On the range of random walk on graphs satisfying a uniform condition

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Abstract. We consider the range of random walks up to time n, R_n, on graphs satisfying a uniform condition. This condition is characterized by potential theory. Not only all vertex transitive graphs but also many non-regular graphs satisfy the condition. We show certain weak laws of R_n from above and below. We also show that there is a graph such that it satisfies the condition and a sequence of the mean of R_n/n fluctuates. By noting the construction of the graph, we see that under the condition, the weak laws are best in a sense.

1. Introduction

The range of random walk R_n is simply the number of sites which the random walk visits up to time n. One of the most fundamental problems is whether the process $\{R_n\}_n$ satisfies law of large numbers. Dvoretzky and Erdös (1951), Spitzer (1976) considered the ranges of random walks on $\mathbb{Z}^d$ and derived strong law of large numbers. They used the spacial homogeneity of $\mathbb{Z}^d$ heavily. We may need to take alternative techniques to consider the range of walks on graphs which do *not* have such spacial homogeneity.

In this paper we consider the range of random walk on graphs satisfying a uniform condition (U). See Definition 1.1 for the definition of the uniform condition. This condition is characterized by potential theory, specifically, effective resistances. Not only all vertex transitive graphs but also some *non-regular* graphs satisfy (U). See Section 4 for detail. We state certain weak laws of R_n from above and below in Theorem 1.2. Under a stronger assumption, certain strong laws holds for R_n. In Theorem 1.3, we state the existence of a graph such that it satisfies (U) and a sequence of the mean of R_n/n fluctuates. This construction shows that under (U), the two convergences are best in a sense.

Now we describe the settings. Let (X, μ) be an weighted graph. That is, X is an infinite weighted graph and X is endowed with a weight μ_{xy}, which is a symmetric nonnegative function on $X \times X$ such that $\mu_{xy} > 0$ if and only if x and y are connected. We write $x \sim y$ if x and y are connected by an edge. Let $\mu_x = \sum_{y \in X} \mu_{xy}$, $x \in X$. Let $\mu(A) = \sum_{x \in A} \mu_x$ for $A \subset X$.

In this paper we assume that $\sup_{x \in X} \deg(x) < +\infty$ and $0 < \inf_{x,y \in X, x \sim y} \mu_{xy} \leq \sup_{x,y \in X, x \sim y} \mu_{xy} < +\infty$. *Whenever we do not refer to weights, we assume that $\mu_{xy} = 1$ for any $x \sim y$.*

Let $\{S_n\}_{n \geq 0}$ be a Markov chain on X whose transition probabilities are given by $P(S_{n+1} = y | S_n = x) = \mu_{xy}/\mu_x$, $n \geq 0$, $x, y \in X$. We write $P = P_x$ if $P(S_0 = x) = 1$. We say that (X, μ) is recurrent (resp. transient) if $(\{S_n\}_{n \geq 0}, \{P_x\}_{x \in X})$ is recurrent (transient). Let the random walk range $R_n = |\{S_0, \ldots, S_{n-1}\}|$.

Let $T_A = \inf\{n \geq 0 : S_n \in A\}$ and $T_A^+ = \inf\{n \geq 1 : S_n \in A\}$ for $A \subset X$. For $x, y \in X$, $n \geq 0$ and $B \subset X$, let $p_n^B(x, y) = P_x(S_n = y, T_{B^c} > n)/\mu_y$ and $g^B(x, y) = \sum_{n \geq 0} p_n^B(x, y)$. Let $p_n(x, y) = p_n^X(x, y)$ and $g(x, y) = g^X(x, y)$.

Let $F_1 = \inf_{x \in X} P_x(T_x^+ < +\infty)$ and $F_2 = \sup_{x \in X} P_x(T_x^+ < +\infty)$.

Let d be the graph metric on X. Let $B(x, n) = \{y \in X : d(x, y) < n\}$, $x \in X$, $n \in \mathbb{N}_{\geq 1}$. Let $V(x, n) = \mu(B(x, n))$. Let $\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y \in X, x \sim y} (f(x) - f(y))^2 \mu_{xy}$ for $f : X \to \mathbb{R}$. Let us define the effective resistance by $R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f|_A = 1, f|_B = 0\}$ for $A, B \subset X$ with $A \cap B = \emptyset$.

Let $\rho(x, n) = R_{\text{eff}}(\{x\}, B(x, n)^c)$, $x \in X, n \in \mathbb{N}$. Let $\rho(x) = \lim_{n \to \infty} \rho(x, n)$. If (X, μ) is recurrent (resp. transient), then, $\rho(x) = +\infty$ (resp. $\rho(x) < +\infty$) for any $x \in X$.

Now we define a uniform condition for weighted graphs.

Definition 1.1 (uniform condition)**.** We say that an weighted graph (X, μ) satisfies (U) if $\rho(x, n)$ converges *uniformly* to $\rho(x)$, $n \to \infty$.

Not only vertex transitive graphs (e.g. $\mathbb{Z}^d$, the M-regular tree T_M, Cayley graphs of groups) but also some non-regular graphs (e.g. graphs which are roughly isometric with $\mathbb{Z}^d$, Sierpiński gasket or carpet) satisfy (U) if all weights are equal to 1. See Section 4 for detail.

Now we describe the main results.

Theorem 1.2. *Let (X, μ) be an weighted graph satisfying (U). Then, for any $x \in X$ and any $\epsilon > 0$, we have that*

$$\lim_{n \to \infty} P_x(R_n \geq n(1 - F_1 + \epsilon)) = 0, \tag{1.1}$$

and,

$$\lim_{n \to \infty} P_x(R_n \leq n(1 - F_2 - \epsilon)) = 0. \tag{1.2}$$

These convergences are uniform with respect to x. The convergence in (1.1) is exponentially fast.

If (X, μ) satisfies an assumption which is stronger than (U), then, certain strong laws hold for R_n, that is,

$$1 - F_2 \leq \liminf_{n \to \infty} \frac{R_n}{n} \leq \limsup_{n \to \infty} \frac{R_n}{n} \leq 1 - F_1, \ P_x\text{-a.s.}$$

See Corollary 2.3 for detail.

