1999).

Let $A = (A_{i,j})_{i,j \leq N}$ be a $N \times N$ symmetric matrix with nonnegative entries. We assume that $A_{i,j} > 0$ for $i \neq j$, and that $\sum_j A_{i,j}$ does not depend on i. Let $\alpha > 1$ be given. We consider the process $(X_n)_{n \geq 0}$ living in $E = \{1, \ldots, N\}$, with transition probabilities given by

$$P(X_{n+1} = j \mid \mathcal{F}_n) = \frac{A_{X_n,j}(1 + Z_n(j))^\alpha}{\sum_{k \leq N} A_{X_n,k}(1 + Z_n(k))^\alpha},$$

where Z_n is defined like in the introduction. The case of the VRRW on a complete graph is obtained by taking

$$A_{i,j} = 1 - \delta_{ij}. \tag{2.1}$$

For $i \leq N$, set $v_n(i) = Z_n(i)/(n+1)$. Note that if $A_{i,i} = 0$ and $X_k = i$ then $X_{k+1} \neq i$. In particular for any such i, and $n \geq 1$, $v_n(i) \leq (1/2 + 1/(n+1)) \leq 3/4$. In other words, for all $n \geq 1$, v_n belongs to the *reduced simplex*

$$\Delta := \left\{ v \in \mathbb{R}_+^N \ : \ v_i \leq 3/4 \text{ if } A_{ii} = 0 \text{ and } \sum_i v_i = 1 \right\}.$$

In the following, we might sometimes view an element $f = (f_i)_{i \leq N} \in \mathbb{R}^N$ as a function on E, and so we will also use the notation $f(i)$ for f_i.

Now for $\epsilon \in [0,1]$ and $v \in \Delta$ we let $K(\epsilon, v)$ denote the transition matrix defined by

$$K_{i,j}(\epsilon, v) := \frac{A_{i,j}(\epsilon + v_j)^\alpha}{\sum_k A_{i,k}(\epsilon + v_k)^\alpha}, \tag{2.2}$$

for all $i, j \leq N$. To shorten notation we let

$$K(v) := K(0, v). \tag{2.3}$$

Two obvious, but key, observations are that

$$P(X_{n+1} = j \mid \mathcal{F}_n) = K_{X_n,j}((n+1)^{-1}, v_n)),$$

and

$$\lim_{n \to \infty} K((n+1)^{-1}, v) = K(v).$$

Hence, relying on Benaïm (1997) and Benaïm and Raimond (2010), the behavior of (v_n) can be analyzed through the ordinary differential equation $\dot{v} = -v + \pi(v)$, where $\pi(v)$ is the invariant probability measure of the Markov chain with transition matrix $K(v)$.

3. The limit set theorem

3.1. *The limiting differential equation and its equilibria.* For $v \in \Delta$, set $v^\alpha = (v_1^\alpha, \ldots, v_N^\alpha)$, and

$$H(v) := \sum_{i,j} A_{i,j} v_i^\alpha v_j^\alpha = \langle Av^\alpha, v^\alpha \rangle. \tag{3.1}$$

Note that H is positive on Δ. Hence one can define

$$\pi_i(v) = \frac{v_i^\alpha (Av^\alpha)_i}{H(v)}, \quad i = 1, \ldots, N.$$

From the relation

$$v_j^\alpha (Av^\alpha)_i K_{i,j}(v) = A_{i,j} v_i^\alpha v_j^\alpha,$$

it follows that $K(v)$ is reversible with respect to $\pi(v)$.

For $r = 0, 1$ set $T_r\Delta = \left\{ v \in \mathbb{R}^N : \sum_i v_i = r \right\}$ and let $\imath : T_1\Delta \to \Delta$ be the map defined by $\imath(v) = \mathsf{argmin}\{\|y - v\| : y \in \Delta\}$. Since Δ is convex $\imath$ is a Lipschitz retraction from $T_1\Delta$ onto Δ.

Let now $F : T_1\Delta \to T_0\Delta$ be the vector field defined as

$$F(v) = -v + \pi(\imath(v)). \tag{3.2}$$

Note that F is Lipschitz. Thus by standard results, F induces a global flow $\Phi : \mathbb{R} \times T_1\Delta \to T_1\Delta$ where for all $x \in T_1\Delta$, $t \mapsto \Phi(t, x) := \Phi_t(x)$ is the solution to $\dot{v} = F(v)$ with initial condition $v(0) = x$.

For any subset $I \subset \{1, \ldots, N\}$, we let

$$\Delta_I := \{v \in \Delta : v_i = 0 \quad \forall i \in E \setminus I\},$$

denote the *I-face* of Δ. We let $int\Delta_I = \{v \in \Delta_I : v_i > 0 \quad \forall i \in I\}$ denote the (relative) interior of Δ_I. For $v \in \Delta$ we let $supp(v) = \{i \in E : v_i \neq 0\}$, so that v always lies in the $supp(v)$-face of Δ.

Lemma 3.1. *The flow* Φ *leaves* Δ *positively invariant:* $\forall t \geq 0$, $\Phi_t(\Delta) \subset \Delta$; *and for each* $I \subset E$, *the face* Δ_I *is locally invariant:* $\forall v \in \Delta_I$, $\forall t \in \mathbb{R}$, $\Phi_t(v) \in \Delta \Leftrightarrow \Phi_t(v) \in \Delta_I$.

