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On the recurrence of some random walks in random environment

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Abstract. This work is motivated by the study of some two-dimensional random walks in random environment (RWRE) with transition probabilities independent of one coordinate of the walk. These are non-reversible models and can not be treated by electrical network techniques. The proof of the recurrence of such RWRE needs new estimates for quenched return probabilities of a one-dimensional recurrent RWRE. We obtained these estimates by constructing suitable valleys for the potential. They imply that k independent walkers in the same one-dimensional (recurrent) environment will meet in the origin infinitely often, for any k. We also consider direct products of one-dimensional recurrent RWRE with another RWRE or with a RW. We point out that models involving one-dimensional recurrent RWRE are more recurrent than the corresponding models involving simple symmetric walk.

1. Introduction

Since the early works of Solomon Solomon (1975) and Sinai Sinaĭ (1982) (see also Kesten (1986) and Golosov (1984)), one-dimensional random walks in random environment (RWRE) have been studied by many authors. For an introduction

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to this model, we refer to Zeitouni (2004). In the present work, we consider a one-dimensional RWRE $(X_n)_n$ with random environment given by a sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ of independent identically distributed (i.i.d.) random variables with values in $(0,1)$ defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $z \in \mathbb{Z}$. Given ω, under P_ω^z, $(X_n)_{n\geq 0}$ is a Markov chain such that $P_\omega^z(X_0 = z) = 1$ and with the following transition probabilities

$$P_\omega^z(X_{n+1} = x+1 | X_n = x) = \omega_x = 1 - P_\omega^z(X_{n+1} = x-1 | X_n = x). \tag{1.1}$$

For $i \in \mathbb{Z}$ we define $\rho_i = \rho_i(\omega) := \frac{1-\omega_i}{\omega_i}$ and we assume throughout the paper that

$$\mathsf{E}[\log \rho_0] = 0, \quad \mathsf{Var}(\log \rho_0) > 0, \tag{1.2}$$

$$\mathsf{P}(\varepsilon \leq \omega_0 \leq 1-\varepsilon) = 1 \text{ for some } \varepsilon \in \left(0, \tfrac{1}{2}\right). \tag{1.3}$$

The first part of (1.2) ensures that the RWRE is recurrent for P-a.e. ω, its second part excludes the case of a deterministic environment. Such RWREs are often called “Sinai’s walk” due to the results in Sinaĭ (1982). Assumption (1.3) (called uniform ellipticity) is a common technical condition in the context of RWRE. Our main results on the one-dimensional RWRE $(X_n)_n$ are the following. We write P_ω for P_ω^0.

Theorem 1.1. *For $0 \leq \alpha < 1$ and for P-a.e. ω, we have*

$$\sum_{n\in\mathbb{N}} P_\omega(X_{2n} = 0) \cdot n^{-\alpha} = \infty. \tag{1.4}$$

Theorem 1.2. *For all $\alpha > 0$ and for P-a.e. ω, we have*

$$\sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n} = 0)\Big)^\alpha = \infty. \tag{1.5}$$

In particular, d independent particles performing recurrent RWRE in the same environment (and starting from the origin) are meeting in the origin infinitely often, almost surely.

Remark 1.3. It was shown in Gallesco (2013) that d independent particles in the same environment meet infinitely often, and the tail of the meeting time was investigated (more precisely, the random walks considered in Gallesco (2013) are on $\mathbb{Z}^+$). We show here that the d particles even meet infinitely often in the origin.

For the next statement, we consider d independent environments.

Corollary 1.4. *For $d \in \mathbb{N}$, consider d i.i.d. random environments $\omega^{(1)}$, $\omega^{(2)}, \ldots$, $\omega^{(d)}$ fulfilling (1.2) and (1.3). Then, for $\mathsf{P}^{\otimes d}$-a.e. $(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)})$, we have*

$$\sum_{n\in\mathbb{N}} \prod_{k=1}^{d} P_{\omega^{(k)}}(X_{2n} = 0) = \infty. \tag{1.6}$$

In particular, d independent particles performing recurrent RWRE in i.i.d. environments (and starting from the origin) are meeting in the origin infinitely often, almost surely.

We point out that a proof of Corollary 1.4 can also be found in Zeitouni (2004) after Lemma A.2. The proof there uses the Nash-Williams inequality in the context of electrical networks.

In Comets and Popov (2003), Comets and Popov also consider the return probabilities of the one-dimensional recurrent RWRE on $\mathbb{Z}$. In contrast to our setting, they consider the corresponding jump process in continuous time $(\xi_t^z)_{t\geq 0}$ started at $z \in \mathbb{Z}$ and with jump rates $(\omega_x^+, \omega_x^-)_{x\in\mathbb{Z}}$ to the right and left neighbouring sites. One advantage of this process in continuous time is that it is not periodic as the RWRE in discrete time. They show the following (under two conditions on the environment $(\omega_x^+, \omega_x^-)_{x\in\mathbb{Z}}$):

Theorem (cf. Corollary 2.1 and Theorem 2.2 in Comets and Popov (2003)) *We have* $\frac{\log P_\omega(\xi_t^0=0)}{\log t} \xrightarrow{t\to\infty} -\widehat{a}_e$ *in distribution where* $\widehat{a}_e$ *has the density* f *given by* $f(z) = 2 - z - (z+2)\cdot e^{-2z}$ *if* $z \in (0,1)$ *and* $f(z) = ([e^2-1]\cdot z - 2)\cdot e^{-2z}$ *if* $z \geq 1$.

Since we can embed the recurrent RWRE $(X_n)_{n\in\mathbb{N}_0}$ in discrete-time into the corresponding jump process in continuous time, we can expect the return probabilities to behave similarly as in the continuous setting. In particular, for P-a.e. environment ω, we expect

$$P_\omega(X_{2n}=0) =: n^{-a(\omega,n)} \quad \text{with} \quad \liminf_{n\to\infty} a(\omega,n) = 0,\ \limsup_{n\to\infty} a(\omega,n) = \infty.$$

Theorem 1.1, 1.2 and Corollary 1.4 allow us to establish the recurrence of the multidimensional RWRE $(M_n)_n$ in the cases (I)-(III) below. Except model (I) (which is the direct product of $(X_n)_n$ with a RW), the models considered here are 2-dimensional RWRE with transition probabilities independent of the vertical position of the walk. The study of such models was initiated by Matheron and de Marsily in Matheron and de Marsily (1984) to modelise transport in a stratified porus medium (see also Bouchaud et al. (1990)) and also by Campanino and Petritis in Campanino and Petritis (2003).

Let $\delta \in (0,1)$. We establish recurrence of the RWRE $(M_n)_n$ on $\mathbb{Z}^2$ in the three following cases:

(I) $d=2$ and $(M_n)_n$ is the direct product of the Sinai walk $(X_n)_n$ and of some recurrent random walk on $\mathbb{Z}$; more precisely

$$P_\omega(M_{n+1} = (x+1, y+z) | M_n = (x,y)) = \omega_x \cdot \nu(\{z\})$$
$$\text{and} \quad P_\omega(M_{n+1} = (x-1, y+z) | M_n = (x,y)) = (1-\omega_x)\cdot \nu(\{z\}),$$

where ν is a distribution on $\mathbb{Z}$ (with zero expectation) belonging to the domain of attraction of a β-stable random variable with $\beta \in (1,2]$.

(II) $d=2$ and $(M_n)_n$ either moves horizontally with respect to the Sinai walk (with probability δ) or moves vertically with respect to some recurrent random walk (with probability $1-\delta$):

$$P_\omega(M_{n+1} = (x+1, y) | M_n = (x,y)) = \delta\omega_x = \delta - P_\omega(M_{n+1} = (x-1,y)|M_n = (x,y)),$$
$$P_\omega(M_{n+1} = (x, y+z) | M_n = (x,y)) = (1-\delta)\cdot\nu(z),$$

where ν is a probability distribution on $\mathbb{Z}$ (with zero expectation) belonging to the domain of attraction of a β-stable distribution with $\beta \in (1,2]$.

