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Kernel density estimation for stationary random fields

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Abstract. In this paper, under natural and easily verifiable conditions, we prove the $\mathbb{L}^1$-convergence and the asymptotic normality of the Parzen-Rosenblatt density estimator for stationary random fields of the form $X_k = g\left(\varepsilon_{k-s}, s \in \mathbb{Z}^d\right)$, $k \in \mathbb{Z}^d$, where $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are independent and identically distributed real random variables and g is a measurable function defined on $\mathbb{R}^{\mathbb{Z}^d}$. Such kind of processes provides a general framework for stationary ergodic random fields. A Berry-Esseen's type central limit theorem is also given for the considered estimator.

1. Introduction and main results

Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an unknown marginal density f. The kernel density estimator f_n of f introduced by Rosenblatt (1956) and Parzen (1962) is defined for all positive integer n and any real x by

$$f_n(x) = \frac{1}{nb_n}\sum_{i=1}^{n} \mathrm{K}\left(\frac{x - X_i}{b_n}\right)$$

where K is a probability kernel and the bandwidth b_n is a parameter which converges slowly to zero such that nb_n goes to infinity. The literature dealing with the asymptotic properties of f_n when the observations $(X_i)_{i\in\mathbb{Z}}$ are independent is very extensive (see Silverman (1986)). Parzen (1962) proved that when $(X_i)_{i\in\mathbb{Z}}$ are independent and identically distribut (i.i.d) and the bandwidth b_n goes to zero such that nb_n goes to infinity then $(nb_n)^{1/2}(f_n(x_0) - \mathbb{E}f_n(x_0))$ converges in distribution to the normal law with zero mean and variance $f(x_0)\int_{\mathbb{R}} \mathrm{K}^2(t)dt$. Under the same conditions on the bandwidth, this result was extended by Wu and Mielniczuk (2002)

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for causal linear processes with i.i.d. innovations and by Dedecker and Merlevède (2002) for strongly mixing sequences.

In this paper, we are interested by the kernel density estimation problem in the setting of dependent random fields indexed by $\mathbb{Z}^d$ where d is a positive integer. The question is not trivial since $\mathbb{Z}^d$ does not have a natural ordering for $d \geq 2$. In recent years, there is a growing interest in asymptotic properties of kernel density estimators for random fields. One can refer for example to Carbon et al. (1996, 1997), Cheng et al. (2008), El Machkouri (2011), Hallin et al. (2001), Tran (1990) and Wang and Woodroofe (2014). In Tran (1990), the asymptotic normality of the kernel density estimator for strongly mixing random fields was obtained using the Bernstein's blocking technique and coupling arguments. Using the same method, the case of linear random fields with i.i.d. innovations was handled in Hallin et al. (2001). In El Machkouri (2011), the central limit theorem for the Parzen-Rosenblatt estimator given in Tran (1990) was improved using the Lindeberg's method (see Lindeberg (1922)) which seems to be better than the Bernstein's blocking technique approach. In particular, a simple criterion on the strong mixing coefficients is provided and the only condition imposed on the bandwith is $n^d b_n \to \infty$ which is similar to the usual condition imposed in the independent case (see Parzen (1962)). In El Machkouri (2011), the regions where the random field is observed are reduced to squares but a carrefull reading of the proof allows us to state that the main result in El Machkouri (2011) still holds for very general regions Λ_n, namely those which the cardinality $|\Lambda_n|$ goes to infinity such that $|\Lambda_n| b_n$ goes to zero as n goes to infinity (see Assumption **(A3)** below). Cheng et al. (2008) investigated the asymptotic normality of the kernel density estimator for linear random fields with i.i.d. innovations using a martingale approximation method (initiated by Cheng and Ho (2006)) but it seems that there is a mistake in their proof (see Remark 6 in Wang and Woodroofe (2014)). Since the mixing property is often unverifiable and might be too restrictive, it is important to provide limit theorems for nonmixing and possibly nonlinear random fields. We consider in this work a field $(X_i)_{i\in\mathbb{Z}^d}$ of identically distributed real random variables with an unknown marginal density f such that

$$X_i = g\left(\varepsilon_{i-s};\, s \in \mathbb{Z}^d\right), \quad i \in \mathbb{Z}^d, \tag{1.1}$$

where $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ are i.i.d. random variables and g is a measurable function defined on $\mathbb{R}^{\mathbb{Z}^d}$. In the one-dimensional case ($d = 1$), the class (1.1) includes linear as well as many widely used nonlinear time series models as special cases. More importantly, it provides a very general framework for asymptotic theory for statistics of stationary time series (see e.g. Wu (2005) and the review paper Wu (2011)).
We introduce the physical dependence measure first introduced by Wu (2005). Let $(\varepsilon_j^{'})_{j\in\mathbb{Z}^d}$ be an i.i.d. copy of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ and consider for all positive integer n the coupled version X_i^* of X_i defined by $X_i^* = g\left(\varepsilon_{i-s}^*;\, s \in \mathbb{Z}^d\right)$ where $\varepsilon_j^* = \varepsilon_j \,\mathbb{1}_{\{j\neq 0\}} + \varepsilon_0^{'} \,\mathbb{1}_{\{j=0\}}$ for all j in $\mathbb{Z}^d$. In other words, we obtain X_i^* from X_i by just replacing ε_0 by its copy $\varepsilon_0^{'}$. Let i in $\mathbb{Z}^d$ and $p > 0$ be fixed. If X_i belongs to $\mathbb{L}_p$ (that is, $\mathbb{E}|X_i|^p$ is finite), we define the physical dependence measure $\delta_{i,p} = \|X_i - X_i^*\|_p$ where $\|\,.\,\|_p$ is the usual $\mathbb{L}^p$-norm and we say that the random field $(X_i)_{i\in\mathbb{Z}^d}$ is p-stable if $\sum_{i\in\mathbb{Z}^d} \delta_{i,p} < \infty$. For $d \geq 2$, the reader should keep in mind the following two examples already given in El Machkouri et al. (2013) :

<u>*Linear random fields*</u>: Let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be i.i.d random variables with ε_i in $\mathbb{L}^p$, $p\geq 2$. The linear random field X defined for all i in $\mathbb{Z}^d$ by

$$X_i=\sum_{s\in\mathbb{Z}^d}a_s\varepsilon_{i-s}$$

with $(a_s)_{s\in\mathbb{Z}^d}$ in $\mathbb{R}^{\mathbb{Z}^d}$ such that $\sum_{i\in\mathbb{Z}^d}a_i^2<\infty$ is of the form (1.1) with a linear functional g. For all i in $\mathbb{Z}^d$, $\delta_{i,p}=|a_i|\|\varepsilon_0-\varepsilon_0^{'}\|_p$. So, X is p-stable if $\sum_{i\in\mathbb{Z}^d}|a_i|<\infty$. Clearly, if H is a Lipschitz continuous function, under the above condition, the subordinated process $Y_i=\mathrm{H}(X_i)$ is also p-stable since $\delta_{i,p}=O(|a_i|)$.
<u>*Volterra field*</u>: Another class of nonlinear random field is the Volterra process which plays an important role in the nonlinear system theory (Casti (1985), Rugh (1981)): consider the second order Volterra process

$$X_i=\sum_{s_1,s_2\in\mathbb{Z}^d}a_{s_1,s_2}\varepsilon_{i-s_1}\varepsilon_{i-s_2},$$

where a_{s_1,s_2} are real coefficients with $a_{s_1,s_2}=0$ if $s_1=s_2$ and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are i.i.d. random variables with ε_i in $\mathbb{L}^p$, $p\geq 2$. Let

$$A_i=\sum_{s_1,s_2\in\mathbb{Z}^d}(a_{s_1,i}^2+a_{i,s_2}^2)\quad\text{and}\quad B_i=\sum_{s_1,s_2\in\mathbb{Z}^d}(|a_{s_1,i}|^p+|a_{i,s_2}|^p).$$

By the Rosenthal inequality, there exists a constant $C_p>0$ such that

$$\delta_{i,p}=\|X_i-X_i^*\|_p\leq C_pA_i^{1/2}\|\varepsilon_0\|_2\|\varepsilon_0\|_p+C_pB_i^{1/p}\|\varepsilon_0\|_p^2.$$

From now on, for all finite subset Λ of $\mathbb{Z}^d$, we denote $|\Lambda|$ the number of elements in Λ and we observe $(X_i)_{i\in\mathbb{Z}^d}$ on a sequence $(\Lambda_n)_{n\geq1}$ of finite subsets of $\mathbb{Z}^d$ which only satisfies $|\Lambda_n|$ goes to infinity as n goes to infinity. It is important to note that we do not impose any condition on the boundary of the regions Λ_n. The density estimator f_n of f is defined for all positive integer n and any real x by

$$f_n(x)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\mathrm{K}\left(\frac{x-X_i}{b_n}\right)$$

where b_n is the bandwidth parameter and K is a probability kernel. Our aim is to provide sufficient conditions for the $\mathbb{L}_1$-distance between f_n and f to converge to zero (Theorem 1.1) and for $(|\Lambda_n|b_n)^{1/2}(f_n(x_i)-\mathbb{E}f_n(x_i))_{1\leq i\leq k}$, $(x_i)_{1\leq i\leq k}\in\mathbb{R}^k$, $k\in\mathbb{N}\backslash\{0\}$, to converge in law to a multivariate normal distribution (Theorem 1.4) under minimal conditions on the bandwidth parameter. We give also a Berry-Esseen's type central limit theorem for the considered estimator (Theorem 1.5). In the sequel, we denote $|i|=\max_{1\leq k\leq d}|i_k|$ for all $i=(i_1,...,i_d)\in\mathbb{Z}^d$ and we denote also δ_i for $\delta_{i,2}$. The following assumptions are required.