Theorem 1.3. *There exists an infinite weighted graph* (X, μ) *with a reference point* o *which satisfies* $F_1 < F_2$, (U),

$$\liminf_{n\to\infty} \frac{E_o[R_n]}{n} = 1 - F_2, \ \ and, \ \limsup_{n\to\infty} \frac{E_o[R_n]}{n} = 1 - F_1. \tag{1.3}$$

Remark 1.4. (i) If X is vertex transitive, then, $F_1 = F_2$ and hence $R_n/n \to 1 - F_1 \in [0, 1]$ in probability. On the other hand, by noting Theorem 1.3, there exists an infinite weighted graph (X, μ) with a reference point o which satisfies (U) and R_n/n does *not* converge to any $a \in [0, 1]$ in probability under P_o.
(ii) If we replace F_1 (resp. F_2) with a real number larger than F_1 (resp. smaller than F_2), (1.1) (resp. (1.2)) fails for an weighted graph in Theorem 1.3. In this sense, the convergences (1.1) and (1.2) are best.

The main difficulty of the proof of Theorem 1.2 is that $P_x \neq P_y$ can happen for $x \neq y$. On the other hand, we use the fact in order to show Theorem 1.3.

2. Proof of Theorem 1.2

First, we show the following lemma.

Lemma 2.1. *Let* (X, μ) *be an weighted graph satisfying* (U). *Then,*

$$\lim_{n\to\infty} \sup_{x\in X} P_x(n < T_x^+ < +\infty) = 0.$$

Proof: By Kumagai (2010) Theorem 1.14, $\rho(x, n)^{-1} = \mu_x P_x(T_x^+ > T_{B(x,n)^c})$, $x \in X$, $n \geq 1$. Letting $n \to \infty$, we have $\rho(x)^{-1} = \mu_x P_x(T_x^+ = +\infty)$.
Since $\rho(x, 1)^{-1} = \mu_x$,

$$P_x(T_{B(x,n)^c} < T_x^+ < +\infty) = \mu_x^{-1}(\rho(x, n)^{-1} - \rho(x)^{-1})$$
$$\leq \mu_x(\rho(x) - \rho(x, n)).$$

Since $\mu_x \leq \sup_{y\in X} \deg(y) \sup_{y,z\in X, y\sim z} \mu_{yz} < +\infty$ and (X, μ) satisfies (U), we see that

$$\lim_{n\to\infty} \sup_{x\in X} P_x(T_{B(x,n)^c} < T_x^+ < +\infty) = 0. \tag{2.1}$$

Since $\sup_x \deg(x) < +\infty$ and $\sup_{y,z\in X, y\sim z} \mu_{yz} < +\infty$, we have that $\sup_{x\in X} V(x, n) < +\infty$, $n \geq 1$. Since $\rho(x, n)^{-1} \geq \inf_{y,z\in X, y\sim z} \mu_{yz}/n > 0$, we have that $\sup_{x\in X} \rho(x, n) < +\infty$, $n \geq 1$.
Thus we can let $f(n) = \sup_{x\in X} \rho(x, n) \sup_{x\in X} V(x, n)$, $n \geq 1$.
By Kumagai (2010) Lemma 3.3(v),

$$P_x(T_{B(x,n)^c} \geq nf(n)) \leq \frac{E_x[T_{B(x,n)^c}]}{nf(n)} \leq \frac{\rho(x, n)V(x, n)}{nf(n)} \leq \frac{1}{n}.$$

Hence,

$$\lim_{n\to\infty} \sup_{x\in X} P_x(T_{B(x,n)^c} \geq nf(n)) = 0. \tag{2.2}$$

We have that

$$P_x(nf(n) < T_x^+ < +\infty) \leq P_x(T_{B(x,n)^c} < T_x^+ < +\infty) + P_x(T_{B(x,n)^c} \geq nf(n)).$$

By noting (2.1) and (2.2), we have that

$$\lim_{n\to\infty} \sup_{x\in X} P_x(nf(n) < T_x^+ < +\infty) = 0.$$

This completes the proof of Lemma 2.1. □

Let $Y_{i,j}$ be the indicator function of $\{S_i \neq S_{i+k} \text{ for any } 1 \leq k \leq j\}$. Let $Y_{i,\infty}$ be the indicator function of $\{S_i \neq S_{i+k} \text{ for any } k \geq 1\}$.

Proof of Theorem 1.2: We show this assertion in a manner which is partially similar to the proof of Theorem 1 in Benjamini et al. (2007). However $P_x \neq P_y$ can happen for $x \neq y$ and hence the random variables $\{Y_{k+aM,M}\}_{a\in\mathbb{N}}$ are *not* necessarily independent. The details are different from the proof of Theorem 1 in Benjamini et al. (2007).

First, we will show (1.1). Let $\epsilon > 0$. Let M be a positive integer such that $\sup_{x\in X} P_x(M < T_x^+ < +\infty) < \epsilon/4$. We can take such M by Lemma 2.1.

By considering a last exit decomposition (as in Benjamini et al. (2007)),

$$R_n = 1 + \sum_{i=0}^{n-2} Y_{i,n-1-i} \leq M + \sum_{i=0}^{n-1-M} Y_{i,n-1-i} \leq M + \sum_{i=0}^{n-1-M} Y_{i,M}.$$

Hence for $n > 2M/\epsilon$,

$$P_x(R_n \geq n(1 - F_1 + \epsilon)) \leq P_x\left(\sum_{i=0}^{n-1-M} Y_{i,M} > n\left(1 - F_1 + \frac{\epsilon}{2}\right)\right)$$

$$= P_x\left(\sum_{a=0}^{M} \sum_{i\equiv a \bmod (M+1)} Y_{i,M} > n\left(1 - F_1 + \frac{\epsilon}{2}\right)\right)$$

$$\leq \sum_{a=0}^{M} P_x\left(\sum_{i\equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right).$$

Therefore it is sufficient to show that for each $a \in \{0, 1, \ldots, M\}$,

$$P_x\left(\sum_{i\equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) \to 0, \ n \to \infty, \text{ exponentially fast.}$$

(2.3)

For any $t > 0$, we have that

$$P_x\left(\sum_{i\equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right)$$

$$\leq \exp\left(-t\frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) E_x\left[\exp\left(t \sum_{i\equiv a \bmod (M+1)} Y_{i,M}\right)\right]. \quad (2.4)$$

By using the Markov property of $\{S_n\}_n$,

$$E_x\left[\exp\left(t \sum_{i\equiv a \bmod (M+1)} Y_{i,M}\right)\right] = E_x\left[\prod_{i\equiv a \bmod (M+1)} \exp(tY_{i,M})\right]$$

$$\leq \left(\sup_{y\in X} E_y[\exp(tY_{0,M})]\right)^{n/(M+1)}$$

$$= \left(1 + (\exp(t) - 1)\sup_{y\in X} P_y(T_y^+ > M)\right)^{n/(M+1)}.$$

By noting the definition of M and F_1,

$$\sup_{y \in X} P_y(T_y^+ > M) \leq \sup_{y \in X} P_y(M < T_y^+ < +\infty) + \sup_{y \in X} P_y(T_y^+ = +\infty)$$

$$\leq \frac{\epsilon}{4} + 1 - F_1.$$

Hence, for any $t \geq 0$ and $x \in X$,

$$E_x \left[\exp \left(t \sum_{i \equiv a \bmod (M+1)} Y_{i,M} \right) \right] \leq \left(1 + (\exp(t) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right)^{n/(M+1)}.$$