Proof: For all $v \in \Delta$, $\pi(v)$ lies in Δ. Indeed for the Markov chain having transition matrix $K(v)$ the empirical occupation measure lies in Δ and by the ergodic theorem (for finite Markov chains) the same is true for $\pi(v)$. Hence $F(v)$ points inward Δ for all $v \in \Delta$, proving that Δ is positively invariant. Since $F_i(v) = 0$ when $v_i = 0$, each face is locally invariant. $\square$

Let

$$\mathcal{C} = \{v \in \Delta : F(v) = 0\},$$

denote the *equilibria set* of F. Relying on stochastic approximation theory Benaïm (1997, 1999); Benaïm and Raimond (2010) it will be shown below (Proposition 3.5) that $(v_n, n \geq 1)$ converges almost surely to $\mathcal{C}$.

The next result is similar to the case $\alpha = 1$ (see for instance Pemantle (1992)):

Lemma 3.2. *The map* $H : \Delta \to \mathbb{R}$ *is a* strict Lyapunov function, *meaning that* $\langle \nabla H(v), F(v) \rangle$ *is positive for all* $v \in \Delta \setminus \mathcal{C}$.

Proof: One has $\partial_i H(v) = 2\alpha v_i^{\alpha-1}(Av^\alpha)_i$. Thus

$$
\begin{aligned}
\langle \nabla H(v), F(v) \rangle &= \sum_i 2\alpha v_i^{\alpha-1}(Av^\alpha)_i \left(-v_i + \frac{v_i^\alpha (Av^\alpha)_i}{H(v)} \right) \\
&= \frac{2\alpha}{H(v)} \left(-\left(\sum_i v_i^{\alpha-1}(Av^\alpha)_i v_i \right)^2 + \sum_i \left(v_i^{\alpha-1}(Av^\alpha)_i \right)^2 v_i \right) \\
&\geq 0,
\end{aligned}
$$

with equality only when $v_i^{\alpha-1}(Av^\alpha)_i$ does not depend on $i \in \mathrm{supp}(v)$, i.e. only when v is an equilibrium. $\qquad\square$

Remark 3.3. *The barycenter $v = (1/N, \ldots, 1/N)$ is always an equilibrium.*

Lemma 3.4. *$H(\mathcal{C})$ has empty interior.*

Proof: The computation of $\partial_i H(v)$ shows that

$$(\partial_i - \partial_j)H(v) = 2\alpha(v_i^{\alpha-1}(Av^\alpha)_i - v_j^{\alpha-1}(Av^\alpha)_j).$$

Hence, for all v in the relative interior of Δ, $F(v) = 0 \Leftrightarrow \nabla H(v) = 0$. In other word $\mathcal{C} \cap int\Delta = \nabla H^{-1}(0) \cap int\Delta$. By Sard's theorem, it follows that $H(\mathcal{C} \cap int\Delta)$ has measure zero, hence empty interior. Similarly, for each face I, $H(\mathcal{C} \cap int\Delta_I)$ has empty interior. This proves the lemma. $\qquad\square$

Proposition 3.5. *The set of limit points of (v_n) is a connected subset of $\mathcal{C}$.*

Proof: By proposition 3.3 and Theorem 3.4 in Benaïm (1997) or Proposition 4.6 in Benaïm and Raimond (2010), we get that the limit set of $(v(n))$ is an internally chain transitive set for Φ. Since H is a strict Lyapunov function and $H(\mathcal{C})$ has empty interior, it follows from Proposition 6.4 in Benaïm (1999) that such a limit set is contained in $\mathcal{C}$. $\qquad\square$

In particular, when all the equilibria of F are isolated, then v_n converges a.s. toward one of them, as $n \to \infty$.

Remark 3.6. *When A is not symmetric, the convergence result given by Proposition 3.5 fails to hold. Indeed, an example is constructed in Benaïm (1997) with $N = 3$ and $\alpha = 1$ for which the limit set of $(v_n)_{n\geq 1}$ equals $\partial\Delta$. This behavior persists for $\alpha = 1 + \epsilon$ and $\epsilon > 0$ small enough.*

3.2. *Stable and unstable equilibria.* An equilibrium v is called *linearly stable* provided all the eigenvalues of $DF(v)$, the differential of F at point v, have negative real parts. It is called *linearly unstable* if one of its eigenvalues has a positive real part.

Now we will see with the next result that to study the stability or instability of the equilibria, it suffices in fact to consider only those which belong to the interior of Δ. In the following we let $(e_1, \ldots, e_N)$ denote the canonical basis of $\mathbb{R}^N$.

Lemma 3.7. *Let v be an equilibrium. Then, for $i, j \in supp(v)$, we have*

$$D_{e_i-e_j}F(v) = (\alpha - 1)(e_i - e_j) + \alpha \sum_\ell \frac{v_\ell^\alpha(A_{\ell,i}v_i^{\alpha-1} - A_{\ell,j}v_j^{\alpha-1})}{H(v)}e_\ell,$$

and for $i \notin supp(v)$,

$$D_{e_i-v}F(v) = -(e_i - v).$$

Furthermore, the eigenvalues of $DF(v)$ are all reals.