- **(A1)** The marginal density function f of each X_k is Lipschitz.
- **(A2)** K is Lipschitz, $\int_{\mathbb{R}}\mathrm{K}(u)\,du=1$, $\int_{\mathbb{R}}u^2|\mathrm{K}(u)|\,du<\infty$ and $\int_{\mathbb{R}}\mathrm{K}^2(u)\,du<\infty$.
- **(A3)** $b_n\to0$ and $|\Lambda_n|\to\infty$ such that $|\Lambda_n|b_n\to\infty$.
- **(A4)** $\sum_{i\in\mathbb{Z}^d}|i|^{\frac{5d}{2}}\delta_i<\infty$.

Theorem 1.1. *If* **(A1)**, **(A2)**, **(A3)** *and* **(A4)** *hold, then there exists* $\kappa>0$ *such that for all integer* $n\geq1$,

$$\mathbb{E}\int_{\mathbb{R}}|f_n(x)-f(x)|\,dx\leq\kappa\left(b_n+\frac{1}{\sqrt{|\Lambda_n|b_n}}\right)^{\frac{2}{3}}.\tag{1.2}$$

Remark 1. One can optimize the inequality (1.2) by taking $b_n = |\Lambda_n|^{-\frac{1}{3}}$. Then, we obtain $\mathbb{E}\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx = O\left(|\Lambda_n|^{-\frac{2}{9}}\right)$.

Remark 2. The convergence in probability of $\int_{\mathbb{R}} |f_n(x)-f(x)|\, dx$ to 0 was obtained (without rate) by Hallin et al. (Hallin et al. (2004), Theorem 2.1) for rectangular region Λ_n. The authors defined the so-called stability coefficients $(v(m))_{m\geq 1}$ by $v(m) = \|X_0 - \overline{X}_0\|_2^2$ where $\overline{X}_0 = \mathbb{E}\left(X_0|\mathcal{H}_m\right)$ and $\mathcal{H}_m = \sigma\left(\varepsilon_s\,,\, |s| \leq m\right)$. Under minimal conditions on the bandwidth b_n, with our notations, their result holds as soon as $v(m) = o(m^{-4d})$. Arguing as in the proof of Lemma 3.3 below, one can relate the stability coefficients with the physical dependence measure ones by the inequality $v(m) \leq C \sum_{|i|>m} \delta_i^2$, $m \geq 1$, $C > 0$.

In the sequel, we consider the sequence $(m_n)_{n\geq 1}$ defined by

$$m_n = \max\left\{v_n, \left[\left(\frac{1}{b_n^3}\sum_{|i|>v_n} |i|^{\frac{5d}{2}}\,\delta_i\right)^{\frac{1}{3d}}\right] + 1\right\} \tag{1.3}$$

where $v_n = \left[b_n^{-\frac{1}{2d}}\right]$ and $[\,.\,]$ denotes the integer part function. The following technical lemma is a spatial version of a result by Bosq et al. (Bosq et al. (1999), pages 88-89).

Lemma 1.2. *If* **(A4)** *holds then*

$$m_n \to \infty, \quad m_n^d b_n \to 0 \quad \textit{and} \quad \frac{1}{(m_n^d b_n)^{3/2}} \sum_{|i|>m_n} |i|^{\frac{5d}{2}}\,\delta_i \to 0.$$

For all z in $\mathbb{R}$ and all i in $\mathbb{Z}^d$, we denote

$$\mathrm{K}_i(z) = \mathrm{K}\left(\frac{z - X_i}{b_n}\right) \quad \text{and} \quad \overline{\mathrm{K}}_i(z) = \mathbb{E}\left(\mathrm{K}_i(z)|\mathcal{F}_{n,i}\right) \tag{1.4}$$

where $\mathcal{F}_{n,i} = \sigma\left(\varepsilon_{i-s}\,;\, |s| \leq m_n\right)$. So, denoting $M_n = 2m_n + 1$, $(\overline{\mathrm{K}}_i(z))_{i\in\mathbb{Z}^d}$ is an M_n-dependent random field (i.e. $\overline{\mathrm{K}}_i(z)$ and $\overline{\mathrm{K}}_j(z)$ are independent as soon as $|i - j| \geq M_n$).

Lemma 1.3. *For all $p > 1$, all x in $\mathbb{R}$, all positive integer n and all $(a_i)_{i\in\mathbb{Z}^d}$ in $\mathbb{R}^{\mathbb{Z}^d}$,*

$$\left\|\sum_{i\in\Lambda_n} a_i\left(\mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x)\right)\right\|_p \leq \frac{8m_n^d}{b_n}\left(p\sum_{i\in\Lambda_n} a_i^2\right)^{1/2} \sum_{|i|>m_n} \delta_{i,p}.$$

In order to establish the asymptotic normality of f_n, we need additional assumptions:

- **(B1)** The marginal density function of each X_k is positive, continuous and bounded.
- **(B2)** K is Lipschitz, $\int_{\mathbb{R}} \mathrm{K}(u)\, du = 1$, $\int_{\mathbb{R}} |\mathrm{K}(u)|\, du < \infty$ and $\int_{\mathbb{R}} \mathrm{K}^2(u)\, du < \infty$.
- **(B3)** There exists $\kappa > 0$ such that $\sup_{\substack{(x,y)\in\mathbb{R}^2 \\ i\in\mathbb{Z}^d\setminus\{0\}}} f_{0,i}(x, y) \leq \kappa$ where $f_{0,i}$ is the joint density of (X_0, X_i).

Theorem 1.4. *Assume that* **(A3)**, **(A4)**, **(B1)**, **(B2)** *and* **(B3)** *hold. For all positive integer k and any distinct points $x_1, ..., x_k$ in $\mathbb{R}$,*

$$(|\Lambda_n|b_n)^{1/2} \begin{pmatrix} f_n(x_1) - \mathbb{E}f_n(x_1) \\ \vdots \\ f_n(x_k) - \mathbb{E}f_n(x_k) \end{pmatrix} \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, \Gamma) \tag{1.5}$$

where Γ is a diagonal matrix with diagonal elements $\gamma_{ii} = f(x_i) \int_{\mathbb{R}} \mathrm{K}^2(u)du$.

Remark 3. A replacement of $\mathbb{E}f_n(x_i)$ by $f(x_i)$ for all $1 \leq i \leq k$ in (1.5) is a classical problem in density estimation theory. Let $s \geq 2$ be a positive integer and $\kappa > 0$. If the sth derivative $f^{(s)}$ of f exists such that $|f^{(s)}| \leq \kappa$ and the kernel K satisfies $\int_{\mathbb{R}} u^r \mathrm{K}(u)du = 0$ for $r = 1, 2, ..., s-1$ and $0 < \int_{\mathbb{R}} |u|^s |\mathrm{K}(u)|du < \infty$ then $|\mathbb{E}f_n(x_i) - f(x_i)| = O(b_n^s)$ and thus the centering $\mathbb{E}f_n(x_i)$ may be changed to $f(x_i)$ without affecting the above result provided that $|\Lambda_n|b_n^{2s+1}$ converges to zero.

Remark 4. If $(X_i)_{i\in\mathbb{Z}^d}$ is a linear random field of the form $X_i = \sum_{j\in\mathbb{Z}^d} a_j \varepsilon_{i-j}$ where $(a_j)_{j\in\mathbb{Z}^d}$ are real numbers such that $\sum_{j\in\mathbb{Z}^d} a_j^2 < \infty$ and $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ are i.i.d. real random variables with zero mean and finite variance then $\delta_i = |a_i| \|\varepsilon_0 - \varepsilon_0^{'}\|_2$ and Theorem 1.4 holds provided that $\sum_{i\in\mathbb{Z}^d} |i|^{\frac{5d}{2}} |a_i| < \infty$. For Λ_n rectangular, Hallin et al. (2001) obtained the same result when $|a_j| = O(|j|^{-\gamma})$ with $\gamma > \max\{d+3, 2d+0.5\}$ and $|\Lambda_n| b_n^{(2\gamma-1+6d)/(2\gamma-1-4d)}$ goes to infinity. So, in the particular case of linear random fields, our assumption **(A4)** is more restrictive than the condition obtained by Hallin et al. (2001) but our result is valid for a larger class of random fields and under only minimal conditions on the bandwidth (see Assumption **(A3)**). Finally, for causal linear random fields, Wang and Woodroofe (2014) obtained also a sufficient condition on the coefficients $(a_j)_{j\in\mathbb{N}^d}$ for the kernel density estimator to be asymptotically normal. Their condition is less restrictive than the condition $\sum_{i\in\mathbb{Z}^d} |i|^{\frac{5d}{2}} |a_i| < \infty$ but they assumed also $\mathbb{E}(|\varepsilon_0|^p) < \infty$ for some $p > 2$.

Now, we are going to investigate the rate of convergence in (1.5). For all positive integer n and all x in $\mathbb{R}$, we denote $\mathrm{D}_n(x) = \sup_{t\in\mathbb{R}} |\mathbb{P}(U_n(x) \leq t) - \Phi(t)|$ where Φ is the distribution function of the standard normal law and

$$U_n(x) = \frac{\sqrt{|\Lambda_n|b_n}\,(f_n(x) - \mathbb{E}f_n(x))}{\sqrt{f(x)\int_{\mathbb{R}} \mathrm{K}^2(t)dt}}.$$

Theorem 1.5. *Let n in $\mathbb{N}\backslash\{0\}$ and x in $\mathbb{R}$ be fixed. Assume that $\int_{\mathbb{R}} |\mathrm{K}(t)|^\tau dt < \infty$ for some $2 < \tau \leq 3$. If there exist $\alpha > 1$ and $p \geq 2$ such that $\sum_{i\in\mathbb{Z}^d} |i|^{d\alpha} \delta_{i,p} < \infty$ then there exists a constant $\kappa > 0$ such that $\mathrm{D}_n(x) \leq \kappa |\Lambda_n|^{-\theta}$ where*

$$\theta = \theta(\alpha, \tau, p) = \left(\frac{1}{2} - \frac{1}{\tau}\right) \frac{3p(1-\tau) + 2p(\alpha-1)}{(\tau-1)(p+1) + p(\alpha-1)}.$$

