The record method for two and three dimensional parameters random fields

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Abstract. Let S be a regular set of $\mathbb{R}^d$ and $X : S \to \mathbb{R}$ be a Gaussian field with regular paths. In order to give bound to the tail of the distribution of the maximum, we use the record method of Mercadier. We present some new form in dimension 2 and extend it to dimension 3 using the result of the expectation of the absolute value of quadratic forms by Li and Wei. Comparison with other methods is conducted.

1. Introduction

The problem of computing the tail of the maximum of random processes (from $\mathbb{R}$ to $\mathbb{R}$) and random fields (from $\mathbb{R}^d$ to $\mathbb{R}$, $d > 1$) has a lot of applications in spatial statistics, image processing, oceanography, genetics etc ..., see for example Cressie and Wikle (2011). It is exactly solved only for about ten processes with parameter of dimension 1, see page 4 in Azaïs and Wschebor (2009) for a complete list. In the other cases, one has to use some approximations.

Starting from the work of Pickands (1969), this problem has received contribution from Piterbarg (1996a,b) (the double sum method), Adler (1981), Adler and Taylor (2007) (the Euler characteristic method), Sun (1993), Takemura and Kuriki (2002) (the tube method), Azaïs and Delmas (2002) (the Rice method) and Azaïs and Wschebor (2008) (the direct method).

Most of these results give equivalents, expansion or bounds using some "unknown constants". In some statistical applications, we often want to guaranty the level of the test to be less than some precise value α. If the statistics of the test is the maximum of the random process, this demands an exact and explicit upper bound for its tail. Among the cited references, only the direct method permits to attain

this goal in the particular case of isotropic fields. In addition the result is somewhat complex in high dimension and it is not sharp (see Section 5).

With respect to these methods, the record method which is the main subject of this paper and which is detailed in Section 2 has the advantage of simplicity and also the advantage of giving a bound which is non asymptotic: it is true for every level and not for large u only.

It has been introduced for one-parameter random processes by Rychlik (1990) and extended to two-parameter random fields by Mercadier (2006) to study the tail of the maximum of smooth Gaussian random fields on rather regular sets.

It has two versions, one is an exact implicit formula: Theorem 2 in Mercadier (2006) that is interesting for numerical purpose and that will not be considered here; the other one, which is the main topics of this paper, is a bound for the tail, see inequality (1.1) hereunder.

This bound has the advantage of its simplicity. In particular it avoids the computation of the expectation of the absolute value of the Hessian determinant as in the direct method of Azaïs and Wschebor (2008) but it works only in dimension 2.

For practical applications, the dimensions 2 and 3 (for the parameter set) are the most relevant so there is a need of an extension to dimension 3 and this is done in Section 3 using results on quadratic forms by Li and Wei (2009).

The bound given by (1.1) also has the drawback of demanding a parameterization of the boundary. For example, if we consider the version of Theorem 9.5 in Azaïs and Wschebor (2009) of the result of Mercadier, under some mild conditions on the set $S \subset \mathbb{R}^2$ and on the Gaussian process X, we have

$$
\begin{aligned}
\mathrm{P}\{M_S \geq u\} \leq &\mathrm{P}\{Y(O) \geq u\} + \int_0^L \mathrm{E}(|Y'(l)| \mid Y(l) = u)p_{Y(l)}(u)\, dl \\
&+ \int_S \mathrm{E}(|X_{11}''(t)^- X_2'(t)^+| \mid X(t) = u,\ X_1'(t) = 0)p_{X(t),X_1'(t)}(u,0)\, dt,
\end{aligned}
$$

(1.1)

where

- M_S is the maximum of $X(t)$ on the set S.
- $Y(l) = X(\rho(l))$ with $\rho : [0,L] \to \partial S$ is a parameterization of the boundary ∂S by its length.
- $X_{ij}'' = \dfrac{\partial^2 X}{\partial x_i \partial x_j}$.
- $p_Z(x)$: the value of the density function of random vector Z at point x.
- $x^+ = \sup(x,0), \quad x^- = \sup(-x,0)$.

The proof is based on considering the point with minimal ordinate (second coordinate) on the level curve. As we will see, this point can be considered as a "record point".

So the second direction of generalizations is to propose nicer and stronger forms of the inequality (1.1). This is done in Section 2. The result on quadratic form is presented in Section 4 and some numerical experiment is presented in Section 5.

Notation.

- S is some rather regular set included in $\mathbb{R}^2$ or $\mathbb{R}^3$. ∂S is its boundary; $\overset{\circ}{S}$ is its interior.

- $M_S = \max\limits_{s \in S} X(s)$ where $X(s)$ is some rather regular process.
- σ_i is the surface measure of dimension i. It can be defined as a Hausdorff measure.
- X', X'' are the first and second derivatives of the process $X(t)$. In particular if α is some direction then X'_α is the derivative along the direction α.
- $M \preceq 0$ means that the square matrix M is semi-definite negative.
- $S^{+\epsilon}$ is the tube around S, i.e

$$S^{+\epsilon} = \{s \in \mathbb{R}^2 : \operatorname{dist}(s, S) \leq \epsilon\}.$$

- d_H is the Hausdorff distance between sets, defined by

$$d_H(S, T) = \inf\{\epsilon : \ S \subset T^{+\epsilon}, T \subset S^{+\epsilon}\}.$$

- $\varphi(x)$ and $\Phi(x)$ are the density and distribution function of a standard normal variable.
  $\overline{\Phi}(x) = 1 - \Phi(x)$.
- For a point $s \in \mathbb{R}^2 \, (\mathbb{R}^3)$, s_i is the i-th coordinate of s.

2. The record method in dimension 2 revisited

We will work essentially under the following assumption:

Assumption 1: $\{X(t), \ t \in NS \subset \mathbb{R}^2\}$ is a stationary Gaussian field, defined in a neighborhood NS of S with $\mathcal{C}^1$ paths and such that there exists some direction, that will be assumed (without loss of generality) to be the direction of the first coordinate, in which the second derivative $X''_{11}(t)$ exists.

We assume moreover the following normalizing conditions that can always be obtained by a scaling

$$\mathrm{E}(X(t)) = 0, \ \mathrm{Var}(X(t)) = 1, \ \mathrm{Var}(X'(t)) = I_2.$$

Finally we assume that $\mathrm{Var}(X''_{11}(t)) > 1$ which is true as soon as the spectral measure of the process restricted to the first axis is not concentrated on two opposite atoms.

In some cases we will assume in addition

Assumption 2: $X(t)$ is isotropic, i.e $\mathrm{Cov}(X(s), X(t)) = \rho(\|t - s\|^2)$, with $\mathcal{C}^2$ paths and S is a convex polygon.

Under Assumption 1 and 2 plus some light additional hypotheses, the Euler Characteristic (EC) method Adler and Taylor (2007) gives

$$\mathrm{P}\{M_S \geq u\} = \mathrm{P}_E(u) + \mathrm{Rest},$$

with

$$\mathrm{P}_E(u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} u\varphi(u),$$

where the rest is super exponentially smaller.

The direct method Azaïs and Wschebor (2008) gives

$$P\{M_S \geq u\} \leq P_M(u)$$

$$= \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} \int_u^\infty \left[c\varphi(x/c) + x\Phi(x/c) \right] \varphi(x)dx + \frac{\sigma_2(S)}{2\pi}$$

$$\times \int_u^\infty \left[x^2 - 1 + \frac{(8\rho''(0))^{3/2} \exp(-x^2.(24\rho''(0) - 2)^{-1})}{\sqrt{24\rho''(0) - 2}} \right] \varphi(x)dx,$$

$$(2.1)$$

where $c = \sqrt{\mathrm{Var}(X''_{11}) - 1} = \sqrt{12\rho''(0) - 1}$.

The record method Mercadier (2006) gives

$$P\{M_S \geq u\} \leq \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c) \right] \varphi(u).$$

A careful examination of these equations shows that the main terms are almost the same except that in the record method the coefficient of $\sigma_1(\partial S)$ is twice too large. When S is a rectangle, it is easy to prove that this coefficient 2 can be removed, see for example Exercise 9.2 in Azaïs and Wschebor (2009).