Hence, the right hand side of the inequality (2.4) is less than or equal to

$$\left[\exp \left(-t \left(1 - F_1 + \frac{\epsilon}{2} \right) \right) \left\{ 1 + (\exp(t) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right\} \right]^{n/(M+1)}.$$

It is easy to see that for sufficiently small $t_1 = t_1(F_1, \epsilon) > 0$,

$$\left\{ 1 + (\exp(t_1) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right\} < \exp \left(t_1 \left(1 - F_1 + \frac{\epsilon}{2} \right) \right).$$

Thus we have (2.3) and this convergence is uniform with respect to x. This completes the proof of (1.1).

Second, we will show (1.2). Let $\epsilon > 0$. Let M be a positive integer. By a last exit decomposition,

$$P_x(R_n \leq n(1 - F_2 - \epsilon)) = P_x(n - R_n \geq n(F_2 + \epsilon))$$

$$= P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,n-1-i}) \geq n(F_2 + \epsilon) \right)$$

$$\leq P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,\infty}) \geq n(F_2 + \epsilon) \right).$$

Now we have $1 - Y_{i,\infty} = 1 - Y_{i,M} + Y_{i,M} - Y_{i,\infty}$ and

$$P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,\infty}) \geq n(F_2 + \epsilon) \right) \leq P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,M}) \geq n \left(F_2 + \frac{\epsilon}{2} \right) \right)$$

$$+ P_x \left(\sum_{i=0}^{n-2} (Y_{i,M} - Y_{i,\infty}) \geq \frac{n\epsilon}{2} \right). \qquad (2.5)$$

We have that $Y_{i,M} - Y_{i,\infty}$ is the indicator function of

$$\{ S_i \neq S_{i+k} \text{ for any } 1 \leq k \leq M, \ S_i = S_{i+k} \text{ for some } k > M \},$$

and hence, $E_x[Y_{i,M} - Y_{i,\infty}] \leq \sup_{y \in X} P_y(M < T_y^+ < +\infty)$.

Then for any n,

$$P_x \left(\sum_{i=0}^{n-2} (Y_{i,M} - Y_{i,\infty}) \geq \frac{n\epsilon}{2} \right) \leq \frac{2}{n\epsilon} \sum_{i=0}^{n-2} E_x[Y_{i,M} - Y_{i,\infty}]$$

$$\leq \frac{2}{\epsilon} \sup_{y \in X} P_y(M < T_y^+ < +\infty). \qquad (2.6)$$

On the other hand,

$$P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,M}) \geq n \left(F_2 + \frac{\epsilon}{2} \right) \right)$$

$$\leq \sum_{a=0}^{M} P_x \left(\sum_{i \equiv a \bmod M+1} (1 - Y_{i,M}) \geq \frac{n}{M+1} \left(F_2 + \frac{\epsilon}{2} \right) \right).$$

By the Markov property of $\{S_n\}_n$, we have that for any $t > 0$ and any $a \in \{0, 1, \ldots, M\}$,

$$P_x \left(\sum_{i \equiv a \bmod M+1} (1 - Y_{i,M}) \geq \frac{n}{M+1} \left(F_2 + \frac{\epsilon}{2} \right) \right)$$

$$\leq \exp \left(-t \frac{n}{M+1} \left(F_2 + \frac{\epsilon}{2} \right) \right) E_x \left[\prod_{i \equiv a \bmod M+1} \exp(t(1 - Y_{i,M})) \right]$$

$$\leq \exp \left(-t \frac{n}{M+1} \left(F_2 + \frac{\epsilon}{2} \right) \right) \left(\sup_{y \in X} E_y \left[\exp(t(1 - Y_{0,M})) \right] \right)^{n/(M+1)}$$

$$= \left[\exp \left(-t \left(F_2 + \frac{\epsilon}{2} \right) \right) \left\{ 1 + (\exp(t) - 1) \sup_{y \in X} P_y(T_y^+ \leq M) \right\} \right]^{n/(M+1)}.$$

Since $\sup_{y \in X} P_y(T_y^+ \leq M) \leq F_2$, we have that for sufficiently small $t_2 = t_2(F_2, \epsilon) > 0$,

$$\exp \left(-t_2 \left(F_2 + \frac{\epsilon}{2} \right) \right) \left\{ 1 + (\exp(t_2) - 1) \sup_{y \in X} P_y(T_y^+ \leq M) \right\} < 1.$$

Therefore for any $a \in \{0, 1, \ldots M\}$,

$$P_x \left(\sum_{i \equiv a \bmod M+1} (1 - Y_{i,M}) \geq \frac{n}{M+1} \left(F_2 + \frac{\epsilon}{2} \right) \right) \to 0, n \to \infty.$$

Thus we see that

$$P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,M}) \geq n \left(F_2 + \frac{\epsilon}{2} \right) \right) \to 0, n \to \infty. \qquad (2.7)$$

This convergence is uniform with respect to x.

By using (2.5), (2.6) and (2.7), we have

$$\limsup_{n \to \infty} P_x(R_n \leq n(1 - F_2 - \epsilon)) \leq \frac{2}{\epsilon} \sup_{y \in X} P_y(M < T_y^+ < +\infty).$$

By letting $M \to \infty$, it follows from Lemma 2.1 that

$$\limsup_{n \to \infty} P_x(R_n \leq n(1 - F_2 - \epsilon)) = 0.$$

This convergence is uniform with respect to x. This completes the proof of (1.2). □

Remark 2.2. If $F_1 = F_2$, then (1.2) is easy to see by noting (1.1) and $E_x[R_n] \geq n(1 - F_2)$, $n \geq 1$, $x \in X$.

Corollary 2.3. *If $\sup_x P_x(M < T_x^+ < +\infty) = O(M^{-1-\delta})$ for some $\delta > 0$, then, certain strong laws hold. More precisely, for any $x \in X$,*

$$1 - F_2 \leq \liminf_{n \to \infty} \frac{R_n}{n} \leq \limsup_{n \to \infty} \frac{R_n}{n} \leq 1 - F_1, \ P_x\text{-}a.s.$$

Proof: By noting the Borel-Cantelli lemma, we see that it suffices to show that for any $x \in X$ and $\epsilon > 0$,

$$\sum_{n \geq 1} P_x(R_n \geq n(1 - F_1 + \epsilon)) < +\infty, \tag{2.8}$$

and,

$$\sum_{n \geq 1} P_x(R_n \leq n(1 - F_2 - \epsilon)) < +\infty. \tag{2.9}$$

(2.8) follows from that the convergence (1.1) is exponentially fast.