(III) An odd-even oriented model: $d=2$ and $(M_n)_n$ either moves horizontally with respect to Sinai's walk (with probability δ) or moves vertically (with probability $1-\delta$) with respect to ν if the first coordinate of the current position of the walk is even and to $\tilde{\nu} := \nu(-\cdot)$ otherwise; i.e.

$$P_\omega(M_{n+1} = (x+1, y) | M_n = (x,y)) = \delta\omega_x = \delta - P_\omega(M_{n+1} = (x-1,y)|M_n = (x,y)),$$
$$P_\omega(M_{n+1} = (x, y+z) | M_n = (x,y)) = (1-\delta)\nu((-1)^x z),$$

where ν is a probability distribution on $\mathbb{Z}$ (admitting a first moment) belonging to the domain of attraction of a stable distribution.

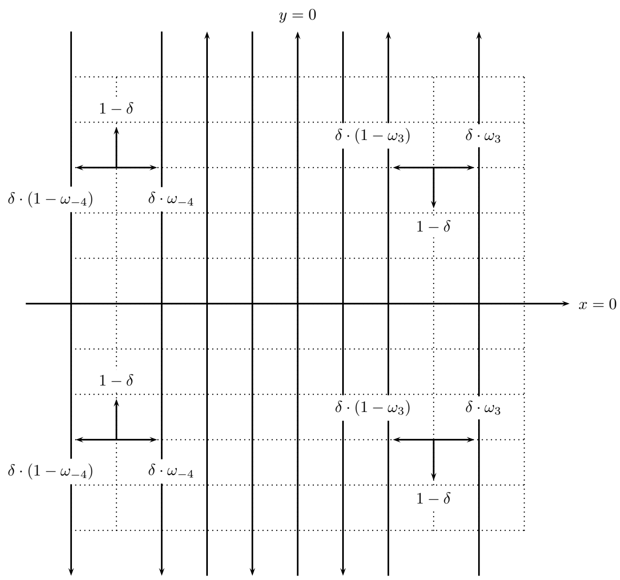

FIGURE 1.1. Transition probabilities in case (III) in the particular case where $\nu = \delta_1$. This is an example of an oriented RWRE. Every even vertical line is oriented upward and every odd vertical line is oriented downward.

If ω_x is replaced by $1/2$ (i.e. if we replace Sinai's walk by the simple symmetric walk), the walks given in (I)-(III) are transient when ν is in the domain of attraction of a β-stable distribution with $\beta < 2$. Hence, in this study, Sinai's walk gives rise to more recurrent models than the simple symmetric random walk does.

The structure of our paper is the following: In Section 2, we introduce the potential of the one-dimensional RWRE and we recall some known results. Section 3 contains the proofs of our main results for one-dimensional RWRE. In Section 4, we state our recurrence results for multidimensional RWRE involving the RWRE $(X_n)_n$ (models (I)-(III)) and we compare our results with the case when $(X_n)_n$ is replaced by a simple random walk.

2. Preliminaries

As usual, we use P_ω^o instead of P_ω^0 and will even drop the superscript o where no confusion is to be expected. We can now define the potential V as

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{for } x = 1, 2, \dots \\ 0 & \text{for } x = 0 \\ \sum_{i=x+1}^{0} -\log \rho_i & \text{for } x = -1, -2, \dots \ . \end{cases} \tag{2.1}$$

Note that $V(x)$ is a sum of i.i.d. random variables which are centered and whose absolute value is bounded due to assumptions (1.2) and (1.3). One of the crucial facts for the RWRE is that, for fixed ω, the random walk is a reversible Markov chain and can therefore be described as an electrical network. The conductances are given by $C_{(x,x+1)} = e^{-V(x)}$ and the stationary reversible measure which is unique up to multiplication by a constant is given by

$$\mu_\omega(x) = e^{-V(x)} + e^{-V(x-1)} \tag{2.2}$$

The reversibility means that, for all $n \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$, we have

$$\mu_\omega(x) \cdot P_\omega^x(X_n = y) = \mu_\omega(y) \cdot P_\omega^y(X_n = x). \tag{2.3}$$

For the random time of the first arrival in x

$$\tau(x) := \inf\{n \geq 0 : \ X_n = x\}, \tag{2.4}$$

the interpretation of the RWRE $(X_n)_n$ as an electrical network helps us to compute the following probability for $x < y < z$ (for a proof see for example formula (2.1.4) in Zeitouni (2004)):

$$P_\omega^y(\tau(z) < \tau(x)) = \frac{\sum_{j=x}^{y-1} e^{V(j)}}{\sum_{j=x}^{z-1} e^{V(j)}}. \tag{2.5}$$

Further (cf. (2.4) and (2.5) in Shi and Zindy (2007) and Lemma 7 in Golosov (1984)), we have for $k \in \mathbb{N}$ and $y < z$

$$P_\omega^y(\tau(z) < k) \leq k \cdot \exp\left(-\max_{y \leq i < z}\big[V(z-1) - V(i)\big]\right) \tag{2.6}$$

and similarly for $x < y$

$$P_\omega^y(\tau(x) < k) \leq k \cdot \exp\left(-\max_{x < i \leq y}\big[V(x+1) - V(i)\big]\right). \tag{2.7}$$

To get bounds for large values of $\tau(\cdot)$, we can use that for $x < y < z$ we have (cf. Lemma 2.1 in Shi and Zindy (2007))

$$E_\omega^y[\tau(z) \cdot \mathbf{1}_{\{\tau(z) < \tau(x)\}}] \leq (z-x)^2 \cdot \exp\left(\max_{x \leq i \leq j \leq z}\big(V(j) - V(i)\big)\right). \tag{2.8}$$

Further, the Komlós-Major-Tusnády strong approximation theorem (cf. Theorem 1 in Komlós et al. (1975), see also formula (2) in Comets and Popov (2003)) will help us to compare the shape of the potential with the path of a two-sided Brownian motion:

Theorem 2.1. *In a possibly enlarged probability space, there exists a version of our environment process ω and a two-sided Brownian motion $(B(t))_{t \in \mathbb{R}}$ with diffusion constant $\sigma := (\mathsf{Var}(\log \rho_0))^{\frac{1}{2}}$ (i.e. $Var(B(t)) = \sigma^2|t|$) such that for some $K > 0$ we have*

$$\mathsf{P}\left(\limsup_{x \to \pm\infty} \frac{|V(x) - B(x)|}{\log|x|} \leq K\right) = 1. \tag{2.9}$$

3. Dimension 1 : Proofs of Theorem 1.1, 1.2 and Corollary 1.4

For $L \in \mathbb{N}$ and $0 < \delta < 1$, we introduce the set $\Gamma(L, \delta)$ of environments defined by

$$\Gamma(L,\delta) := \{R_1^{\pm}(L) \le \delta L,\, R_2^{\pm}(L) \le \delta L,\, T^{\pm}(L) \le L^2\},$$

where

$$T^+(L) := \inf\{z \ge 0 :\ V(z) - \min_{0\le y\le z} V(y) \ge L\},$$

$$T^-(L) := \sup\{z \le 0 :\ V(z) - \min_{n\le y\le z} V(y) \ge L\},$$

$$R_1^+(L) := -\min_{0\le y\le T^+(L)} V(y), \quad R_1^-(L) := -\min_{T^-(L)\le y\le 0} V(y),$$

$$T_b^+(L) := \inf\{z \ge 0 :\ V(z) = -R_1^+(L)\}, \quad T_b^-(L) := \sup\{z \le 0 :\ V(z) = -R_1^-(L)\},$$

$$R_2^+(L) := \max_{0\le y\le T_b^+(L)} V(y), \quad R_2^-(L) := \max_{T_b^-(L)\le y\le 0} V(y).$$

We then consider the valley of the potential V between $T^-(L)$ and $T^+(L)$ (see Figure 3.2). Here, the $+$-sign and the $-$-sign indicate whether we deal with properties of the valley on the positive or negative half-line, respectively. Note that the definition of the set $\Gamma(L, \delta)$ is compatible with the scaling of a Brownian motion in space and time.