Proof: For any $i, j \leq N$, and $v \in \Delta$,

$$\partial_j(Av^\alpha)_i = \alpha A_{i,j}v_j^{\alpha-1},$$

and then by using that A is symmetric, we get

$$\partial_j H(v) = 2\alpha v_j^{\alpha-1}(Av^\alpha)_j.$$

Thus

$$
\begin{aligned}
\partial_j \pi_i(v) &= \delta_{i,j}\alpha \frac{v_i^{\alpha-1}(Av^\alpha)_i}{H(v)} + \alpha \frac{v_i^\alpha A_{i,j} v_j^{\alpha-1}}{H(v)} \\
&\quad -2\alpha \frac{v_i^\alpha (Av^\alpha)_i}{H^2(v)} v_j^{\alpha-1}(Av^\alpha)_j.
\end{aligned}
$$

Now assume that v is an equilibrium, and let $i, j \in \mathrm{supp}(v)$. We get with Lemma 3.2

$$
\partial_j \pi_i(v) = \delta_{i,j}\alpha + \alpha \frac{v_i^\alpha A_{i,j} v_j^{\alpha-1}}{H(v)} - 2\alpha v_i,
$$

and then

$$
\partial_j F_i(v) = \delta_{i,j}(\alpha - 1) + \alpha \frac{v_i^\alpha A_{i,j} v_j^{\alpha-1}}{H(v)} - 2\alpha v_i.
$$

On the other hand if $v_i = 0$ or $v_j = 0$, then

$$
\partial_j \pi_i(v) = 0,
$$

and thus

$$
\partial_j F_i(v) = -\delta_{i,j}.
$$

The first part of the lemma follows. To see that eigenvalues are real, we may assume without loss of generality that $v \in int(\Delta)$. Note that $\partial_j F_i(v) v_j = \partial_i F_j(v) v_i$. Therefore, the transpose of $DF(v)$ is self adjoint with respect to the dot product $(x, y) = \sum_i v_i x_i y_i$, and this concludes the proof of the lemma. $\qquad\square$

As announced above we deduce

Corollary 3.8. *An equilibrium on a face is linearly stable (respectively unstable), if and only if, it is so for the restriction of F to this face.*

Proof: Indeed, assume that v is an equilibrium on a face Δ_I associated to some subset I. Then the previous lemma shows that for any $i \notin I$, $e_i - v$ is a stable direction. So the result of the corollary follows from our definitions of stable and unstable equilibria. $\qquad\square$

In other words to study the stability or instability of equilibria, and we will see in the next two subsections why this question is important, it suffices to consider those belonging to the interior of Δ.

3.3. *Non convergence towards unstable equilibria.* The purpose of this section is to prove the following result.

Theorem 3.9. *Let v^* be a linearly unstable equilibrium. Then the probability that v_n converges towards v^* is equal to 0.*

Proof: Let us recall now that for $g \in \mathbb{R}^N$ and $i \in E$, we use the notation $g(i) = g_i$. For $u, v \in \mathbb{R}^N$, we also set $uv := \sum_i u_i v_i$, and $\|u\| = \sup_i |u_i|$. Furthermore, C will denote a non-random constant that may vary from lines to lines.

For $v \in \Delta$, let $Q(v)$ be the pseudo-inverse of $K(v)$ defined by:

$$
(I - K(v))Q(v)g = Q(v)(I - K(v))g = g - (\pi(v)g)\mathbf{1},
$$

for all $g \in \mathbb{R}^N$, with I is the identity matrix and $\mathbf{1}(i) = 1$ for all $i \in E$. Then by a direct application of the implicit function theorems (see Lemma 5.1 in Benaïm (1997)) one has

Lemma 3.10. *For any $v \in \Delta$ and $i \in E$, $Q(v)$, $K(v)Q(v)$, and $(\partial/\partial_{v_i})(K(v)Q(v))$, are bounded operators on $\ell^\infty(E)$, the space of bounded functions on E, and their norms are uniformly bounded in $v \in \Delta$.*

Now for all $n \geq 1$ and $i \in E$, we can write

$$v_{n+1}(i) - v_n(i) = \frac{1}{n+1}\big(-v_n(i) + e_i(X_{n+1})\big).$$

Note that $K\big((n+1)^{-1}, v_n\big) = K(\tilde{v}_n)$, where

$$\tilde{v}_n(i) = \frac{v_n(i) + 1/(n+1)}{1 + N/(n+1)} \quad \text{for } 1 \leq i \leq N.$$

Note also that $\|\tilde{v}_n - v_n\| \leq C/n$. Let next z_n be defined by

$$z_n(i) = v_n(i) + \frac{K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n)}{n} \quad \text{for } 1 \leq i \leq N.$$

Then Lemma 3.10 implies that $\|z_n - v_n\| \leq C/n$. Moreover, we can write:

$$z_{n+1} - z_n = \frac{F(z_n)}{n+1} + \frac{\epsilon_{n+1}}{n+1} + \frac{r_{n+1}}{n+1}, \tag{3.3}$$

where ϵ_{n+1} and r_{n+1} are such that for all $1 \leq i \leq N$,

$$\epsilon_{n+1}(i) := Q(\tilde{v}_n)e_i(X_{n+1}) - K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n),$$

and $r_{n+1} = \sum_{k=1}^4 r_{n+1,k}$, with

$$
\begin{aligned}
r_{n+1,1} &= F(v_n) - F(z_n) \\
r_{n+1,2} &= \pi(\tilde{v}_n) - \pi(v_n) \\
r_{n+1,3}(i) &= K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n)\left(1 - \frac{n+1}{n}\right) \\
r_{n+1,4}(i) &= K(\tilde{v}_{n+1})Q(\tilde{v}_{n+1})e_i(X_{n+1}) - K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_{n+1}),
\end{aligned}
$$

for $1 \leq i \leq N$.

By using the facts that F and π are Lipschitz functions on Δ, $\|v_n - \tilde{v}_n\| + \|v_n - z_n\| \leq C/n$, and by applying Lemma 3.10, we deduce that

$$\|r_{n+1}\| \leq C/n.$$

Moreover, we have

$$\mathsf{E}[\epsilon_{n+1} \mid \mathcal{F}_n] = 0.$$

Since v^* is linearly unstable there exists, by Lemma 3.7, $f \in T_0\Delta$ and $\lambda > 0$ such that

(i): $D_f F(v) = \lambda f$ and,
(ii): $f_i = 0$ for $i \in E \setminus supp(v^*)$.

