A permuted random walk exits faster

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Abstract. Let σ be a permutation of $\{0, \ldots, n\}$. We consider the Markov chain X which jumps from $k \neq 0, n$ to $\sigma(k+1)$ or $\sigma(k-1)$, equally likely. When X is at 0 it jumps to either $\sigma(0)$ or $\sigma(1)$ equally likely, and when X is at n it jumps to either $\sigma(n)$ or $\sigma(n-1)$, equally likely. We show that the identity permutation maximizes the expected hitting time of n, when the walk starts at 0. More generally, we prove that the hitting time of a random walk on a strongly connected d-regular directed graph is maximized when the graph is the line $[0, n] \cap \mathbb{Z}$ with $d-2$ self-loops at every vertex and $d-1$ self-loops at 0 and n.

1. Introduction

Let σ be a permutation of $\{0, \ldots, n\}$ and $(\xi_i)_i$ be i.i.d. uniform random variables in $\{-1, 1\}$. We define the process X^σ by setting $X_0^\sigma = 0$ and $X_{t+1}^\sigma = \sigma(X_t^\sigma + \xi_{t+1})$ if $X_t^\sigma \neq 0, n$. Otherwise, if $X_t^\sigma = 0$, then X_{t+1}^σ is uniformly random in the set $\{\sigma(0), \sigma(1)\}$ and if $X_t^\sigma = n$, then it is uniformly random in the set $\{\sigma(n), \sigma(n-1)\}$.

In this paper we address the question of maximizing the hitting time of n starting from 0 by the process X^σ. In particular we show that the identity permutation gives the slowest hitting time of n starting from 0, i.e. in this case X^σ is a simple random walk on $\{0, \ldots, n\}$ with a self-loop at 0 and at n.

Theorem 1.1. *Let σ be a permutation of $\{0, \ldots, n\}$ and X^σ the Markov chain defined above. If $\tau_n = \inf\{t \geq 0 : X_t^\sigma = n\}$, then*

$$\mathbb{E}_0[\tau_n] \leq n^2 + n.$$

Equality is achieved if and only if σ is the identity permutation.

In Lemma 3.1 in Section 3, we show that the process X^σ can be viewed as a random walk on a strongly connected directed graph where every vertex has outdegree and indegree equal to 2. In Section 2 we prove a more general result (Theorem 1.4) concerning directed graphs in which every vertex has outdegree and indegree equal to d. Then in Section 3 we give the proof of Theorem 1.1 by applying Theorem 1.4 for $d = 2$.

Definition 1.2. Let $d \geq 2$ and $n \in \mathbb{N}$. We define $L(d, n)$ to be the graph on $[0, n] \cap \mathbb{Z}$ with the following properties:
1) 0 and n have $d - 1$ self loops each and all other vertices have $d - 2$ self loops
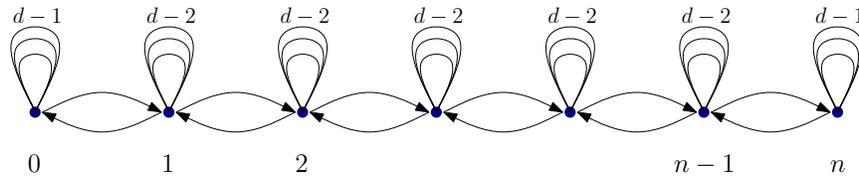2) for every $k \neq 0$ and $\ell \neq n$ there is a directed edge from k to $k - 1$ and from ℓ to $\ell + 1$.

FIGURE 1.1. The graph $L(d, n)$

Definition 1.3. A directed graph G is *connected* if for all vertices x and y there is a path from x to y ignoring directions. A directed graph G is *strongly connected* if for every pair of vertices x, y there is a directed path from x to y. We denote this by $x \rightsquigarrow y$. We denote by $V(G)$ the vertex set of a graph G and we write $\text{outdeg}_G(x)$ and $\text{indeg}_G(x)$ for the outdegree and indegree of the vertex x in the graph G. A directed graph G is called *regular* if $\text{outdeg}_G(x) = \text{indeg}_G(x)$ for all $x \in V(G)$ and *d-regular* if $\text{outdeg}_G(x) = \text{indeg}_G(x) = d$ for all $x \in V(G)$.