The goal of this section is to extend the result above to more general sets and to fields satisfying Assumption 1 only. The main result of this section is the following

Theorem 2.1. *Let X satisfy the Assumption 1 and suppose that S is the Hausdorff limit of connected polygons S_n. Then,*

$$P\{M_S \geq u\} \leq \overline{\Phi}(u) + \frac{\liminf_n \sigma_1(\partial S_n)\varphi(u)}{2\sqrt{2\pi}} + \frac{\sigma_2(S)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c) \right] \varphi(u),$$

$$(2.2)$$

where $c = \sqrt{\mathrm{Var}(X''_{11}) - 1}$.

Remark 2.2. The choice of the direction of ordinates is arbitrary and is a consequence of the arbitrary choice of the second derivative X''_{11}. When the process $X(t)$ admits derivative in all direction, the choice that gives the sharpest bound consists in chosing as first axis, the direction α such that $\mathrm{Var}(X''_{\alpha\alpha})$ is minimum.

Unfortunately the proof it is based on an exotic topological property of the set S that will be called "emptyable".

Definition 2.3. The compact set S is emptyable if there exists a point $O \in S$ which has minimal ordinate, and such that for every $s \in S$ there exists a continuous path inside S from O to s with non decreasing ordinate.

In other word, suppose that S is filled with water and that gravity is in the usual direction; S is emptyable if after making a small hole at O, all the water will empty out, see Figure 2.1.

Proof: Step 1 : Suppose for the moment that X has $\mathcal{C}^\infty$ paths and that S is an emptyable polygon. Considering the event $\{M_S \geq u\}$, we have

$$P\{M_S \geq u\} = P\{X(O) \geq u\} + P\{X(O) < u, M_S \geq u\}. \qquad (2.3)$$

It is clear that if $X(O) < u$ and $M_S \geq u$, because S is connected, the level curve

$$\mathcal{C}(u) = \{t \in S : \ X(t) = u\}$$

is not empty, and there is at least one point T on $\mathcal{C}(u)$ with minimal ordinate. There are two possibilities:
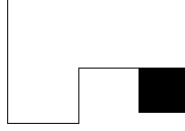
FIGURE 2.1. Example of non-emptyable set. The non-emptyable part is displayed in black.

- T is in the interior of S. In that case, suppose that there exists a point $s \in S$ with smaller ordinate than T ($s_2 < T_2$), such that $X(s) \geq u$. Then, due to the emptyable property, on the continuous path from O to s there would exist one point s' with smaller ordinate than T, and with $X(s') = u$. This is in contradiction with the definition of T. So we have proved that for every $s \in S$ such that $s_2 < T_2$, then $X(s) < u$. It is in this sense that T can be considered as a record point. It implies that

$$X'_1(T) = 0, \ X'_2(T) \geq 0 \text{ and } X''_{11}(T) \leq 0.$$

The probability that there exists such a point is clearly bounded, by the Markov inequality, by

$$\mathrm{E}\left(\mathrm{card}\{t \in S: \ X(t) = u, \ X'_1(t) = 0, \ X'_2(t) \geq 0, \ X''_{11}(t) \leq 0\}\right).$$

Applying the Rice formula to the field $Z = (X, X'_1)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, we get that

$$\mathrm{P}\left\{\exists t \in \overset{\circ}{S}: \ X(t) = u, \ t \text{ has minimal ordinate on } \mathcal{C}(u)\right\}$$

$$\leq \ \int_{\overset{\circ}{S}} \mathrm{E}\left(|\det(Z'(t))|\mathbb{I}_{X'_2(t) \geq 0}\mathbb{I}_{X''_{11}(t) \leq 0} \mid Z(t) = (u, 0)\right) p_{Z(t)}(u, 0) \, dt$$

$$= \ \sigma_2(S)\frac{\varphi(u)}{\sqrt{2\pi}} \, \mathrm{E}\left(X''^{-}_{11}(t)X'^{+}_2(t) \mid X(t) = u, X'_1(t) = 0\right)$$

$$= \ \sigma_2(S)\frac{\varphi(u)}{\sqrt{2\pi}} \, \mathrm{E}\left(X'^{+}_2(t)\right) \, \mathrm{E}\left(X''^{-}_{11}(t) \mid X(t) = u, X'_1(t) = 0\right)$$

$$= \ \sigma_2(S)\frac{\varphi(u)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c)\right]. \tag{2.4}$$

Note that the validity of the Rice formula holds true because the paths are of class $\mathcal{C}^\infty$ and that $X(t)$ and $X'_1(t)$ are independent. The computations above also use the fact that $X'_2(t)$ is independent of $(X(t), X'_1(t))$ and the main point that is, under the conditioning,

$$\det(Z'(t)) = -X''_{11}(t)X'_2(t).$$

- T is on the boundary of S that is the union of the edges $(F_1, \ldots, F_n)$. It is with probability 1 not located on a vertex. Suppose that, without loss of generality, it belongs to F_1. Using the reasoning we have done in the preceding case, because of the emptyable property, it is easy to see that

$$X(T) = u, \ X'_\alpha(T) \geq 0 \text{ and } X'_\beta(T) \leq 0,$$

where α is the upward direction on F_1 and β is the inward horizontal direction.

Then, apply the Markov inequality and Rice formula in the edge F_1,

$$
\begin{aligned}
& \mathrm{P}\{\exists\, t \in F_1 :\ X(t) = u,\ t \text{ has minimal ordinate on } \mathcal{C}(u)\} \\
\leq\ & \mathrm{P}\{\exists\, t \in F_1 :\ X(t) = u,\ X'_\alpha(t) \geq 0,\ X'_\beta(t) \leq 0\} \\
\leq\ & \mathrm{E}\left(\mathrm{card}\{t \in F_1 :\ X(t) = u,\ X'_\alpha(t) \geq 0,\ X'_\beta(t) \leq 0\}\right) \\
=\ & \int_{F_1} \mathrm{E}\left(|X'_\alpha(t)|\mathbb{I}_{X'_\alpha(t)\geq 0}\mathbb{I}_{X'_\beta(t)\leq 0} \mid X(t) = u\right) p_{X(t)}(u)\, dt \\
=\ & \sigma_1(F_1)\varphi(u)\,\mathrm{E}\left(X'^{+}_\alpha(t)\,\mathbb{I}_{X'_\beta(t)\leq 0}\right).
\end{aligned}
$$

Denote by θ_1 the angle (α, β). X'_β can be expressed as

$$
\cos\theta_1\, X'_\alpha + \sin\theta_1\, Y,
$$

where Y is a standard normal variable that is independent of X'_α. Then

$$
\begin{aligned}
& \mathrm{E}(X'^{+}_\alpha(t)\,\mathbb{I}_{X'_\beta(t)\leq 0}) \\
=\ & \mathrm{E}(X'^{+}_\alpha\,\mathbb{I}_{\cos\theta_1 X'_\alpha + \sin\theta_1 Y \leq 0}) \\
=\ & \frac{1 - \cos\theta_1}{2\sqrt{2\pi}}.
\end{aligned}
$$

Summing up, the term corresponding to the boundary of S is at most equal to

$$
\varphi(u) \sum_{i=1}^{n} \frac{(1 - \cos\theta_i)\sigma_1(F_i)}{2\sqrt{2\pi}} = \frac{\varphi(u)\sigma_1(\partial S)}{2\sqrt{2\pi}}, \tag{2.5}
$$

since $\sum_{i=1}^{n} \sigma_1(F_i) \cos\theta_i$ is just the length of the oriented projection of the boundary of S on the x-axis, so it is zero.

Hence, summing up (2.4), (2.5) and substituting into (2.3), we obtain the desired upper bound in our particular case.

Step 2: Suppose now that S is a general connected polygon such that the vertex O with minimal ordinate is unique. We define S_1 as the maximal emptyable subset of S that contains O. It is easy to prove that S_1 is still a polygon with some horizontal edges and that $S \backslash S_1$ consists of several polygons with horizontal edges, say $S_2^1, \ldots, S_2^m$, see Figure 2.2.
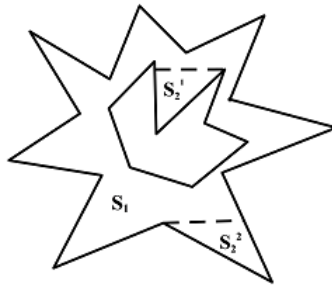

FIGURE 2.2.   Example on construction of S_1.