Such an f being fixed, we claim that on the event $\{v_n \to v^*\}$,

$$\liminf_{n \to \infty} \big\{\mathsf{E}[(\epsilon_{n+2}f)^2 \mid \mathcal{F}_{n+1}] + \mathbb{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n]\big\} > 0. \tag{3.4}$$

To prove this claim, we first introduce some notation: for μ a probability measure on E, and $g \in \mathbb{R}^N$, denote by $\mathsf{V}_\mu(g)$ the variance of g with respect to μ

$$\mathsf{V}_\mu(g) := \frac{1}{2} \sum_{1 \leq j,k \leq N} \mu(j)\mu(k)\big(g(j) - g(k)\big)^2.$$

Then for any $n \geq 0$ and $i \leq N$, let $\mu_{n,i}$ be the probability measure defined by $\mu_{n,i}(j) = K_{i,j}(\tilde{v}_n)$. Set also $V_n(i) := \mathsf{V}_{\mu_{n,i}}(Q(\tilde{v}_n)f)$. Then we have that

$$\mathsf{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n] = V_n(X_n).$$

Furthermore, when v_n converges toward v^*, $K(\tilde{v}_n)$ and $Q(\tilde{v}_n)$ converge respectively toward $K(v^*)$ and $Q(v^*)$. Thus, for any $i \in E$,

$$\liminf_{n \to \infty} V_n(i) \geq V^*(i) := \mathsf{V}_{\mu_i^*}(Q(v^*)f),$$

where $\mu_i^*(j) = K_{i,j}(v^*)$, for all $j \leq N$. Next by using the fact that when $X_n = i$ and $A_{i,i} = 0$, then $X_{n+1} \neq X_n$, we get that

$$\liminf \big\{ \mathsf{E}[(\epsilon_{n+2}f)^2 \mid \mathcal{F}_{n+1}] + \mathsf{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n] \big\} \geq \min_i c^*(i),$$

where

$$c^*(i) \quad := \quad \min_{j \in \mathcal{A}_i}(V^*(i) + V^*(j)),$$

and

$$\mathcal{A}_i := \left\{ \begin{array}{ll} E & \text{if } A_{i,i} \neq 0 \\ E \smallsetminus \{i\} & \text{if } A_{i,i} = 0. \end{array} \right.$$

Now by using that

$$Q(v^*)f - K(v^*)Q(v^*)f = f - (v^*f)\mathbf{1}$$

(recall that $\pi(v^*) = v^*$), we see that $Q(v^*)f$ has constant coordinates on $\mathrm{supp}(v^*)$, if and only if, f has constant coordinates on $\mathrm{supp}(v^*)$. But since $f \in T_0\Delta$ and $f_i = 0$ for $i \notin \mathrm{supp}(v^*)$; this cannot be the case. Since $\mu_i^*(j) > 0$, when $j \neq i$ and $j \in \mathrm{supp}(v^*)$, it follows already that $c^*(i) > 0$, for all $i \notin \mathrm{supp}(v^*)$. Now let $i \in \mathrm{supp}(v^*)$ be given. If $A_{i,i} \neq 0$, then again we have $V^*(i) > 0$, and thus $c^*(i) > 0$. Now assume that $A_{i,i} = 0$. If $\#\mathrm{supp}(v^*) \geq 3$, then there can be at most one value of i, for which $V^*(i) = 0$, and thus in this case we have $c^*(i) > 0$ as well. Let us then consider the case when $\#\mathrm{supp}(v^*) = 2$, and say $\mathrm{supp}(v^*) = \{i, j\}$. Recall that if $A_{i,i} \neq 0$, then $V^*(i) > 0$. However, we cannot have $A_{i,i} = A_{j,j} = 0$, since otherwise $v_i^* = v_j^* = 1/2$ and by lemma 3.7 v^* is linearly stable. Finally we have proved that in any case $\min_i c^*(i) > 0$. Theorem 3.9 is then a consequence of (3.4) and Corollary 3.IV.15 p.126 in Duflo (1996). $\square$

3.4. *Convergence towards stable equilibria and localization.*

Theorem 3.11. *Let v^* be a linearly stable equilibrium. Then the probability that v_n converges towards v^* is positive.*

Proof: follows from Corollary 6.5 in Benaïm (1997) since any linearly stable equilibrium is a minimal attractor. $\square$

Theorem 3.12. *Let $v^* \in \Delta$ be a linearly stable equilibrium. Then a.s. on the event $\{\lim_{n \to \infty} v_n = v^*\}$, the set $E \setminus \mathrm{supp}(v^*)$ is visited only finitely many times.*

The proof follows directly from the next two lemmas:

Lemma 3.13. *There exists $\nu > 0$ such that on the event $\{\lim_{n\to\infty} v_n = v^*\}$,*

$$\lim_{n\to\infty} n^\nu \|v_n - v^*\| = 0.$$

Proof: This is similar to Lemma 8 in Benaïm and Tarrès (2011). We give here an alternative and more direct proof relying on Benaïm (1999). Since v^* is a linearly stable equilibrium there exists a neighborhood B of v^*, a constant $C > 0$ and $\lambda > 0$, such that $\|\Phi_t(v) - v^*\| \leq Ce^{-\lambda t}$, for all $v \in B$ (see e.g Robinson (1999), Theorem 5.1). Let $\tau_n = \sum_{k=1}^n 1/k$ and let $V : \mathbb{R}^+ \to \Delta$ denote the continuous time piecewise affine process defined by a) $V(\tau_n) = z_n$ and b) V is affine on $[\tau_n, \tau_{n+1}]$. By (3.3) and Doob's inequalities, the interpolated process V is almost surely a $-1/2$ *asymptotic pseudo trajectory* of Φ, meaning that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\sup_{0\leq h\leq T} \|\Phi_h(V(t) - V(t+h))\| \right) \leq -1/2$$

for all $T > 0$. For a proof of this later assertion see Benaïm (1999), Proposition 8.3. Now, by Lemma 8.7 in Benaïm (1999)

$$\limsup_{t\to\infty} \frac{1}{t} \log(\|V(t) - v^*\|) \leq -\min(1/2, \lambda)$$

on the event $\{v_n \to v^*\}$. This proves that $\|z_n - v^*\| = O(n^{-\min(1/2,\lambda)})$, which concludes the proof of the lemma. $\qquad\square$

