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Constructing Stieltjes classes for M-indeterminate absolutely continuous probability distributions

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Abstract. If P is a moment-indeterminate probability distribution, then it is desirable to present explicitly other distributions possessing the same moments as P. In this paper, a method to construct an infinite family of probability densities - called the Stieltjes class - all with the same moments is presented. The method is applicable for densities with support $(0, \infty)$ which satisfy the lower bound: $f(x) \geq A \exp\{-ax^{\alpha}\}$ for some $A > 0$, $a > 0$ and some $\alpha \in (0, 1/2)$.

1. Introduction

Let P be an absolutely continuous probability distribution with finite moments of all orders. If the moment problem for P has a unique solution, then P is said to be *M-determinate*, otherwise it is *M-indeterminate*. The systematic treatment of the moment problem is presented in the classical sources Akhiezer, 1965 and Shohat and Tamarkin, 1943. There is a number of known criteria both for M-determinacy and M-indeterminacy. However, in the case of M-indeterminacy, those criteria do not advise how to find explicitly distributions different from P and with the same moments.

Let us mention that it is exactly the M-indeterminacy property that was crucial for constructing sets of random variables with given marginal distributions and possessing prescribed uncorrelatedness sets, see Ostrovska (2005).

In this paper, absolutely continuous M-indeterminate distributions are considered. The goal is to start with a given probability density f of an M-indeterminate distribution P and construct a Stieltjes class, that is, an infinite family of different distributions all with the same moments. The answer is given in terms of the density f and a function called 'perturbation'.

The term 'Stieltjes class' was introduced in Stoyanov (2004), while the idea is traced back to works by Stieltjes (1995), Heyde (1963), and Berg (1988). Further

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results on the Stieltjes classes along with various methods for their construction are available in Lin and Stoyanov (2009); Ostrovska and Stoyanov (2005); Pakes (2007); Stoyanov and Tolmatz (2005). The subject is currently under scrutiny and new researches are constantly coming out. See, for example, Kleiber (2013b,a). The method proposed in this article has an advantage of being applicable easily for construction of the Stieltjes classes even for "casewise" given density functions, when the methods involving integration are ineffective.

Let us recall the necessary notations and definitions introduced in Stoyanov (2004).

Definition 1.1. Let f be a probability density possessing finite moments of all orders. Let h be a measurable function on $(-\infty, \infty)$ such that $\text{vraisup}|h(x)| = 1$. If, for any polynomial Q,

$$\int_{\mathbb{R}} Q(x)h(x)f(x)\,dx = 0,$$

then h is called a *perturbation* of f.

In this case, we also say that the product hf has its all moments vanishing.

Definition 1.2. Given a probability density f and its perturbation h, the set

$$\mathbf{S} = \mathbf{S}(f, h) := \{f_\varepsilon(x) : f_\varepsilon(x) = f(x)[1 + \varepsilon\, h(x)],\ \ x \in \mathbf{R}, \varepsilon \in [-1, 1], \}$$

is said to be a *Stieltjes class* for density f based on perturbation h.

Clearly, $\mathbf{S}$ is an infinite family of probability densities all having the same moments as f. Notice that, for a given probability density f, there exist different Stieltjes classes based on different perturbation functions h. Trivially, a convex combination of perturbations is again a perturbation.

For any Stieltjes class $\mathbf{S} = \mathbf{S}(f, h)$, the following quantity $D_{\mathbf{S}} := \int_{-\infty}^{\infty} |h(u)|f(u)\,du$ is said to be the *index of dissimilarity* of the class. Obviously, $D_{\mathbf{S}} \in [0, 1]$. This index can be regarded as a global characteristic of the class $\mathbf{S}$. It shows the amount of "indeterminacy" within the class $\mathbf{S}(f, h)$, see Stoyanov (2004).

2. Main result and its proof

Let us have an absolutely continuous probability distribution P with support $[0, \infty)$ and density f. It can be observed directly from Definition 1.2 that the crucial step in the construction of a Stieltjes class for f is finding a perturbation function h. Under certain conditions on f, this can be achieved with the help of the following theorem.

Theorem 2.1. *Suppose that for some constants $A > 0$, $a > 0$ and $\alpha \in (0, 1/2)$, density f does not fall below the lower bound:*

$$f(x) \geq A \exp\{-ax^\alpha\}, \quad x > 0. \tag{2.1}$$

Let $g(z)$ be a function which is analytic in $\{z : Im z \geq 0\} \setminus \{0\}$ with $g(x)$ being real for $x > 0$, and whose values are within the upper bound:

$$|g(z)| \leq B \exp\{-a|z|^\alpha\}, \quad z \in \{z : Im z \geq 0\} \setminus \{0\} \ \ for\ \ some\ \ B > 0. \tag{2.2}$$

Then, the function

$$h(x) := \frac{Im\, g(-x)}{f(x)},\ \ x \in [0, \infty)$$

is bounded and, moreover, the product $f(x)h(x),\ x \in [0,\infty)$ *has all moments vanishing.*

Comment. A density f satisfying (2.1) corresponds to a distribution which is M-indeterminate. This follows from the well-known Krein criterion, see Stoyanov (2013), Section 10.