Remark 5. If $\tau = 3$, $p = 2$ and $\sum_{i\in\mathbb{Z}^d} |i|^{d\alpha} \delta_i < \infty$ for some $\alpha > 4$ then

$$D_n(x) \leq \kappa |\Lambda_n|^{-\theta(\alpha)} \quad \text{where} \quad \theta(\alpha) = \frac{2\alpha - 8}{3(4+2\alpha)} \xrightarrow[\alpha\to\infty]{} \frac{1}{3}.$$

2. Numerical illustration

In this section, we give some simulations with a view to illustrate the results given in this paper. We assume $d = 2$ and we consider the autoregressive random field $(X_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by

$$X_{i,j} = \alpha X_{i-1,j} + \beta X_{i,j-1} + \varepsilon_{i,j} \tag{2.1}$$

where $\alpha = 0.2$, $\beta = 0.7$ and $(\varepsilon_{i,j})_{(i,j)\in\mathbb{Z}^2}$ are iid random variables uniformly distributed over the interval $[-5, 5]$. Since $|\alpha| + |\beta| < 1$, the equation (2.1) has a stationary solution $X_{i,j}$ (see Kulkarni (1992)) defined by

$$X_{i,j} = \sum_{k_1\geq 0}\sum_{k_2\geq 0}\binom{k_1+k_2}{k_1}\alpha^{k_1}\beta^{k_2}\varepsilon_{i-k_1,j-k_2} \tag{2.2}$$

and each $X_{i,j}$ is uniformly distributed over the interval $[-5\gamma, 5\gamma]$ with

$$\gamma = \sum_{k_1\geq 0}\sum_{k_2\geq 0}\binom{k_1+k_2}{k_1}\alpha^{k_1}\beta^{k_2} = \frac{1}{1-(\alpha+\beta)} = 10.$$

We simulate the $\varepsilon_{i,j}$'s over the rectangular grid $[0, 2t]^2 \cap \mathbb{Z}^2$ where t is a positive integer and the data $X_{i,j}$ over the grid $\Lambda_t = [t+1, 2t]^2 \cap \mathbb{Z}^2$ following (2.2). We take the data $X_{i,j}$ for (i, j) in the region Λ_t as our data set and we calculate from this data set the kernel density estimator

$$\hat{f}_t(x) = \frac{1}{t^2 \times b_t}\sum_{(i,j)\in\Lambda_t}\mathrm{K}\left(\frac{x - X_{i,j}}{b_t}\right) \tag{2.3}$$

where x is fixed in $\mathbb{R}$, b_t is the bandwith parameter and K is the Epanachnikov kernel defined by $\mathrm{K}(s) = \frac{3}{4}(1 - s^2)$ if $s \in]-1, 1[$ and $\mathrm{K}(s) = 0$ if $s \notin]-1, 1[$.
In order to illustrate the result obtained in Theorem 1.1, we calculate (Monte Carlo method) $\int_{-100}^{100}|\hat{f}_t(x) - f(x)|dx$ where f is the true density function of $X_{0,0}$ and the bandwith b_t is being set to $|\Lambda_t|^{-1/3}$ with $|\Lambda_t|$ denoting the number of elements in Λ_t. Hence, we derive its expectation $\mathbb{E}\int_{-100}^{100}|\hat{f}_t(x) - f(x)|dx$ by taking the arithmetic mean value of 100 replications of $\int_{-100}^{100}|\hat{f}_t(x) - f(x)|dx$. The results are given for several values of t in the following table

t	$\|\Lambda_t\| = t^2$	$b_t = \|\Lambda_t\|^{-1/3}$	$\mathbb{E}\int_{-100}^{100}\|\hat{f}_t(x) - f(x)\|dx$
10	100	0.215	0.0171
20	400	0.136	0.0163
50	2500	0.074	0.0157
100	10000	0.046	0.0153

and we observe the L^1-convergence of $\hat{f}_t$ to the true density function f of $X_{0,0}$. In order to illustrate the asymptotic normality of the estimator (2.3), we put $x = -1$, $t = 20$ and $b_{20} = 0.7$ and we calculate the expectation $\mathbb{E}\left(\hat{f}_t(-1)\right)$ of $\hat{f}_t(-1)$ by taking again the arithmetic mean value of 100 replications of $\hat{f}_t(-1)$. Finally, noting that $\int_{\mathbb{R}}\mathrm{K}^2(x)dx = 4/5$ and $f(-1) = 1/100$, we consider 1500 replications of

$$\frac{\sqrt{400\times 0.7}\left(\hat{f}_{20}(-1) - \mathbb{E}\left(\hat{f}_{20}(-1)\right)\right)}{\sqrt{1/100\times 4/5}}$$

and we obtain the following histogram (see figure 2.1) which seems to fit well to the target distribution, that is the standard normal law $\mathcal{N}(0,1)$.

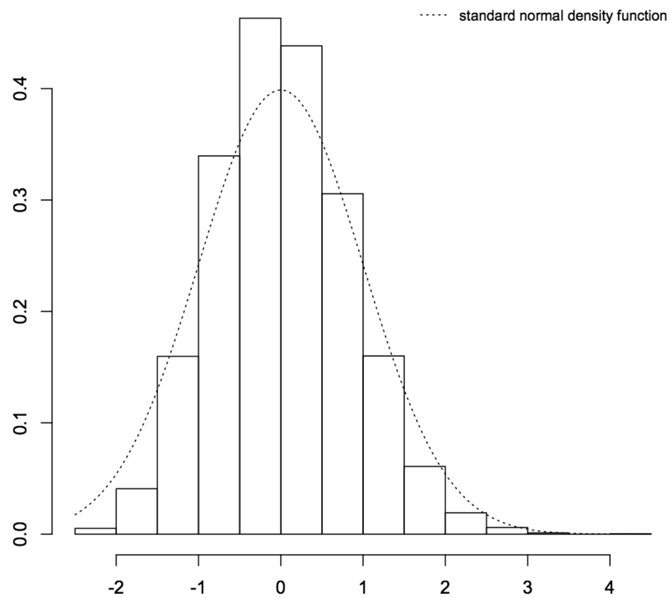

FIGURE 2.1. Asymptotic normality of the kernel density estimator.

In the simulation given in Figure 2.1, we fixed the bandwith $b_{20} = 0.7$ arbitrarily since we do not investigate in this work any procedure for a data-driven choice of the bandwith parameter. Such a study is an important task and will be done in a forthcoming paper.

3. Proofs

The proof of all lemmas of this section are postponed to the appendix. In the sequel, the letter κ denotes a positive constant which the value is not important.

3.1. *Proof of Theorem 1.1.* For all positive integer n, denote $\mathrm{J}_n = \int_{\mathbb{R}} |f_n(x) - f(x)|\,dx$. For all real $A \geq 1$, we have $\mathrm{J}_n = \mathrm{J}_{n,1}(A) + \mathrm{J}_{n,2}(A)$ where

$$\mathrm{J}_{n,1}(A) = \int_{|x|>A} |f_n(x) - f(x)|\,dx \quad \text{and} \quad \mathrm{J}_{n,2}(A) = \int_{|x|\leq A} |f_n(x) - f(x)|\,dx.$$

Moreover

$$\mathbb{E}\mathrm{J}_{n,1}(A) \leq \int_{|x|>A} \mathbb{E}|f_n(x)|dx + \frac{1}{A^2}\int_{\mathbb{R}} x^2 f(x)dx$$

and

$$\begin{aligned}\int_{|x|>A}\mathbb{E}|f_n(x)|dx &\leq \int_{|x|>A}\int_{\mathbb{R}}|\mathrm{K}(t)|f(x-b_nt)dtdx\\ &= \int_{|t|>\frac{A}{2}}|\mathrm{K}(t)|\int_{|x|>A}f(x-b_nt)dxdt\\ &\quad+\int_{|t|\leq\frac{A}{2}}|\mathrm{K}(t)|\int_{|x|>A}f(x-b_nt)dxdt\\ &\leq \int_{|t|>\frac{A}{2}}|\mathrm{K}(t)|\int_{|y+b_nt|>A}f(y)dydt\\ &\quad+\int_{|t|\leq\frac{A}{2}}|\mathrm{K}(t)|\int_{|y|>A(1-\frac{b_n}{2})}f(y)dydt\\ &\leq \frac{4}{A^2}\int_{\mathbb{R}}t^2|\mathrm{K}(t)|dt+\frac{4}{A^2}\int_{\mathbb{R}}|\mathrm{K}(t)|dt\int_{\mathbb{R}}y^2f(y)dy.\end{aligned}$$

Consequently, we obtain

$$\mathbb{E}\mathrm{J}_{n,1}(A)\leq\frac{\kappa}{A^2}. \tag{3.1}$$

Now, $\mathrm{J}_{n,2}(A)\leq\mathrm{J}_{n,2}^{(1)}(A)+\mathrm{J}_{n,2}^{(2)}(A)$ where

$$\mathrm{J}_{n,2}^{(1)}(A)=\int_{|x|\leq A}|f_n(x)-\mathbb{E}f_n(x)|\,dx\quad\text{and}\quad\mathrm{J}_{n,2}^{(2)}(A)=\int_{|x|\leq A}|\mathbb{E}f_n(x)-f(x)|\,dx.$$

Since

$$\begin{aligned}|\mathbb{E}f_n(x)-f(x)| &= \left|\int_{\mathbb{R}}\mathrm{K}(t)\left(f(x-b_nt)-f(x)\right)dt\right|\\ &\leq \int_{\mathbb{R}}|\mathrm{K}(t)|\,|f(x-b_nt)-f(x)|\,dt\\ &\leq \kappa b_n\int_{\mathbb{R}}|t||\mathrm{K}(t)|dt,\end{aligned}$$

we obtain

$$\mathrm{J}_{n,2}^{(2)}(A)\leq\kappa Ab_n. \tag{3.2}$$

Keeping in mind the notation (1.4) and denoting $\overline{f}_n(x)=\frac{1}{|\Lambda_n|b_n}\sum_{i\in\Lambda_n}\overline{\mathrm{K}}_i(x)$, we have $\mathrm{J}_{n,2}^{(1)}(A)\leq\mathrm{I}_{n,1}(A)+\mathrm{I}_{n,2}(A)$ where

$$\mathrm{I}_{n,1}(A)=\int_{|x|\leq A}|f_n(x)-\overline{f}_n(x)|\,dx\quad\text{and}\quad\mathrm{I}_{n,2}(A)=\int_{|x|\leq A}|\overline{f}_n(x)-\mathbb{E}\overline{f}_n(x)|\,dx.$$