So we write

$$P\{M_S \geq u\} \leq P\{X(O) \geq u\} + P\{M_{S_1} \geq u, X(O) < u\} + \sum_{i=1}^{m} P\{M_{S_1} < u, M_{S_2^i} \geq u\}.$$
(2.6)

Suppose for the moment that all the S_2^i, $i = 1, \ldots, m$ are emptyable. Then, to give bounds to the event

$$\{M_{S_1} < u, \ M_{S_2^i} \geq u\},$$

we can apply the reasoning of the preceding proof but inverting the direction: in S_2^i, we search points on the level curve with **maximum ordinate**. Let E be the common edge of S_1 and S_2^i. Clearly, when $\{M_{S_1} < u, \ M_{S_2^i} \geq u\}$, the level curve is non empty and by the same arguments as in Step 1, there exists one record point $T \in S_2^i$ satisfying whether (excepting events with zero probability)

- T is in the interior of S_2^i and

$$X(T) = u, \ X_1'(T) = 0, \ X_2'(T) \leq 0, \ X_{11}''(T) \leq 0.$$

  From Markov inequality and Rice formula, the probability such that there exist some points satisfying the above conditions is at most equal to

$$\frac{\varphi(u)\sigma_2(S_2^i)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c) \right].$$
(2.7)

- T lies on some edges of S_2^i. Note that t can not belong to E since $E \subset S_1$. Then, to give an upper bound for the probability of this event, we sum up the upper bounds for the probabilities of the events "T is on each edge except for E". Now, as in Step 1 (see (2.5)), if we consider the record point with the maximal ordinate on the level curve in S_2^i, then the sum of the upper bounds corresponding to to all the edges on the boundary of S_2^i (including E) is

$$\frac{\varphi(u)\sigma_1(\partial S_2^i)}{2\sqrt{2\pi}},$$

  and the upper bound corresponding to the edge E is

$$\frac{\varphi(u)\sigma_1(E)}{\sqrt{2\pi}}.$$

Therefore,

$$P\left(\{\text{the record point } T \in \partial S_2^i\} \cap \{M_{S_1} < u\}\right) \leq \frac{\varphi(u)}{2\sqrt{2\pi}} \left[\sigma_1(\partial S_2^i) - 2\sigma_1(E)\right]. \quad (2.8)$$

From (2.7) and (2.8) we have

$$\begin{aligned} P\{M_{S_1} < u, \ M_{S_2^i} \geq u\} \leq & \frac{\varphi(u)\sigma_2(S_2^i)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c) \right] \\ & + \frac{\varphi(u)[\sigma_1(\partial S_2^i) - 2\sigma_1(E)]}{2\sqrt{2\pi}}. \end{aligned}$$
(2.9)

Summing up all the bounds as in (2.9), considering the upper bound for $P\{X(O) < u, M_{S_1} \geq u\}$ as in Step 1 and substituting into (2.6), we get the result.

In the general case, when some S_2^i is not emptyable, we can decompose S_2^i as we did for S, and search for the record point as above. The procedure of decomposing must stop since the number of vertices is decreasing, for example, the one of S_2^i is smaller than the one of S. By summing up all the bounds, the result follows.

Step 3: Passing to the limit. The extension to process with non $\mathcal{C}^\infty$ paths is direct by an approximation argument. Let $\overline{X}_\epsilon(t)$ be the Gaussian field obtained by convolution of $X(t)$ with a size ϵ convolution kernel (for example a Gaussian density with variance $\epsilon^2 I_2$). We can apply the preceding bound to the process

$$X_\epsilon(t) := \frac{1}{\sqrt{\mathrm{Var}(\overline{X}_\epsilon(t))}} \overline{X}_\epsilon\left(\Sigma_\epsilon^{-1/2} t\right),$$

where $\Sigma_\epsilon = \mathrm{Var}(\overline{X}'_\epsilon(t))$. Since $\mathrm{Var}(\overline{X}_\epsilon(t)) \to 1$, $\Sigma_\epsilon \to I_2$ and $\max_{t \in S} X_\epsilon(t) \to M_S$, we are done.

The passage to the limit for S_n tending to S in the Hausdorff topology is direct. $\qquad\square$

Some examples.

- If S is compact convex with non-empty interior then it is easy to construct a sequence of polygons S_n converging to S and such that $\liminf_n \sigma_1(\partial S_n) = \sigma_1(\partial S)$, giving

$$P\{M_S \geq u\} \leq P_R(u) = \overline{\Phi}(u) + \frac{\sigma_1(\partial S)}{2\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c)\right] \varphi(u).$$

$$(2.10)$$

- More generaly, if S is compact and has a boundary that is piecewise-$\mathcal{C}^2$ except for a finite number of points and the closure of the interior of S equals to S, we get (2.10) by the same tools.

- Let us now get rid of the condition $\overline{\overset{\circ}{S}} = S$ but still assuming the piecewise-$\mathcal{C}^2$ condition. Define the "outer Minkowski content" of a closed subset $S \subset \mathbb{R}^2$ as (see Cuevas et al. (2012))

$$\mathrm{OMC}(S) = \lim_{\epsilon \to 0} \frac{\sigma_2(S^{+\epsilon} \setminus S)}{\epsilon},$$

whenever the limit exists (for more treatment in this subject, see Ambrosio et al. (2008)). This definition of the perimeter differs from the quantity $\sigma_1(\partial S)$. A simple counter-example is a set corresponding to the preceding example with some "whiskers" added. Using approximation by polygons, we get

$$P\{M_S \geq u\} \leq P_R(u) = \overline{\Phi}(u) + \frac{\mathrm{OMC}(S)}{2\sqrt{2\pi}} \varphi(u) + \frac{\sigma_2(S)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c)\right] \varphi(u).$$

$$(2.11)$$

- The next generalization concerns compact r-convex sets with a positive r in the sense of Cuevas et al. (2012). These sets satisfy

$$S = \bigcap_{\overset{\circ}{B}(x,r) \cap S = \emptyset} \mathbb{R}^2 \setminus \overset{\circ}{B}(x,r).$$

This condition is slightly more general than the condition of having positive reach in the sense of Federer (1959). Suppose in addition that S satisfies the interior local connectivity property: there exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and for all $x \in S$, $\mathrm{int}\,(B(x,\alpha) \cap S)$ is a non-empty connected set. Then we can construct a sequence of approximating polygons in the following way.

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a uniform distribution on S and S_n be the r-convex hull of this sample, i.e

$$S_n = \bigcap_{\overset{\circ}{B}(x,r)\cap\{X_1,X_2,\ldots,X_n\}=\emptyset} \mathbb{R}^2 \setminus \overset{\circ}{B}(x,r),$$

which can be approximated by polygons with an arbitrary error. By Theorem 6 in Cuevas et al. (2012), S_n is a fully consistent estimator of S, it means that $d_H(S_n, S)$ and $d_H(\partial S_n, \partial S)$ tend to 0 as n tends to infinity. This implies $\sigma_2(S_n) \to \sigma_2(S)$ and $\text{OMC}(S_n) \to \text{OMC}(S)$. Hence, we obtain (2.11).

- A complicated case: a "Swiss cheese". Here, we consider an unit square and inside it, we remove a sequence of disjoint disks of radius r_i such that $\pi\sum_{i=1}^{\infty} r_i^2 < 1$ to obtain the set S. When $\sum_{i=1}^{\infty} r_i < \infty$ the bound (2.2) makes sense directly. But examples can be constructed from the Sierpinski carpet (see Figure 2.3) such that $\sum_{i=1}^{\infty} r_i = \infty$: divide the square into 9 subsquares of the same size and instead of removing the central square, remove the disk inscribed in this square and do the same procedure for the remaining 8 subsquares, ad infinitum.
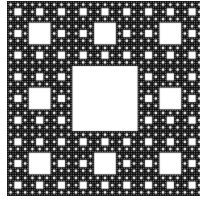

FIGURE 2.3.   Sierpinski carpet (source: Wikipedia).