Theorem 1.4. *Fix $d \geq 2$ and let G be a strongly connected d-regular directed graph on $n + 1$ vertices (allowing (multiple) self-loops and multiple edges). If τ_x is the first hitting time of $x \in V(G)$ by a simple random walk on G, then we have*

$$\max_{x,y} \mathbb{E}_x[\tau_y] \leq \frac{d}{2} n(n + 1).$$

Equality is achieved if and only if G is isomorphic to $L(d, n)$.

To date, much of the work on Markov chains has focused on random walks on undirected graphs. Random walks on directed graphs have received relatively less attention and there are many interesting unexplored questions in this area. In particular, the first known bounds for mixing time parameters of a simple random walk on a directed graph have been studied in Fill (1991) and for the Eulerian case in Montenegro (2009).

Although the methods and ideas of the proofs are completely different, at a philosophical level the problem of maximizing the hitting time of $\{0, n\}$ by X^σ over all permutations σ is related to applications of rearrangement inequalities as in Burchard and Schmuckenschläger (2001) and Peres and Sousi (2012). We state a related result that was proved in Aizenman and Simon (1982): among all open sets of equal area, the ball maximizes the exit time by a Brownian motion. An analogous statement for a discrete lazy random walk is proved in Sousi and Winkler (2014).

2. Directed graphs

In this section we give the proof of Theorem 1.4. We start by stating a standard result about Eulerian graphs whose proof can be found in the discussion following Theorem 12 in Bollobás (1998). We then state and prove some preliminary results about directed graphs that will be used in the proof.

Lemma 2.1. *Let G be a regular directed graph. If G is connected, then it is strongly connected.*

Lemma 2.2. *Let G be a finite strongly connected graph. Suppose there exists a vertex i such that $indeg_G(i) \geq outdeg_G(i)$ and for all $x \neq i$ we have $indeg_G(x) \leq outdeg_G(x)$. If τ_i^+ is the first return time to i by a simple random walk on G, then*

$$\mathbb{E}_i[\tau_i^+] \leq \frac{\sum_x outdeg_G(x) + outdeg_G(i) - indeg_G(i)}{outdeg_G(i)}.$$

Proof: Since in every directed graph the sum of the outdegrees must equal the sum of the indegrees, we get

$$\text{indeg}_G(i) - \text{outdeg}_G(i) = \sum_{x \neq i}(\text{outdeg}_G(x) - \text{indeg}_G(x)). \tag{2.1}$$

Consider the set $A = \{x : \text{indeg}_G(x) < \text{outdeg}_G(x)\}$. We start adding fictitious edges from i to all vertices $j \in A$ until the total number of edges that come into j equals $\text{outdeg}_G(j)$. We call the new graph G' as shown in Figure 2.2.
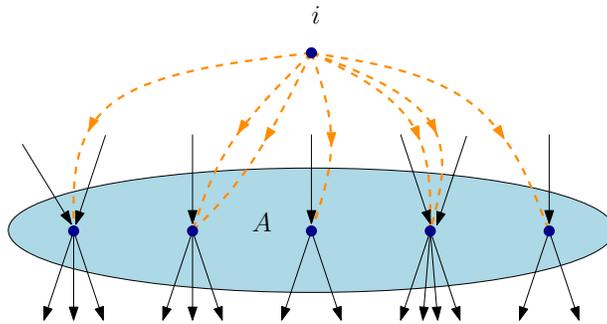

FIGURE 2.2. The graph G', where the orange dashed lines are the new fictitious edges and the black lines are edges in the original graph.

In view of (2.1) we have $\text{indeg}_G(i) = \text{indeg}_{G'}(i) = \text{outdeg}_{G'}(i)$. Also $\text{indeg}_{G'}(x) = \text{outdeg}_{G'}(x) = \text{outdeg}_G(x)$ for all $x \in V(G) \setminus \{i\}$, and hence if π' denotes the stationary distribution of a simple random walk on the directed graph G', we obtain

$$\pi'(i) = \frac{\text{indeg}_G(i)}{\sum_{x \neq i} \text{outdeg}_G(x) + \text{indeg}_G(i)}.$$

Since the directed graph G' is strongly connected, the simple random walk on G' is irreducible, and hence the expected return time to i in the graph G' is $1/\pi'(i)$.