By Lemma 1.3, we have

$$\left\|f_n(x)-\overline{f}_n(x)\right\|_2\leq\frac{\kappa\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\delta_i}{\sqrt{|\Lambda_n|b_n}(m_n^db_n)^{3/2}}.$$

Applying Lemma 1.2, we obtain

$$\mathbb{E}\mathrm{I}_{n,1}(A)\leq\frac{\kappa A}{\sqrt{|\Lambda_n|b_n}}. \tag{3.3}$$

Now, $\left\|\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)\right\|_2^2$ equals to

$$\frac{1}{|\Lambda_n|^2 b_n}\left(|\Lambda_n|\mathbb{E}\left(\overline{Z}_0^2(x)\right) + \sum_{\substack{j\in\mathbb{Z}^d\backslash\{0\}\\|j|<M_n}} |\Lambda_n \cap (\Lambda_n - j)|\mathbb{E}\left(\overline{Z}_0(x)\overline{Z}_j(x)\right)\right) \tag{3.4}$$

where we recall that $\overline{Z}_i(x) = \frac{1}{\sqrt{b_n}}\left(\overline{\mathrm{K}}_i(x) - \mathbb{E}\overline{\mathrm{K}}_i(x)\right)$ and $M_n = 2m_n + 1$.

Lemma 3.1. *Let x, s and t be fixed in $\mathbb{R}$. Then $\mathbb{E}\left(\overline{Z}_0^2(x)\right)$ converges to $f(x)\int_{\mathbb{R}} \mathrm{K}^2(u)du$ and $\sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(M_n^{-d})$.*

Combining (3.4) and Lemma 3.1, we derive $\left\|\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)\right\|_2^2 = O\left((|\Lambda_n|b_n)^{-1}\right)$. Hence,

$$\mathbb{E}\mathrm{I}_{n,2}(A) \leq \frac{\kappa A}{\sqrt{|\Lambda_n|b_n}}. \tag{3.5}$$

Combining (3.1), (3.2), (3.3) and (3.5), we obtain

$$\mathbb{E}\mathrm{J}_n \leq \kappa\left(\frac{1}{A^2} + A\left(b_n + \frac{1}{\sqrt{|\Lambda_n|b_n}}\right)\right).$$

Optimizing in A, we derive (1.2). The proof of Theorem 1.1 is complete.

3.2. *Proof of Theorem 1.4*. Without loss of generality, we consider only the case $k = 2$ and we refer to x_1 and x_2 as x and y ($x \neq y$). Let λ_1 and λ_2 be two constants such that $\lambda_1^2 + \lambda_2^2 = 1$ and note that

$$\lambda_1(|\Lambda_n|b_n)^{1/2}(f_n(x) - \mathbb{E}f_n(x)) + \lambda_2(|\Lambda_n|b_n)^{1/2}(f_n(y) - \mathbb{E}f_n(y)) = \sum_{i\in\Lambda_n}\frac{\Delta_i}{|\Lambda_n|^{1/2}},$$

$$\lambda_1(|\Lambda_n|b_n)^{1/2}(\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)) + \lambda_2(|\Lambda_n|b_n)^{1/2}(\overline{f}_n(y) - \mathbb{E}\overline{f}_n(y)) = \sum_{i\in\Lambda_n}\frac{\overline{\Delta}_i}{|\Lambda_n|^{1/2}},$$

where $\Delta_i = \lambda_1 Z_i(x) + \lambda_2 Z_i(y)$ and $\overline{\Delta}_i = \lambda_1\overline{Z}_i(x) + \lambda_2\overline{Z}_i(y)$ and for all z in $\mathbb{R}$,

$$Z_i(z) = \frac{1}{\sqrt{b_n}}\left(\mathrm{K}_i(z) - \mathbb{E}\mathrm{K}_i(z)\right) \quad \text{and} \quad \overline{Z}_i(z) = \frac{1}{\sqrt{b_n}}\left(\overline{\mathrm{K}}_i(z) - \mathbb{E}\overline{\mathrm{K}}_i(z)\right)$$

where $\mathrm{K}_i(z)$ and $\overline{\mathrm{K}}_i(z)$ are defined by (1.4). Applying Lemma 1.2 and Lemma 1.3, we know that

$$\frac{1}{|\Lambda_n|^{1/2}}\left\|\sum_{i\in\Lambda_n}\left(\Delta_i - \overline{\Delta}_i\right)\right\|_2 \leq \frac{\kappa(|\lambda_1| + |\lambda_2|)}{(m_n^d b_n)^{3/2}}\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\delta_i = o(1). \tag{3.6}$$

So, it suffices to prove the asymptotic normality of the sequence $\left(|\Lambda_n|^{-1/2}\sum_{i\in\Lambda_n}\overline{\Delta}_i\right)_{n\geq 1}$. We are going to follow the Lindeberg's type proof of Theorem 1 in Dedecker (1998). We consider the notations

$$\eta = (\lambda_1^2 f(x) + \lambda_2^2 f(y))\sigma^2 \quad \text{and} \quad \sigma^2 = \int_{\mathbb{R}}\mathrm{K}^2(u)du. \tag{3.7}$$

Lemma 3.2. $\mathbb{E}(\overline{\Delta}_0^2)$ *converges to η and* $\sup_{i\in\mathbb{Z}^d\backslash\{0\}}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| = o(M_n^{-d})$.

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1, ..., i_d)$ and $j = (j_1, ..., j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{\text{lex}} j$ means that either $i_1 < j_1$ or for some k in $\{2, 3, ..., d\}$, $i_k < j_k$ and $i_l = j_l$ for $1 \leq l < k$. We let φ denote the unique function from $\{1, ..., |\Lambda_n|\}$ to Λ_n such that $\varphi(k) <_{\text{lex}} \varphi(l)$ for $1 \leq k < l \leq |\Lambda_n|$. For all real random field $(\zeta_i)_{i\in\mathbb{Z}^d}$ and all integer k in $\{1, ..., |\Lambda_n|\}$, we denote

$$S_{\varphi(k)}(\zeta) = \sum_{i=1}^{k} \zeta_{\varphi(i)} \quad \text{and} \quad S^c_{\varphi(k)}(\zeta) = \sum_{i=k}^{|\Lambda_n|} \zeta_{\varphi(i)}$$

with the convention $S_{\varphi(0)}(\zeta) = S^c_{\varphi(|\Lambda_n|+1)}(\zeta) = 0$. From now on, we consider a field $(\xi_i)_{i\in\mathbb{Z}^d}$ of i.i.d. standard normal random variables independent of $(X_i)_{i\in\mathbb{Z}^d}$. We introduce the fields Y and γ defined for all i in $\mathbb{Z}^d$ by

$$Y_i = \frac{\overline{\Delta}_i}{|\Lambda_n|^{1/2}} \quad \text{and} \quad \gamma_i = \frac{\sqrt{\eta}\xi_i}{|\Lambda_n|^{1/2}}$$

where η is defined by (3.7). Note that Y is an M_n-dependent random field where $M_n = 2m_n + 1$ and m_n is defined by (1.3). Let h be any function from $\mathbb{R}$ to $\mathbb{R}$. For $0 < k \leq l \leq |\Lambda_n|$, we introduce $h_{k,l}(Y) = h(S_{\varphi(k)}(Y) + S^c_{\varphi(l)}(\gamma))$. With the above convention we have that $h_{k,|\Lambda_n|+1}(Y) = h(S_{\varphi(k)}(Y))$ and also $h_{0,l}(Y) = h(S^c_{\varphi(l)}(\gamma))$. In the sequel, we will often write $h_{k,l}$ instead of $h_{k,l}(Y)$. We denote by $B^4_1(\mathbb{R})$ the unit ball of $C^4_b(\mathbb{R})$: h belongs to $B^4_1(\mathbb{R})$ if and only if it belongs to $C^4(\mathbb{R})$ and satisfies $\max_{0\leq i\leq 4} \|h^{(i)}\|_\infty \leq 1$. It suffices to prove that for all h in $B^4_1(\mathbb{R})$,

$$\mathbb{E}\left(h\left(S_{\varphi(|\Lambda_n|)}(Y)\right)\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(h\left(\sqrt{\eta}\xi_0\right)\right).$$

We use Lindeberg's decomposition:

$$\mathbb{E}\left(h\left(S_{\varphi(|\Lambda_n|)}(Y)\right) - h\left(\sqrt{\eta}\xi_0\right)\right) = \sum_{k=1}^{|\Lambda_n|} \mathbb{E}\left(h_{k,k+1} - h_{k-1,k}\right).$$

Now, we have $h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}$ and by Taylor's formula we obtain

$$h_{k,k+1} - h_{k-1,k+1} = Y_{\varphi(k)} h'_{k-1,k+1} + \frac{1}{2} Y^2_{\varphi(k)} h''_{k-1,k+1} + R_k$$
$$h_{k-1,k+1} - h_{k-1,k} = -\gamma_{\varphi(k)} h'_{k-1,k+1} - \frac{1}{2} \gamma^2_{\varphi(k)} h''_{k-1,k+1} + r_k$$

where $|R_k| \leq Y^2_{\varphi(k)}(1 \wedge |Y_{\varphi(k)}|)$ and $|r_k| \leq \gamma^2_{\varphi(k)}(1 \wedge |\gamma_{\varphi(k)}|)$. Since $(Y, \xi_i)_{i\neq\varphi(k)}$ is independent of $\xi_{\varphi(k)}$, it follows that

$$\mathbb{E}\left(\gamma_{\varphi(k)} h'_{k-1,k+1}\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\gamma^2_{\varphi(k)} h''_{k-1,k+1}\right) = \mathbb{E}\left(\frac{\eta}{|\Lambda_n|} h''_{k-1,k+1}\right)$$