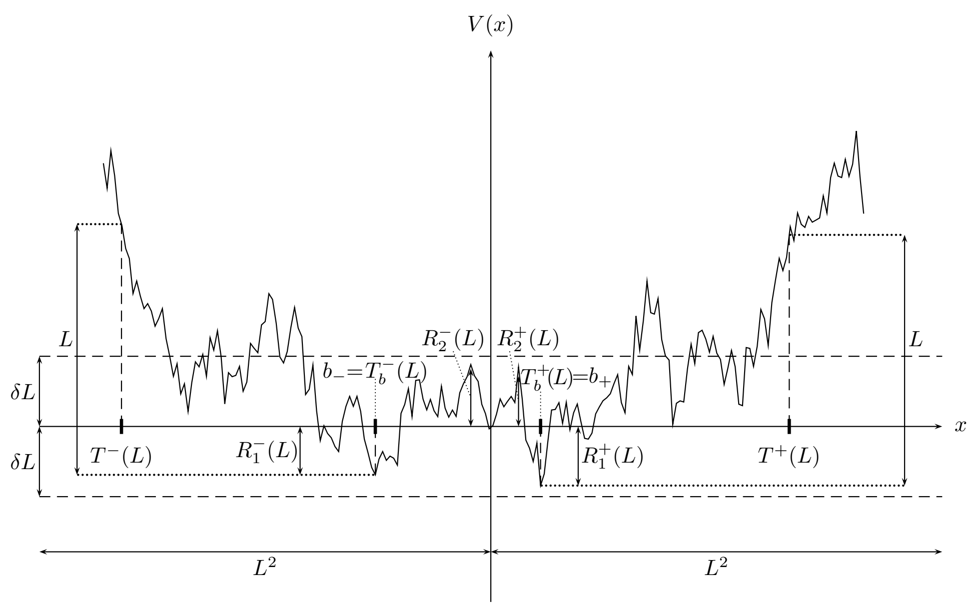

Figure 3.2. Shape of a valley of an environment in $\Gamma(L, \delta)$

Remark 3.1. We have constructed the valleys in such a way that the return probability of the random walk to the origin is bounded from below (for even time points) as long as the random walk has not left the valley. If $\omega \in \Gamma(L, \delta)$, the random walk $(X_n)_{n\in\mathbb{N}_0}$ in the environment ω satisfies the following:

(1) Since we have $V(T^-(L)) - V(T_b^-(L)) \geq L$ and $V(T^+(L)) - V(T_b^+(L)) \geq L$, the random walk $(X_n)_{n\in\mathbb{N}_0}$ stays within $\{T^-(L), T^-(L)+1, \ldots, T^+(L)\}$ with high probability for at least $\exp((1-2\delta)L)$ steps (cf. (3.8)).
(2) Within the valley $\{T^-(L), T^-(L)+1, \ldots, T^+(L)\}$, the random walk prefers to stay at positions x with a small potential $V(x)$, i.e. at positions close to the bottom points $T_b^-(L)$ and $T_b^+(L)$.
(3) The return probability for the random walk from the bottom points $T_b^-(L)$ and $T_b^+(L)$ to the origin is mainly given by the potential differences $R_2^-(L) + R_1^-(L) \leq 2\delta L$ and $R_2^+(L) + R_1^+(L) \leq 2\delta L$ respectively, i.e. by the height of the potential the random walk has to overcome from the bottom points back to the origin (cf. (3.3)).

Proposition 3.2. *For every $\delta \in (0, \frac{1}{5})$, there exists $C = C(\delta)$ such that, for every L, for every $\omega \in \Gamma(L,\delta)$ and every n satisfying $e^{3\delta L} \leq n \leq e^{(1-2\delta)L}$, we have*

$$P_\omega^o(X_{2n} = 0) \geq C \cdot \exp(-3\delta L). \tag{3.1}$$

Proof of Proposition 3.2: The return probability to the origin for the time points of interest is mainly influenced by the shape of the "valley" of the environment ω between $T^-(L)$ and $T^+(L)$. For the positions of the two deepest bottom points of this valley on the positive and negative side, we write $b_\pm := T_b^\pm(L)$ and we assume for the following proof that we have (cf. (2.4) for the definition of $\tau(\cdot)$)

$$P_\omega^o\big(\tau(b_+) < \tau(b_-)\big) \geq \frac{1}{2}. \tag{3.2}$$

(Due to the symmetry of the RWRE, the proof also works in the opposite case if we switch the roles of b_+ and b_-). We have

$$\begin{aligned}
&P_\omega^o(X_{2n} = 0) \geq P_\omega^o\left(X_{2n} = 0,\ \tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \\
&\geq P_\omega^o\left(\tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \cdot \widehat{\inf}_{\ell\in\left\{\left\lceil\frac{4n}{3}\right\rceil,\ldots,2n\right\}} P_\omega^{b_+}(X_\ell = 0) \\
&= P_\omega^o\left(\tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \cdot \frac{\mu_\omega(0)}{\mu_\omega(b_+)} \cdot \widehat{\inf}_{\ell\in\left\{\left\lceil\frac{4n}{3}\right\rceil,\ldots,2n\right\}} P_\omega^o(X_\ell = b_+)
\end{aligned} \tag{3.3}$$

where we used (2.3) in the third step and with the short notation

$$\widehat{\inf}_{\ell\in\left\{\left\lceil\frac{4n}{3}\right\rceil,\ldots,2n\right\}} P_\omega^x(X_\ell = y) := \inf_{\ell\in\left\{\left\lceil\frac{4n}{3}\right\rceil,\ldots,2n\right\}\cap\left(2\mathbb{Z}+(x+y)\right)} P_\omega^x(X_\ell = y).$$

Let us now have a closer look at the factors in the lower bound in (3.3) separately:

First factor in (3.3): We can bound the first factor from below by

$$\begin{aligned}
&P_\omega^o\left(\tau(b_+) \le \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right)\\
&= 1 - P_\omega^o\left(\tau(b_+) > \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) - P_\omega^o\big(\tau(b_+) \ge \tau(b_-)\big)\\
&\ge 1 - \tfrac{3}{2n} \cdot E_\omega^o\left[\tau(b_+) \cdot \mathbf{1}_{\{\tau(b_+)<\tau(b_-)\}}\right] - P_\omega^o\big(\tau(b_+) \ge \tau(b_-)\big)\\
&\ge 1 - \tfrac{3}{2n} \cdot (b_+ - b_-)^2 \cdot \exp\left(\max_{b_- \le i \le j \le b_+} \big(V(j) - V(i)\big)\right) - \frac{1}{2}\ ,
\end{aligned}$$

where we used (2.8) and assumption (3.2) for the last step. Therefore, we get for $\omega \in \Gamma(L, \delta)$ and $\exp(3\delta L) \le n$ that

$$\begin{aligned}
P_\omega^o\left(\tau(b_+) \le \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) &\ge \frac{1}{2} - \frac{3 \cdot 4 \cdot L^4}{2 \cdot \exp(3\delta L)} \cdot \exp(2\delta L) &(3.4)\\
&\ge \frac{1}{2} - 6 \cdot L^4 \cdot \exp(-\delta L). &(3.5)
\end{aligned}$$

Second factor in (3.3): Due to Assumption (1.3) and to (2.2), we get for $\omega \in \Gamma(L, \delta)$:

$$\begin{aligned}
&\frac{\mu_\omega(0)}{\mu_\omega(b_+)} = \frac{\frac{1}{\omega_0}}{e^{-V(b_+)} + e^{-V(b_+-1)}} = \frac{\frac{1}{\omega_0}}{e^{-V(b_+)} \cdot (1 + \rho_{b_+})}\\
&\ge \frac{\frac{1}{1-\varepsilon}}{1 + \frac{1-\varepsilon}{\varepsilon}} \cdot e^{V(b_+)} = \frac{\varepsilon}{1-\varepsilon} \cdot e^{V(b_+)} \ge \frac{\varepsilon}{1-\varepsilon} \cdot \exp(-\delta L). \qquad (3.6)
\end{aligned}$$

Here we used that $V(b_+) \ge -\delta L$ holds for $\omega \in \Gamma(L, \delta)$.