Lemma 3.14. *For any $I \subseteq \{1, \ldots, N\}$, and $\nu \in (0, 1)$, a.s. on the event*

$$E_\nu(I) := \{ \lim_{n\to\infty} v_n(i)\, n^\nu = 0 \quad \forall i \in I \},$$

the set I is visited only finitely many times.

Proof: For $m \geq 1$, set

$$E_{m,\nu}(I) := \{ |v_k(i)| \leq k^{-\nu} \quad \forall k \geq m \quad \forall i \in I \}.$$

Note that on $E_{m,\nu}(I)$, at each time $k \geq m$, the probability to jump to some vertex $i \in I$, is bounded above by $p_k := N^{1+\alpha}\, k^{-\alpha\nu}$. Let now $(\xi_k)_{k\geq m}$ denotes some sequence of independent Bernoulli random variables with respective parameters $(p_k)_{k\geq m}$. Then for any $n \geq m$, on $E_{m,\nu}(I)$ the number of jumps on I between time m and n is stochastically dominated by

$$Z_n := \sum_{k=m}^n \xi_k.$$

However, it is well known that a.s. $\limsup Z_n/n^{1-\nu'} < \infty$, for any $\nu' < \alpha\nu \wedge 1$. We deduce that a.s. for any $\nu' < \alpha\nu \wedge 1$,

$$E_{m,\nu}(I) \subseteq E_{\nu'}(I).$$

Since $E_\nu(I) \subseteq \cup_m E_{m,\nu(I)}$, we deduce that a.s. for any $\nu' < \alpha\nu \wedge 1$,

$$E_\nu(I) \subseteq E_{\nu'}(I).$$

Since $\alpha > 1$, it follows by induction that a.s.

$$E_\nu(I) \subseteq E_\beta(I),$$

for any $\beta \in (1/\alpha, 1)$. But a simple application of the Borel-Cantelli lemma shows that for any such β, a.s. on $E_\beta(I)$, the set I is visited only finitely many times. This concludes the proof of the lemma. $\qquad\square$

4. The case of the VRRW on a complete graph

In this section we study in detail the case of the VRRW on a complete graph described in the introduction. In other words, A is given by (2.1).

Since the case $N = 2$ is trivial, we assume in all this section that $N \geq 3$.

We first study the stability of the centers of the faces. As already explained, this reduces to analyze the center of Δ.

Lemma 4.1. *Let* $v = (1/N, \dots, 1/N)$ *be the center of* Δ. *Then* v *is a linearly stable (respectively unstable) equilibrium if* $\alpha < (N-1)/(N-2)$, *(respectively* $\alpha > (N-1)/(N-2)$).

Proof: Lemma 3.7 shows that for all $i \neq j$,

$$D_{e_i - e_j} F(v) = \left(-1 + \alpha \left(\frac{N-2}{N-1}\right)\right)(e_i - e_j).$$

The lemma follows immediately.   □

By combining this lemma with Corollary 3.8, we get

Lemma 4.2. *Let* v *be the center of the face* Δ_I *associated with some subset* I *with cardinality* $k \leq N$. *Then* v *is a linearly stable (respectively unstable) equilibrium if* $\alpha < (k-1)/(k-2)$, *(respectively* $\alpha > (k-1)/(k-2)$).

It remains to study the stability of the other equilibria. We will see that they are all unstable, which will conclude the proof of Theorem 1.1.

First we need the following lemma, which shows that coordinates of equilibriums take at most two different values.

Lemma 4.3. *Let* v *be an equilibrium in the interior of* Δ, *which is different from its center. Then* $\#\{v_i \ : \ i \leq N\} = 2$.

Proof: Let $a = \sum_i v_i^\alpha$ and $b = \sum_i v_i^{2\alpha}$. Since v is an equilibrium, we have for all i,

$$v_i = v_i^\alpha \frac{a - v_i^\alpha}{a^2 - b}.$$

Since all coordinates of v are positive by hypothesis, this is equivalent to

$$f(v_i) = a - b/a,$$

for all i, where $f(x) = -x^{2\alpha - 1}/a + x^{\alpha - 1}$. Now observe that

$$f'(x) = x^{\alpha - 2}\big(-(2\alpha - 1)x^\alpha/a + (\alpha - 1)\big),$$

does not vanish on $(0, 1)$ if $a \geq (2\alpha - 1)/(\alpha - 1)$, and vanishes in exactly one point otherwise. Thus for any fixed $\lambda \in \mathbb{R}$, the equation $f(x) = \lambda$, has at most one solution in $(0, 1)$, if $a \geq (2\alpha - 1)/(\alpha - 1)$, and at most two otherwise. The lemma follows.   □