Hence, we obtain

$$\begin{aligned}\mathbb{E}\left(h(S_{\varphi(|\Lambda_n|)}(Y)) - h\left(\sqrt{\eta}\xi_0\right)\right) &= \sum_{k=1}^{|\Lambda_n|}\mathbb{E}(Y_{\varphi(k)}h^{'}_{k-1,k+1})\\ &+\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(\left(Y^2_{\varphi(k)}-\frac{\eta}{|\Lambda_n|}\right)\frac{h^{''}_{k-1,k+1}}{2}\right)\\ &+\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(R_k+r_k\right).\end{aligned}$$

Let $1 \leq k \leq |\Lambda_n|$ be fixed. Since $\mathbb{E}|\overline{\Delta}_0| = O\left(\sqrt{b_n}\right)$ and $\left(\overline{\Delta}_0^2 b_n\right)_{n\geq 1}$ is uniformly integrable, we derive

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|R_k| \leq \mathbb{E}\left(\overline{\Delta}_0^2\left(1\wedge\frac{|\overline{\Delta}_0|}{|\Lambda_n|^{1/2}}\right)\right) = o(1)$$

and

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}|r_k| \leq \frac{\eta^{3/2}\mathbb{E}|\xi_0|^3}{|\Lambda_n|^{1/2}} = O\left(|\Lambda_n|^{-1/2}\right).$$

Consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(|R_k|+|r_k|\right) = o(1).$$

Now, it is sufficient to show

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\left(\mathbb{E}(Y_{\varphi(k)}h^{'}_{k-1,k+1}) + \mathbb{E}\left(\left(Y^2_{\varphi(k)}-\frac{\eta}{|\Lambda_n|}\right)\frac{h^{''}_{k-1,k+1}}{2}\right)\right) = 0. \qquad (3.8)$$

First, we focus on $\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(Y_{\varphi(k)}h^{'}_{k-1,k+1}\right)$. Let the sets $\{V_i^k\,;\, i\in\mathbb{Z}^d\,,\, k\in\mathbb{N}\backslash\{0\}\}$ be defined as follows: $V_i^1 = \{j\in\mathbb{Z}^d\,;\, j<_{\text{lex}} i\}$ and for $k\geq 2$, $V_i^k = V_i^1\cap\{j\in\mathbb{Z}^d\,;\,|i-j|\geq k\}$. For all n in $\mathbb{N}\backslash\{0\}$ and all k in $\{1,...,|\Lambda_n|\}$, we define

$$\mathrm{E}_k^{(n)} = \varphi(\{1,..,k\})\cap V_{\varphi(k)}^{M_n} \quad\text{and}\quad S_{\varphi(k)}^{M_n}(Y) = \sum_{i\in\mathrm{E}_k^{(n)}} Y_i.$$

For all function h from $\mathbb{R}$ to $\mathbb{R}$, we define $h^{M_n}_{k-1,l} = h\left(S^{M_n}_{\varphi(k)}(Y)+S^c_{\varphi(l)}(\gamma)\right)$. Our aim is to show that

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left(Y_{\varphi(k)}h^{'}_{k-1,k+1} - Y_{\varphi(k)}\left(S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y)\right)h^{''}_{k-1,k+1}\right) = 0. \quad (3.9)$$

First, we use the decomposition

$$Y_{\varphi(k)}h^{'}_{k-1,k+1} = Y_{\varphi(k)}h^{'M_n}_{k-1,k+1} + Y_{\varphi(k)}\left(h^{'}_{k-1,k+1}-h^{'M_n}_{k-1,k+1}\right).$$

Applying again Taylor's formula,

$$Y_{\varphi(k)}(h^{'}_{k-1,k+1}-h^{'M_n}_{k-1,k+1}) = Y_{\varphi(k)}\left(S_{\varphi(k-1)}(Y)-S^{M_n}_{\varphi(k)}(Y)\right)h^{''}_{k-1,k+1}+R^{'}_k,$$

where

$$|R_k^{'}| \leq 2 \left| Y_{\varphi(k)} \left(S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) \left(1 \wedge |S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y)| \right) \right|.$$

Since $(Y_i)_{i\in\mathbb{Z}^d}$ is M_n-dependent, we have $\mathbb{E}\left(Y_{\varphi(k)} h_{k-1,k+1}^{'M_n}\right) = 0$ and consequently (3.9) holds if and only if $\lim_{n\to\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R_k^{'}| = 0$. In fact, considering the sets $W_n = \{-M_n+1, ..., M_n-1\}^d$ and $W_n^* = W_n \backslash \{0\}$, it follows that

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R_k^{'}| &\leq 2\mathbb{E}\left(|\overline{\Delta}_0| \left(\sum_{i\in W_n^*} |\overline{\Delta}_i| \right) \left(1 \wedge \frac{1}{|\Lambda_n|^{1/2}} \sum_{i\in W_n^*} |\overline{\Delta}_i| \right) \right) \\ &\leq 2M_n^d \sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}(|\overline{\Delta}_0 \overline{\Delta}_i|) \\ &= o(1) \qquad \text{(by Lemma 3.2)}. \end{aligned}$$

In order to obtain (3.8) it remains to control

$$\mathrm{F}_1 = \mathbb{E}\left(\sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}^{''} \left(\frac{Y_{\varphi(k)}^2}{2} + Y_{\varphi(k)} \left(S_{\varphi(k-1)}(Y) - S_{\varphi(k)}^{M_n}(Y) \right) - \frac{\eta}{2|\Lambda_n|} \right) \right).$$

Applying again Lemma 3.2, we have

$$\begin{aligned} \mathrm{F}_1 \leq & \left| \mathbb{E}\left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}^{''} \left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| \\ &+ \left| \eta - \mathbb{E}\left(\overline{\Delta}_0^2\right) \right| + 2 \sum_{j\in V_0^1 \cap W_n} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_j| \\ \leq & \left| \mathbb{E}\left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}^{''} \left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| + o(1). \end{aligned}$$

So, it suffices to prove that

$$\mathrm{F}_2 = \left| \mathbb{E}\left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}^{''} \left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right|$$

goes to zero as n goes to infinity. In fact, we have $\mathrm{F}_2 \leq \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left(\mathrm{J}_k^{(1)}(n) + \mathrm{J}_k^{(2)}(n) \right)$ where $\mathrm{J}_k^{(1)}(n) = \left| \mathbb{E}\left(h_{k-1,k+1}^{''M_n} \left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}\left(\overline{\Delta}_0^2\right) \right) \right) \right| = 0$ since $h_{k-1,k+1}^{''M_n}$ and $\overline{\Delta}_{\varphi(k)}$

are independent. Moreover,

$$
\begin{aligned}
\mathrm{J}_k^{(2)}(n) &= \left|\mathbb{E}\left(\left(h''_{k-1,k+1} - h''^{M_n}_{k-1,k+1}\right)\left(\overline{\Delta}^2_{\varphi(k)} - \mathbb{E}\left(\overline{\Delta}^2_0\right)\right)\right)\right| \\
&\leq \mathbb{E}\left(\left(2 \wedge \sum_{\substack{|i|<M_n \\ i\neq 0}} \frac{|\overline{\Delta}_i|}{|\Lambda_n|^{1/2}}\right)\overline{\Delta}^2_0\right) \\
&\leq \frac{1}{\sqrt{|\Lambda_n|b_n}}\,\mathbb{E}\left(|\overline{\Delta}_0|\sqrt{b_n} \times \sum_{\substack{|i|<M_n \\ i\neq 0}} |\overline{\Delta}_0\overline{\Delta}_i|\right) \\
&= o(1)
\end{aligned}
$$

since $(|\overline{\Delta}_0|\sqrt{b_n})_{n\geq 1}$ is uniformly integrable and $\sum_{\substack{|i|<M_n \\ i\neq 0}} \mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| = o(1)$ by Lemma 3.2. The proof of Theorem 1.4 is complete.

3.3. *Proof of Theorem 1.5*. Let n be a fixed positive integer and let x be fixed in $\mathbb{R}$. We have $U_n(x) = \overline{U}_n(x) + R_n(x)$ where

$$
\overline{U}_n(x) = \frac{\sqrt{|\Lambda_n|b_n}\left(\overline{f}_n(x) - \mathbb{E}\overline{f}_n(x)\right)}{\sqrt{f(x)\int_{\mathbb{R}} \mathrm{K}^2(t)dt}} \quad \text{and} \quad R_n(x) = \frac{\sqrt{|\Lambda_n|b_n}\left(f_n(x) - \overline{f}_n(x)\right)}{\sqrt{f(x)\int_{\mathbb{R}} \mathrm{K}^2(t)dt}}.
$$

Denote $\overline{D}_n(x) = \sup_{t\in\mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t) - \Phi(t)|$ and let $p \geq 2$ be fixed. Arguing as in Theorem 2.2 in El Machkouri (2010), we have

$$
D_n(x) \leq \overline{D}_n(x) + \|R_n\|_p^{\frac{p}{p+1}}. \tag{3.10}
$$

Denoting $\sigma^2 = f(x)\int_{\mathbb{R}} \mathrm{K}^2(t)dt$ and $\sigma_n^2 = \mathbb{E}\left(\overline{U}_n^2\right)$, we have

$$
\begin{aligned}
\overline{D}_n(x) &= \sup_{t\in\mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t) - \Phi(t)| \\
&\leq \sup_{t\in\mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t) - \Phi\left(t/\sigma_n\right)| + \sup_{t\in\mathbb{R}} |\Phi\left(t/\sigma_n\right) - \Phi\left(t\right)| \\
&= \sup_{t\in\mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t\sigma_n) - \Phi\left(t\right)| + \sup_{t\in\mathbb{R}} |\Phi\left(t/\sigma_n\right) - \Phi\left(t\right)|.
\end{aligned}
$$