By noting (2.5), (2.6) and (2.7), we have that there exists $a = a(F_2, \epsilon) \in (0, 1)$ such that for any n and $M < n$,

$$P_x(R_n \leq n(1 - F_2 - \epsilon)) \leq \frac{2}{\epsilon}O(M^{-1-\delta}) + a^{n/(M+1)}.$$

If we let $M = n^{1-\delta/2} - 1$ for each n, then, we see (2.9). $\square$

Since the convergence in (1.1) is exponentially fast, we can extend Theorem 1 in Benjamini et al. (2007), which treats the range of the random walk bridge on vertex transitive graphs.

Corollary 2.4. *Let (X, μ) be an weighted graph satisfying (U). Let $x \in X$. We assume that $\limsup_{n \to \infty} P_x(S_{2n} = x)^{1/n} = 1$. Let $\epsilon > 0$. Then,*

$$\lim_{n \to \infty} P_x\left(R_n \geq n(1 - F_1 + \epsilon)|S_n = x\right) = 0.$$

The limit is taken on n such that $P_x(S_n = x) > 0$. This convergence is exponentially fast.

3. Proof of Theorem 1.3

To begin with, we state a very rough sketch of the proof.

Let N_1, N_2 be integers such that $3 < N_1 < N_2 < (N_1 - 1)^2$. First, we prepare a finite tree with degree N_1 and denote it $X^{(1)}$. Second, we surround $X^{(1)}$ with finite trees with degree N_2. We denote the graph we obtain by $X^{(2)}$. Third, we surround $X^{(2)}$ with finite trees with degree N_1. We denote the graph we obtain by $X^{(3)}$. Repeating this construction, we obtain an increasing sequence of finite trees $(X^{(n)})_n$. $X^{(2n+1)} \setminus X^{(2n)}$ (resp. $X^{(2n+2)} \setminus X^{(2n+1)}$) is a ring-like object consisting of the N_1 (resp. N_2) -trees. Let r_{2n+1} (resp. r_{2n+2}) be the "width" of the ring. Assume $r_i \ll r_{i+1}$ for any i. Let X be the infinite graph of the limit of $(X^{(n)})_n$. This satisfies (U), because N_1 and N_2 are not too far apart. Lemma 3.3 states this formally. X also satisfies $F_1 < F_2$ and (1.3), because $r_i \ll r_{i+1}$ for any i.

In this section, we assume that any weight is equal to 1, that is, $\mu_{xy} = 1$ for any $x \sim y$.

Let X be an infinite tree. For a connected subgraph Y of X, we denote the restriction of $\mathcal{E}$, deg, and ρ to Y by $\mathcal{E}_Y$, $\deg_Y$, and ρ_Y respectively. For a connected subgraph $Y \subset X$, we let $\text{diam}(Y) = \sup_{y_1, y_2 \in Y} d(y_1, y_2)$. Here d is the graph distance on X.

Let $x \in X$. Let

$$D_x(y) = \{z \in X : \text{the path between } x \text{ and } z \text{ contains } y\}, \ y \in X.$$

We remark that $y \in D_x(y)$ and $D_x(x) = X$. Let $I_x(y,n) = \rho_{D_x(y)}(y,n)^{-1}$, $y \in X$. We remark that $I_x(x,n) = \rho_X(x,n)^{-1} = \rho(x,n)^{-1}$. Then we have the following.

Lemma 3.1. *Let X be an infinite tree. Let $x, y \in X$. Let $n \geq 1$. Let y_i, $1 \leq i \leq \deg_{D_x(y)}(y)$, be the neighborhoods of y in $D_x(y)$. Then,*

$$I_x(y, n+1) = \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{1 + I_x(y_i, n)}.$$

Proof: Let $f : D_x(y) \to \mathbb{R}$ such that $f(y) = 1$ and $f = 0$ on $D_x(y) \backslash B_{D_x(y)}(y, n+1)$. Then, $f = 0$ on $D_x(y_i) \setminus B_{D_x(y_i)}(y, n)$ for any $1 \leq i \leq \deg_{D_x(y)}(y)$. Hence,

$$\mathcal{E}_{D_x(y)}(f, f) = \sum_{i=1}^{\deg_{D_x(y)}(y)} (1 - f(y_i))^2 + \mathcal{E}_{D_x(y_i)}(f, f)$$

$$\geq \sum_{i=1}^{\deg_{D_x(y)}(y)} (1 - f(y_i))^2 + f(y_i)^2 I_x(y_i, n) \geq \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{1 + I_x(y_i, n)}.$$

Thus we see that

$$I_x(y, n+1) \geq \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{1 + I_x(y_i, n)}.$$

Let $f_i : D_x(y_i) \to \mathbb{R}$ be a function such that $f_i(y_i) = 1$ and $f_i = 0$ on $D_x(y_i) \setminus B_{D_x(y_i)}(y_i, n)$, $1 \leq i \leq \deg_{D_x(y)}(y)$. Let $f : D_x(y) \to \mathbb{R}$ be the function defined by $f(y) = 1$ and $f = f_i/(1 + \mathcal{E}_{D_x(y_i)}(f_i, f_i))^{1/2}$ on $D_x(y_i)$. Then, $f = 0$ on $D_x(y) \setminus B_{D_x(y)}(y, n)$ and,

$$I_x(y, n+1) \leq \mathcal{E}_{D_x(y)}(f, f) = \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{\mathcal{E}_{D_x(y_i)}(f_i, f_i)}{1 + \mathcal{E}_{D_x(y_i)}(f_i, f_i)}.$$

Since each f_i is taken arbitrarily, we have

$$I_x(y, n+1) \leq \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{1 + I_x(y_i, n)}.$$

These complete the proof of Lemma 3.1.   □

Lemma 3.2. *Let $3 \leq N_1 < N_2$. Let X be an infinite tree such that $\deg(x) \in [N_1, N_2]$ for any $x \in X$. Then, $N_1 - 2 \leq I_x(y, n) \leq N_2$ for any $x, y \in X$ and any $n \geq 1$.*

Proof: We show this assertion by induction on n. If $n = 1$, then, by noting the definition of $D_x(y)$ and $I_x(y, 1)$, $I_x(y, 1) = \deg_{D_x(y)}(y) \in [N_1 - 1, N_2]$. Thus the assertion holds.