Third factor in (3.3): For the last factor in (3.3), we can compare the RWRE with the process $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which behaves as the original RWRE but is reflected at the positions $T^- := T^-(L)$ and $T^+ := T^+(L)$, i.e. we have for $x \in \{T^-, T^- + 1, \dots, T^+\}$

$$\begin{aligned}
&P_\omega^x(\widetilde{X}_0 = x) = 1,\\
&P_\omega^x(\widetilde{X}_{n+1} = y \pm 1 | \widetilde{X}_n = y) = P_\omega^x(X_{n+1} = y \pm 1 | X_n = y),\\
&\qquad\qquad\qquad\qquad\qquad \forall y \in \{T^- + 1, \dots, T^+ - 1\},\\
&P_\omega^x(\widetilde{X}_{n+1} = y + 1 | \widetilde{X}_n = y) = 1 \qquad \text{for } y = T^-,\\
&P_\omega^x(\widetilde{X}_{n+1} = y - 1 | \widetilde{X}_n = y) = 1 \qquad \text{for } y = T^+.
\end{aligned}$$

Therefore, we have for $\ell \in \left\{ \left\lceil \frac{4n}{3} \right\rceil, \dots, 2n \right\} \cap \big(2\mathbb{Z} + b_+\big)$

$$\begin{aligned}
&P_\omega^o(X_\ell = b_+) \ge P_\omega^o(X_\ell = b_+,\ \min\{\tau(T^-), \tau(T^+)\} > 2n)\\
\ge\ & P_\omega^o(\widetilde{X}_\ell = b_+) - P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \le 2n)\\
\ge\ & P_\omega^o\big(\widetilde{X}_\ell = b_+,\ \tau(b_+) \le \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\big) - P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \le 2n)\\
\ge\ & P_\omega^o\big(\tau(b_+) \le \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\big) \cdot \widehat{\inf}_{k \in \left\{ \left\lceil \frac{\ell}{2} \right\rceil, \dots, \ell \right\}} P_\omega^{b_+}(\tilde{X}_k = b_+)\\
&-P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \le 2n). \qquad (3.7)
\end{aligned}$$

Using (2.6) and (2.7), we see that the last term in (3.7) with the negative sign decreases exponentially for $n \le e^{(1-2\delta)L}$, i.e.

$$\begin{aligned} &P_\omega^o(\min\{\tau(T^-),\tau(T^+)\} \le 2n) \le P_\omega^o\left(\min\{\tau(T^-),\tau(T^+)\} \le 2e^{(1-2\delta)L}\right) \\ &\le P_\omega^o\left(\tau(T^-) \le 2\, e^{(1-2\delta)L}\right) + P_\omega^o\left(\tau(T^+) \le 2\, e^{(1-2\delta)L}\right) \\ &\le 4e^{(1-2\delta)L}e^{-L} = 4e^{-2\delta L}. \end{aligned} \tag{3.8}$$

In order to derive a lower bound for the first term in (3.7), we first notice that the analogous calculation as in (3.5) shows for $\omega \in \Gamma(L,\delta)$ that

$$\begin{aligned} P_\omega^o\left(\tau(b_+) \le \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\right) &\ge 1 - \frac{2}{\ell}\cdot 4 \cdot L^4\, e^{2\delta L} - \frac{1}{2} \\ &\ge \frac{1}{2} - 6\, L^4\, e^{-\delta L} \end{aligned} \tag{3.9}$$

since $\ell \ge \lceil \frac{4n}{3} \rceil \ge \frac{4}{3}\, e^{3\delta L}$ for $n \ge e^{3\delta L}$. For the second factor of (3.7), we show the following

Lemma 3.3. *For $\omega \in \Gamma(L,\delta)$ and for all $\ell \in 2\mathbb{N}$, we have*

$$P_\omega^{b_+}(\widetilde{X}_\ell = b_+) \ge \frac{1}{2}\cdot\frac{1}{|T^-| + T^+ + 1}\, e^{-\delta L}.$$

Proof of Lemma 3.3: Using the reversibility (cf. (2.3)) of $(\widetilde{X}_\ell)_{\ell\in\mathbb{N}_0}$, we get

$$\begin{aligned} P_\omega^{b_+}(\widetilde{X}_\ell = b_+) &= \sum_{x=T^-}^{T^+} P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x)\cdot P_\omega^{x}(\widetilde{X}_{\ell/2} = b_+) \\ &= \sum_{x=T^-}^{T^+} P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x)\cdot \frac{\widetilde{\mu}_\omega(b_+)}{\widetilde{\mu}_\omega(x)}\cdot P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x), \end{aligned} \tag{3.10}$$

where $\widetilde{\mu}_\omega(\cdot)$ denotes a reversible stationary measure of the reflected random walk $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which is unique up to multiplication by a constant. To see that $(\widetilde{X}_\ell)_{\ell\in\mathbb{N}_0}$ is also reversible, it is enough to note that $(\widetilde{X}_\ell)_{\ell\in\mathbb{N}_0}$ can again be described as an electrical network with the following conductances:

$$\widetilde{C}_{(x,x+1)}(\omega) = \begin{cases} C_{(x,x+1)}(\omega) = e^{-V(x)} & \text{for } x = T^-, T^- + 1, \dots, T^+ - 1 \\ 0 & \text{for } x = T^- - 1, T^+. \end{cases}$$

Therefore, a reversible measure for the reflected random walk is given by (cf. (2.2))

$$\widetilde{\mu}_\omega(x) = \begin{cases} \mu_\omega(x) = e^{-V(x)} + e^{-V(x-1)} & \text{for } x = T^- + 1, T^- + 2, \dots, T^+ - 1, \\ e^{-V(T^-)} & \text{for } x = T^-, \\ e^{-V(T^+ - 1)} & \text{for } x = T^+. \end{cases}$$

Since $0 \le b_+ < T^+$, this implies

$$\begin{aligned} \frac{\widetilde{\mu}_\omega(b_+)}{\widetilde{\mu}_\omega(x)} &\ge \frac{e^{-V(b_+)} + e^{-V(b_+ - 1)}}{e^{-V(x)} + e^{-V(x-1)}} \\ &\ge \frac{e^{-V(b_+)}}{2\cdot e^{(-\min\{V(b_+),V(b_-)\})}} \ge \frac{e^{-\delta L}}{2} \end{aligned} \tag{3.11}$$

for $T^- \le x \le T^+$ and for $\omega \in \Gamma(L,\delta)$. By applying (3.11) to (3.10), we get

$$\begin{aligned} P_\omega^{b_+}(\widetilde{X}_\ell = b_+) &\geq \frac{1}{2} \cdot \sum_{x=T^-}^{T^+} \left(P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x) \right)^2 \cdot e^{-\delta L} \\ &\geq \frac{1}{2} \cdot \frac{1}{|T^-| + T^+ + 1} \cdot e^{-\delta L}, \end{aligned} \tag{3.12}$$

by the Cauchy-Schwarz inequality since $\sum_{x=T^-}^{T^+} P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x) = 1$. □

We can now return to the proof of Proposition 3.2 and finish our lower bound for the third factor in (3.3). By applying (3.8), (3.9) and Lemma 3.3 to (3.7), we get that there exists $\tilde{L}_0 = \tilde{L}_0(\delta)$ such that for all $L \geq \tilde{L}_0$, for $e^{3\delta L} \leq n \leq e^{(1-2\delta)L}$ and $\omega \in \Gamma(L,\delta)$ (since $|T^-|, T^+ \leq L^2$), we have

$$\begin{aligned} &\widehat{\inf}_{\ell \in \left\{ \left\lceil \frac{4n}{3} \right\rceil, \dots, 2n \right\}} P_\omega^o(X_\ell = b_+) \\ &\geq \left(\frac{1}{2} - 6 \cdot L^4 \cdot e^{-\delta L} \right) \cdot \frac{1}{2} \cdot \frac{1}{2L^2 + 1} e^{-\delta L} - 4 \cdot e^{-2\delta L} \geq e^{-\frac{3}{2}\delta L}. \end{aligned} \tag{3.13}$$