Proof: Choose real numbers $0 < r < R$ and consider in the upper half-plane a closed contour $L := l_1 \cup l_2 \cup l_3 \cup l_4$, consisting of two segments: $l_1 = [r, R]$ and $l_3 = [-R, -r]$, and two arcs (semi-circles): $l_2 = \{z : |z| = R,\ 0 < \arg z < \pi\}$, and $l_4 = \{z : |z| = r,\ 0 < \arg z < \pi\}$.

By the Cauchy Theorem

$$\oint_L z^n g(z)\, dz \;=\; 0, \quad n = 0, 1, 2, \ldots,$$

where we take positive (i.e. counterclockwise) direction of the path.

Clearly,

$$\oint_L = \int_{l_1} + \int_{l_2} + \int_{l_3} + \int_{l_4} =: I_1 + I_2 + I_3 + I_4 \ \text{ with } \ I_j = \int_{l_j},\ j = 1, 2, 3, 4.$$

We are going to show that, under condition (2.1), the integrals along the arcs tend to 0 as $r \to 0$ and $R \to \infty$. Indeed, we note that

$$|I_2| \le \pi R^{n+1} \cdot B \exp\{-aR^\alpha\} \to 0 \ \text{ as } \ R \to \infty.$$

Likewise, we get:

$$|I_4| \le \pi r^{n+1} B \exp\{-ar^\alpha\} \to 0 \ \text{ as } \ r \to 0.$$

Therefore, passing to the limit as $R \to \infty, r \to 0$, we get

$$\int_0^\infty x^n g(x)\, dx + (-1)^n \int_0^\infty x^n g(-x)\, dx = 0, \quad n = 0, 1, 2, \ldots \tag{2.3}$$

Taking the imaginary part of (2.3), one obtains:

$$\int_0^\infty x^n \left[\mathrm{Im} g(-x)\right]\, dx = 0. \tag{2.4}$$

We set:

$$h(x) := \frac{\mathrm{Im}\, g(-x)}{f(x)} \ \text{ for } \ x > 0.$$

Then (2.4) implies immediately that $h(x)f(x)$ has all moments vanishing. It remains to show that $h(x)$ is bounded on $[0, \infty)$. To do this, we use the bounds (2.1) and (2.2) to obtain:

$$|h(x)| \le \frac{B}{A}.$$

This completes the proof.

□

The next statement provides a practical way to find perturbation functions for a given density satisfying (2.1).

Theorem 2.2. *Let $f(x)$ be a probability density on $[0, \infty)$ possessing finite moments of all orders and satisfying the bound (2.1). Let $g(z)$ be a function which is analytic in $\{z : Im z \geq 0\} \setminus \{0\}$ with $g(x)$ being real for $x > 0$, and which satisfies the bound:*

$$|g(z)| \leq B \exp\{-b|z|^\beta\}, \quad z \in \{z : Im z \geq 0\} \setminus \{0\} \tag{2.5}$$

for some $B > 0, b > 0$ and $\beta \in (\alpha, 1/2)$.

Then the function

$$h(x) := \frac{Im\, g(-x)}{f(x)} \tag{2.6}$$

is bounded on $[0, \infty)$ and, moreover, $f(x)h(x)$ has all moments vanishing.

The proof of Theorem 2.2 is a direct consequence of Theorem 2.1, because estimate (2.5) implies (2.2) with a proper choice of constant B.

Corollary 2.3. *Set $M_h = \sup_{x\in\mathbb{R}} |h(x)|$, where h is given by (2.6). Then, the function $h(x)/M_h$, $x \in [0, \infty)$ can serve as a perturbation for the density f.*

3. Particular Stieltjes classes

In this section, examples of Stieltjes classes for some popular M-indeterminate probability densities are presented. It should be emphasized that, since only an estimate for the density $f(x)$ is needed, we can easily find different classes writing the corresponding $g(z)$ with appropriate parameters. It is worth to point out that some important densities, such as those for powers of exponential, normal, and power logistic (cf. Koutras et al. (2014)) distributions possess the lower bound of the form (2.1).

In what follows, C stands for a positive constant whose value is determined by the norming condition for the probability density f and a perturbation function h.

Example 3.1. Let $f(x)$ be a probability density of the form:

$$f(x) = C \exp\{-ax^\alpha\}, \quad x > 0, \quad a > 0, \ \alpha \in (0, 1/2).$$

Case 1. Select $g(z) = \exp\{-\frac{a}{\cos\pi\alpha} z^\alpha\}$. Clearly, for $z = |z|e^{i\varphi}, \quad 0 \leq \varphi \leq \pi$, one has:

$$|g(z)| = \exp\{-\frac{a}{\cos\pi\alpha}|z|^\alpha \cos\varphi\alpha\} \leq \exp\{-a|z|^\alpha\}$$

as required by (2.2). Now,

$$\text{Im} g(-x) = -\exp\{ax^\alpha\} \sin\left(ax^\alpha \sin\pi\alpha\right), \tag{3.1}$$

implying that the perturbation is $h(x) = \sin\left(ax^\alpha \sin\pi\alpha\right)$.