In our case,

$$\sum_{i=1}^{\infty} r_i^2 = \frac{1}{4}\sum_{i=1}^{\infty} \frac{8^{i-1}}{3^{2i}} = \frac{1}{4}$$

This proves that the obtained set S has positive Lebesgue measure and is not fractal. We have on the other hand

$$\sum_{i=1}^{\infty} r_i = \frac{1}{2}\sum_{i=1}^{\infty} \frac{8^{i-1}}{3^i} = \infty.$$

Let S_n be the set obtained after removing the n-th disk. Since $S \subset S_n$, an upper bound for $\text{P}\{M_S \geq u\}$ is $\text{P}\{M_{S_n}$ that is at most equal to

$$\overline{\Phi}(u) + \frac{\varphi(u)}{2\sqrt{2\pi}}(4 + 2\pi\sum_{i=1}^{n} r_i) + (1 - \pi\sum_{i=1}^{n} r_i^2)\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u)/(2\pi).$$

Hence,

$$P\{M_S \geq u\} \leq \overline{\Phi}(u) + \min_n \left[\frac{\varphi(u)}{2\sqrt{2\pi}}(4 + 2\pi\sum_{i=1}^n r_i) \right.$$

$$\left. + (1 - \pi\sum_{i=1}^n r_i^2)\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u)/(2\pi) \right].$$

Remark.

1. In comparison with other results, all the examples considered here are new. Firstly the conditions on the process are minimal and weaker than the ones of the other methods. Secondly the considered sets are not covered by any other methods. Even for the first example, because we do not assume that the number of irregular points is finite, which is needed, for example, for the convex set to be a stratified manifold as in Adler and Taylor (2007).
2. Theorem 2.1 can be extended directly to non connected sets using sub-additivity

$$P\{M_{S_1 \cup S_2} \geq u\} \leq P\{M_{S_1} \geq u\} + P\{M_{S_2} \geq u\}.$$

This implies that the coefficient of $\overline{\Phi}(u)$ in (2.2) must be the number of components.

Is the bound sharp?

- Under Assumption 2, Adler and Taylor (2007) show that

$$\liminf_{u \to +\infty} -2u^{-2}\log|P\{M_S \geq u\} - P_E(u)| \geq 1 + 1/c^2.$$

From

$$0 \leq P_R(u) - P_E(u) = \frac{\sigma_2(S)}{2\pi}\varphi(u)\left[c\varphi(u/c) - u\overline{\Phi}(u/c)\right]$$

and the elementary inequality for $x > 0$,

$$\varphi(x)\left(\frac{1}{x} - \frac{1}{x^3}\right) < \overline{\Phi}(x) < \varphi(x)\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right),$$

it is easy to see that

$$\liminf_{u \to +\infty} -2u^{-2}\log(P_R(u) - P_E(u)) \geq 1 + 1/c^2.$$

So the upper bound $P_R(u)$ is as sharp as $P_E(u)$.

- Let S be a compact and simply connected domain in $\mathbb{R}^2$ having a piecewise-$\mathcal{C}^3$ boundary. Assume that all the discontinuity point are convex, in the sense that if we parametrize the boundary in the direction of positive rotation, then at each discontinuity point, the angle of the tangent has a positive discontinuity. Then, it is easy to see that the quantity

$$\kappa(S) = \sup_{t \in S} \sup_{s \in S,\ s \neq t} \frac{\mathrm{dist}(s - t, C_t)}{\|s - t\|^2}$$

is finite, where dist is the Euclidean distance and C_t is the cone generated by the set of directions

$$\left\{ \lambda \in \mathbb{R}^2 : \|\lambda\| = 1, \exists s_n \in S \text{ such that } s_n \to t \text{ and } \frac{s_n - t}{\|s_n - t\|} \to \lambda \right\}.$$

In order to apply the Theorem 8.12 in Azaïs and Wschebor (2009), besides the Assumption 1, we make some additional assumptions on the field X such that it satisfies the conditions (A1)-(A5) page 185 in Azaïs and Wschebor (2009). Assume that

- X has $\mathcal{C}^3$ paths.
- The covariance function $r(t)$ satisfies $|r(t)| \neq 1$ for all $t \neq 0$.
- For all $s \neq t$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.

With these hypotheses, we can see that

- The conditions (A1)-(A3) are easily verified.
- The condition (A4) which states that the maximum is attained at a single point, can be deduced from Proposition 6.11 in Azaïs and Wschebor (2009) since for $s \neq t$, $(X(s), X(t), X'(s), X'(t))$ has a nondegenerate distribution.
- The condition (A5) which states that almost surely there is no point $t \in S$ such that $X'(t) = 0$ and $\det(X''(t)) = 0$, can be deduced from Proposition 6.5 in Azaïs and Wschebor (2009) applied to the process $X'(t)$.

Since all the required conditions are met, by Theorem 8.12 in Azaïs and Wschebor (2009), we have

$$\liminf_{x \to +\infty} -2x^{-2} \log \left[P_M(x) - P\{M_S \geq x\} \right] \geq 1 + \inf_{t \in S} \frac{1}{\sigma_t^2 + \kappa_t^2} > 1, \qquad (2.12)$$

where

$$\sigma_t^2 = \sup_{s \in S \setminus \{t\}} \frac{\text{Var}\left(X(s) \mid X(t), X'(t) \right)}{(1 - r(s, t))^2}$$

and

$$\kappa_t = \sup_{s \in S \setminus \{t\}} \frac{\text{dist}\left(\frac{\partial}{\partial t} r(s, t), C_t \right)}{1 - r(s, t)}.$$

Note that the condition $\kappa(S)$ is finite implies that $\kappa(t)$ is also finite for every $t \in S$. (2.12) is true also for P_R, since as $x \to +\infty$, $P_R(x)$ is smaller than $P_M(x)$ (see Section 5 for the easy proof). As a consequence P_R is super exponentially sharp.

- Suppose that S is a circle in $\mathbb{R}^2$. Then $\{X(t) : t \in S\}$ can be viewed as a periodic process on the line. In that case, it is easy to show, see for example Exercise 4.2 in Azaïs and Wschebor (2009), that as $u \to \infty$

$$P(M_S \geq u) = \frac{\sigma_1(S)}{\sqrt{2\pi}} \varphi(u) + O\left(\varphi\left(u\left(1 + \delta \right) \right) \right) = \frac{\text{OMC}(S)}{2\sqrt{2\pi}} \varphi(u) + O\left(\varphi\left(u\left(1 + \delta \right) \right) \right)$$

for some $\delta > 0$; while Theorem 2.1 gives with a standard approximation of the circle by polygons

$$P(M_S \geq u) \leq P_R(u) = \overline{\Phi}(u) + \frac{\text{OMC}(S)}{2\sqrt{2\pi}} \varphi(u),$$

which is too large. This shows that the bound P_R is not always super exponentially sharp.

3. The record method in dimension 3

For example, with the direct method, some difficulties arise in dimension 3 because we need to compute

$$\mathrm{E}\big| \det\left(X''(t)\right) \big|,$$

under some conditional law. This can be conducted only in the isotropic case using random matrices theory, see Azaïs and Wschebor (2008) and Fyodorov (2004), and even in this case the result is complicated. In dimension 2, the record method is a trick that permits to spare a dimension in the size of the determinant that we have to consider because the conditioning implies a factorization. For example in equation (2.4) we have used the fact that

$$\det\left(Z'(t)\right) = X''_{11}(t)X'_2(t),$$

under the condition. In this section we will use the same kind of trick to pass from a (3,3) matrix to a (2,2) matrix and then a (2,2) determinant is just a quadratic form so we can use, to compute the expectation of its absolute value, the Fourier method of Berry and Dennis (2000) or Li and Wei (2009). This computation is detailed in Section 4 and is one of the main contributions of the paper.

Before stating the main theorem of this section, we recall a result from elementary geometry (see Prasolov and Sharygin (1989), Chapter 5).