To finish the proof of Proposition 3.2, we can collect our lower bounds in (3.5), (3.6), and (3.13) and conclude with (3.3) that there exists $\tilde{L}_1 := \tilde{L}_1(\delta)$ such that for every $L \geq \tilde{L}_1$, for $e^{3\delta L} \leq n \leq e^{(1-2\delta)L}$ and for $\omega \in \Gamma(L,\delta)$ we have

$$\begin{aligned} P_\omega(X_{2n} = 0) &\geq \left(\frac{1}{2} - 6 \cdot L^4 e^{-\delta L} \right) \cdot \frac{\varepsilon}{1-\varepsilon} e^{-\delta L} \cdot e^{-\frac{3}{2}\delta L} \\ &\geq e^{-3\delta L}. \end{aligned}$$

This shows (3.1) since we have $P_\omega(X_{2n} = 0) \geq \varepsilon^{2n} > 0$ for all $n \in \mathbb{N}$ due to assumption (1.3). ■

Proposition 3.4. *For $0 < \delta < 1$, we have*

$$\mathsf{P}(\omega : \ \omega \in \Gamma(L,\delta) \textit{ for infinitely many } L) = 1. \tag{3.14}$$

Proof of Proposition 3.4: Let $(B(t))_{t \in \mathbb{R}}$ be the two-sided Brownian motion from Theorem 2.1. Since $\Gamma(L,\delta)$ is increasing in δ, we can assume that $0 < \delta < \frac{1}{2}$. For $y \in \mathbb{R}$ we define

$$\widehat{T}^+(y) := \inf\{t \geq 0 : \ B(t) = y\} \ \text{ and } \ \widehat{T}^-(y) := \sup\{t \leq 0 : \ B(t) = y\}$$

as the first hitting times of y on the positive and negative side of the origin, respectively. Additionally, for $L \in \mathbb{N}$, $i \in \mathbb{N}$, $y \in \mathbb{R}$, we can introduce the following sets

$$F_L^+(y) := \{\widehat{T}^+(y \cdot L) < \widehat{T}^+(-y \cdot L)\} \ \text{ and } \ F_L^-(y) := \{\widehat{T}^-(y \cdot L) < \widehat{T}^-(-y \cdot L)\}$$

on which the Brownian motion reaches the value $y \cdot L$ before $-y \cdot L$. Further we define

$$\begin{aligned} G_L^+(i) &:= \left\{ B(t) \geq (2i-1) \cdot \tfrac{\delta}{4} \cdot L \ \text{ for } \widehat{T}^+\left(2i \cdot \tfrac{\delta}{4} \cdot L\right) \leq t \leq \widehat{T}^+\left((2i+2) \cdot \tfrac{\delta}{4} \cdot L\right) \right\}, \\ G_L^-(i) &:= \left\{ B(t) \geq (2i-1) \cdot \tfrac{\delta}{4} \cdot L \ \text{ for } \widehat{T}^-\left((2i+2) \cdot \tfrac{\delta}{4} \cdot L\right) \leq t \leq \widehat{T}^-\left(2i \cdot \tfrac{\delta}{4} \cdot L\right) \right\} \end{aligned}$$

on which the Brownian motion does not decrease much between the first hitting times of the two levels of interest. Using these sets, we can define the sets

$$
\begin{aligned}
A^+(L,\delta) &:= F_L^+(\delta) \cap \left\{ \widehat{T}^+(1.1\cdot L) \le L^2,\ \min_{\widehat{T}^+(\delta\cdot L)\le t\le \widehat{T}^+(1.1\cdot L)} B(t) \ge \frac{\delta}{4}\cdot L \right\},\\
A^-(L,\delta) &:= F_L^-(\delta) \cap \left\{ -\widehat{T}^-(1.1\cdot L) \le L^2,\ \min_{\widehat{T}^-(1.1\cdot L)\le t\le \widehat{T}^-(\delta\cdot L)} B(t) \ge \frac{\delta}{4}\cdot L \right\},\\
D^+(L,\delta) &:= G_L^+(0)\cap G_L^+(1)\cap G_L^+(2)\\
&\qquad \cap \left\{ \widehat{T}^+(1.2\cdot L) \le 0.9\cdot L^2,\ \min_{\widehat{T}^+\left(\frac{3\cdot\delta}{2}\cdot L\right)\le t\le \widehat{T}^+(1.2\cdot L)} B(t) \ge \frac{3\delta}{4}\cdot L \right\},\\
D^-(L,\delta) &:= G_L^-(0)\cap G_L^-(1)\cap G_L^-(2)\\
&\qquad \cap \left\{ -\widehat{T}^-(1.2\cdot L) \le 0.9\cdot L^2,\ \min_{\widehat{T}^-(1.2\cdot L)\le t\le \widehat{T}^-\left(\frac{3\delta}{2}\cdot L\right)} B(t) \ge \frac{3\delta}{4}\cdot L \right\}
\end{aligned}
$$

which will be used for an approximation of our previously constructed valleys ω belonging to $\Gamma(L,\delta)$ which we illustrated in Figure 3.2 on page 488. Here, we added the factors 1.1, 1.2 and 0.9 in contrast to the construction before in order to have some space for the approximation. For the Brownian motion, we can directly compute that we have

$$
\mathsf{P}\big(D^+(1,\delta)\cap D^-(1,\delta)\big) > 0. \tag{3.15}
$$

Thereby, for all $L\in\mathbb{N}$, due to the scaling property of the Brownian motion, $(B(L^2\cdot t)/L)_{t\in\mathbb{R}}$ is again a two-sided Brownian motion with diffusion constant σ, this implies

$$
\mathsf{P}\big(D^+(L,\delta)\cap D^-(L,\delta)\big) = \mathsf{P}\big(D^+(1,\delta)\cap D^-(1,\delta)\big) > 0. \tag{3.16}
$$

First, we notice that for $L_0\in\mathbb{N}$ we have

$$
\mathsf{P}\left(\bigcap_{L=L_0}^{\infty}\Big(A^+(L,\delta)\cap A^-(L,\delta)\Big)^c\right) \le \mathsf{P}\left(\bigcap_{k=\ell+1}^{\infty}\Big(A^+(L_k,\delta)\cap A^-(L_k,\delta)\Big)^c\right) \tag{3.17}
$$

for arbitrary $\ell\in\mathbb{N}_0$, where we define

$$
L_k := \max\left\{10, \left\lceil \tfrac{2}{\delta}\right\rceil\right\}\cdot (L_{k-1})^2
$$

for $k\in\mathbb{N}$ inductively. Note that for $n>\ell+1$ with

$$
\mathcal{F}_n := \sigma\left(\big(B(t)\big)_{-(L_{n-1})^2\le t\le (L_{n-1})^2}\right).
$$

Setting $\Theta_n^{\pm}(\delta) := \Big\{ \big(B(t \pm (L_{n-1})^2) - B(\pm(L_{n-1})^2)\big)_{t\in\mathbb{R}} \notin D^{\pm}(L_n,\delta) \Big\}$, the following holds:

$$\begin{aligned}
&\mathsf{P}\left(\bigcap_{k=\ell+1}^{n} \Big(A^+(L_k,\delta) \cap A^-(L_k,\delta)\Big)^c \right) \\
&\le \mathsf{E}\left[\prod_{k=\ell+1}^{n-1} \mathbf{1}_{\left(A^+(L_k,\delta)\cap A^-(L_k,\delta)\right)^c} \cdot \mathbf{1}_{\left\{\max\limits_{-(L_{n-1})^2 \le t \le (L_{n-1})^2} |B(t)| < (L_{n-1})^2\right\}} \right. \\
&\quad \left. \cdot\ \mathsf{E}\Big[\mathbf{1}_{\Theta_n^+(\delta)\cup\Theta_n^-(\delta)} \Big| \mathcal{F}_n \Big] \right] + \mathsf{P}\left(\max_{-(L_{n-1})^2 \le t \le (L_{n-1})^2} |B(t)| \ge (L_{n-1})^2 \right) \\
&\le \Big(1 - \mathsf{P}\Big(D^+(L_n,\delta) \cap D^-(L_n,\delta)\Big)\Big) \cdot \mathsf{P}\left(\bigcap_{k=\ell+1}^{n-1} \Big(A^+(L_k,\delta) \cap A^-(L_k,\delta)\Big)^c \right) \\
&\quad + \mathsf{P}\left(\max_{-(L_{n-1})^2 \le t \le (L_{n-1})^2} |B(t)| \ge (L_{n-1})^2 \right) \\
&\le \Big(1 - \mathsf{P}\Big(D^+(1,\delta) \cap D^-(1,\delta)\Big)\Big)^{n-\ell} \\
&\qquad + \sum_{k=\ell+1}^{n} \mathsf{P}\left(\max_{-(L_{k-1})^2 \le t \le (L_{k-1})^2} |B(t)| \ge (L_{k-1})^2 \right). \qquad (3.18)
\end{aligned}$$