Let v be an equilibrium in the interior of Δ, which is different from its center. Lemma 4.3 shows that its coordinates take exactly two different values, say u_1 and u_2. Since the action of permutation of the coordinates commutes with F, we can always assume w.l.g. that $v_i = u_1$, for $i \leq k$, and $v_i = u_2$, for $i > k$, for some $k \leq N/2$. Denote now by E_k the set of such equilibria (those in the interior of Δ, not equal to the center of Δ, and having their first k coordinates equal as well as their last $N - k$ coordinates). For $v \in E_k$, we also set $t(v) = v_N/v_1$. We have the

Lemma 4.4. *Assume that $\alpha \geq (N-1)/(N-2)$, and let $v \in E_k$, with $k \leq N/2$. If $k > 1$ or if $t(v) < 1$, then v is linearly unstable.*

Proof: It follows from Lemma 3.7 that for any $i < j \leq k$,

$$D_{e_j - e_i} F(v) = \lambda_1 (e_j - e_i),$$

and for $k + 1 \leq i < j \leq N$,

$$D_{e_j - e_i} F(v) = \lambda_2 (e_j - e_i),$$

with

$$\lambda_m = \left((\alpha - 1) - \alpha \frac{u_m^{2\alpha - 1}}{H(v)} \right),$$

for $m = 1, 2$. Then it just suffice to observe that if $u_1 < u_2$, $\lambda_1 > 0$, whereas if $u_1 > u_2$, $\lambda_2 > 0$. Thus if any of the hypotheses of the lemma is satisfied, there $DF(v)$ has at least one positive eigenvalue, and so v is linearly unstable. □

On the other hand we have the following:

Lemma 4.5. *Let $v \in E_k$ be given. If $k = 1$ and $\alpha \geq (N-1)/(N-2)$, then $t(v) < 1$ and v is unstable. Similarly, if $\alpha < (N-1)/(N-2)$, then v is linearly unstable.*

Proof: Following our previous notation, set $u_1 := v_1$ and $u_2 := v_N$, and recall that $u_1 \neq u_2$, since v is not equal to the center of Δ. Recall also that $t(v) = u_2/u_1$. Note that since $ku_1 + (N-k)u_2 = 1$, we have $u_1 = 1/(k + (N-k)t)$. Now the fact that v is an equilibrium means that $F(v) = 0$, which is equivalent to say that the function φ defined by

$$\varphi(t) := -(N - k - 1)t^{2\alpha - 1} + (N - k)t^\alpha - kt^{\alpha - 1} + (k - 1),$$

vanishes at point $t(v)$. We now study the function φ. First $\varphi'(t) = t^{\alpha - 2}\psi(t)$, with

$$\psi(t) = -(2\alpha - 1)(N - k - 1)t^\alpha + \alpha(N - k)t - (\alpha - 1)k.$$

In particular ψ is strictly concave, $\psi(0) < 0$ and $\lim_{t \to \infty} \psi(t) = -\infty$. Now two cases may appear. Either ψ vanishes in at most one point, in which case φ is nonincreasing, thus vanishes only in 1. But this case is excluded, since $t(v) \neq 1$ by hypothesis. In the other case ψ vanishes in exactly two points, which means that there exist $t_1 < t_2$ such that φ is decreasing in $(0, t_1) \cup (t_2, \infty)$, and increasing in (t_1, t_2). Now consider first the case when $k = 1$, which implies $\varphi(0) = 0$. If $\alpha \geq (N-1)/(N-2)$, then $\varphi'(1) \leq 0$, thus φ has at most one zero in $(0, 1)$ and no zero in $(0, \infty)$. Together with Lemma 4.4 this proves the first part of the lemma. If $\alpha > (N-1)/(N-2)$, then $\varphi'(1) > 0$, and thus φ has no zero in $(0, 1)$ and exactly one zero in $(0, \infty)$, in which the derivative of φ is negative. The fact that this zero corresponds to an unstable equilibrium will then follow from (4.1) below. But before we prove this fact, let us consider now the case $k > 1$ and $\alpha < (N-1)/(N-2)$, which imply that $\varphi(0) > 0$ and $\varphi'(1) > 0$. Thus φ vanishes in exactly one point in $(0, 1)$ and another one in $(1, \infty)$, and again the derivative of φ in these points is negative. So all that remains to do is to prove the following fact

$$\text{If } v \in E_k \text{ is such that } \varphi'(t(v)) < 0, \text{ then } v \text{ is unstable.} \tag{4.1}$$

Let us prove it now. For $t \in (0, \infty)$, set $u_1(t) = 1/(k + (N - k)t)$ and $u_2(t) = tu_1(t)$. Then let $v(t) \in (0, 1)^N$, be the point whose first k coordinates are equal to $u_1(t)$ and whose last $N - k$ coordinates are equal to $u_2(t)$. Then we have

$$H(v(t)) = k(k - 1)u_1(t)^{2\alpha} + 2k(N - k)t^\alpha u_1(t)^{2\alpha} + (N - k)(N - k - 1)t^{2\alpha} u_1(t)^{2\alpha},$$

and after some computation we find that

$$F(v(t)) = -\frac{tu_1(t)^{2\alpha+1}}{H(v(t))} \, \varphi(t) \, e_k,$$

where e_k is the vector, whose first k coordinates are equal to $-(N-k)$ and whose last $N-k$ coordinates are equal to k. Then notice that $D_{e_k}F(v(t)) = c(d/dt)F(v(t))$, for some constant $c > 0$. But recall that when $v(t)$ is an equilibrium, $\varphi(t) = 0$. Thus

$$\frac{d}{dt}F(v(t)) = -\frac{tu_1(t)^{2\alpha+1}}{H(v(t))} \, \varphi'(t) \, e_k.$$

So if $\varphi'(t) < 0$, e_k is an unstable direction, proving (4.1). This concludes the proof of the lemma. $\square$