Lemma 3.1. *Let $Omnp$ be a trihedral. Denote by a, b and c the plane angles $\widehat{mOn}$, $\widehat{nOp}$ and $\widehat{pOm}$, respectively. Denote by A, B and C the angles between two faces containing the line Op, Om and On, respectively. Then,*

   a. $\sin a : \sin A = \sin b : \sin B = \sin c : sinC$.
   b. $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

Our main result is the following

Theorem 3.2. *Let S be a compact and convex subset of $\mathbb{R}^3$ with non-empty interior and let X satisfy Assumption 1. Suppose, in addition that X is isotropic with respect to the first and second coordinate, i.e*

$$\mathrm{Cov}(X(t_1,t_2,t_3); X(s_1,s_2,t_3)) = \rho((t_1 - s_1)^2 + (t_2 - s_2)^2) \text{ with } \rho \text{ of class } \mathcal{C}^2.$$

Then, for every real u,

$$
\begin{aligned}
\mathrm{P}\{M_S \geq u\} \leq & \overline{\Phi}(u) + \frac{2\lambda(S)}{\sqrt{2\pi}}\varphi(u) \\
& + \frac{\sigma_2(S)\varphi(u)}{4\pi}\left[\sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)\right. \\
& \qquad\qquad \left. + u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)\right] \\
& + \frac{\sigma_3(S)\varphi(u)}{(2\pi)^{3/2}}\left[u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp\left(-u^2.(24\rho''(0)-2)^{-1}\right)}{\sqrt{24\rho''(0)-2}}\right],
\end{aligned}
$$

where λ is the caliper diameter of S which is defined by placing S between two parallel planes (or calipers), measuring the distance between the planes, and averaging over all rotations of S.

Remark 3.3. From the definition above, we can calculate that the caliper diameter of a ball is just its usual diameter, and the one of a cube $[0, a] \times [0, b] \times [0, c]$ is equal to half of $a + b + c$.

Proof: By the same limit argument as in Theorem 2.1, we can assume that $X(t)$ has $\mathcal{C}^\infty$ paths and that S is a convex polyhedron. Let O be the vertex of S that has minimal third coordinate, we can assume also that this vertex is unique. It is clear that if $X(O) < u$ and $M_S \geq u$ then the level set

$$\mathcal{C}(u) = \{t \in S : X(t) = u\}$$

is non empty and there exists at least one point T having minimal third coordinate on this set. Then, $\mathrm{P}\{M_S \geq u\}$

$$= \mathrm{P}\{X(O) \geq u\} + \mathrm{P}\{X(O) < u, M_S \geq u\}$$
$$\leq \mathrm{P}\{X(O) \geq u\} + \mathrm{P}\{\exists T \in S : X(T) = u, \tag{3.1}$$
$$T \text{ has minimal third coordinate on } \mathcal{C}(u)\}.$$

Now, we consider three possibilities:

• Firstly, if T is in the interior of S, then by the same arguments as in Theorem 2.1, for all the point $s \in S$ with the third coordinate smaller than the one of T, we have $X(s) < X(T)$; it means that, at T, $X(t)$ has a local maximum with respect to the first and second coordinates and is non-decreasing with respect to the third coordinate. Therefore, setting

$$A(t) = \begin{pmatrix} X_{11}''(t) & X_{12}''(t) \\ X_{12}''(t) & X_{22}''(t) \end{pmatrix},$$

we have

$$X(T) = u, \ X_1'(T) = 0, \ X_2'(T) = 0, \ A(T) \preceq 0 \text{ and } X_3'(T) \geq 0.$$

Then, apply the Rice formula to the field $Z(t) = (X(t), X_1'(t), X_2'(t))$ and the Markov inequality,

$$\mathrm{P}\{\exists T \in \overset{\circ}{S} : X(T) = u, T \text{ has minimal third coordinate on } \mathcal{C}(u)\}$$
$$\leq \ \mathrm{P}\{\exists t \in \overset{\circ}{S} : X(t) = u, \ X_1'(t) = 0, \ X_2'(t) = 0, \ X_3'(t) \geq 0, \ A(t) \preceq 0\}$$
$$\leq \ \mathrm{E}\big(\mathrm{card}\{t \in \overset{\circ}{S} : X(t) = u, \ X_1'(t) = 0, \ X_2'(t) = 0, \ X_3'(t) \geq 0, \ A(t) \preceq 0\}\big)$$
$$= \ \mathrm{E}\big(\mathrm{card}\{t \in \overset{\circ}{S} : Z(t) = (u, 0, 0), \ X_3'(t) \geq 0, \ A(t) \preceq 0\}\big)$$
$$= \ \int_{\overset{\circ}{S}} \mathrm{E}\left(|\det(Z'(t))| \mathbb{I}_{X_3'(t) \geq 0} \mathbb{I}_{A(t) \preceq 0} \mid Z(t) = (u, 0, 0)\right) p_{Z(t)}(u, 0, 0) \, dt.$$

Under the condition $Z(t) = (u, 0, 0)$, it is clear that $\det(Z'(t)) = X_3'(t) \det(A(t))$. So, we obtain the bound

$$\sigma_3(S) \frac{\varphi(u)}{2\pi} \mathrm{E}\left(|\det(A(t))|.\mathbb{I}_{A(t) \preceq 0} X_3'^+(t) \mid Z(t) = (u, 0, 0)\right),$$

which is equal to

$$\sigma_3(S) \frac{\varphi(u)}{(2\pi)^{3/2}} \mathrm{E}\left(|\det(A(t))|.\mathbb{I}_{A(t) \preceq 0} \mid Z(t) = (u, 0, 0)\right),$$

since $X_3'(t)$ is independent of $Z(t)$ and $A(t)$.

From Corollary 4.2 of Section 4, we know that

$$\mathrm{E}\left(|\det(A(t))|.\mathbb{I}_{A(t)\preceq 0} \mid Z(t) = (u,0,0)\right)$$

$$\leq u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp\left(-u^2.(24\rho''(0)-2)^{-1}\right)}{\sqrt{24\rho''(0)-2}}.$$

Hence,

$$\mathrm{P}\{\exists\, T \in \mathring{S}:\, X(T) = u,\, T \text{ has minimal third coordinate on } \mathcal{C}(u)\}$$

$$\leq \frac{\sigma_3(S)\varphi(u)}{(2\pi)^{3/2}}\left[u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp\left(-u^2.(24\rho''(0)-2)^{-1}\right)}{\sqrt{24\rho''(0)-2}}\right]. \qquad (3.2)$$

• Secondly, T is in the interior of a face S_1, for instance. On S_1, we choose the base $\{\alpha, \beta\}$ such that along these vectors, the second coordinate is not decreasing and α is the direction of the intersection line between the plane S_1 and the horizontal plane (set of all the points with zero third coordinate), when S_1 is parallel to or just the horizontal plane, α will be chosen as the direction of the x-axis. Let us denote vector γ in the horizontal plane that is perpendicular to α and goes into S. It is easy to see that

$$X(T) = u,\ X_\alpha'(T) = 0,\ X_\beta'(T) \geq 0,\ X_\gamma'(T) \leq 0 \text{ and } X_\alpha''(T) \leq 0.$$

Apply Markov inequality and Rice formula to the field $Y(t) = (X(t), X_\alpha'(t))$ on the plane S_1,

$$\mathrm{P}\{\exists\, T \in \mathring{S}_1:\, X(T) = u,\, T \text{ has the minimal third ordinate on } \mathcal{C}(u)\}$$

$$\leq\ \mathrm{P}\{\exists\, t \in \mathring{S}_1:\, X(t) = u,\ X_\alpha'(t) = 0,\ X_\beta'(t) \geq 0,\ X_\gamma'(t) \leq 0,\ X_{\alpha\alpha}'' \leq 0\}$$

$$\leq\ \mathrm{E}(\mathrm{card}\{t \in \mathring{S}_1:\, X(t) = u,\ X_\alpha'(t) = 0,\ X_\beta'(t) \geq 0,\ X_\gamma'(t) \leq 0,\ X_{\alpha\alpha}'' \leq 0\})$$

$$=\ \int_{\mathring{S}_1}\mathrm{E}\left(|\det(Y'(t))|\mathbb{I}_{X_\beta'(t)\geq 0}\mathbb{I}_{X_\gamma'(t)\leq 0}\mathbb{I}_{X_{\alpha\alpha}''(t)\leq 0} \mid Y(t) = (u,0)\right)p_{Y(t)}(u,0)\,dt$$

$$=\ \frac{\sigma_2(S_1)\varphi(u)}{\sqrt{2\pi}}\mathrm{E}\left(|X_{\alpha\alpha}''^-(t)|X_\beta'^+(t)\mathbb{I}_{X_\gamma'(t)\leq 0} \mid Y(t) = (u,0)\right).$$

Here $\det(Y'(t)) = X_\alpha'(t)X_{\alpha\beta}''(t) - X_\beta'(t)X_{\alpha\alpha}''(t)$, and under the condition $X_\alpha'(t) = 0$, it is equal to $-X_\beta'(t)X_{\alpha\alpha}''(t)$.

