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# An explicit formula for the heat kernel of a left invariant operator

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**Abstract.** The main aim of this note is to find an explicit integral formula for the heat kernel of a certain second-order left invariant differential operator on a solvable Lie group, being a semi-direct product $\mathbb{R} \ltimes \mathbb{R}$, by means of a skew-product formula for diffusions. An explicit formula of a different kind for the special case of the operator considered in this note (the operator without the drift term) was found recently by Calin, Chang and Li.

As a corollary from our main result we get a simple formula for the return probability from which the asymptotic behaviour for small and large values of time follows easily.

## 1. Introduction

Consider $\mathbb{R}^2$ with multiplication given by

$$(x, y) \cdot (x', y') = (x + x', y + e^x y').$$

Then $G = (\mathbb{R}^2, \cdot)$ is a semi-direct product $\mathbb{R} \ltimes \mathbb{R}$, and is a solvable Lie group. A left-invariant Haar measure $\mathcal{H}$ on $G$ is

$$\mathcal{H}(dxdy) = e^{-x} dxdy. \tag{1.1}$$

It is easy to see that

$$X = \partial_x \text{ and } Y = e^x \partial_y$$

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are left invariant vector fields on $G$ that generate the Lie algebra $\mathfrak{g}$ of $G$. Let, for $\mu \in \mathbb{R}$,

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\left(X^2 + 2\mu X + Y^2\right) = \frac{1}{2}\left(\partial_x^2 + 2\mu\partial_x + e^{2x}\partial_y^2\right). \tag{1.2}$$

Let

$$T_t f(x,a) = \int_G p_t^{(\mu)}(x,y;x',y')f(x',y')\mathcal{H}(dx'dy') \tag{1.3}$$

be the semigroup of operators on $G$ generated by $\mathcal{L}^{(\mu)}$. The kernel $p_t^{(\mu)}$ is called the *heat kernel* for $\mathcal{L}^{(\mu)}$.

In a recent paper Calin et al. (2013) found an explicit expression for the fundamental solution (heat kernel) $p_t^{(0)}(x,y;x',y')$ of the parabolic equation $\mathcal{L}^{(0)} - \partial_t = 0$ using a geometric method involving Hamiltonian formalism. The authors of Calin et al. (2013) were interested in such a group since it has total involutivity property.

The aim of this note is to present an alternative approach which, we think, is much simpler and allows us to consider more general operators with the first order term $\mu X$. Namely, we use a probabilistic skew-product formula (see Sec. 2) for the diffusion generated by $\mathcal{L}^{(\mu)}$ in order to give an explicit integral expression for the transition probabilities $p_t^{(\mu)}(x,y;x',y')$ of the stochastic process generated by $\mathcal{L}^{(\mu)}$.

The second main probabilistic ingredient in our proof (Theorem 3.1) is an important result obtained by Yor (1992) which gives an explicit formula for the joint distribution of the Brownian motion (with drift) and some exponential functional of this Brownian motion - time integral from 0 to $t$ of the geometric Brownian motion. We want to mention that exponential functionals of this type play an important role in financial mathematics. In particular, their distributions has many applications in the European and the Asian options (see e.g. Matsumoto and Yor (2005b); Yor (2001); Carr and Schröder (2003); Donati-Martin et al. (2001)).

The method presented here is not new. For example, Matsumoto (2001) applied the skew-product formula (although he did not use that name) in order to find some integral representations of the heat kernels for the Laplace-Beltrami operators on hyperbolic spaces. Penney and the author used the skew-product formula for obtaining the upper bound for the heat kernel and the Poisson kernel for a second order left-invariant differential operators on a certain class of Lie groups in Penney and Urban (2013b) and Penney and Urban (2013a), respectively.

*Remark* 1.1. Since the operator $\mathcal{L}^{(\mu)}$ commutes with left translation, the same is true for the operators $T_t$. Thus, from (1.3),

$$p_t^{(\mu)}(x,y;x',y') = p_t^{(\mu)}(e;(x,y)^{-1}(x',y')),$$

where $e = (0,0)$ is the identity element in the group $G$. Therefore, it is enough to consider only $p_t^{(\mu)}(e;x,y)$.

The main result is the following.