We assume that $N_1 - 2 \leq I_x(y, n) \leq N_2$ for any $x, y \in X$.

Let $x, y \in X$. Since $I_x(y, n+1) \leq I_x(y, n)$, we have $I_x(y, n+1) \leq N_2$. Let y_i, $1 \leq i \leq \deg_{D_x(y)}(y)$, be the neighborhoods of y in $D_x(y)$. By noting Lemma 3.1 and the assumption of induction,

$$I_x(y, n+1) = \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{I_x(y_i, n) + 1} \geq \deg_{D_x(y)}(y) \frac{N_1 - 2}{N_1 - 1} \geq N_1 - 2.$$

These complete the proof of Lemma 3.2. □

Lemma 3.3. *Let* $3 \leq N_1 < N_2 < (N_1 - 1)^2$. *Let* X *be an infinite tree such that* $\deg(x) \in [N_1, N_2]$ *for any* $x \in X$. *Then,* X *satisfies* (U).

Proof: By using Lemma 3.1 and Lemma 3.2, we have that for any $n, k \geq 1$ and any $x, y \in X$,

$$I_x(y, n+1) - I_x(y, n+k+1) = \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{I_x(y_i, n) + 1} - \frac{I_x(y_i, n+k)}{I_x(y_i, n+k) + 1}$$

$$\leq \frac{N_2}{(N_1 - 1)^2} \sup_{z \in X} \left(I_x(z, n) - I_x(z, n+k) \right).$$

Here y_i, $1 \leq i \leq \deg_{D_x(y)}(y)$, be the neighborhoods of y in $D_x(y)$.

Repeating this argument, we have that for any $n, k \geq 1$,

$$I_x(x, n) - I_x(x, n+k) \leq \left(\frac{N_2}{(N_1 - 1)^2} \right)^{n-1} \sup_{z \in X} (I_x(z, 1) - I_x(z, k+1))$$

$$\leq \left(\frac{N_2}{(N_1 - 1)^2} \right)^{n-1} N_2.$$

Since $N_2 < (N_1 - 1)^2$, $\rho_X(x, n)^{-1}$ converges uniformly to $\rho_X(x)^{-1}$, $n \to \infty$.

By Lemma 3.2, $\rho_X(x, n)^{-1} \geq N_1 - 2$ for any $n \geq 1$ and hence $\rho_X(x)^{-1} \geq N_1 - 2$. Therefore,

$$\rho_X(x) - \rho_X(x, n) = \rho_X(x)\rho_X(x, n)(\rho_X(x, n)^{-1} - \rho_X(x)^{-1})$$

$$\leq \frac{\rho_X(x, n)^{-1} - \rho_X(x)^{-1}}{(N_1 - 2)^2}.$$

Hence $\rho_X(x, n)$ converges uniformly to $\rho_X(x)$, $n \to \infty$. This completes the proof of Lemma 3.3. □

Let $N \geq 3$. Let T_N be the infinite N-regular tree. Let $\tilde{T}_N(o)$ be the infinite tree T such that $\deg(o) = N - 1$ for $o \in T$ and $\deg(x) = N$ for any $x \in T \setminus \{o\}$. For the simple random walk on T_N, we let $g_N = P_x(T_x^+ = +\infty)$ and $g_N(n) = P_x(T_x^+ > n)$ for some (or any) $x \in T_N$.

Definition 3.4. Let Y be a finite tree. Let $E(Y) = \{y \in Y : \deg(y) = 1\}$. Let $N \geq 3$. We define an infinite tree Y_N as follows : We prepare Y and $|E(Y)|$ copies of $\tilde{T}_N(o)$. Let Y_N be the infinite tree obtained by attaching $o \in \tilde{T}_N(o)$ to each $y \in E(Y)$.

Lemma 3.5. *Let* $N \geq 3$. *Let* Y *be a finite tree with a reference point* o *such that* $\deg(y) \geq 3$ *for any* $y \in Y \setminus E(Y)$. *Let* Y_N *be the infinite tree in Definition 3.4. We*

assume that Y_N satisfies (U). Let R_n be the range of the simple random walk up to time $n - 1$ on Y_N. Then,

$$\lim_{n \to \infty} \frac{E_o[R_n]}{n} = g_N.$$

Proof: By considering a last exit decomposition as in the proof of Theorem 1.2,

$$E_o[R_n] = 1 + \sum_{i=0}^{n-2} P_o \left(S_i \neq S_j \text{ for any } j \in \{i+1, \ldots, n-1\} \right)$$

$$= 1 + \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ > n - 1 - i)$$

$$= 1 + \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(n - 1 - i < T_y^+ < +\infty)$$

$$+ \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty).$$

Since Y_N satisfies (U),

$$\frac{1}{n} \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(n - 1 - i < T_y^+ < +\infty)$$

$$\leq \frac{1}{n} \sum_{i=0}^{n-2} \sup_{y \in Y_N} P_y(n - 1 - i < T_y^+ < +\infty) \to 0, \ n \to \infty.$$

Hence it is sufficient to show that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty) = g_N. \tag{3.1}$$

By the assumption, Y_N is an infinite tree such that $\deg(y) \geq 3$ for any $y \in Y_N$ and $\sup_{y \in Y_N} \deg(y) < +\infty$. Then, by Woess (2000) Example 3.8, Y_N is roughly isometric to the 3-regular tree T_3. Therefore Y_N is a transient graph.

Let $x, y \in Y_N$. Since $P_x(S_i = y)/\deg(y) = P_y(S_i = x)/\deg(x)$,

$$\frac{\deg(x)}{\deg(y)} P_x(S_i = y)^2 = P_x(S_i = y) P_y(S_i = x) \leq P_x(S_{2i} = x).$$

Therefore,

$$P_x(S_i = y) \to 0, \ i \to \infty, \text{ for any } x, y \in Y_N. \tag{3.2}$$

Let $\epsilon > 0$. Then, there exists a positive integer m_0 such that $g_N(m_0) \leq g_N + \epsilon/2$. By using the definition of Y_N and that the distribution of the random walk up to time $n - 1$ starting at $y \in Y_N$ is determined by $B_{Y_N}(y, n)$, we have that

$$P_y(T_y^+ > m_0) = g_N(m_0) \text{ for any } y \in Y_N \setminus B(o, 2(\mathrm{diam}(Y) + m_0)).$$

Hence $P_y(T_y^+ = +\infty) \leq g_N + \epsilon/2$ for any $y \in Y_N \setminus B(o, 2(\mathrm{diam}(Y) + m_0))$.