Therefore by conditioning on the first step of the random walk and using the Markov property we obtain

$$\frac{\sum_{x \neq i} \text{outdeg}_G(x) + \text{indeg}_G(i)}{\text{indeg}_G(i)} = \frac{\text{outdeg}_G(i)}{\text{outdeg}_{G'}(i)} \mathbb{E}_i\left[\tau_i^+\right]$$
$$+ \sum_{j \in A} \frac{(\text{indeg}_{G'}(j) - \text{indeg}_G(j))}{\text{outdeg}_{G'}(i)} (1 + \mathbb{E}_j[\tau_i']),$$

$$(2.2)$$

where τ_i' is the hitting time of vertex i in G'. However, since $\text{indeg}_{G'}(j) = \text{outdeg}_G(j)$ for $j \in A$, we have by (2.1)

$$\sum_{j \in A} (\text{indeg}_{G'}(j) - \text{indeg}_G(j)) = \text{indeg}_G(i) - \text{outdeg}_G(i).$$

Plugging this into (2.2), using $\mathbb{E}_j[\tau_i'] \geq 1$ for all $j \neq i$, and $\text{outdeg}_{G'}(i) = \text{indeg}_G(i)$, we get by rearranging

$$\mathbb{E}_i\left[\tau_i^+\right] \leq \frac{\sum_x \text{outdeg}_G(x) + \text{outdeg}_G(i) - \text{indeg}_G(i)}{\text{outdeg}_G(i)}$$

which concludes the proof of the lemma. $\square$

In the next results, we will usually need to construct new graphs that come from a directed graph G with a distinguished vertex u. In order to avoid repetitions of the same construction in many of the statements and proofs we now give the definition of the new graph.

Definition 2.3. Let G be a directed graph and $u \in V(G)$ a distinguished vertex. We write I_u for the set of vertices having a directed edge to u. For each $i \in I_u$, we construct the graph G_i as follows: first we remove u from the graph G together with all the edges incident to it and then we connect every $j \in I_u \setminus \{i\}$ to i using multiple edges if there are multiple edges in the original graph between j and u so that $\text{outdeg}_{G_i}(j) = \text{outdeg}_G(j)$. We define $A_i = \{x : i \rightsquigarrow x \text{ in } G_i\}$ and the graph (A_i, E_i) to be the subset of G_i induced by A_i. We write $\text{indeg}_{A_i}(x)$ and $\text{outdeg}_{A_i}(x)$ for the indegree and outdegree of $x \in A_i$ in the graph (A_i, E_i).

Lemma 2.4. *Let G be a strongly connected regular directed graph. Let u be a distinguished vertex of G. Fix $i \in I_u$ and suppose that $A_i \neq \varnothing$. The graph (A_i, E_i) is strongly connected and contains i. Furthermore if r_i is the number of directed edges from i to u in the graph G, then*

$$\text{outdeg}_{A_i}(x) = \text{outdeg}_G(x) \quad \text{and} \quad \text{indeg}_{A_i}(x) \leq \text{indeg}_G(x), \quad \text{for all } x \in A_i \setminus \{i\}$$
$$\text{outdeg}_{A_i}(i) = \text{outdeg}_G(i) - r_i \quad \text{and} \quad \text{indeg}_{A_i}(i) \geq \text{outdeg}_{A_i}(i).$$