Applying the Berry-Esseen's type theorem for m_n-dependent random fields established by Chen and Shao (Chen and Shao (2004), Theorem 2.6), we obtain

$$
\sup_{t\in\mathbb{R}} |\mathbb{P}(\overline{U}_n(x) \leq t\sigma_n) - \Phi\left(t\right)| \leq \frac{\kappa\int_{\mathbb{R}} |K(t)|^\tau f(x - tb_n)dt\ m_n^{(\tau-1)d}}{\sigma^\tau(|\Lambda_n|b_n)^{\frac{\tau}{2}-1}}. \tag{3.11}
$$

Arguing as in Yang et al. (Yang et al. (2012), p. 456), we have

$$\begin{aligned}\sup_{t\in\mathbb{R}}|\Phi\left(t/\sigma_n\right)-\Phi\left(t\right)| &\leq (2\pi e)^{-\frac{1}{2}}(\sigma_n-1)\,\mathbb{1}_{\sigma_n\geq 1}+(2\pi e)^{-\frac{1}{2}}\left(\frac{1}{\sigma_n}-1\right)\mathbb{1}_{0<\sigma_n<1}\\ &\leq (2\pi e)^{-\frac{1}{2}}\max\left\{|\sigma_n-1|,\frac{|\sigma_n-1|}{\sigma_n}\}\right\}\\ &\leq \kappa\max\left\{|\sigma_n-1|,\frac{|\sigma_n-1|}{\sigma_n}\}\right\}\times(\sigma_n+1)\\ &\leq \kappa|\sigma_n^2-1|.\end{aligned}$$

So, we derive

$$\overline{D}_n(x)\leq\frac{\kappa\int_{\mathbb{R}}|K(t)|^{\tau}f(x-tb_n)dt\;m_n^{(\tau-1)d}}{\sigma^{\tau}(|\Lambda_n|b_n)^{\frac{\tau}{2}-1}}+\kappa|\sigma_n^2-1|. \tag{3.12}$$

Using (3.4), we have also

$$|\sigma_n^2-1|\leq\frac{1}{\sigma^2}\left|\mathbb{E}(\overline{Z}_0^2(x))-\sigma^2\right|+\sum_{\substack{j\in\mathbb{Z}^d\backslash\{0\}\\|j|<M_n}}\left|\mathbb{E}\left(\overline{Z}_0(x)\overline{Z}_j(x)\right)\right|. \tag{3.13}$$

Noting that $\|\mathrm{K}_0(x)\|_1=O(b_n)$ and $\|\mathrm{K}_0(x)\|_2=O(\sqrt{b_n})$ and using the following lemma,

Lemma 3.3. *For all $p>1$, any positive integer n and any x in $\mathbb{R}$,*

$$\|\mathrm{K}_0(x)-\overline{\mathrm{K}}_0(x)\|_p\leq\frac{\sqrt{2p}}{b_n}\sum_{|j|>m_n}\delta_{j,p},$$

we obtain

$$\begin{aligned}\left|\mathbb{E}(\overline{Z}_0^2(x))-\mathbb{E}(Z_0^2(x))\right| &= \frac{1}{b_n}\left|\mathbb{E}(\overline{\mathrm{K}}_0^2(x))-\mathbb{E}(\mathrm{K}_0^2(x))\right|\\ &\leq\frac{1}{b_n}\|\mathrm{K}_0(x)\|_2\|\mathrm{K}_0(x)-\overline{\mathrm{K}}_0(x)\|_2\\ &\leq\frac{\kappa}{b_n^{3/2}}\sum_{|j|>m_n}\delta_j\end{aligned}$$

and

$$\begin{aligned}\left|\mathbb{E}(Z_0^2(x))-\sigma^2\right| &= \left|\frac{1}{b_n}\left(\mathbb{E}(\mathrm{K}_0^2(x))-(\mathbb{E}(\mathrm{K}_0(x))^2\right)-f(x)\int_{\mathbb{R}}\mathrm{K}^2(t)dt\right|\\ &\leq\left|\frac{1}{b_n}\mathbb{E}(\mathrm{K}_0^2(x))-f(x)\int_{\mathbb{R}}\mathrm{K}^2(t)dt\right|+\frac{1}{b_n}\left(\mathbb{E}(\mathrm{K}_0(x)\right)^2\\ &\leq\int_{\mathbb{R}}\mathrm{K}^2(v)|f(x-vb_n)-f(x)|dv+O(b_n)\\ &\leq\kappa\,b_n\int_{\mathbb{R}}|v|\mathrm{K}^2(v)dv+O(b_n)\\ &=O(b_n).\end{aligned}$$

Hence,

$$\left|\mathbb{E}(\overline{Z}_0^2(x))-\sigma^2\right|\leq\frac{\kappa}{b_n^{3/2}}\sum_{|j|>m_n}\delta_j+O(b_n). \tag{3.14}$$

Now, let $i \neq 0$ be fixed. We have

$$\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(x)| \leq \frac{1}{b_n}\mathbb{E}|\overline{\mathrm{K}}_0(x)\overline{\mathrm{K}}_i(x)| + \frac{3}{b_n}\left(\mathbb{E}|\mathrm{K}_0(x)|\right)^2. \tag{3.15}$$

Moreover, keeping in mind that $||\alpha|-|\beta|| \leq |\alpha-\beta|$ for all (α,β) in $\mathbb{R}^2$ and applying the Cauchy-Schwarz inequality, we obtain

$$\left|\mathbb{E}|\overline{\mathrm{K}}_0(x)\overline{\mathrm{K}}_i(x)| - \mathbb{E}|\mathrm{K}_0(x)\mathrm{K}_i(x)|\right| \leq 2\|\mathrm{K}_0(x)\|_2\|\mathrm{K}_0(x)-\overline{\mathrm{K}}_0(x)\|_2$$

and applying Lemma 3.3, we derive

$$\left|\mathbb{E}|\overline{\mathrm{K}}_0(x)\overline{\mathrm{K}}_i(x)| - \mathbb{E}|\mathrm{K}_0(x)\mathrm{K}_i(x)|\right| \leq \frac{\kappa}{\sqrt{b_n}}\sum_{|j|>m_n}\delta_j. \tag{3.16}$$

Combining (3.15) and (3.16), we have

$$\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(x)| \leq \frac{\kappa}{b_n^{3/2}}\sum_{|j|>m_n}\delta_j + \frac{1}{b_n}\mathbb{E}|\mathrm{K}_0(x)\mathrm{K}_i(x)| + \frac{3}{b_n}\left(\mathbb{E}|\mathrm{K}_0(x)|\right)^2. \tag{3.17}$$

Using Assumption **(B3)**, we obtain

$$\begin{aligned}\mathbb{E}\big|\mathrm{K}_0(x)\mathrm{K}_i(x)\big| &= \iint_{\mathbb{R}^2}\left|\mathrm{K}\left(\frac{x-u}{b_n}\right)\mathrm{K}\left(\frac{x-v}{b_n}\right)\right| f_{0,i}(u,v)dudv\\ &\leq \kappa b_n^2\left(\int_{\mathbb{R}}|\mathrm{K}(w)|dw\right)^2.\end{aligned}$$

Since $\mathbb{E}|\mathrm{K}_0(x)| = O(b_n)$, we derive from (3.17) that

$$\sum_{\substack{j\in\mathbb{Z}^d\setminus\{0\}\\ |j|<M_n}}\left|\mathbb{E}\left(\overline{Z}_0(x)\overline{Z}_j(x)\right)\right| \leq \frac{\kappa M_n^d}{b_n^{3/2}}\sum_{|j|>m_n}\delta_j + O(M_n^d b_n). \tag{3.18}$$

Finally, combining (3.12), (3.13), (3.14) and (3.18), for all $\alpha > 1$, we obtain

$$\overline{D}_n(x) \leq \frac{\kappa m_n^{d(\tau-1)}}{\sigma^\tau(|\Lambda_n|b_n)^{\frac{\tau}{2}-1}} + \frac{\kappa}{m_n^{d(\alpha-1)}b_n^{3/2}}\sum_{|j|>m_n}|j|^{d\alpha}\delta_j + O(m_n^d b_n). \tag{3.19}$$

Since there exist $\alpha > 1$ and $p \geq 2$ such that $\sum_{i\in\mathbb{Z}^d}|i|^{d\alpha}\delta_{i,p} < \infty$, we derive from Lemma 1.3 that

$$\|R_n(x)\|_p \leq \frac{\kappa\sqrt{p}}{\sigma m_n^{d(\alpha-1)}b_n^{3/2}}\sum_{i\in\mathbb{Z}^d}|i|^{d\alpha}\delta_{i,p}. \tag{3.20}$$

Combining (3.10), (3.19) and (3.20), we obtain

$$D_n(x) \leq \kappa\left(m_n^{d(\tau-1)}\left(b_n + \frac{1}{(|\Lambda_n|b_n)^{\frac{\tau}{2}-1}}\right) + \left(\frac{1}{m_n^{d(\alpha-1)}b_n^{3/2}}\right)^{\frac{p}{p+1}}\right) \tag{3.21}$$

for all $2 < \tau \leq 3$, all $p \geq 2$ and all $\alpha > 1$ such that $\sum_{i\in\mathbb{Z}^d}|i|^{d\alpha}\delta_{i,p} < \infty$. Optimizing in m_n we derive

$$D_n(x) \leq \kappa\, b_n^{\theta_1}\left(b_n + \frac{1}{(|\Lambda_n|b_n)^{\frac{\tau}{2}-1}}\right)^{\theta_2}$$

where

$$\theta_1 = \frac{3p(1-\tau)}{2(\tau-1)(p+1)+2p(\alpha-1)} \quad \text{and} \quad \theta_2 = \frac{p(\alpha-1)}{(\tau-1)(p+1)+p(\alpha-1)}.$$

Finally, choosing $b_n = |\Lambda_n|^{\frac{2}{\tau}-1}$, we obtain $D_n(x) \leq \kappa |\Lambda_n|^{-\theta}$ where

$$\theta = \left(\frac{1}{2} - \frac{1}{\tau}\right) \frac{3p(1-\tau) + 2p(\alpha-1)}{(\tau-1)(p+1) + p(\alpha-1)}.$$

The proof of Theorem 1.5 is complete.