By (3.2), we have that $P_o(S_i \in B(o, 2(\mathrm{diam}(Y) + m_0))) \to 0, \ i \to \infty$. Hence there exists a positive integer n_0 such that $P_o(S_i \in B(o, 2(\mathrm{diam}(Y) + m_0))) \leq \epsilon/2$ for any $i \geq n_0$. Then, for any $n > n_0$,

$$\frac{1}{n} \sum_{i=n_0}^{n-1} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty)$$

$$\leq \frac{1}{n} \sum_{i=n_0}^{n-1} P_o(S_i \in B(o, 2(\mathrm{diam}(Y) + m_0))) + \sup_{y \notin B(o, 2(\mathrm{diam}(Y)+m_0))} P_y(T_y^+ = +\infty).$$

$$\leq \frac{n - n_0}{n} \frac{\epsilon}{2} + g_N + \frac{\epsilon}{2} \leq g_N + \epsilon.$$

We remark that

$$\frac{1}{n} \sum_{i=0}^{n_0 - 1} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty) \leq \frac{n_0}{n} \to 0, \ n \to \infty.$$

Since $\epsilon > 0$ is taken arbitrarily, we see that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty) \leq g_N. \tag{3.3}$$

Let $\epsilon > 0$. Since Y_N satisfies (U), there exists a positive integer m_1 such that $\sup_{y \in Y_N} P_y(m_1 < T_y^+ < +\infty) \leq \epsilon$. We have that for any $y \in Y_N \setminus B(o, 2(\mathrm{diam}(Y) + m_1)$, $P_y(T_y^+ > m_1) = g_N(m_1)$. Hence,

$$g_N \leq g_N(m_1) = P_y(T_y^+ = +\infty) + P_y(m_1 < T_y^+ < +\infty) \leq P_y(T_y^+ = +\infty) + \epsilon$$

for any $y \in Y_N \setminus B(o, 2(\mathrm{diam}(Y) + m_1))$.

By (3.2), there exists a positive integer n_1 such that $P_o(S_i \in B(o, 2(\mathrm{diam}(Y) + m_1))) \leq \epsilon$, for any $i \geq n_1$. Then,

$$\frac{1}{n} \sum_{i=0}^{n-1} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty)$$

$$\geq \frac{1}{n} \sum_{i=n_1}^{n-1} \sum_{y \notin B(o, 2(\mathrm{diam}(Y)+m_1))} P_o(S_i = y) P_y(T_y^+ = +\infty) \geq \frac{n - n_1}{n} (1 - \epsilon)(g_N - \epsilon).$$

By letting $n \to \infty$ and recalling that $\epsilon > 0$ is taken arbitrarily,

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ = +\infty) \geq g_N. \tag{3.4}$$

(3.3) and (3.4) imply (3.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Theorem 1.3: First, we will construct an increasing sequence of finite trees $(X^{(n)})_n$ by induction on n. Second, we will show that the limit infinite graph X of $(X^{(n)})_n$ satisfies (U), $F_1 < F_2$ and (1.3).

Let $3 \leq N_1 < N_2 < (N_1 - 1)^2$. Let $X^{(1)}$ be a finite tree such that $\deg(x) = N_1$ for any $x \in X^{(1)} \setminus E(X^{(1)})$ and $X^{(1)} = B(o, k_1)$ for a point $o \in X^{(1)}$ and a positive integer k_1.

We assume that $X^{(2n-1)}$ is constructed and $X^{(2n-1)} = B_{X^{(2n-1)}}(o, k_{2n-1})$ for a positive integer k_{2n-1}. By Lemma 3.5, there exists $k_{2n} > 2k_{2n-1}$ such that for the simple random walk on $(X^{(2n-1)})_{N_2}$ starting at o,

$$\frac{E_o[R_{k_{2n}}]}{k_{2n}} \geq g_{N_2} - \frac{1}{n}. \tag{3.5}$$

Then we let $X^{(2n)} = (X^{(2n-1)})_{N_2} \cap B_{(X^{(2n-1)})_{N_2}}(o, k_{2n})$.

We assume that $X^{(2n)}$ is constructed and $X^{(2n)} = B_{X^{(2n)}}(o, k_{2n})$ for a positive integer k_{2n}. By Lemma 3.5, there exists $k_{2n+1} > 2k_{2n}$ such that for the simple random walk on $(X^{(2n)})_{N_1}$ starting at o,

$$\frac{E_o[R_{k_{2n+1}}]}{k_{2n+1}} \leq g_{N_1} + \frac{1}{n}. \tag{3.6}$$

Then we let $X^{(2n+1)} = (X^{(2n)})_{N_1} \cap B_{(X^{(2n)})_{N_1}}(o, k_{2n+1})$.

Let X be the infinite graph obtained by the limit of a sequence of $(X^{(n)})$. Then $\deg_X(x) \in \{N_1, N_2\}$ and by Lemma 3.3 X satisfies (U).

Now we show (1.3). We remark that the distribution of the simple random walk up to time $k - 1$ on X starting at o is determined by $B_X(o, k)$, $k \geq 1$. By the definition of X, (3.5) and (3.6) hold also for the simple random walk on X. Hence,

$$\liminf_{n \to \infty} \frac{E_o[R_n]}{n} \leq g_{N_1}, \text{and,} \limsup_{n \to \infty} \frac{E_o[R_n]}{n} \geq g_{N_2}. \tag{3.7}$$

By considering a last exit decomposition as in the proof of Theorem 1.2, and, noting that X satisfies (U), we have

$$1 - F_2 = \inf_{x \in X} P_x(T_x^+ = +\infty) \leq \liminf_{n \to \infty} \frac{E_o[R_n]}{n}, \tag{3.8}$$

and,

$$\limsup_{n \to \infty} \frac{E_o[R_n]}{n} \leq \sup_{x \in X} P_x(T_x^+ = +\infty) = 1 - F_1. \tag{3.9}$$

In order to see (1.3), it is sufficient to show that for any $x \in X$,

$$g_{N_1} \leq P_x(T_x^+ = +\infty) \leq g_{N_2}. \tag{3.10}$$

Let $x \in X$. We recall that $\deg_X(x) = N_1$ or $\deg_X(x) = N_2$. Then we can assume that T_{N_1} is a subtree of X and X is a subtree of T_{N_2} and $x \in T_{N_1}$.