Proof: First we establish that if $x \in A_i$, then there is a directed path from x to i in graph (A_i, E_i). Indeed, in the original graph G, there is a path from x to i, since G was assumed to be strongly connected. If this path does not use the vertex u, then we have nothing to show. If it does, then if it uses the edge (i, u), then we are done again. If not, then it uses an edge of the form (i_ℓ, u), in which case since $i_\ell \in I_u$ is connected to i by at least one edge in G_i, it follows that $x \rightsquigarrow i$. Clearly by the definition of the set A_i all the vertices in the path from x to i are in A_i. Furthermore, $i \in A_i$, since its out-neighbours are in A_i by definition. Since
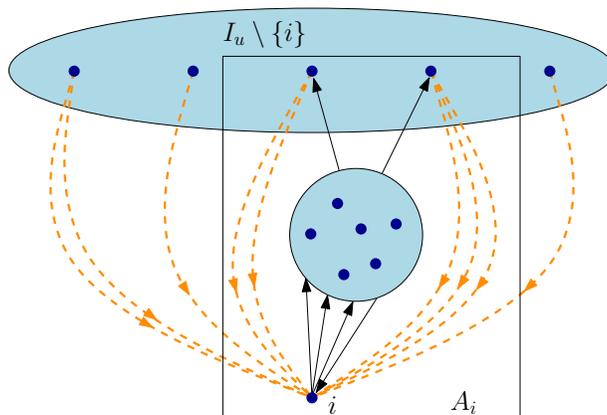
FIGURE 2.3. The graph (A_i, E_i), where the orange dashed lines represent the new edges and the black lines are edges in the original graph.

all vertices in A_i are connected to i in both directions, it follows that the graph defined by A_i is strongly connected.

Again by definition it follows that all the out-neighbours of $x \in A_i \setminus \{i\}$ are in A_i. Hence if $x \in A_i \setminus \{i\}$ we have $\text{outdeg}_{A_i}(x) = \text{outdeg}_G(x)$ and $\text{outdeg}_{A_i}(i) = \text{outdeg}_G(i) - r_i$. Also by definition of A_i we have that $\text{indeg}_{A_i}(x) \leq \text{indeg}_G(x)$ for all $x \in A_i \setminus \{i\}$. Using these inequalities together with the assumption that G is regular and the fact that

$$\sum_{x \in A_i \setminus \{i\}} (\text{outdeg}_{A_i}(x) - \text{indeg}_{A_i}(x)) = \text{indeg}_{A_i}(i) - \text{outdeg}_{A_i}(i)$$

we deduce that $\text{indeg}_{A_i}(i) \geq \text{outdeg}_{A_i}(i)$ and this finishes the proof of the lemma.
□

Lemma 2.5. *Let G be a strongly connected d-regular directed graph on n vertices. We fix a vertex u and write I_u for the set of vertices that have an edge leading to u. If μ is a probability measure supported on I_u and τ_u is the first hitting time of u by a simple random walk on G, then*

$$\mathbb{E}_\mu[\tau_u] \leq nd - d.$$

Proof: Let X be a simple random walk on G with $X_0 \sim \mu$ and $I_u = \{i_1, \ldots, i_k\}$ with $k \leq d$, since there could be multiple edges.

For any $i \in I_u$ we write r_i for the number of directed edges from i to u. Every time the random walk is at a vertex $i \in I_u$ it has probability r_i/d of jumping directly to u. If it does not jump, then it starts walking in the remaining graph until the first time that it hits I_u again. Define $(\xi_k^{(i)})_k$ to be the lengths of i.i.d. "excursions" starting from $i \in I_u$ until the first time that they come back to the set I_u without hitting u independently for different i.

It is clear that adding directed edges from every $\ell \in I_u \setminus \{i\}$ to i cannot affect $\xi_1^{(i)}$. Hence we can upper bound $\xi_1^{(i)}$ by the return time to i in the graph (A_i, E_i) constructed in Definition 2.3. In this graph we have $\text{outdeg}_{A_i}(i) = d - r_i$.

Lemma 2.4 gives that A_i satisfies the assumptions of Lemma 2.2. Therefore since $|A_i| < n$ and $r_i \geq 1$ for all $i \in I_u$ we deduce

$$\mathbb{E}_i\left[\xi_1^{(i)}\right] \leq \frac{dn - d - 1}{d - r_i}. \tag{2.3}$$

We now define independent collections of random variables

$$B_1^{(i_1)}, B_2^{(i_1)}, \ldots \quad \text{i.i.d. } \mathcal{B}(r_{i_1}/d)$$
$$B_1^{(i_2)}, B_2^{(i_2)}, \ldots \quad \text{i.i.d. } \mathcal{B}(r_{i_2}/d)$$
$$\vdots$$
$$B_1^{(i_k)}, B_2^{(i_k)}, \ldots \quad \text{i.i.d. } \mathcal{B}(r_{i_k}/d),$$

where $\mathcal{B}(p)$ stands for the Bernoulli distribution of parameter p.