4. Appendix

Proof of Lemma 1.2. We follow the proof by Bosq et al. (Bosq et al. (1999), pages 88-89). First, m_n goes to infinity since $v_n = \left[b_n^{-\frac{1}{2d}}\right]$ goes to infinity and $m_n \geq v_n$. For all positive integer m, we consider $r(m) = \sum_{|i|>m} |i|^{\frac{5d}{2}} \delta_i$. Since **(A4)** holds, $r(m)$ converges to zero as m goes to infinity. Moreover, $m_n^d b_n \leq \max\left\{\sqrt{b_n}, \kappa\left(r(v_n)^{1/3} + b_n\right)\right\} \xrightarrow[n\to\infty]{} 0$ and $m_n^d \geq \frac{1}{b_n}\left(r\left(v_n\right)\right)^{1/3} \geq \frac{1}{b_n}\left(r\left(m_n\right)\right)^{1/3}$ since $v_n \leq m_n$. Finally, we obtain

$$\frac{1}{\left(m_n^d b_n\right)^{3/2}} \sum_{|i|>m_n} |i|^{\frac{5d}{2}} \delta_i \leq \sqrt{r(m_n)} \xrightarrow[n\to\infty]{} 0.$$

The proof of Lemma 1.2 is complete.

Proof of Lemma 1.3. Let $p > 1$ be fixed. We follow the proof of Proposition 1 in El Machkouri et al. (2013). For all i in $\mathbb{Z}^d$ and all x in $\mathbb{R}$, we denote $R_i = \mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x)$. Since there exists a measurable function H such that $R_i = \mathrm{H}(\varepsilon_{i-s}; s \in \mathbb{Z}^d)$, we are able to define the physical dependence measure coefficients $(\delta_{i,p}^{(n)})_{i\in\mathbb{Z}^d}$ associated to the random field $(R_i)_{i\in\mathbb{Z}^d}$. We recall that $\delta_{i,p}^{(n)} = \|R_i - R_i^*\|_p$ where $R_i^* = \mathrm{H}(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$ and $\varepsilon_j^* = \varepsilon_j \mathbb{1}_{\{j\neq 0\}} + \varepsilon_0^{'} \mathbb{1}_{\{j=0\}}$ for all j in $\mathbb{Z}^d$. In other words, we obtain R_i^* from R_i by just replacing ε_0 by its copy $\varepsilon_0^{'}$ (see Wu (2005)). Let $\tau : \mathbb{Z} \to \mathbb{Z}^d$ be a bijection. For all $l \in \mathbb{Z}$, for all $i \in \mathbb{Z}^d$, we denote $P_l R_i := \mathbb{E}(R_i|\mathcal{F}_l) - \mathbb{E}(R_i|\mathcal{F}_{l-1})$ where $\mathcal{F}_l = \sigma\left(\varepsilon_{\tau(s)}; s \leq l\right)$ and $R_i = \sum_{l\in\mathbb{Z}} P_l R_i$. Consequently, $\left\|\sum_{i\in\Lambda_n} a_i R_i\right\|_p = \left\|\sum_{l\in\mathbb{Z}} \sum_{i\in\Lambda_n} a_i P_l R_i\right\|_p$ and applying the Burkholder inequality (cf. Hall and Heyde (1980), page 23) for the martingale difference sequence $\left(\sum_{i\in\Lambda_n} a_i P_l R_i\right)_{l\in\mathbb{Z}}$, we obtain

$$\left\|\sum_{i\in\Lambda_n} a_i R_i\right\|_p \leq \left(2p \sum_{l\in\mathbb{Z}} \left\|\sum_{i\in\Lambda_n} a_i P_l R_i\right\|_p^2\right)^{\frac{1}{2}} \leq \left(2p \sum_{l\in\mathbb{Z}} \left(\sum_{i\in\Lambda_n} |a_i| \, \|P_l R_i\|_p\right)^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i\in\Lambda_n} |a_i| \, \|P_l R_i\|_p\right)^2 \leq \sum_{i\in\Lambda_n} a_i^2 \, \|P_l R_i\|_p \times \sum_{i\in\Lambda_n} \|P_l R_i\|_p. \tag{4.2}$$

Let l in $\mathbb{Z}$ and i in $\mathbb{Z}^d$ be fixed.

$$\|P_l R_i\|_p = \|\mathbb{E}(R_i|\mathcal{F}_l) - \mathbb{E}(R_i|\mathcal{F}_{l-1})\|_p = \left\|\mathbb{E}(R_0|T^i\mathcal{F}_l) - \mathbb{E}(R_0|T^i\mathcal{F}_{l-1})\right\|_p$$

where $T^i\mathcal{F}_l = \sigma\left(\varepsilon_{\tau(s)-i}; s \le l\right)$. Hence,

$$\begin{aligned}
\|P_l R_i\|_p &= \left\|\mathbb{E}\left(\mathrm{H}\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right)|T^i\mathcal{F}_l\right) - \mathbb{E}\left(\mathrm{H}\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{i-\tau(l)\}};\varepsilon'_{\tau(l)-i}\right)|T^i\mathcal{F}_l\right)\right\|_p \\
&\le \left\|\mathrm{H}\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right) - \mathrm{H}\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\setminus\{i-\tau(l)\}};\varepsilon'_{\tau(l)-i}\right)\right\|_p \\
&= \left\|\mathrm{H}\left((\varepsilon_{i-\tau(l)-s})_{s\in\mathbb{Z}^d}\right) - \mathrm{H}\left((\varepsilon_{i-\tau(l)-s})_{s\in\mathbb{Z}^d\setminus\{i-\tau(l)\}};\varepsilon'_{0}\right)\right\|_p \\
&= \left\|R_{i-\tau(l)} - R^*_{i-\tau(l)}\right\|_p \\
&= \delta^{(n)}_{i-\tau(l),p}.
\end{aligned}$$

Consequently, $\sum_{i\in\mathbb{Z}^d}\|P_l R_i\|_p \le \sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p}$ and combining (4.1) and (4.2), we obtain

$$\left\|\sum_{i\in\Lambda_n} a_i R_i\right\|_p \le \left(2p\sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p}\sum_{i\in\Lambda_n}a_i^2\sum_{l\in\mathbb{Z}}\|P_l R_i\|_p\right)^{\frac{1}{2}}.$$

Similarly, for all i in $\mathbb{Z}^d$, we have $\sum_{l\in\mathbb{Z}}\|P_l R_i\|_p \le \sum_{j\in\mathbb{Z}^d}\delta^{(n)}_{j,p}$ and we derive

$$\left\|\sum_{i\in\Lambda_n} a_i R_i\right\|_p \le \left(2p\sum_{i\in\Lambda_n}a_i^2\right)^{\frac{1}{2}}\sum_{i\in\mathbb{Z}^d}\delta^{(n)}_{i,p}. \tag{4.3}$$

Since $\overline{\mathrm{K}}^*_i = \mathbb{E}\left(\mathrm{K}^*_i(x)\middle|\mathcal{F}^*_{n,i}\right)$ where $\mathcal{F}^*_{n,i} = \sigma\left(\varepsilon^*_{i-s}\,;\,|s|\le m_n\right)$ and $\left(\mathrm{K}_i(x) - \overline{\mathrm{K}}_i(x)\right)^* = \mathrm{K}^*_i(x) - \overline{\mathrm{K}}^*_i(x)$, we derive $\delta^{(n)}_{i,p} \le 2\|\mathrm{K}_i(x) - \mathrm{K}^*_i(x)\|_p$. Since K is Lipschitz, we obtain

$$\delta^{(n)}_{i,p} \le \frac{2\delta_{i,p}}{b_n} \tag{4.4}$$

where $\delta_{i,p} = \|X_i - X^*_i\|_p$. Morever, we have also

$$\delta^{(n)}_{i,p} \le 2\|\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x)\|_p. \tag{4.5}$$

Combining (4.5) and Lemma 3.3, we derive

$$\delta^{(n)}_{i,p} \le \frac{\sqrt{8p}}{b_n}\sum_{|j|>m_n}\delta_{j,p}. \tag{4.6}$$

Combining (4.4) and (4.6), we obtain

$$\sum_{i\in\mathbb{Z}^d}\delta^{(n)}_{i,p} \le \frac{m_n^d\sqrt{8p}}{b_n}\sum_{|j|>m_n}\delta_{j,p} + \frac{2}{b_n}\sum_{|j|>m_n}\delta_{j,p} \le \frac{2\sqrt{8p}m_n^d}{b_n}\sum_{|j|>m_n}\delta_{j,p}.$$

The proof of Lemma 1.3 is complete.

Proof of Lemma 3.1. Let s and t be fixed in $\mathbb{R}$. Since

$$\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right) = \mathbb{E}\left(\mathrm{K}_0(s)\overline{\mathrm{K}}_0(t)\right),$$

we have

$$\left|\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right) - \mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right)\right| \le \|\mathrm{K}_0(s)\|_2\|\mathrm{K}_0(t) - \overline{\mathrm{K}}_0(t)\|_2.$$

Keeping in mind that $\|\mathrm{K}_0(s)\|_2 = O(\sqrt{b_n})$ and using Lemma 3.3, we have

$$\left|\mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right) - \mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right)\right| \leq \frac{\kappa}{\sqrt{b_n}} \sum_{|j|>m_n} \delta_j.$$

Since $b_n|\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t)| = |\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right) - \mathbb{E}\left(\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_0(t)\right)|$, we have

$$M_n^d|\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\overline{Z}_0(s)\overline{Z}_0(t)| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j|>m_n} |j|^{\frac{5d}{2}}\,\delta_j. \tag{4.7}$$

Moreover, keeping in mind Assumptions **(A1)**, **(A2)** and **(A4)**, we have

$$\begin{aligned}\lim_n \frac{1}{b_n}\mathbb{E}\left(\mathrm{K}_0(s)\mathrm{K}_0(t)\right) &= \lim_n \int_{\mathbb{R}} \mathrm{K}\left(v\right)\mathrm{K}\left(v + \frac{t-s}{b_n}\right) f(s - vb_n)dv \\ &= u(s,t)\, f(s)\, \int_{\mathbb{R}} \mathrm{K}^2(u)du\end{aligned} \tag{4.8}$$

where $u(s,t) = 1$ if $s = t$ and $u(s,t) = 0$ if $s \neq t$. We have also

$$\lim_n \frac{1}{b_n}\mathbb{E}\mathrm{K}_0(s)\mathbb{E}\mathrm{K}_0(t) = \lim_n b_n \int_{\mathbb{R}} \mathrm{K}(v)f(s - vb_n)dv \int_{\mathbb{R}} \mathrm{K}(w)f(t - wb_n)dw = 0. \tag{4.9}$$