To see that the first step in (3.18) holds, note that for

$$\begin{aligned}
\omega \in &\left\{ \max_{-(L_{n-1})^2 \le t \le (L_{n-1})^2)} |B(t)| < (L_{n-1})^2 \right\} \\
&\cap \Big\{ \big(B(t + (L_{n-1})^2) - B((L_{n-1})^2)\big)_{t\in\mathbb{R}} \in D^+(L_n,\delta) \Big\} \qquad (3.19)
\end{aligned}$$

we have

$$\begin{aligned}
\min_{0\le t\le (L_n)^2} B(t) \quad &\ge \quad \min_{0\le t\le (L_{n-1})^2} B(t) \\
&\qquad\qquad + \min_{(L_{n-1})^2 \le t \le (L_n)^2} B(t + (L_{n-1})^2) - B((L_{n-1})^2) \\
&> \quad -(L_{n-1})^2 - \frac{\delta}{4} \cdot L_n > -\delta \cdot L_n
\end{aligned}$$

since $(L_{n-1})^2 \le \delta L_n/2$ and

$$\begin{aligned}
\max_{0\le t\le (L_n)^2} B(t) \quad &\ge \quad B\big((L_{n-1})^2\big) \\
&\qquad + \max_{(L_{n-1})^2 \le t \le (L_n)^2 - (L_{n-1})^2} B(t + (L_{n-1})^2) - B((L_{n-1})^2) \\
&\ge \quad -(L_{n-1})^2 + 1.2 \cdot L_n \ge 1.1 \cdot L_n
\end{aligned}$$

since $(L_{n-1})^2 \leq L_n/10$. In particular, we have $\widehat{T}^+(\delta \cdot L_n) < \widehat{T}^+(-\delta \cdot L_n)$ and $\widehat{T}^+(1.1 \cdot L_n) \leq (L_n)^2$ on the considered set. Similarly, again on the set in (3.19), we see that we have

$$\begin{aligned}
\widehat{T}^+(\delta \cdot L_n) &> \inf\{t \geq (L_{n-1})^2 :\ (B(t+(L_{n-1})^2) - B((L_{n-1})^2) \geq \tfrac{\delta}{2} \cdot L_n\}, \\
\widehat{T}^+(\delta \cdot L_n) &< \inf\{t \geq (L_{n-1})^2 :\ (B(t+(L_{n-1})^2) - B((L_{n-1})^2) \geq \tfrac{3\cdot\delta}{2} \cdot L_n\},
\end{aligned}$$

since $(L_{n-1})^2 \leq \delta L_n/2$, this implies

$$\min_{\widehat{T}^+(\delta\cdot L) \leq t \leq \widehat{T}^+(1.1\cdot L)} B(t) \geq \frac{\delta}{4} \cdot L_n$$

by construction of $D^+(L_n, \delta)$. Altogether, we can conclude that $\omega \in A^+(L_n, \delta)$ holds for our choice of ω in (3.19). The argument for the negative part runs completely analogously. Further in (3.18), we used the Markov property of the Brownian motion in the second step. Additionally, we iterated the first two steps $n - \ell - 1$ times and used (3.16) for the last step. To control the last sum in (3.18), let us recall that due to the reflection principle (see e.g. Chapter III, Proposition 3.7 in Revuz and Yor (1999)), we have

$$\forall T, x > 0, \quad \mathsf{P}\left(\max_{t\in[0,T]} \frac{B(t)}{\sigma\sqrt{T}} \geq x\right) = \mathsf{P}\left(|Z| \geq x\right) = 2\mathsf{P}\left(Z \geq x\right) \leq \frac{1}{x} \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

for a random variable $Z \sim \mathcal{N}(0,1)$ (the last estimate can be found for example in Lemma 12.9 in Appendix B of Mörters and Peres (2010)). Due to this upper bound, we can conclude that

$$\sum_{k=\ell+1}^{n} \mathsf{P}\left(\max_{-(L_{k-1})^2 \leq t \leq (L_{k-1})^2} |B(t)| \geq (L_{k-1})^2\right) \tag{3.20}$$

$$\leq 4 \cdot \sum_{k=\ell+1}^{n} \mathsf{P}\left(\max_{0 \leq t \leq (L_{k-1})^2} \frac{B(t)}{\sigma \cdot L_{k-1}} \geq \frac{L_{k-1}}{\sigma}\right)$$

$$\leq 8 \cdot \sum_{k=\ell+1}^{\infty} \frac{\sigma}{L_{k-1}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(L_{k-1})^2}{2\sigma^2}} \xrightarrow{\ell\to\infty} 0. \tag{3.21}$$

By combining the upper bounds in (3.17), (3.18), and (3.21), we get for all $\ell \in \mathbb{N}_0$

$$\mathsf{P}\left(\omega \notin \left(A^+(L,\delta) \cap A^-(L,\delta)\right) \text{ for all } L \geq L_0\right)$$

$$\leq \lim_{n\to\infty} \left(1 - \mathsf{P}\Big(D^+(1,\delta) \cap D^-(1,\delta)\Big)\right)^{n-\ell}$$

$$+ \sum_{k=\ell+1}^{\infty} \mathsf{P}\left(\max_{-(L_{k-1})^2 \leq t \leq (L_{k-1})^2} |B(t)| \geq (L_{k-1})^2\right) \xrightarrow{\ell\to\infty} 0.$$

Since $L_0 \in \mathbb{N}$ was chosen arbitrarily, we can conclude that for $0 < \delta < \frac{1}{2}$ we have

$$\mathsf{P}\left(\omega :\ \omega \in \left(A^+(L,\delta) \cap A^-(L,\delta)\right) \text{ for infinitely many } L\right) = 1.$$

Using the Komlós-Major-Tusnády strong approximation Theorem (cf. Theorem 2.1), we see that for $0 < \delta < \frac{1}{2}$ we have

$$\begin{aligned}&\left\{\omega :\ \omega \in \left(A^{+}(L,\delta) \cap A^{-}(L,\delta)\right) \text{ for infinitely many } L\right\}\\ \subseteq\ &\left\{\omega :\ \omega \in \Gamma(L, 2\delta) \text{ for infinitely many } L\right\},\end{aligned}$$

which is enough to conclude that (3.14) holds for all $0 < \delta < 1$. ■

With the help of Proposition 3.2 and Proposition 3.4, we can now turn to the proofs of Theorems 1.1 and 1.2 and Corollary 1.4:

Proof of Theorem 1.1: For a fixed $0 \le \alpha < 1$, we choose $0 < \delta < \frac{1}{6}$ such that $\alpha < (1-5\delta)/(1-2\delta)$. For $\omega \in \Gamma(L,\delta)$, the inequality in (3.1) implies that

$$\begin{aligned}\sum_{n\in\mathbb{N}} P_\omega(X_{2n}=0)\cdot n^{-\alpha} &\ge \sum_{\lceil e^{3\delta L}\rceil \le n \le \lfloor e^{(1-2\delta)L}\rfloor} P_\omega(X_{2n}=0)\cdot n^{-\alpha}\\ &\ge \left(e^{(1-2\delta)L} - e^{3\delta L} - 1\right)\cdot C\cdot e^{-3\delta L}\cdot\left(e^{(1-2\delta)L}\right)^{-\alpha}\\ &= C\cdot\left(e^{(1-5\delta)L} - 1 - e^{-3\delta L}\right)\cdot e^{-\alpha(1-2\delta)L} \xrightarrow{L\to\infty} \infty.\end{aligned}$$