We can realize the random walk X until the first time that it hits u in the following way: at time 0 if $B_1^{(X_0)} = 1$, then it jumps directly to u. Otherwise it makes an "excursion" of length $\xi_1^{(X_0)}$ until the first time that it comes back to I_u. We define $\ell(1) = X_0$ and $\zeta_1 = \xi_1^{(X_0)}$. Inductively we define

$$\ell(k+1) = X_{\zeta_k} \quad \text{and} \quad \zeta_{k+1} = \sum_{s=1}^{k+1} \xi_s^{(\ell(s))}.$$

In words, ζ_k is the time which has passed until the end of the k-th "excursion" and $\ell(k)$ is the position of the random walk at the end of the $(k-1)$-th "excursion". At the end of the $(k-1)$-th "excursion" X hits u directly with probability $r_{\ell(k)}/d$. If it does not, then we attach another "excursion" of length $\xi_k^{(\ell(k))}$ and we continue in the same way until the first time that X hits u. We finally define

$$T = \min\{s : B_s^{(\ell(s))} = 1\},$$

i.e. T is the number of used Bernoulli random variables until the first time that a Bernoulli is equal to 1. Hence we can now write

$$\mathbb{E}_\mu[\tau_u] = \mathbb{E}[\zeta_{T-1}] + 1. \tag{2.4}$$

By the definition of ζ we get

$$\mathbb{E}[\zeta_{T-1}] = \mathbb{E}\left[\sum_{k=1}^{T-1} \xi_k^{(\ell(k))}\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[\xi_k^{(\ell(k))} \mathbf{1}(T > k)\right]$$
$$= \sum_{k=1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[\xi_k^{(\ell(k))} \mathbf{1}(T > k) \,\Big|\, \xi_k^{(\ell(k))}, (\ell(j))_{j \leq k}\right]\right]$$
$$= \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{(d - r_{\ell(1)})}{d} \cdots \frac{(d - r_{\ell(k)})}{d} \mathbb{E}\left[\xi_k^{(\ell(k))} \,\Big|\, (\ell(j))_{j \leq k}\right]\right]. \tag{2.5}$$

Using (2.3) we now immediately get that

$$\mathbb{E}\left[\xi_k^{(\ell(k))} \,\Big|\, (\ell(j))_{j \leq k}\right] \leq \frac{dn - d - 1}{d - r_{\ell(k)}},$$

since $\xi_k^{(\ell(k))}$ is the length of an "excursion" started from the vertex $\ell(k)$. Hence plugging that into (2.5) and using that $r_i \geq 1$ for all i we get

$$\mathbb{E}[\zeta_{T-1}] \leq \sum_{k=1}^{\infty} (dn - d - 1)\frac{(d-1)^{k-1}}{d^k} = dn - d - 1.$$

This together with (2.4) gives

$$\mathbb{E}_\mu[\tau_u] \leq dn - d$$

and this concludes the proof of the lemma. □

Before proving Theorem 1.4 we state and prove a lemma that shows how we can rewire a graph after removing one vertex.

Lemma 2.6. *Suppose that G is a strongly connected d-regular directed graph and we remove a vertex $y \in G$ and its incident edges. Then there is a way of rewiring the remaining components to obtain a new graph, each component of which is strongly connected and d-regular. Moreover, this rewiring has the property that $\forall x \neq y$ and $x \notin I_y$, $\mathbb{E}_x[\tau_{I_y}]$ is equal in G and the rewired graph.*

Proof: Let $I = I_y$ be the vertices having a directed edge to y. Note that $|I| \leq d$, since we are allowing multiple edges and self-loops.

Let J be the vertices having a directed edge from y. Note that as above $|J| \leq d$. Removing the vertex y and its incident edges removes the edges from y to vertices in J as well as edges from vertices in I to y. Therefore in order to restore the in and out degrees of all the vertices in the new graphs (obtained by removing y) to d we shall add edges going from I to J as well as self-loops. We describe how to achieve this whilst ensuring the components become strongly connected.