Assume $\deg_X(x) = N_1$. By using Kumagai (2010) Theorem 1.16 and that T_{N_1} is a subtree of X, we have

$$g_{N_1} = N_1^{-1} \rho_{T_{N_1}}(x)^{-1} \leq N_1^{-1} \rho_X(x)^{-1} = P_x(T_x^+ = +\infty).$$

Let x_i, $1 \leq i \leq N_2$, be the neighborhoods of x in T_{N_2} and $x_i \in T_{N_1}$ for $1 \leq i \leq N_1$. Let $f : T_{N_2} \to \mathbb{R}$ be a function such that $f(x) = 1$ and it has a compact support in T_{N_2}. Then, f has compact support also in $D_x(x_i)$ for each i. Here $D_x(x_i)$ is defined in T_{N_2}. Then,

$$\mathcal{E}_{T_{N_2}}(f, f) - \mathcal{E}_X(f, f) \geq \sum_{i=N_1+1}^{N_2} (1 - f(x_i))^2 + \mathcal{E}_{D_x(x_i)}(f, f)$$

$$\geq \sum_{i=N_1+1}^{N_2} (1 - f(x_i))^2 + f(x_i)^2 \rho_{D_x(x_i)}(x_i)^{-1}$$

$$\geq \sum_{i=N_1+1}^{N_2} \frac{\rho_{D_x(x_i)}(x_i)^{-1}}{1 + \rho_{D_x(x_i)}(x_i)^{-1}}.$$

Since $D_x(x_i)$ is graph isomorphic to $\tilde{T}_{N_2}(o)$, $\rho_{D_x(x_i)}(x_i) = \rho_{\tilde{T}_{N_2}(o)}(o)$. Hence,

$$\mathcal{E}_{T_{N_2}}(f,f) - \mathcal{E}_X(f,f) \geq (N_2 - N_1) \frac{\rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}.$$

Since $f|_X(x) = 1$ and $f|_X$ has compact support on X,

$$\mathcal{E}_{T_{N_2}}(f,f) - \rho_X(x)^{-1} \geq (N_2 - N_1) \frac{\rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}.$$

Since f is taken arbitrarily,

$$\rho_{T_{N_2}}(x)^{-1} - \rho_X(x)^{-1} \geq (N_2 - N_1) \frac{\rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}.$$

We see that $\rho_{T_{N_2}}(x)^{-1} = N_2 \dfrac{\rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_2}(o)}(o)^{-1}}$ in the same manner as in the proof of Lemma 3.1. Hence, $N_1 \rho_{T_{N_2}}(x)^{-1} \geq N_2 \rho_X(x)^{-1}$. By using Kumagai (2010) Theorem 1.16, we see that

$$P_x(T_x^+ = +\infty) = N_1^{-1} \rho_X(x)^{-1} \leq N_2^{-1} \rho_{T_{N_2}}(x)^{-1} = g_{N_2}.$$

Assume $\deg_X(x) = N_2$. We can show (3.10) in the same manner as above and sketch the proof.

By using Kumagai (2010) Theorem 1.16 and that X is a subtree of T_{N_2}, we have

$$P_x(T_x^+ = +\infty) = N_2^{-1} \rho_X(x)^{-1} \leq N_2^{-1} \rho_{T_{N_2}}(x)^{-1} = g_{N_2}.$$

Let x_i, $1 \leq i \leq N_2$, be the neighborhoods of x in X and $x_i \in T_{N_1}$ for $1 \leq i \leq N_1$. Let $f : X \to \mathbb{R}$ be a function such that $f(x) = 1$ and it has a compact support in X. Then, f has compact support also in $D_x(x_i)$ for each i. Here $D_x(x_i)$ is defined in X. Then,

$$\mathcal{E}_X(f,f) - \mathcal{E}_{T_{N_1}}(f,f) \geq \sum_{i=N_1+1}^{N_2} (1 - f(x_i))^2 + \mathcal{E}_{D_x(x_i)}(f,f)$$

$$\geq \sum_{i=N_1+1}^{N_2} \frac{\rho_{D_x(x_i)}(x_i)^{-1}}{1 + \rho_{D_x(x_i)}(x_i)^{-1}}.$$

We can regard $\tilde{T}_{N_1}(o)$ as a subtree of $D_x(x_i)$ and can assume $x_i = o$. Hence $\rho_{D_x(x_i)}(x_i)^{-1} \geq \rho_{\tilde{T}_{N_1}(o)}(o)^{-1}$ and

$$\mathcal{E}_X(f,f) - \mathcal{E}_{T_{N_1}}(f,f) \geq (N_2 - N_1) \frac{\rho_{\tilde{T}_{N_1}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_1}(o)}(o)^{-1}}.$$

Therefore,

$$\rho_X(x)^{-1} - \rho_{T_{N_1}}(x)^{-1} \geq (N_2 - N_1) \frac{\rho_{\tilde{T}_{N_1}(o)}(o)^{-1}}{1 + \rho_{\tilde{T}_{N_1}(o)}(o)^{-1}} = (N_2 - N_1) \frac{\rho_{T_{N_1}}(x)^{-1}}{N_1}.$$

and then we have $N_1 \rho_X(x)^{-1} \geq N_2 \rho_{T_{N_1}}(x)^{-1}$. By using Kumagai (2010) Theorem 1.16, we see that

$$g_{N_1} = N_1^{-1} \rho_{T_{N_1}}(x)^{-1} \leq N_2^{-1} \rho_X(x)^{-1} = P_x(T_x^+ = +\infty).$$

Thus the proof of (3.10) completes and we obtain (1.3).

By using Woess (2000) Lemma 1.24 and $N_1 < N_2$, we see that $g_{N_1} = (N_1 - 2)/(N_1 - 1) < g_{N_2} = (N_2 - 2)/(N_2 - 1)$. By using (3.7), (3.8), (3.9) and (3.10), we see $g_{N_1} = 1 - F_2$ and $g_{N_2} = 1 - F_1$. Hence $F_1 < F_2$.

Thus we see that X satisfies (U), $F_1 < F_2$, and, (1.3). $\square$

4. Examples of graphs satisfying the uniform condition

In this section, we give some examples of graphs satisfying (U). We assume that all weights are equal to 1.

Here we follow Kumagai (2010) Definition 1.8 for the definition of rough isometry introduced by Kanai (1985).

Definition 4.1. Let X_i be weighted graphs and d_i be the graph metric of X_i, $i = 1, 2$. We say that a map $T : X_1 \to X_2$ is a $((A, B, M)\text{-})rough\ isometry$ if there exist constants $A > 1$, $B > 0$, and, $M > 0$ satisfying the following inequalities.

$$A^{-1}d_1(x, y) - B \le d_2(T(x), T(y)) \le Ad_1(x, y) + B, \ x, y \in X_1.$$

$$d_2(T(X_1), z) \le M, \ z \in X_2.$$

We say that X_1 is *roughly isometric* to X_2 if there exists a rough isometry between them. We say that a property is *stable under rough isometry* if whenever X_1 satisfies the property and is roughly isometric to X_2, then X_2 also satisfies the property.