**Theorem 1.2.** *Let $p_t^{(\mu)}$ be the heat kernel (with respect to the left-invariant Haar measure $\mathcal{H}$) for the operator $\mathcal{L}^{(\mu)}$ defined in (1.2). Then*

$$p_t^{(\mu)}(e;x,y) = \frac{2^{1/2}\Gamma(3/2)e^{(2+\mu)x-\mu^2t/2+\pi^2/(2t)}}{\pi^2 t^{1/2}} \int_0^\infty \frac{e^{-\xi^2/2t}\sinh(\xi)\sin\left(\frac{\pi\xi}{t}\right)}{\left(1+y^2+e^{2x}+2e^x\cosh(\xi)\right)^{3/2}}d\xi.$$

*Remark* 1.3. Notice that since $G$ is non-unimodular the heat kernel $p_t^{(\mu)}(x,y;x',y')$ is not symmetric with respect to the left-invariant Haar measure $\mathcal{H}(dxdy)$. Specifically,

$$p_t^{(\mu)}(x',y';x,y) = e^{-x}e^{x'}p_t^{(\mu)}(x,y;x',y').$$

The value $p_t^{(\mu)}(e;e)$ is the analogue of the probability of return in $t$ steps to the origin (neutral element) $e=(0,0)$ of the group $G$ for the (discrete time) random walk. It is a very important and general problem to know, for a given Lie group $H$ and a left-invariant second order differential operator $L$ on $H$ satisfying Hörmander condition, the asymptotic of the heat kernel $p_t$ for $L$ at identity $e$ both for small and large time $t$. The behavior of $p_t(e;e)$ is closely related with the geometry of the underlying group $H$. There is a lot of literature devoted to this subject. See e.g. the survey by Varopoulos (2005) and references therein.

As a corollary from Theorem 1.2 we obtain, with a help of McKean's result McKean (1970) about heat kernel for Laplace-Beltrami operator on the upper half-plane, a simple explicit formula for "return probability" $p_t^{(\mu)}(e;e)$, from which we deduce the asymptotic for $t\to 0$ and $t\to\infty$.

**Theorem 1.4.** *We have*

$$p_t^{(\mu)}(e;e) = \frac{1}{2(\pi t)^{3/2}}e^{-\mu^2 t/2}\int_0^\infty \frac{\xi e^{-\xi^2/(2t)}}{(\cosh\xi)^{1/2}}d\xi.$$

**Corollary 1.5.** *The asymptotic behavior of $p_t^{(\mu)}(e;e)$, for small and large $t$, is as follows,*

$$p_t^{(\mu)}(e;e) \sim \begin{cases} \frac{1}{2\pi^{3/2}}t^{-1/2}, & \text{as } t\to 0,\\ \frac{c}{2\pi^{3/2}}t^{-3/2}e^{-\mu^2 t/2}, & \text{as } t\to\infty,\end{cases}$$

*where* $c=\int_0^\infty \xi(\cosh\xi)^{-1/2}d\xi$.

Similar results on solvable groups of upper triangular $2\times 2$ matrices are proved by Konakov et al. (2011).

*Remark* 1.6. The diffusion process $(X_t,Y_t)$ associated to the infinitesimal generator $\mathcal{L}^{(\mu)}$ is the solution of the following system of stochastic differential equations,

$$\begin{aligned} dX_t =& dW_t^1 + \mu dt,\\ dY_t =& e^{X_t}dW_t^2,\end{aligned}$$

where $W_t^1$ and $W_t^2$ are independent standard Brownian motions. This process is closely related to processes considered in financial mathematics. Specifically, one can consider a model in which the process $e^{X_t}$, which is a geometric Brownian motion, is the volatility process and $Y_t$ is the log returns process.

## 2. Skew-product formula

Consider the product of two manifolds $M\times N$. A generic element of $M\times N$ is denoted by $(x,y)$. Let $L_1$ be a differential operator acting on $M$ and let, for every $x\in M$, $L_2(x)$ be an operator acting on $N$. Let $\mathcal{L}$ be a *skew-product* of $L_1$ and $L_2(x)$. That is $\mathcal{L}$ is the operator, on the product $M\times N$, and acts on $f: M\times N\to\mathbb{R}$ by formula,

$$\mathcal{L}f(x,y) = L_1 f(\cdot,y)\big|_x + L_2(x)f(x,\cdot)\big|_y, \tag{2.1}$$

i.e., $L_1$ in (2.1) differentiate only the fist variable of the function $f$, whereas $L_2(x)$ acts on the second variable. Then it is natural to expect that the semigroup $T_t$ generated by $\mathcal{L}$ is given by the following *skew-product formula*:

$$T_t f(x, y) = \mathbf{E}_x \left( U^\sigma(0, t) f(\sigma(t), y) \right), \tag{2.2}$$

where the expectation $\mathbf{E}_x$ is taken with respect to the diffusion $\sigma(t)$ on $M$ generated by $L_1$ (a subscript $x$ denotes that $\sigma$ starts from $x \in M$, i.e., $\sigma(0) = x$) and $U^\sigma(s, t)$ is a family of evolution operators (acting on variable $y \in N$) generated by time dependent operator $L_2(\sigma(t))$ (see Tanabe (1979); van Casteren (2011) for information on evolutions). The idea of such a decomposition of the diffusion on $M \times N$ goes back to Malliavin (1978); Malliavin and Malliavin (1974) (see also Taylor (1992)).