Let B_J be the set of vertices $j \in J$ such that there is no directed path from j to any $i \in I \setminus \{j\}$ without visiting vertex y. Let $j \in B_J$. Since the only way to hit y is by passing through the set I (and we know that there must be a path from j to y as G is strongly connected) it must be that $j \in I$. Next, consider the set B_I defined as the set of vertices $i \in I$ such that there is no directed path from any $j \in J \setminus \{i\}$ to i without using y. Let $i \in B_I$. Choose (arbitrarily) $j \in J$. There must exist a path γ from j to i since G is strongly connected. Consider the last time γ visits y. The next vertex k visited by γ must be in J and furthermore the path then hits i without visiting y. It follows that there is indeed a path from $k \in J$ to i without visiting y. We deduce that it must be the case that $k = i$, i.e. $i \in J$. Therefore, we deduce that $B_J \subset I$ and $B_I \subset J$.

The above inclusions and the definition of the sets B_J and B_I now immediately give that if $i, j \in B_J \cup B_I$, then there are no paths from i to j which do not pass through vertex y. Hence after the removal of y, each point k in $B_J \cup B_I$ creates a distinct component G_k. Since in each G_k the indegrees and outdegrees of every vertex other than k are equal to d, it follows that in graph G the number of edges from y to k is equal to the number of edges from k to y, and thus we add self-loops to k in G_k so that G_k becomes d-regular. By Lemma 2.1 it follows that each G_k is strongly connected.

Suppose without loss of generality that $I \setminus B_I = \{i_1, \ldots, i_m\}$. Let J_1 be the subset of $J \setminus B_J$ containing only those vertices j_r which in graph G have a directed path from j_r to i_1 that does not visit vertex y.
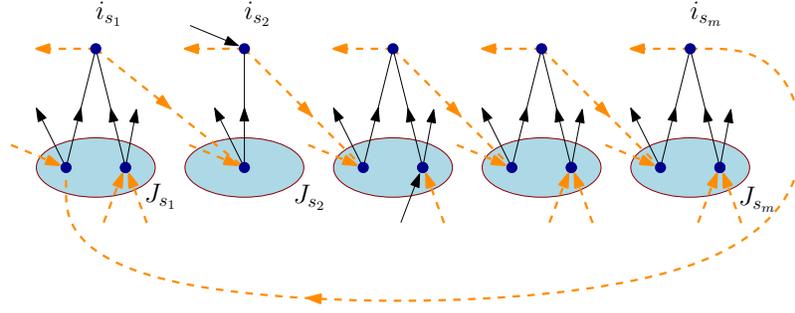
FIGURE 2.4. The graph G', where the orange dashed lines are the new edges and the black lines are edges in the original graph G.

Suppose we have defined the sets $J_1, \ldots, J_\ell$. We next define $J_{\ell+1}$ to be the set of $j_r \in J \setminus (\cup_{s \le \ell} J_s \cup B_J)$ which in G have a directed path from j_r to $i_{\ell+1}$ that does not visit vertex y.

Next we consider only the non-empty subsets J_r, which we index by $s_1, \ldots, s_\ell$ so that if $j \in J_{s_1}$, then it is connected to i_{s_1}. We then add a directed edge from i_{s_1} to an element of J_{s_2} (chosen arbitrarily) and from i_{s_2} to an element of J_{s_3} and so on. Finally we add a directed edge from i_{s_m} to an element of J_{s_1}.

At the end of this procedure, we add directed edges from $I \setminus B_I$ to $J \setminus B_J$ so that in the resulting graph every vertex has indegree equal to outdegree equal to d. This is possible, since the indegree of y is equal to its outdegree and the number of edges from y to $k \in B_J \cup B_I$ is equal to the number of edges from k to y. We call the resulting graph G' as shown in Figure 2.4.

We now claim that this new graph G' is connected. Indeed, all the vertices in J_r are connected to each other by the definition of the set J_r. Let $j_1 \in J_{s_k}$ and $j_2 \in J_{s_{k+1}}$. Then since we connect j_1 to $i_{s_{k+1}}$ it follows that j_1 is connected to j_2. Furthermore, as each $i \in I \setminus B_I$ is connected to at least one $j \in J \setminus B_J$, the graph is connected. Since the graph is also regular we apply Lemma 2.1 to deduce that it is strongly connected.