Since Proposition 3.4 shows that for P-a.e. environment ω we find L arbitrarily large such that $\omega \in \Gamma(L,\delta)$, we can conclude that (1.4) holds for P-a.e. environment ω. ■

Proof of Theorem 1.2: Given $\alpha > 0$, we choose δ such that $0 < \delta < \min\left\{\frac{1}{2+3\alpha}, \frac{1}{5}\right\}$, which yields $1-2\delta-3\alpha\delta > 0$ and $1-2\delta > 3\delta$. For $\omega \in \Gamma(L,\delta)$, the inequality in (3.1) implies

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\left(P_\omega(X_{2n}=0)\right)^{\alpha} \ge \sum_{\lceil e^{3\delta L}\rceil \le n \le \lfloor e^{(1-2\delta)L}\rfloor}\left(P_\omega(X_{2n}=0)\right)^{\alpha}\\ &\ge \left(e^{(1-2\delta)L} - e^{3\delta L} - 1\right)\cdot\left(C\cdot e^{-3\delta L}\right)^{\alpha}\\ &= C^{\alpha}\cdot\left(e^{(1-2\delta-3\alpha\delta)L} - e^{(3\delta-3\alpha\delta)L} - e^{-3\alpha\delta L}\right) \xrightarrow{L\to\infty} \infty.\end{aligned}$$

Again since Proposition 3.4 shows that for P-a.e. environment ω we find L arbitrarily large such that $\omega \in \Gamma(L,\delta)$, we can conclude that (1.5) holds for P-a.e. environment ω. ■

Proof of Corollary 1.4: Due to the independence of the environments $\omega^{(1)}, \omega^{(2)}, \dots,$ $\omega^{(d)}$, we can extend the proof of Proposition 3.4 to get

$$\mathsf{P}^{\otimes d}\left(\text{For infinitely many } L\in\mathbb{N},\ \text{we have } \omega^{(i)} \in \Gamma(L,\delta) \text{ for } i = 1,2,\dots d\right) = 1 \tag{3.22}$$

for all $0 < \delta < 1$. Indeed (3.18) becomes

$$
\begin{aligned}
&\mathsf{P}\left(\forall k=\ell+1,\cdots,n,\ \exists i,\ \omega_i \notin A^+(L_k,\delta)\cap A^-(L_k,\delta)\right)\\
\le &\left(1-\mathsf{P}\Big(\forall i,\ B^{(i)}\in D^+\left(L_n,\delta\right)\cap D^-\left(L_n,\delta\right)\Big)\right)\\
&\cdot \mathsf{P}\left(\forall k=\ell+1,\cdots,n-1,\ \exists i,\ \omega_i \notin A^+(L_k,\delta)\cap A^-(L_k,\delta)\right)\\
&+\mathsf{P}\left(\max_{i=1,\cdots,d}\ \max_{-(L_{n-1})^2\le t\le (L_{n-1})^2}|B^{(i)}(t)|\ge (L_{n-1})^2\right).
\end{aligned}
$$

Now, using Proposition 3.2, we have for $(\omega^{(1)},\omega^{(2)},\ldots,\omega^{(d)})$ with $\omega^{(i)}\in\Gamma(L,\delta)$ for $i=1,2,\ldots d$

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}\prod_{k=1}^{d}P_{\omega^{(k)}}(X_{2n}=0)&\ge \sum_{\lceil e^{3\delta L}\rceil\le n\le\lfloor e^{(1-2\delta)L}\rfloor}\prod_{k=1}^{d}P_{\omega^{(k)}}(X_{2n}=0)\\
&\ge \left(e^{(1-2\delta)L}-e^{3\delta L}-1\right)\cdot C^d\cdot e^{-3\delta dL}\\
&=C^d\cdot\left(e^{(1-2\delta-3\delta d)L}-e^{(3\delta-3\delta d)L}-e^{-3\delta dL}\right)\xrightarrow{L\to\infty}\infty
\end{aligned}
$$

for $0<\delta<\frac{1}{2+3d}$. Since (3.22) holds for arbitrarily small δ, we can conclude that (1.6) holds for $\mathsf{P}^{\otimes d}$-a.e. environment $(\omega^{(1)},\omega^{(2)},\ldots,\omega^{(d)})$. ■

4. Recurrence properties of the RWRE (I)-(III)

4.1. *Direct products involving a one dimensional RWRE.*

Proposition 4.1 (Case (I))**.** *Fix a random environment ω which fulfils (1.2) and (1.3). Let $(X_n,Y_n)_{n\in\mathbb{N}_0}$ be a 2-dimensional process where $(X_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ are independent with respect to P_ω, $(X_n)_{n\in\mathbb{N}_0}$ being a RWRE in the environment ω (in the sense of (1.1)) and $(Y_n)_{n\in\mathbb{N}_0}$ a centered random walk such that $Y_0=0$ and $(Y_n/A_n)_n$ converges in distribution to a β-stable distribution with $\beta\in(1,2]$ (for some suitable normalization A_n).*

Then, $(X_n,Y_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment ω.

Proof of Proposition 4.1: We denote by $\mathbb{P}$ the annealed probability measure of the 2-dimensional random walk. Due to the local limit theorem (see Ibragimov and Linnik (1971, Chapter 4)), we have $\mathbb{P}(Y_{d_0 n}=0)\sim C(A_n)^{-1}$ for some $C>0$ and for $d_0:=gcd\{m\ge 1,\,\mathbb{P}(X_m=Y_m=0)\neq 0\}$. Recall that $A_n=n^{\frac{1}{\beta}}L(n)$ with L a slowly varying function. Due to the independence of the two components, we have

$$
\sum_{n\in\mathbb{N}}P_\omega\big((X_n,Y_n)=(0,0)\big)=\sum_{n\in\mathbb{N}}P_\omega\big(X_{d_0 n}=0\big)\cdot P_\omega\big(Y_{d_0 n}=0\big)=\infty,
$$

where the last equation is due to Theorem 1.1 applied with $\frac{1}{\beta}<\alpha<1$. This proves the recurrence of the process $(X_n,Y_n)_{n\in\mathbb{N}_0}$ for P-a.e. environment ω. □

Observe that, if we take $(X_n)_n$ to be the simple symmetric random walk on $\mathbb{Z}$ in Proposition 4.1 instead of Sinai's walk, we have $\mathbb{P}(X_{2n}=0)\sim cn^{-\frac{1}{2}}$ and hence we lose the recurrence as soon as $\beta<2$.

4.2. *Other two-dimensional RWRE governed by a one-dimensional RWRE.* We study now the cases (II) and (III). We consider a process moving horizontally with probability δ and vertically with probability $1-\delta$. We assume that the horizontal displacements follow Sinai's walk and that the vertical ones either follow some recurrent random walk (case (II)) or depend on the parity of the first coordinate of the current position (case (III)).

Proposition 4.2 (Case (II))**.** *Let $\delta \in (0,1)$. Let $\omega = (\omega_x)_x$ be a random environment which fulfils (1.2) and (1.3). We assume that, given ω, $(M_n)_{n\in\mathbb{N}_0}$ is a Markov chain with values in $\mathbb{Z}^2$ such that*

$$P_\omega\big(M_0 = (0,0)\big) = 1,$$

$$P_\omega\big(M_{n+1} = (x+1,y)\big|M_n = (x,y)\big) = \delta \cdot \omega_x,$$

$$P_\omega\big(M_{n+1} = (x-1,y)\big|M_n = (x,y)\big) = \delta \cdot (1-\omega_x),$$

$$P_\omega\big(M_{n+1} = (x,y+z)\big|M_n = (x,y)\big) = (1-\delta) \cdot \nu(z),$$

where ν is a probability distribution on $\mathbb{Z}$ with zero expectation such that $(\nu^{*n}(A_n\cdot))_n$ converges to a β-stable distribution with $\beta \in (1,2]$ (for some suitable increasing sequence $(A_n)_n$ of positive real numbers). Then, $(M_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment ω.