As in Theorem 2.1, it is clear that

$$\mathrm{E}\left(|X_{\alpha\alpha}''^-(t)| \mid Y(t) = (u,0)\right)$$

$$=\ \sqrt{12\rho''(0)-1}\,\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right) + u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right),$$

$$\mathrm{E}\left(X_\beta'^+(t)\mathbb{I}_{X_\gamma'(t)\leq 0} \mid Y(t) = (u,0)\right) = \frac{1 - \cos(\beta,\gamma)}{2\sqrt{2\pi}}.$$

Observe that the angle between β and γ is the angle θ_1 between the face S_1 and the horizontal plane, then the probability that there exists one point with minimal third coordinate on the level set and in the interior of the face S_1 is at most equal

to

$$\frac{\sigma_2(S_1)\varphi(u)(1-\cos\theta_1)}{4\pi}\left[\sqrt{12\rho''(0)-1}\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)\right.$$
$$\left.+u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)\right].$$

Taking the sum of all the bounds at each faces, observing that

$$\sum_{i=1}^{n}\sigma_2(S_i)\cos\theta_i=0,$$

we have the following upper bound for the probability of having a point T with minimal third coordinate on the level set and belonging to the interior of a face:

$$\frac{\sigma_2(S)\varphi(u)}{4\pi}\left[\sqrt{12\rho''(0)-1}\varphi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)+u\Phi\left(\frac{u}{\sqrt{12\rho''(0)-1}}\right)\right]. \quad (3.3)$$

• Thirdly, when T belongs to one edge, for example F_1. Let us define η is the upward direction on this edge, i.e such that along this vector, the third coordinate is not decreasing. On two faces containing F_1, we denote respectively α and β by the direction of the intersection line between the face and the horizontal plane such that it goes inside the face. Then,

$$X(T)=u,\ X_\eta'(T)\geq 0,\ X_\alpha'(T)\leq 0 \text{ and } X_\beta'(T)\leq 0.$$

By Rice formula, the expectation of the number of the points in F_1 satisfying this condition is

$$\int_{F_1}\mathrm{E}\left(X_\eta'^{+}(t)\mathbb{I}_{X_\alpha'(t)\leq 0}\mathbb{I}_{X_\beta'(t)\leq 0}\mid X(t)=u\right)p_{X(t)}(u)\,dt$$
$$=\ \sigma_1(F_1)\,\varphi(u)\,\mathrm{E}\left(X_\eta'^{+}(t)\mathbb{I}_{X_\alpha'(t)\leq 0}\mathbb{I}_{X_\beta'(t)\leq 0}\right),$$

since $X(t)$ is independent of $(X_\alpha'(t),X_\beta'(t),X_\eta'(t))$.

Let a and b be two inward vectors in two faces containing the edge F_1 and perpendicular to η; θ_1 be the angle between α and η; θ_2 be the angle between β and η. It is clear that

$$X_\alpha'(t)=\cos\theta_1\,X_\eta'(t)+\sin\theta_1\,X_a'(t),$$

$$X_\beta'(t)=\cos\theta_2\,X_\eta'(t)+\sin\theta_2\,X_b'(t),$$

and $\mathrm{cov}(X_a'(t),X_b'(t))=\cos\theta_3$, where θ_3 is the angle between two faces containing the edge F_1. Then,

$$\mathrm{E}\left(X_\eta'^{+}\mathbb{I}_{X_\alpha'(t)\leq 0}\mathbb{I}_{X_\beta'(t)\leq 0}\right)$$
$$=\mathrm{E}\left(X_\eta'^{+}\mathbb{I}_{\{\cos\theta_1\,X_\eta'(t)+\sin\theta_1\,X_a'(t)\leq 0\}}\mathbb{I}_{\{\cos\theta_2\,X_\eta'(t)+\sin\theta_2\,X_b'(t)\leq 0\}}\right)$$
$$=\int_0^\infty x\,\varphi(x)\,F(x)\,dx,$$

where

$$F(x) = \mathrm{E}\left(\mathbb{I}_{\{\cos\theta_1\,X'_\eta(t)+\sin\theta_1\,X'_a(t)\le 0\}}\mathbb{I}_{\{\cos\theta_2\,X'_\eta(t)+\sin\theta_2\,X'_b(t)\le 0\}} \mid X'_\eta(t) = x\right)$$

$$= \int_{-\infty}^{-\cot\theta_1.x} \varphi(y)\,\Phi\left(\frac{-\cot\theta_2.x - \cos\theta_3.y}{\sin\theta_3}\right)\,dy.$$

By integration by parts,

$$\int_0^\infty x\,\varphi(x)\,F(x)\,dx = -\int_0^\infty F(x)\,d(\varphi(x))$$

$$= F(0)\varphi(0) + \int_0^\infty \varphi(x)\,F'(x)\,dx,$$

where

$$F'(x) = -\cot\theta_2\,\varphi(-\cot\theta_2.x)\,\Phi\left(\frac{-\cot\theta_1.x + \cos\theta_3\,\cot\theta_2.x}{\sin\theta_3}\right)$$

$$-\cot\theta_1\,\varphi(-\cot(\theta_1\,x))\,\Phi\left(\frac{-\cot\theta_2.x + \cos\theta_3\,\cot\theta_1.x}{\sin\theta_3}\right).$$

It is easy to check that

$$\int_0^\infty \varphi(x)\,\Phi(mx)\,dx = \frac{1}{4} + \frac{\arctan(m)}{2\pi},$$

$$\int_{-\infty}^0 \varphi(x)\,\Phi(mx)\,dx = \frac{1}{4} - \frac{\arctan(m)}{2\pi}.$$

Therefore,

$$F(0)\varphi(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \varphi(y)\Phi\left(\frac{-\cos\theta_3}{\sin\theta_3}y\right)\,dy$$

$$= \frac{1}{(2\pi)^{3/2}}\,(\pi - \theta_3).$$

and

$$\int_0^\infty \varphi(x)\,F'(x)\,dx =$$

$$-\cot\theta_2\int_0^\infty \varphi(x)\varphi(-\cot\theta_2.x)\,\Phi\left(\frac{-\cot\theta_1.x + \cos\theta_3\,\cot\theta_2.x}{\sin\theta_3}\right)\,dx$$

$$-\cot\theta_1\int_0^\infty \varphi(x)\varphi(-\cot\theta_1.x)\,\Phi\left(\frac{-\cot\theta_2.x + \cos\theta_3\,\cot\theta_1.x}{\sin\theta_3}\right)\,dx$$

$$= \frac{-\cos\theta_2}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_2.\cot\theta_1 + \cos\theta_3\cos\theta_2}{\sin\theta_3}\right)\right)$$

$$+\frac{-\cos\theta_1}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_1\cot\theta_2 + \cos\theta_3\cos\theta_1}{\sin\theta_3}\right)\right).$$

Then the probability that there exists one point with minimal third coordinate on the level set $\mathcal{C}(u)$ and belonging to F_1 is at most equal to

$$\frac{\sigma_1(F_1)(\pi - \theta_3)\varphi(u)}{(2\pi)^{3/2}}$$
$$+ \sigma_1(F_1)\left[\frac{-\cos\theta_2}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_2\cot\theta_1 + \cos\theta_3\cos\theta_2}{\sin\theta_3}\right)\right)\right.$$
$$\left.+ \frac{-\cos\theta_1}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_1\cot\theta_2 + \cos\theta_3\cos\theta_1}{\sin\theta_3}\right)\right)\right].$$