By the construction of our rewiring it is clear that $\mathbb{E}_x[\tau_{I_y}]$ is equal in G and G'. Indeed, adding edges out of I and deleting vertices reachable only via I preserve $\mathbb{E}_x[\tau_{I_y}]$.   □

We are now ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4: It is elementary to check that if G is isomorphic to $L(d, n)$, then

$$\max_{x,y} \mathbb{E}_x[\tau_y] = \frac{d}{2} n(n+1).$$

We prove the strict inequality by induction on n. For $n = 1$ it is trivially true. Suppose that for any strongly connected d-regular directed graph on $n + 1$ vertices not isomorphic to $L(d, n)$ we have

$$\max_{x,y} \mathbb{E}_x[\tau_y] < \frac{d}{2} n(n+1).$$

Let G' be a strongly connected d-regular directed graph on $n + 2$ vertices which is not isomorphic to $L(d, n + 1)$. We will show that for all x and y

$$\mathbb{E}_x[\tau_y] < \frac{d}{2}(n + 1)(n + 2). \tag{2.6}$$

Let $I = I_y$ be the vertices having a directed edge to y. Clearly we can write

$$\mathbb{E}_x[\tau_y] = \mathbb{E}_x[\tau_I] + \mathbb{E}_\mu[\tau_y], \tag{2.7}$$

where τ_I is the first hitting time of the set I by the simple random walk on G' and μ is the probability measure on I defined by

$$\mu(i) = \mathbb{P}_x(X_{\tau_I} = i),$$

for each $i \in I$. By Lemma 2.5 we immediately obtain

$$\mathbb{E}_\mu[\tau_y] \le nd + d. \tag{2.8}$$

If $x \in I$, then (2.8) finishes the proof. So from now on we assume that $x \notin I$. Since in order to hit y we must first hit the set I, we are going to look at the graph not containing the vertex y and the edges incident to it. Clearly adding edges coming out of points of I is not going to change the first hitting time of the set I starting from x.

By Lemma 2.6 there exists a way of rewiring the remaining graph so that we obtain a collection $(G_k)_{k \le s}$ of strongly connected d-regular directed graphs with $\mathbb{E}_x[\tau_I]$ the same for both graphs.

Suppose that $H \in \{G_k, k \le s\}$ is such that $x \in H$. If the graph H is not isomorphic to $L(d, n)$, then by the induction hypothesis we get

$$\mathbb{E}_x[\tau_{I \cap H}] < \frac{d}{2}n(n + 1)$$

and this together with (2.7) and (2.8) finishes the proof of (2.6) in this case. If H is isomorphic to $L(d, n)$ (in which case we identify these two graphs), then we shall consider two separate cases: $|I \cap H| = 1$ and $|I \cap H| \ge 2$. We start with the case $|I \cap H| = 1$. If $I \cap H = \{i_1\}$ and from i_1 the only vertex we can reach in one step is y in G', then

$$\mathbb{E}_{i_1}[\tau_y] \le d. \tag{2.9}$$

Since the graph H is a strongly connected graph on at most $n + 1$ vertices with in and out degree of every vertex equal to d, from the induction hypothesis it follows that

$$\mathbb{E}_x[\tau_{I \cap H}] \le \frac{d}{2}n(n + 1).$$

Hence this together with (2.7) and (2.9) gives that in this case

$$\mathbb{E}_x[\tau_y] < \frac{d}{2}(n + 1)(n + 2).$$

If i_1 has another out-neighbour $h \ne y$ in G', then i_1 cannot be an endpoint of the line, so the hitting time of i_1 will be bounded by the maximum hitting time on $L(d, m)$ for $m \le n - 1$ and thus we get

$$\mathbb{E}_x[\tau_{i_1}] < \frac{d}{2}n(n + 1).$$

This finishes the proof of (2.6) in the case $|I \cap H| = 1$. It remains to show that if $|I \cap H| \geq 2$, then

$$\mathbb{E}_x[\tau_{I \cap H}] < \frac{d}{2}n(n+1).$$

Since the subgraph H is isomorphic to $L(d, n)$, then the vertex i^* of $I \cap H$ closest to x satisfies

$$\mathbb{E}_x[\tau_{i^*}] < \frac{d}{2}n(n+1).$$

This together with (2.8) finishes the proof of the theorem. $\qquad \square$

3. Permutation walk

Lemma 3.1. *Let σ be a permutation of $\{0, \ldots, n\}$. Then the Markov chain X^σ is a random walk on a strongly connected 2-regular directed graph.*