In our context $M = \mathbb{R}$, $N = \mathbb{R}$, and the operator $\mathcal{L}^{(\mu)}$ is a skew-product of $L_1 = \frac{1}{2}(\partial_x^2 + 2\mu\partial_x)$ and $L_2(x) = \frac{1}{2}e^{2x}\partial_y^2$. Then $\sigma(t)$ in (2.2) is the Brownian motion with a constant drift $\mu$, that is $\sigma(t) = b^{(\mu)}(t) := b(t) + t\mu$, where $b(t)$ is a standard Brownian motion. The formula (2.2) in this situation can be proved along the lines of the proof of Theorem 3.1 in Damek et al. (2001).

## 3. Proofs

We are going to apply formula (2.2) to get the explicit expression for the heat kernel for the operator $\mathcal{L}^{(\mu)}$ defined in (1.2).

*Proof of Theorem 1.2:* For a trajectory $b^{(\mu)} \in C([0, +\infty), \mathbb{R})$ of the Brownian motion with drift (generated by the operator $L_1 = \frac{1}{2}(\partial_x^2 + 2\mu\partial_x)$ we consider the following time dependent operator

$$L^{b^{(\mu)}(t)} = e^{2b^{(\mu)}(t)}\partial_y^2.$$

(This is $L_2(b^{(\mu)}(t))$ in the notation of the previous section.) Then the fundamental solution $p_{t,s}^{b^{(\mu)}}(y)$ of

$$L^{b^{(\mu)}(t)} - \partial_t = 0$$

can be easily computed (using, for example, the Fourier transformation) and is given by

$$p_{t,s}^{b^{(\mu)}}(y) = \frac{1}{\left(2\pi \int_s^t e^{2b^{(\mu)}(u)}du\right)^{1/2}} \exp\left(-\frac{y^2}{2\int_s^t e^{2b^{(\mu)}(u)}du}\right).$$

In notation of Sec. 2, $U^{b^\mu}(s, t)f(y) = p_{t,s}^{b^{(\mu)}} * f(y)$ is a convolution operator. Let, for $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$I_\varepsilon(x) = [x - \varepsilon, x + \varepsilon].$$

By the skew-product formula (2.2) applied to the function

$$\delta_y(\cdot) \otimes \frac{1}{\nu(I_\varepsilon(x))}\mathbf{1}_{I_\varepsilon(x)}(\cdot),$$

where $\mathbf{1}_A$ is the indicator function of a given set $A \subseteq \mathbb{R}$, and

$$\nu(A) = \int_A e^{-x}dx, \qquad A \subseteq \mathbb{R},$$

i.e., $\nu$ is a projection of the Haar measure $\mathcal{H}$ defined in (1.1) onto the first component of $G$, we get

$$p_t^{(\mu)}(e;x,y) = \lim_{\varepsilon\to 0} \frac{1}{\nu(I_\varepsilon(x))} \mathbf{E}_0 \left( p_{t,0}^{b^{(\mu)}}(y) \mathbf{1}_{I_\varepsilon(x)}(b^{(\mu)}(t)) \right). \tag{3.1}$$

Let, for $t > 0$, $\varphi(t;r,w)$ denote the density function (with respect to the measure $drdw$) on $(0,+\infty)\times\mathbb{R}$ of the joint distribution of the exponential functional

$$A_t^{(\mu)} = \int_0^t e^{2b^{(\mu)}(u)} du$$

and $b^{(\mu)}(t) = b(t) + t\mu$, i.e.,

$$\varphi(t;r,w)drdw = \mathbf{P}_0 \left( A_t^{(\mu)} \in dr,\, b^{(\mu)}(t) \in dw \right).$$

Then, by (3.1),

$$\begin{aligned} &p_t^{(\mu)}(e;x,y) \\ &\quad = \lim_{\varepsilon\to 0} \frac{1}{\nu(I_\varepsilon(x))} \int_{\mathbb{R}} \int_0^{+\infty} \frac{1}{(2\pi r)^{1/2}} \exp\left(-\frac{y^2}{2r}\right) \mathbf{1}_{I_\varepsilon(x)}(w)\varphi(t;r,w)drdw. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\nu(I_\varepsilon(x))} \int_{\mathbb{R}} \mathbf{1}_{I_\varepsilon(x)}(w)\varphi(t;r,w)dw \\ &\qquad = \lim_{\varepsilon\to 0} \frac{1}{\nu(I_\varepsilon(x))} \int_{\mathbb{R}} \mathbf{1}_{I_\varepsilon(x)}(w)\varphi(t;r,w)e^w e^{-w} dw = \varphi(t;r,x)e^x, \end{aligned}$$

we get that

$$p_t^{(\mu)}(e;x,y) = e^x \int_0^\infty \frac{1}{(2\pi r)^{1/2}} \exp\left(-\frac{y^2}{2r}\right) \varphi(t;r,x)dr. \tag{3.2}$$

The density $\varphi(t;r,x)$, by a probabilistic method, was for the first time computed by Yor (1992). An analytic proof is given in Matsumoto and Yor (2005a).