Lemma 4.6. *The number of equilibriums in Δ is finite.*

Proof: Let $v \in \Delta$, such that that $v_i = u_1 > 0$, for $i \leq k$, and $v_i = u_2 > 0$, for $i > k$, for some $k \leq N/2$. Set $t = u_2/u_1$, so that $u_1 = (k + \ell t)^{-1}$ and $u_2 = tu_1$ and where $\ell = N - k$. Recall (see the proof of Lemma 4.5) that v is an equilibrium if and only if $\phi(t) = 0$, where $\phi :]0, \infty[\to \mathbb{R}$ is defined by

$$\phi(t) = (\ell - 1)t^{2\alpha-1} - \ell t^\alpha + kt^{\alpha-1} - (k - 1).$$

Studying ϕ, we show that two situations may occur: 1 is the only solution or there exist two other solutions, distinct from 1. This proves that E_k contains at most 2 equilibriums. And therefore, one can check that the number of equilibriums in the interior of Δ is at most $2^N - 1$. $\square$

Lemmas 4.4 and 4.5 show that any equilibrium in the interior of Δ, which is not equal to the center of Δ is linearly unstable. Together with Lemma 4.2, Lemma 4.6 and Theorems 3.9, 3.11 and 3.12 this concludes the proof of Theorem 1.1. $\square$

5. Almost sure localization

In this section we prove a general result on the localization of strongly reinforced VRRWs. First we recall that given a locally finite graph G, and $w : \mathbb{N} \to (0, \infty)$, we can define the VRRW $(X_n)_{n \geq 0}$ by

$$\mathsf{P}(X_{n+1} = y \mid \mathcal{F}_n) = \frac{w(Z_n(y))}{\sum_{z \sim X_n} w(Z_n(z))} \mathbf{1}_{\{X_n \sim y\}},$$

with $Z_n(\cdot)$ as in the introduction, and where the symbol $X_n \sim y$ means that X_n and y are neighbours in G. We say that the VRRW is strongly reinforced when

$$\sum_{n=0}^{\infty} \frac{1}{w(n)} < \infty. \tag{5.1}$$

It is easy to see that under this hypothesis the walk can localize on two sites with positive probability. But something more can be said:

Theorem 5.1. *Let $w : \mathbb{N} \to (0, \infty)$ be a weight function satisfying (5.1), and let G be a graph with bounded degree. Then a.s. the VRRW on G with weight w visits only a finite number of sites.*

Proof: We adapt an argument of Volkov who proved the result when the graph is a tree (see Volkov (2001)). Let $d > 0$ be such that any vertex in G has degree at most d and let $\varepsilon \in (0, 1)$ be such that $\varepsilon d^3 < 1$.

It will be convenient to introduce a Rubin's construction of the walk. Similar ones were introduced already by Sellke (2008) and Tarrès (2011), see also Basdevant et al. (2014).

To each pair of neighbours $x \sim y$ of G we associate two oriented edges (x, y) and (y, x), and denote by $\mathcal{E}$ the set of oriented edges obtained in this way. Then we consider a sequence $(\xi_\ell^e)_{e \in \mathcal{E}, \ell \geq 0}$ of independent exponential random variables, such that for any e and ℓ, the mean of ξ_ℓ^e equals $1/w(\ell)$. We will also use the notation:

$$\xi_x = (\xi_\ell^{(x,y)})_{y \sim x, \ell \geq 0},$$

for any vertex x of G. Then given the sequence $(\xi_x)_{x \in G}$, and a vertex x_0 of G, we construct a continuous time process $(\tilde{X}_t)_{t \geq 0}$ as follows.

First $\tilde{X}_0 = x_0$. Then for each oriented edge (x_0, y), we let a clock ring at time $\xi_0^{(x_0,y)}$, and let the process jump along the edge corresponding to the clock which rings first (at time $\min_{y \sim x_0} \xi_0^{(x_0,y)}$). At this time we also stop all clocks associated to the edges going out of x.

Now assuming that $(\tilde{X}_t)$ has been constructed up to some jumping time t, and that $\tilde{X}_t = x$, we use the following rule. For any $y \sim x$, if during the time between the last visit to x before t (which we take to be 0 if t is the first time the walk visits x) and t the walk has visited y, then start a new clock associated to the edge (x, y) which will ring at time $\xi_k^{(x,y)}$, with k the number of visits to y before time t. If not, then restart the clock associated to (x, y) which has been stopped after the last visit of the walk in x, or start a new clock which will ring after a time $\xi_0^{(x,y)}$ if both x and y have never been visited yet before time t. Then the walk jumps again along the edge associated to the clock which rings first at the time when it rings, and then we stop all clocks associated to edges going out of x. Then it is a basic and well known fact that a.s. two clocks never ring at the same time, so that this construction is a.s. well defined, and that if $(\tau_k)_{k \geq 0}$ is the sequence of jumping times (with $\tau_0 = 0$), then $(\tilde{X}_{\tau_k})_{k \geq 0}$ has the same law as the process X starting from x_0.

Now for any vertex x and integer $N \geq 1$ we consider the following event

$$E_x(N) := \left\{ \sum_{\ell \geq N} \xi_\ell^{(x,y)} < \min_{z \sim x} \xi_0^{(x,z)} \quad \forall y \sim x \right\}.$$

Note that, denoting by d_x the degree of x,

$$\begin{aligned}
\mathsf{P}(E_x(N)) &= \prod_{y \sim x} \mathsf{E}\left[e^{-d_x w(0) \sum_{\ell \geq N} \xi_\ell^{(x,y)}} \right] \\
&= \prod_{l \geq N} \left(1 - \frac{d_x w(0)}{w(\ell)} \right)^{d_x} \geq \prod_{l \geq N} \left(1 - \frac{d w(0)}{w(\ell)} \right)^d
\end{aligned}$$

using in the last inequality that $d_x \leq d$. So that, when (5.1) is satisfied, we can fix N such that $\mathsf{P}(E_x(N)) \geq 1 - \varepsilon$, for all x.