Summing up all the terms at all the edges, we obtain the bound

$$\varphi(u)\sum_{i=1}^{n}\frac{\sigma_1(F_i)(\pi - \theta_{3i})}{(2\pi)^{3/2}}$$
$$+\varphi(u)\sum_{i=1}^{n}\sigma_1(F_i)$$
$$\times\left[\frac{-\cos\theta_{2i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{2i}\cot\theta_{1i} + \cos\theta_{3i}\cos\theta_{2i}}{\sin\theta_{3i}}\right)\right)\right.$$
$$\left.+ \frac{-\cos\theta_{1i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{1i}\cot\theta_{2i} + \cos\theta_{3i}\cos\theta_{1i}}{\sin\theta_{3i}}\right)\right)\right].$$

By definition,

$$\sum_{i=1}^{n}\sigma_1(F_i)(\pi - \theta_{3i}) = 4\pi\lambda(S).$$

Now, we prove that

$$I = \sum_{i=1}^{n}\sigma_1(F_i)\left[\frac{\cos\theta_{2i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{2i}\cot\theta_{1i} + \cos\theta_{3i}\cos\theta_{2i}}{\sin\theta_{3i}}\right)\right)\right.$$
$$\left.+ \frac{\cos\theta_{1i}}{\sqrt{2\pi}}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\sin\theta_{1i}\cot\theta_{2i} + \cos\theta_{3i}\cos\theta_{1i}}{\sin\theta_{3i}}\right)\right)\right] = 0.$$

Indeed, let us assume that the three vectors η, α and β have the same original point I. We use the Lemma 3.1 for the trihedral $I\eta\alpha\beta$ with the observation that

$$\theta_1 = \widehat{\eta I\alpha},\ \theta_1 = \widehat{\eta I\beta}$$

and θ_3 is the angle between two faces containing the line $I\eta$. Denote θ_4 and h by the angle $\widehat{\alpha I\beta}$ and the angle between two faces containing the line $I\alpha$. h can be viewed as the angle between the face containing α, η and the horizontal plane.

We have

$$\frac{-\sin\theta_1.\cot\theta_2 + \cos\theta_3.\cos\theta_1}{\sin\theta_3}$$

$$= \frac{-\sin\theta_1.\cos\theta_2 + \sin\theta_2.\cos\theta_3.\cos\theta_1}{\sin\theta_2.\sin\theta_3}$$

$$= \frac{-\sin\theta_1.\cos\theta_2 + \sin\theta_2.\cos\theta_1.(\cos\theta_4 - \cos\theta_1.\cos\theta_2)/(\sin\theta_1.\sin\theta_2)}{\sin\theta_4.\sin h}$$

$$= \frac{-\sin^2\theta_1.\cos\theta_2 + \cos\theta_1.(\cos\theta_4 - \cos\theta_1.\cos\theta_2)}{\sin\theta_4.\sin\theta_1.\sin h}$$

$$= \frac{-\cos\theta_2 + \cos\theta_1.\cos\theta_4}{\sin\theta_4.\sin\theta_1.\sin h}$$

$$= \frac{-\sin\theta_4.\sin\theta_1.\cos h}{\sin\theta_4.\sin\theta_1.\sin h} = \frac{-\cos h}{\sin h}.$$

Since h is constant for each face,

$$I = \sum_{S\in\{S_1,\ldots,S_k\}} \sum_{F\subset S} \sigma_1(F)\cos\theta_{1,F}\left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{-\cos h}{\sin h}\right)\right) = 0.$$

Therefore, we have the following upper bound for the probability of having a point T with minimal third coordinate on the level set and belonging to an edge:

$$\frac{2\lambda(S)\varphi(u)}{(2\pi)^{1/2}}. \tag{3.4}$$

From (3.2), (3.3), (3.4) and the fact that $\mathrm{P}\{X(O) > u\} = \overline{\Phi}(u)$, the result follows.
□

4. Computation of the absolute value of the determinant of the Hessian matrices

As we see in the proof of Theorem 3.2, we deal with the following

$$\mathrm{E}(|\det(X''(t))|\mathbb{I}_{X''(t)\preceq 0} \mid X(t) = u,\ X_1'(t) = 0,\ X_2'(t) = 0).$$

To evaluate this quantity, we have the following statement that is one of our main results in this paper:

Theorem 4.1. *Let X be a standard stationary isotropic centered two-dimensional Gaussian field. One has*

$$\mathrm{E}\left(|\det(X''(t))| \mid (X, X_1', X_2')(t) = (u, 0, 0)\right)$$

$$= u^2 - 1 + 2\frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0) - 2)^{-1})}{\sqrt{24\rho''(0) - 2}}. \tag{4.1}$$

Proof: Under the condition, the vector $(X_{11}'', X_{12}'', X_{22}'')$ has the same distribution with $(Y_1, Y_2, Y_3) + (-u, 0, -u)$, where (Y_1, Y_2, Y_3) is a centered Gaussian vector with the covariance matrix:

$$\Sigma = \begin{pmatrix} 12\rho''(0) - 1 & 0 & 4\rho''(0) - 1 \\ 0 & 4\rho''(0) & 0 \\ 4\rho''(0) - 1 & 0 & 12\rho''(0) - 1 \end{pmatrix}.$$

Then, the LHS in (4.1) can be written as

$$\mathrm{E}(|X_{11}''(t)X_{22}''(t) - X_{12}''(t)^2| \mid (X, X_1', X_2')(t) = (u, 0, 0))$$
$$= \mathrm{E}(|(Y_1 - u)(Y_3 - u) - Y_2^2|)$$
$$= \mathrm{E}(|Y_1 Y_3 - Y_2^2 - u(Y_1 + Y_3) + u^2|)$$
$$= \mathrm{E}(| < Y, BY > + < b, Y > + u^2|),$$

where $B = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -u \\ 0 \\ -u \end{pmatrix}$.

Here, from Theorem 2.1 in Li and Wei (2009), the expectation is equal to

$$\mathrm{E}(| < Y, BY > + < b, Y > + u^2|) = \frac{2}{\pi} \int_0^\infty t^{-2}(1 - F(t) - \overline{F}(t)) dt,$$

where

$$F(t) = \frac{\exp(itu^2 - 2^{-1}t^2 < b, (I - 2it\Sigma B)^{-1}\Sigma b >)}{2 \det(I - 2it\Sigma B)^{1/2}}.$$

It is clear that

$$F(t) = \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{2(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}},$$

and

$$\overline{F}(t) = \frac{\exp(-itu^2[1 + it(16\rho''(0) - 2)]^{-1})}{2(1 - 8it\rho''(0))[1 + it(16\rho''(0) - 2)]^{1/2}} = F(-t).$$

So, the expectation is equal to

$$\frac{2}{\pi} \int_0^\infty \frac{1}{t^2}(1 - F(t) - \overline{F}(t)) dt = \mathrm{Re}(\frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{t^2}(1 - 2.F(t)) dt)$$

$$= \mathrm{Re}(\frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{t^2}(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}}) dt).$$

Here, we apply the residue theorem to compute

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{t^2}(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}}) dt$$
$$= \quad 2i. \,(\text{sum of residues in upper half plane}) + i. \,(\text{sum of residues on x-axis}) .$$

The residues come from two poles at $i.(8\rho''(0))^{-1}$ and 0 and we see that:
The residue at 0 is equal to

$$\frac{d}{dt}\left(1 - \frac{\exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)]^{1/2}}\right)\Bigg|_{t=0} = -i.u^2 + i.$$

And the residue at $i.(8\rho''(0))^{-1}$ is equal to

$$\frac{(1 + 8it\rho''(0))[1 - it(16\rho''(0) - 2)] - \exp(itu^2[1 - it(16\rho''(0) - 2)]^{-1})}{t^2.8i\rho''(0).[1 - it(16\rho''(0) - 2)]^{1/2}}\Bigg|_{t=i.(8\rho''(0))^{-1}}$$

$$= \frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0) - 2)^{-1})}{\sqrt{24\rho''(0) - 2.i}}.$$