Case 2. Choose $g(z) = z^u \exp\{-bz^\beta\}, \quad u, b > 0, \beta \in (\alpha, 1/2)$. Estimate (2.5) is obviously satisfied. As $g(-x) = x^u \exp\{-ibx^\beta \pi u e^{i\pi\beta}\}$, one obtains:

$$h(x) = Cx^u \exp\{ax^\alpha - bx^\beta \cos\pi\beta\} \sin(\pi u - bx^\beta \sin\pi\beta).$$

Case 3. Let $g(z) = \exp\{-bz^\beta \ln z\}, \beta \in (\alpha, 1/2)$. Estimate (2.5) is satisfied because:

$$\begin{aligned}|g(z)| &= \exp\{-b|z|^\beta \ln|z| \cdot \cos(\varphi\beta)\} \exp\{b\varphi \sin(\varphi\beta)|z|^\beta\} \\ &\leq C_2 \exp\{-b|z|^\beta\} \text{ for some } C_2 > 0.\end{aligned}$$

Therefore, in view of Theorem 2.2,

$$h(x) = C\exp\{ax^\alpha - bx^\beta(\cos(\pi\beta)\ln x - \pi\sin(\pi\beta))\} \sin\left(bx^\beta(\pi\cos\pi\beta + \ln x\, \sin(\pi\beta))\right).$$

Example 3.2. Let $f(x)$ be a probability density of the form:

$$f(x) = Cx^{-u}\exp\{-ax^\alpha\}, \quad a>0, u\in(0,1), \alpha\in(0,1/2).$$

Obviously, this density satisfies condition (2.1) with any $\alpha' > \alpha$.

Case 1. Given $\rho > 1$, set $g(z) = \exp\{-\frac{a\rho}{\cos\pi\alpha}z^\alpha\}$, yielding $|g(z)| \le \exp\{-a\rho|z|^\alpha\}$, so that Theorem 2.1 is applicable. By plain calculations, the perturbation function is:

$$h(x) = Cx^u\exp\{-a(\rho-1)x^\alpha\}\sin(a\rho x^\alpha\tan\pi\alpha).$$

Case 2. Taking $g(z) = \exp\{-bz^\beta\}, \quad b>0, \quad \beta\in(\alpha,1/2)$, one obtains the perturbation function as:

$$h(x) = C\,x^u\exp\{ax^\alpha - bx^\beta\cos\pi\beta\}\sin(bx^\beta\sin\pi\beta).$$

In the case $b=1$ or $b=1/(\cos\pi\beta)$, we recover perturbation functions $h_1(x)$ and $h_2(x)$ previously found in Stoyanov and Tolmatz (2005).

Case 3. For $\rho>1$, put $g(z) = z^u\exp\{-\frac{a\rho}{\cos(\pi\alpha)}z^\alpha\}$. Then, for some $C_1>0$, the following estimate holds: $|g(z)| \le |z|^u\exp\{-a\rho|z|^\alpha\} \le C_1\exp\{-a|z|^\alpha\}$, because $\rho>1$. By virtue of Theorem 2.1, we find a perturbation function of the form:

$$h(x) = C\,\exp\{-a(\rho-1)x^\alpha\}\sin\left(\pi u - a\rho x^\alpha\tan(\pi\alpha)\right).$$

Remark 3.3. As a limiting case, with $\rho=1$, one has: $h(x) = \sin(\pi u - ax^\alpha\tan(\pi\alpha))$. It can be verified directly that $h(x)f(x)$ has all vanishing moments. We notice that this is the perturbation function $h_B(x)$ found previously in Stoyanov and Tolmatz (2005) by different methods.

Example 3.4. Consider a case-wise defined density of the form:

$$f(x) = \begin{cases} c_1e^{s(x)} & \text{if} \quad x\in[0,x_0], \\ c_2e^{-ax^\alpha} & \text{if} \quad x\in(x_0,+\infty), \end{cases}$$

where $a>0$, $\alpha\in(0,1/2)$, and $s(x)$ is any bounded function on $[0,x_0]$. Clearly, any function $g(z)$ from examples above can be used to find the perturbation $h(x)$. It should be pointed out that our method allows to find Stieltjes classes even though direct computation of integrals for specific densities is rather difficult. We present just one simple perturbation function, while the other cases can be treated in a similar way.

Choose $g(z) = \exp\{-\frac{a}{\cos(\pi\alpha)}z^\alpha\}$. Using (3.1), we obtain:

$$h(x) = \begin{cases} \frac{1}{c_1}\exp\{ax^\alpha - s(x)\}\sin(ax^\alpha\sin(\pi\alpha)) & \text{if} \quad x\in[0,x_0], \\ \frac{1}{c_2}\sin(ax^\alpha\sin(\pi\alpha)) & \text{if} \quad x\in(x_0,\infty). \end{cases}$$

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