To each vertex x, let τ_x be the first time the walk visits x. When $\tau_x < \infty$, we associate to x a type:

- the type 1 if at time τ_x one of its neighbours has been visited more than N times;
- the type 2 if x is not of type 1 and if every neighbour of x has at least one of its neighbours which has been visited more than N times before τ_x;
- the type 3 otherwise.

We also associate two sets of vertices, C_x^1 and C_x^2: when x is of type 1 and on $E_x(N)$, $C_x^1 = \emptyset$, otherwise C_x^1 is the set of all neighbours of x that have not been visited by the walk at time τ_x. Thus C_x^1 contains all the vertices that may be visited for the first time after a jump from x.

Define next $C_x^2 := \cup_{y \in C_x^1} C_{x,y}^2$, with $C_{x,y}^2$ as follows. When y has a neighbour that has been visited more than N times at time τ_x and on $E_y(N)$, $C_{x,y}^2 = \emptyset$, otherwise $C_{x,y}^2$ is the set of all neighbours of y that have not been visited by the walk at time τ_x. Thus C_x^2 contains all the vertices that may be visited for the first time after a jump from a vertex in C_x^1.

Observe that when x is of type 3, conditionally to $\mathcal{F}_{\tau_x}$, the walk has a probability bounded below by some constant $p > 0$, depending only on N, in particular independent of x and $\mathcal{F}_{\tau_x}$, to be stuck on $\{x, y\}$ for some $y \sim x$ forever after τ_x. So the conditional Borel-Cantelli lemma ensures that only finitely many x's can be of type 3.

Now we define inductively a growing sequence of subgraphs $(G_n)_{n \geq 0} \subset G$, an increasing sequence of random times $(\tau_n)_{n \geq 0}$ and a sequence of vertices $(x_n)_{n \geq 0}$ as follows. First $G_0 = \{x_0\} \cup C_{x_0}^1$ and $\tau_0 = 0$. Note that $C_{x_0}^1 = \{y \sim x_0\}$. Now assume that G_k, τ_k and x_k have been defined for all $k \leq n$, for some $n \geq 0$. If the walk never exists the set G_n, then set $G_{n+1} = G_n$ and $\tau_{n+1} = \infty$. Otherwise let τ_{n+1} be the first exit time of G_n, x_{n+1} be the position of the walk at time τ_{n+1} and set $G_{n+1} = G_n \cup \{x_{n+1}\} \cup C_{x_{n+1}}^1$.

For all $n \geq 0$ with $\tau_n < \infty$, set $O_n = \cup_{k=0}^n C_{x_k}^2 \setminus G_n$, so that τ_{n+1} is the first entrance time of the walk in O_n. Observe that if for some n, $O_n = \emptyset$, then $\tau_{n+1} = \infty$ and thus the walk will visit only a finite number of sites. Note also conversely that if the walk visits an infinite number of vertices, then $\tau_n < \infty$, for all n.

Define now a filtration $(\mathcal{G}_n)_{n \geq 0}$ by $\mathcal{G}_n = \sigma(\xi_x : x \in G_n)$. Then, we have that $\sigma(X_k : k \leq \tau_{n+1}) \subset \mathcal{G}_n$, and that x_{n+1} and its type are $\mathcal{G}_n$ measurable. Likewise the number R_n of sites in O_n is also $\mathcal{G}_n$-measurable. Moreover, since $x_{n+1} \in O_n$, we have $R_{n+1} \leq R_n - 1 + |C_{x_{n+1}}^2|$ (where if A is a set we denote by $|A|$ its cardinality). Set, for $n \geq 1$, $\ell_n = -1$ if x_n is of type 1 and on $E_{x_n}(N)$ or if x_n is of type 2 and on $\cap_{y \in C_{x_n}^1} E_y(N)$, and set $\ell_n = d^2 - 1$ otherwise. Thus ℓ_n is $\mathcal{G}_n$-measurable and for $n \geq 0$, $R_{n+1} - R_n \leq -1 + |C_{x_{n+1}}^2| \leq \ell_{n+1}$. Now it is easy to see that if x_{n+1} is of type 1,

$$\mathbb{E}[\ell_{n+1} \mid \mathcal{G}_n] \leq -1 + \epsilon d^2 < 0.$$

Similarly if x_{n+1} is of type 2,

$$\mathbb{E}[\ell_{n+1} \mid \mathcal{G}_n] \leq -1 + (1 - (1 - \epsilon)^d)d^2 \leq -1 + \epsilon d^3 < 0.$$

Thus, setting $A_n := \{x_n \text{ is of type 1 or 2}\}$, we get that $L_n := \sum_{k=1}^n \ell_k \mathbf{1}_{A_k}$ is a supermartingale. Therefore, a.s. on the event A_n occurs an infinite number of times, $\liminf_{n \to \infty} L_n = -\infty$ (using that on this event, L_n cannot converge since on A_n, $|\ell_{n+1}| \geq 1$).

Therefore a.s., if the walk visits an infinite number of vertices, then since x_n is of type 3 for only a finite number of n, a.s. on the event A_n occurs an infinite

number of times, $\liminf_{n\to\infty} R_n = -\infty$, which is absurd since, R_n is nonnegative by definition. This proves the lemma. $\qquad\square$

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