These two residues imply the result.   □

We have the corollary

Corollary 4.2. *Let X be a standard stationary isotropic centered Gaussian field. One has*

$$\mathrm{E}(|\det(X''(t))|.\mathbb{I}_{X''(t)\preceq 0} \mid (X, X_1', X_2')(t) = (u, 0, 0))$$

$$\leq u^2 - 1 + \frac{(8\rho''(0))^{3/2}\exp(-u^2.(24\rho''(0)-2)^{-1})}{\sqrt{24\rho''(0)-2}}.$$

Proof: The result follows from two observations

- $|\det(X''(t))|.\mathbb{I}_{X''(t)\preceq 0} \leq \dfrac{|\det(X''(t))| + \det(X''(t))}{2}.$
- $\mathrm{E}(\det(X''(t))) \mid (X, X_1', X_2')(t) = (u, 0, 0)) \;=\; u^2 - 1.$

<div align="right">□</div>

5. Numerical comparison

In this section, we compare the upper bounds given by the direct method and record method with the approximation given by the EC method. For simplicity we limit our attention to the case where S is the square $[0, L]^2$ and X is a standard stationary isotropic centered Gaussian field with covariance function $\rho(\|s - t\|^2)$. Note that only $\rho''(0)$ plays a role, the exact form of ρ does not need to be specified. More precisely, we consider

1. the approximation given by the EC method

$$\mathrm{P}_E(u) = \overline{\Phi}(u) + \frac{2L}{\sqrt{2\pi}}\varphi(u) + \frac{L^2}{2\pi}u\varphi(u);$$

2. and the upper bound given by the direct method

$$\mathrm{P}_M(u) = \quad \overline{\Phi}(u) + \frac{2L}{\sqrt{2\pi}}\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/\mathrm{c})\right]\varphi(x)dx$$

$$+ \frac{L^2}{2\pi}\int_u^\infty \left[x^2 - 1 + \left(\frac{2(c^2+1)}{3}\right)^{3/2}\sqrt{\pi}\frac{\varphi(x/c)}{c}\right]\varphi(x)dx,$$

    where $c = \sqrt{12\rho''(0) - 1}$,

3. and the one given by the record method

$$\mathrm{P}_R(u) = \overline{\Phi}(u) + \frac{2L}{\sqrt{2\pi}}\varphi(u) + \frac{L^2}{2\pi}\left[c\varphi(u/c) + u\Phi(u/c)\right]\varphi(u).$$

It is easy to see that P_E is always less than P_R and P_M. We will prove that $\mathrm{P}_R(u)$ is smaller than $\mathrm{P}_M(u)$ as u is large. Indeed, if we compare the "dimension 1 terms" (corresponding to $\sigma_1(\partial S)$), we have

$$\begin{aligned}&\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/\mathrm{c})\right]\varphi(x)dx - \varphi(u)\\ =\;&\int_u^\infty \left[c\varphi(x/c) + x\Phi(x/\mathrm{c})\right]\varphi(x)dx - \int_u^\infty x\varphi(x)dx\\ =\;&\int_u^\infty \left[c\varphi(x/c) - x\overline{\Phi}(x/\mathrm{c})\right]\varphi(x)dx \geq 0,\end{aligned}$$

since when $x \geq 0$,

$$\frac{\varphi(x)}{x} \geq \overline{\Phi}(x).$$

So the term in the direct method is always larger when $u \geq 0$.

Let us consider now the two terms corresponding to $\sigma_2(S)$:

- $A_d = u\varphi(u) + \int_u^\infty \left[\left(\frac{2(c^2+1)}{3} \right)^{3/2} \sqrt{\pi} \frac{\varphi(x/c)}{c} \right] \varphi(x)dx = u\varphi(u) + \overline{A}_d.$
- $A_r = \left[c\varphi(u/c) + u\Phi(u/c) \right] \varphi(u) = u\varphi(u) + \overline{A}_r.$

It is easy to show that , as $u \to +\infty$,

$$\overline{A}_d = (const) \int_u^\infty \varphi\left(\frac{x}{c} \right) \varphi(x)dx$$
$$= (const)\overline{\Phi}\left(u\sqrt{\frac{1+c^2}{c^2}} \right) \simeq (const)u^{-1}\varphi\left(u\sqrt{\frac{1+c^2}{c^2}} \right).$$

and that

$$\overline{A}_r \simeq (const)u^{-2}\varphi\left(u\sqrt{\frac{1+c^2}{c^2}} \right).$$

This shows that for u sufficiently large A_r is smaller than A_d.

The numerical comparison is performed in Figure 5.4 for six different situations. It shows that the record method is always better than the direct method. EC method and record method are very close, but it is not possible to identify the better among those two since P_E can be smaller than the true value.

Acknowledgements

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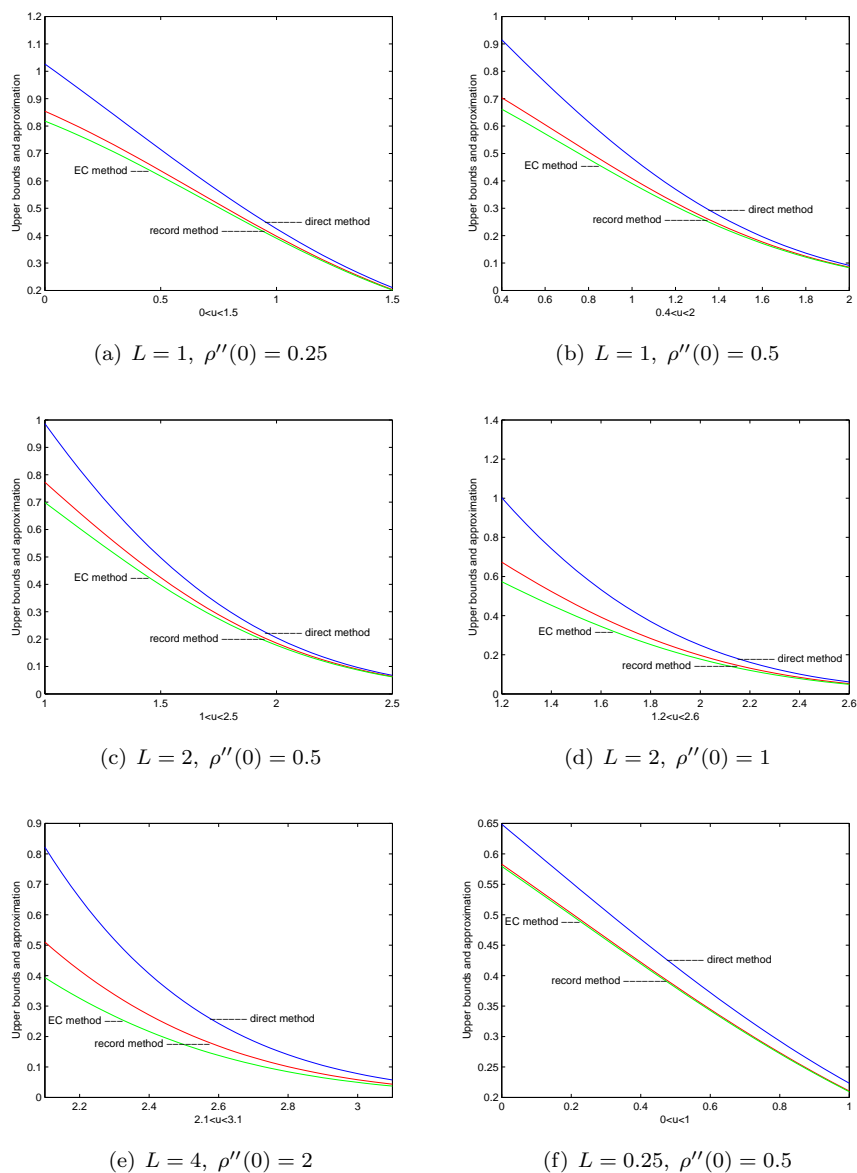
(a) $L = 1$, $\rho''(0) = 0.25$

(b) $L = 1$, $\rho''(0) = 0.5$

(c) $L = 2$, $\rho''(0) = 0.5$

(d) $L = 2$, $\rho''(0) = 1$

(e) $L = 4$, $\rho''(0) = 2$

(f) $L = 0.25$, $\rho''(0) = 0.5$

Figure 5.4. Comparison of the two bounds P_R and P_M and the approximation P_E for several values of $\rho''(0)$ and L.

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