Proof: It is easy to see that the Markov chain X^σ can be represented as a random walk on a 2-regular directed graph (note that we also count self-loops). Hence if we establish that ignoring orientations, the underlying graph is connected, then we can apply Lemma 2.1 and finish the proof. However, this is clear since, from the description of the process, all the odd points are connected to each other and all the even points are connected to each other, since $k - 1$ and $k + 1$ both lead to $\sigma(k)$. Since 0 and 1 both lead to $\sigma(0)$, it follows that the two sets (odd and even points) are connected, and hence this concludes the proof. $\qquad \square$

Proof of Theorem 1.1: From Lemma 3.1 above we have that the Markov chain X^σ can be viewed as a random walk on a strongly connected 2-regular directed graph.

Hence applying Theorem 1.4 shows that

$$\mathbb{E}_0[\tau_n] \leq n^2 + n.$$

From Theorem 1.4 we get that equality is achieved only if the resulting graph is isomorphic to $L(2, n)$ and $\sigma(0) = 0$ and $\sigma(n) = n$. It thus follows that σ has to be the identity permutation and this finishes the proof. $\qquad \square$

Remark 3.2. We note that the statement of Theorem 1.1 remains true if we change the Markov chain as follows: whenever at k the next step is either $\sigma(k) + 1$ or $\sigma(k) - 1$ equally likely. Indeed, it is easy to see that this Markov chain is again a simple random walk on a 2-directed graph which is strongly connected, and hence Theorem 1.4 applies.

4. Open problem

The following problem was communicated to us by Yuval Peres, but we could not trace its origins. We state it here:

Open problem: Let σ be a permutation of $\{-n, \ldots, n\}$. Let $(\xi_i)_i$ be i.i.d. random variables taking values in $\{-1, 1\}$ equally likely and set $X_0^\sigma = 0$ and $X_{t+1}^\sigma = \sigma(X_t^\sigma + \xi_{t+1})$ if $X_t^\sigma \neq n, -n$, otherwise X_{t+1}^σ takes values in $\{\sigma(n), \sigma(n-1)\}$ or $\{\sigma(-n), \sigma(-n+1)\}$ respectively equally likely. Show that the identity permutation maximizes $\mathbb{E}_0\big[\tau_{\{-n,n\}}\big]$, where $\tau_{\{-n,n\}}$ is the first hitting time of the set $\{-n, n\}$ by X^σ.

Remark 4.1. In contrast to Theorem 1.1 where equality is achieved only when σ is the identity permutation, we note that for this problem this is no longer the case. In other words, there exist other permutations for which $\mathbb{E}_0\big[\tau_{\{-n,n\}}\big] = n^2$. Indeed, if $\sigma(x) = -x$ for all $x \in [-n,n] \cap \mathbb{Z}$, then $\mathbb{E}_0\big[\tau_{\{-n,n\}}\big] = n^2$. Also, it is easy to check that if σ is the permutation that transposes 0 and 1, then it also achieves the same upper bound. Nevertheless, if σ only transposes k with $k+1$, then

$$\mathbb{E}_0\big[\tau_{\{-n,n\}}\big] = n\frac{2n-3-2k}{n-1} + n(n-2)$$

for all $0 < k \neq n-1, n$.

By arguing in a similar way as in the proof of Lemma 3.1, it is easy to see that the process X^σ can be viewed as a random walk on a 2 directed graph which is strongly connected. Hence by Theorem 1.4 we immediately get that for any permutation σ

$$\mathbb{E}_0\big[\tau_{\{-n,n\}}\big] \leq 4n^2 + 6n + 2.$$

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