4.1. *Recurrent graphs.*

Proposition 4.2. *The condition (U) is stable under rough isometry between recurrent graphs.*

Proof: Assume that X_1 is a recurrent graph satisfying (U) and X_2 is a (recurrent) graph which is roughly isometric to X_1. We would like to show that X_2 satisfies (U).

Since rough isometry is an equivalence relation, there exists a (A, B, M)-rough isometry $T : X_2 \to X_1$. Fix $n \in \mathbb{N}$ and $x \in X_2$. Let f be a function on X_1 such that $f(T(x)) = 1$ and $f = 0$ on $X_1 \setminus B(T(x), A^{-1}n - B)$. Since T is a (A, B, M)-rough isometry, we have that for any $y \in X_2 \setminus B(x, n)$, $T(y) \in X_1 \setminus B(T(x), A^{-1}n - B)$, and hence, $f \circ T = 0$ on $X_2 \setminus B(x, n)$.

By using Theorem 3.10 in Woess (2000), we see that there exists a constant $c > 0$ such that $\mathcal{E}_{X_1}(f, f) \ge c\mathcal{E}_{X_2}(f \circ T, f \circ T)$. This constant does not depend on (x, n, f). Therefore,

$$\inf \left\{ \mathcal{E}_{X_1}(f, f) : f(T(x)) = 1, f = 0 \text{ on } X_1 \setminus B(T(x), A^{-1}n - B) \right\}$$

$$\ge c \inf \left\{ \mathcal{E}_{X_2}(g, g) : g(x) = 1, g = 0 \text{ on } X_2 \setminus B(x, n) \right\}.$$

Hence, $\rho_{X_2}(x, n) \ge c\rho_{X_1}(T(x), A^{-1}n - B)$. By recalling that X_1 satisfies (U), we see that X_2 satisfies (U). $\square$

Proposition 4.3. *Let X be a graph such that there exists $C > 0$ such that $V(x, n) \le Cn^2$ for any $x \in X$ and $n \ge 1$. Let X' be a graph which is roughly isometric to X. Then, X and X' satisfy (U).*

We can show the above assertion in the same manner as in the proof of Woess (2000), Lemma 3.12 and Lemma 3.13, so we omit the proof.

Proposition 4.4. *Let X be a graph such that*

$$\lim_{n \to \infty} \inf_{x \in X} \sum_{k=0}^{n} p_k(x, x) = +\infty. \tag{4.1}$$

Let X' be a graph which is roughly isometric to X. Then, X and X' satisfy (U).

Proof: By noting Kumagai (2010) Lemma 3.3(iv), we see that $\rho(x, n) = g^{B(x,n)}(x, x)$, $x \in X$, $n \geq 1$. Since $p_k^{B(x,n)}(x, x) = p_k(x, x)$ for $k < n$,

$$\rho(x, n) = g^{B(x,n)}(x, x) \geq \sum_{0 \leq k < n} p_k(x, x).$$

By noting (4.1), we see X satisfies (U). Since X is recurrent, it follows from Proposition 4.2 that X' also satisfies (U). □

By using Section 5 in Barlow et al. (2005), we see that the d-dimensional standard graphical Sierpiński gaskets, $d \geq 2$, and Vicsek trees (See Barlow (2004) for definition) satisfies (4.1). Thus we have

Example 4.5. The graphs which are roughly isometric with the following graphs satisfy (U).
(i) Infinite connected subgraphs in $\mathbb{Z}^2$.
(ii) Infinite connected subgraphs in the planer triangular lattice.
(iii) The d-dimensional standard graphical Sierpiński gaskets, $d \geq 2$.
(iv) Vicsek trees.

4.2. *Transient graphs.*

Proposition 4.6. *Assume that a graph X satisfies (UC_α), $\alpha > 2$, that is, there exist $C > 0$ such that $\sup_{x \in X} p_n(x, x) \leq Cn^{-\alpha/2}$, $n \geq 1$. Let X' be a graph which is roughly isometric to X. Then, X and X' satisfy (U).*

Proof: Let $m > n$. Then, by using Kumagai (2010) Lemma 3.3(iv) and $p_k^{B(x,m)}(x, x) = p_k^{B(x,n)}(x, x) = p_k(x, x)$ for $k < n$,

$$\begin{aligned}
\rho(x, m) - \rho(x, n) &= g^{B(x,m)}(x, x) - g^{B(x,n)}(x, x) \\
&= \sum_{k \geq n} (p_k^{B(x,m)}(x, x) - p_k^{B(x,n)}(x, x)) \\
&\leq \sum_{k \geq n} p_k(x, x).
\end{aligned}$$

Letting $m \to \infty$,

$$\rho(x) - \rho(x, n) \leq \sum_{k \geq n} p_k(x, x), x \in X, n \geq 1.$$

Thus we see that if X satisfies (UC_α) for some $\alpha > 2$, then X satisfies (U).

The stability of the property (UC_α), $\alpha > 2$, under rough isometry follows from Varopoulos (1985) Theorem 1 and 2, and, Kanai (1986) Proposition 2.1. Thus we see that X' also satisfies (U). □

$\mathbb{Z}^d$ satisfies (UC_d). By using Barlow and Bass (1999a,b), we see that if $d \geq 3$, then d-dimensional standard graphical Sierpiński carpet satisfies (UC_α) for some $\alpha > 2$. Therefore we have

Example 4.7. The graphs which are roughly isometric with the following graphs satisfy (U).
(i) $\mathbb{Z}^d$, $d \geq 3$.
(ii) d-dimensional standard graphical Sierpiński carpet, $d \geq 3$.

4.3. *A graph which does not satisfy* (U). Finally, we give an example of a graph which does not satisfy (U).

Remark 4.8. The recurrent tree T treated in Woess (2000), Example 6.16 does not satisfy (U). For any $n \geq 1$, there exists $x_n \in T$ such that $\rho(x_n, n) = \rho_{T_4}(x_n, n)$, where T_4 is the 4-regular tree. Since T_4 is vertex transitive and transient, we have that $\rho_{T_4}(x_n, n) \leq \rho_{T_4}(x_n) = \rho_{T_4}(o) < +\infty$, $n \geq 1$, for a reference point $o \in T_4$. However, T is recurrent and hence $\rho(x, n) \to \infty$, $n \to \infty$, $x \in T$. Thus we see that T does not satisfy (U).

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