**Theorem 3.1.** *Let $\mu\in\mathbb{R}$. Fix $t>0$. Then, for $r>0$ and $x\in\mathbb{R}$, it holds that*

$$\mathbf{P}_0 \left( A_t^{(\mu)} \in dr,\, b^{(\mu)}(t) \in dx \right) = e^{\mu x - \mu^2 t/2} \exp\left(-\frac{1+e^{2x}}{2r}\right) \theta(e^x/r, t) \frac{drdx}{r},$$

*where,*

$$\theta(r,t) = \frac{r}{(2\pi^3 t)^{1/2}} e^{\pi^2/2t} \int_0^\infty e^{-\xi^2/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi.$$

Thus, by Theorem 3.1 and (3.2),

$$\begin{aligned} p_t^{(\mu)}(e;x,y) =& e^x \int_0^\infty \frac{1}{(2\pi r)^{1/2}} \exp\left(-\frac{y^2}{2r}\right) \\ & \times e^{\mu x - \mu^2 t/2} \exp\left(-\frac{1+e^{2x}}{2r}\right) \theta(e^x/r,t) \frac{dr}{r} \\ =& \frac{e^{(2+\mu)x - \mu^2 t/2 + \pi^2/(2t)}}{2\pi^2 t^{1/2}} \int_0^\infty \frac{1}{r^{5/2}} \exp\left(-\frac{1+y^2+e^{2x}}{2r}\right) \\ & \times \int_0^\infty e^{-\xi^2/2t} e^{-e^x\cosh(\xi)/r} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi dr. \end{aligned}$$

By Fubini theorem,

$$p_t^{(\mu)}(e;x,y) = \frac{e^{(2+\mu)x-\mu^2 t/2+\pi^2/(2t)}}{2\pi^2 t^{1/2}} \int_0^\infty e^{-\xi^2/2t}\sinh(\xi)\sin\left(\frac{\pi\xi}{t}\right) \times \int_0^\infty \frac{1}{r^{5/2}}\exp\left(-\frac{1+y^2+e^{2x}+2e^x\cosh(\xi)}{2r}\right) dr d\xi.$$

The integral over $r$ is equal to

$$\left(\frac{1+y^2+e^{2x}+2e^x\cosh(\xi)}{2}\right)^{-3/2}\Gamma(3/2).$$

Hence, Theorem 1.2 follows. □

*Proof of Theorem 1.4:* From Theorem 1.2,

$$p_t^{(\mu)}(e;e) = \frac{\Gamma(3/2)e^{-\mu^2 t/2+\pi^2/(2t)}}{2\pi^2 t^{1/2}} \int_0^\infty \frac{e^{-\xi^2/2t}\sinh(\xi)\sin\left(\frac{\pi\xi}{t}\right)}{(1+\cosh(\xi))^{3/2}} d\xi.$$

Comparing above formula with Matsumoto (2001, Theorem 3.1) we see that $p_t^{(1/2)}(e;e)$ is equal to $h_t(0)$, where $h_t(r)$ is the heat kernel for the Laplace-Beltrami operator on the upper half plane ($r$ is the hyberbolic distance between given points), i.e.,

$$p_t^{(\mu)}(e;e) = e^{-\mu^2 t/2} e^{t/8} h_t(0).$$

Using a well known formula for $h_t(r)$ (which was found by McKean (1970), see also Matsumoto and Yor (2005b, (3.3)), Matsumoto (2001, p. 558)),

$$h_t(r) = \frac{1}{2(\pi t)^{3/2}} e^{-t/8} \int_0^\infty \frac{\xi e^{-\xi^2/(2t)}}{(\cosh\xi - \cosh r)^{1/2}} d\xi$$

the result follows. □

*Proof of Corollary 1.5:* For large $t$, it is enough to note that

$$\lim_{t\to\infty} \int_0^\infty \frac{\xi e^{-\xi^2/(2t)}}{(\cosh\xi)^{1/2}} d\xi = \int_0^\infty \frac{\xi}{(\cosh\xi)^{1/2}} d\xi = c.$$

For small $t$, we change variables and get

$$\int_0^\infty \frac{\xi e^{-\xi^2/(2t)}}{(\cosh\xi)^{1/2}} d\xi = 2t \int_0^\infty \frac{\xi e^{-\xi^2}}{\cosh(\sqrt{2t}\xi)} d\xi.$$

The integral on the right tends (as $t\to 0$) to $\int_0^\infty \xi e^{-\xi^2} d\xi = 1/2$. □

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