Let x be fixed in $\mathbb{R}$. Choosing $s = t = x$ and combining (4.7), (4.8), (4.9) and Lemma 1.2, we obtain $\mathbb{E}(\overline{Z}_0^2(x))$ goes to $f(x)\int_{\mathbb{R}} \mathrm{K}^2(u)du$ as n goes to infinity.
In the other part, let $i \neq 0$ be fixed in $\mathbb{Z}^d$ and let s and t be fixed in $\mathbb{R}$. We have

$$\mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| \leq \frac{1}{b_n}\mathbb{E}\left|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)\right| + \frac{3}{b_n}\mathbb{E}\left|\mathrm{K}_0(s)\right|\mathbb{E}\left|\mathrm{K}_0(t)\right|. \tag{4.10}$$

Keeping in mind that $||\alpha| - |\beta|| \leq |\alpha - \beta|$ for all (α, β) in $\mathbb{R}^2$ and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\mathbb{E}|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)| - \mathbb{E}|\mathrm{K}_0(s)\mathrm{K}_i(t)|\right| &\leq \|\overline{\mathrm{K}}_0(s)\|_2\|\overline{\mathrm{K}}_0(t) - \mathrm{K}_0(t)\|_2 \\ &\quad + \|\mathrm{K}_0(t)\|_2\|\overline{\mathrm{K}}_0(s) - \mathrm{K}_0(s)\|_2\end{aligned} \tag{4.11}$$

Applying again Lemma 3.3, we obtain

$$\frac{M_n^d}{b_n}\left|\mathbb{E}|\overline{\mathrm{K}}_0(s)\overline{\mathrm{K}}_i(t)| - \mathbb{E}|\mathrm{K}_0(s)\mathrm{K}_i(t)|\right| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j|>m_n} |j|^{\frac{5d}{2}}\delta_j. \tag{4.12}$$

Since Assumptions **(A1)** and **(A4)** hold and $M_n^d b_n = o(1)$, we have

$$\frac{M_n^d}{b_n}\mathbb{E}\left|\mathrm{K}_0(s)\right|\mathbb{E}\left|\mathrm{K}_0(t)\right| = M_n^d b_n \int_{\mathbb{R}} |\mathrm{K}(u)|f(s - ub_n)du \int_{\mathbb{R}} |\mathrm{K}(v)|f(t - vb_n)dv = o(1). \tag{4.13}$$

Moreover, using Assumption **(B3)**, we have

$$\begin{aligned}\mathbb{E}\left|\mathrm{K}_0(s)\mathrm{K}_i(t)\right| &= \iint_{\mathbb{R}^2} \left|\mathrm{K}\left(\frac{s-u}{b_n}\right)\mathrm{K}\left(\frac{t-v}{b_n}\right)\right| f_{0,i}(u,v)dudv \\ &\leq \kappa b_n^2 \left(\int_{\mathbb{R}} |\mathrm{K}(w)|dw\right)^2.\end{aligned}$$

So, using again Assumption **(A4)** and $M_n^d b_n = o(1)$, we derive

$$\frac{M_n^d}{b_n}\mathbb{E}\left|\mathrm{K}_0(s)\mathrm{K}_i(t)\right| = o(1). \tag{4.14}$$

Combining (4.10), (4.12), (4.13), (4.14) and Lemma 1.2, we obtain

$$M_n^d \sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(1). \tag{4.15}$$

The proof of Lemma 3.1 is complete.

Proof of Lemma 3.2. Let x and y be two distinct real numbers. Noting that

$$\mathbb{E}(\Delta_0^2) = \lambda_1^2\mathbb{E}(Z_0^2(x)) + \lambda_2^2\mathbb{E}(Z_0^2(y)) + 2\lambda_1\lambda_2\mathbb{E}(Z_0(x)Z_0(y))$$
$$\mathbb{E}(\overline{\Delta}_0^2) = \lambda_1^2\mathbb{E}(\overline{Z}_0^2(x)) + \lambda_2^2\mathbb{E}(\overline{Z}_0^2(y)) + 2\lambda_1\lambda_2\mathbb{E}(\overline{Z}_0(x)\overline{Z}_0(y))$$

and using (4.7) and Lemma 1.2, we obtain

$$\lim_{n\to\infty} M_n^d|\mathbb{E}(\Delta_0^2) - \mathbb{E}(\overline{\Delta}_0^2)| = 0. \tag{4.16}$$

Combining (4.8) and (4.16), we derive that $\mathbb{E}(\overline{\Delta}_0^2)$ converges to

$$\eta = \left(\lambda_1^2 f(x) + \lambda_2^2 f(y)\right)\int_{\mathbb{R}} \mathrm{K}^2(u)du.$$

Let $i \neq 0$ be fixed in $\mathbb{Z}^d$. Combining (4.15) and

$$\begin{aligned}\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| \leq\ & \lambda_1^2\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(x)| + \lambda_2^2\mathbb{E}|\overline{Z}_0(y)\overline{Z}_i(y)| \\ & + \lambda_1\lambda_2\mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(y)| + \lambda_1\lambda_2\mathbb{E}|\overline{Z}_0(y)\overline{Z}_i(x)|,\end{aligned} \tag{4.17}$$

we obtain $M_n^d \sup_{i\in\mathbb{Z}^d\backslash\{0\}} \mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| = o(1)$. The proof of Lemma 3.2 is complete.

Proof of Lemma 3.3. Let $p > 1$ be fixed. We consider the sequence $(\Gamma_n)_{n\geq 0}$ of finite subsets of $\mathbb{Z}^d$ defined by $\Gamma_0 = \{(0,...,0)\}$ and for all n in $\mathbb{N}\backslash\{0\}$, $\Gamma_n = \{i \in \mathbb{Z}^d\,;\,|i| = n\}$. For all integer n, let $a_n = \sum_{j=0}^n |\Gamma_j|$ and let $\tau : \mathbb{N}\backslash\{0\} \to \mathbb{Z}^d$ be the bijection defined by $\tau(1) = (0,...,0)$ and

- for all n in $\mathbb{N}\backslash\{0\}$, if $l \in\,]a_{n-1}, a_n]$ then $\tau(l) \in \Gamma_n$,
- for all n in $\mathbb{N}\backslash\{0\}$, if $(i,j) \in\,]a_{n-1}, a_n]^2$ and $i < j$ then $\tau(i) <_{\text{lex}} \tau(j)$

Let $(m_n)_{n\geq 1}$ be the sequence of positive integers defined by (1.3). For all n in $\mathbb{N}\backslash\{0\}$, we recall that $\mathcal{F}_{n,0} = \sigma\left(\varepsilon_{-s}\,;\,|s| \leq m_n\right)$ (see (1.4)) and we consider also the σ-algebra $\mathcal{G}_n := \sigma\left(\varepsilon_{\tau(j)}\,;\,1 \leq j \leq n\right)$. By the definition of the bijection τ, we have $1 \leq j \leq a_n$ if and only if $|\tau(j)| \leq n$. Consequently $\mathcal{G}_{a_{m_n}} = \mathcal{F}_{n,0}$ and $\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x) = \sum_{l>a_{m_n}} D_l$ with $D_l = \mathbb{E}\left(\mathrm{K}_0(x)|\mathcal{G}_l\right) - \mathbb{E}\left(\mathrm{K}_0(x)|\mathcal{G}_{l-1}\right)$ for all l in $\mathbb{Z}$. Let $p > 1$ be fixed. Since $(D_l)_{l\in\mathbb{Z}}$ is a martingale-difference sequence, applying Burkholder's inequality (cf. Hall and Heyde (1980), page 23), we derive

$$\|\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x)\|_p \leq \left(2p\sum_{l>a_{m_n}} \|D_l\|_p^2\right)^{1/2}.$$

Denoting $\mathrm{K}_0^{'}(x) = \mathrm{K}\left(b_n^{-1}\left(x - g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\backslash\{-\tau(l)\}}; \varepsilon_{\tau(l)}^{'}\right)\right)\right)$, we obtain

$$\begin{aligned}
\|D_l\|_p = \|\mathbb{E}\left(\mathrm{K}_0(x)|\mathcal{G}_l\right) - \mathbb{E}\left(\mathrm{K}_0^{'}(x)|\mathcal{G}_l\right)\|_p &\leq \|\mathrm{K}_0(x) - \mathrm{K}_0^{'}(x)\|_p \\
&\leq \frac{1}{b_n}\left\|g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{-s})_{s\in\mathbb{Z}^d\backslash\{-\tau(l)\}}; \varepsilon_{\tau(l)}^{'}\right)\right\|_p \\
&= \frac{1}{b_n}\left\|g\left((\varepsilon_{-\tau(l)-s})_{s\in\mathbb{Z}^d}\right) - g\left((\varepsilon_{-\tau(l)-s})_{s\in\mathbb{Z}^d\backslash\{-\tau(l)\}}; \varepsilon_0^{'}\right)\right\|_p \\
&= \frac{1}{b_n}\left\|X_{-\tau(l)} - X_{-\tau(l)}^{*}\right\|_p = \frac{\delta_{-\tau(l),p}}{b_n}
\end{aligned}$$

and finally

$$\|\mathrm{K}_0(x) - \overline{\mathrm{K}}_0(x)\|_p \leq \frac{1}{b_n}\left(2p\sum_{l>a_{m_n}}\delta_{-\tau(l),p}^2\right)^{1/2} \leq \frac{\sqrt{2p}}{b_n}\sum_{|j|>m_n}\delta_{j,p}.$$

The proof of Lemma 3.3 is complete.

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