Let us recall that $\nu^{*n}(A_n\cdot)$ is the distribution of $(Z_1+\cdots+Z_n)/A_n$ if $Z_1,...,Z_n$ are i.i.d. random variables with distribution ν.

Proof of Proposition 4.2: Let us write $M_n = (\tilde{X}_n, \tilde{Y}_n)$. We look at the process $(M_n)_{n\in\mathbb{N}_0}$ whenever it has moved in the first component. For this, we define inductively $\tau_0 := 0$ and $\tau_k := \inf\Big\{n > \tau_{k-1} : \tilde{X}_n \neq \tilde{X}_{\tau_{k-1}}\Big\}$ for $k \geq 1$. Additionally, we define $X_n := \tilde{X}_{\tau_n}$ and $Y_n := \tilde{Y}_{\tau_n}$ for $n \in \mathbb{N}_0$. Note that $(X_n)_{n\in\mathbb{N}_0}$ is a usual RWRE on $\mathbb{Z}$ with environment ω. Further, we have

$$Y_n = \sum_{k=1}^{\tau_n - n} Z_k = \sum_{\ell=1}^{n} \sum_{k=\tau_{\ell-1}-\ell+2}^{\tau_\ell - \ell} Z_k,$$

where $(Z_k)_k$ is a sequence of i.i.d. random variables with distribution ν. We know that the random variables $(\tilde{Z}_\ell := \sum_{k=\tau_{\ell-1}-\ell+2}^{\tau_\ell-\ell} Z_k)_\ell$ are identically distributed and centered. Let us prove that their distribution belongs to the domain of attraction of a β-stable distribution. We know that $(\sum_{k=1}^m Z_k/A_m)_m$ converges in distribution to a β-stable centered random variable U and that $A_m = m^{\frac{1}{\beta}} L(m)$, L being a slowly varying function. Observe that $(Y_n/A_{\tau_n - n})_n$ converges in distribution to U and that $(A_{\tau_n-n}/A_n)_n$ converges almost surely to $(\mathbb{E}[\tau_1]-1)^{\frac{1}{\beta}}$. Hence $(Y_n/A_n)_n$ converges in distribution to $(\mathbb{E}[\tau_1]-1)^{\frac{1}{\beta}} U$. Therefore, (since $\mathbb{P}(\tilde{Z}_1 = 0) > 0$) we conclude that $P_\omega(Y_n = 0) \sim C(A_n)^{-1}$. Hence, for P-a.e. environment ω, we have

$$\sum_{n\in\mathbb{N}} P_\omega\big((X_{2n}, Y_{2n}) = (0,0)\big) = \sum_{n\in\mathbb{N}} P_\omega(X_{2n} = 0) \cdot P_\omega(Y_{2n} = 0) = \infty$$

(due to Theorem 1.1 applied with $\frac{1}{\beta} < \alpha < 1$). This implies the recurrence of $(X_n, Y_n)_n$ and so of $(M_n)_n$. $\square$

Finally we consider the case (III). We suppose now that every vertical line is oriented upward if the line is labelled by an even number and downward otherwise. We consider again a process moving horizontally with probability δ and moving vertically with probability $1-\delta$. We assume that the horizontal displacements follow a Sinai walk (as in the previous example) but that the vertical displacements follow the orientation of the vertical line on which the walker is located.

Proposition 4.3 (Case (III), odd-even orientations of vertical lines)**.** *Let $\delta \in (0,1)$ and let ω be a random environment which fulfils (1.2) and (1.3). Given ω, $(M_n)_{n\in\mathbb{N}_0}$ is a Markov chain with values in $\mathbb{Z}^2$ such that*

$$P_\omega\big(M_0 = (0,0)\big) = 1,$$

$$P_\omega\big(M_{n+1} = (x+1,y)\big| M_n = (x,y)\big) = \delta\cdot\omega_x,$$

$$P_\omega\big(M_{n+1} = (x-1,y)\big| M_n = (x,y)\big) = \delta\cdot(1-\omega_x),$$

$$P_\omega\big(M_{n+1} = (x,y+z)\big| M_n = (x,y)\big) = (1-\delta)\cdot\nu((-1)^x z),$$

with ν a probability distribution on $\mathbb{Z}$ admitting a first moment and belonging to the domain of attraction of a stable distribution of index $\beta>1$. Then, $(M_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment ω.

Proof of Proposition 4.3: The proof follows the same scheme as the previous one and uses the same notations $(\tilde X_n, \tilde Y_n)_n$, τ_n and (X_n, Y_n). Again $(X_n)_{n\in\mathbb{N}_0}$ is a RWRE on $\mathbb{Z}$ with environment ω. Let us write $T_n := \tau_n - \tau_{n-1}$, $\tau_0^+ = \tau_0^- = 0$, $\tau_n^+ := \sum_{\ell=1}^n T_{2\ell-1}$ and $\tau_n^- := \sum_{\ell=1}^n T_{2\ell}$. Observe that $\tau_n^+ - n$ (resp. $\tau_n^- - n$) is the number of vertical moves on an even (resp. odd) vertical axis before the $2n$-th horizontal displacement. We have

$$Y_{2n} = \sum_{\ell=1}^n [\xi_{2\ell-1} - \xi_{2\ell}], \quad \text{with } \xi_{2\ell-1} = \sum_{k=\tau_{\ell-1}^+-\ell+2}^{\tau_\ell^+-\ell} Z_{2k-1}, \quad \xi_{2\ell} = \sum_{k=\tau_{\ell-1}^--\ell+2}^{\tau_\ell^--\ell} Z_{2k},$$

where $(Z_k)_k$ is a sequence of i.i.d. random variables with distribution ν. With these notations Z_{2k+1} (resp. $-Z_{2k}$) is the k-th vertical displacement on an even (resp. odd) vertical axis.

The random variables $\xi_{2\ell-1} - \xi_{2\ell}$ are i.i.d.. We already know that $\xi_1 - \xi_2$ is centered. Let us prove that its distribution belongs to the domain of attraction of a β-stable centered distribution, i.e. that Y_{2n} suitably normalized converges to a β-stable random variable. We observe that

$$Y_{2n} = \sum_{k=1}^{\tau_n^+-n} Z_{2k-1} - \sum_{k=1}^{\tau_n^--n} Z_{2k} = U_n + V_n^+ - V_n^- + W_n,$$

with

$$U_n := \sum_{k=1}^{n\mathbb{E}[\tau_1-1]} (Z_{2k-1} - Z_{2k}),$$

$$V_n^+ := \sum_{k=1}^{\tau_n^+-n} (Z_{2k-1} - \mathbb{E}[Z_1]) - \sum_{k=1}^{n\mathbb{E}[\tau_1-1]} (Z_{2k-1} - \mathbb{E}[Z_1]),$$

$$V_n^- := \sum_{k=1}^{\tau_n^- - n} (Z_{2k} - \mathbb{E}[Z_1]) - \sum_{k=1}^{n\mathbb{E}[\tau_1 - 1]} (Z_{2k} - \mathbb{E}[Z_1]),$$

$$W_n := (\tau_n^+ - \tau_n^-)\mathbb{E}[Z_1].$$

We know that there exists an increasing sequence of real numbers $(A_n)_n$ (regularly varying of index $1/\beta$) such that $(V_n := \sum_{k=1}^n (Z_k - \mathbb{E}[Z_1])/A_n)_n$ converges in distribution to a β-stable random variable V. This implies that $(U_n/A_n)_n$ converges in distribution to a β-stable random variable U.

By tightness of $(V_n)_n$, for every $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that for every n we have $\mathbb{P}(|V_n^\pm| \geq K_\varepsilon A_{\tau_n^\pm - n\mathbb{E}[\tau_1]}) < \varepsilon$ (with the convention $A_{-m} = A_m$). Moreover $(\tau_n^\pm - n\mathbb{E}[\tau_1])/n^{3/4} \to 0$ almost surely and so $A_{\tau_n^\pm - n\mathbb{E}[\tau_1]} \ll A_n$ almost surely. It follows that $(V_n^\pm/A_n)_n$ converges in probability to 0.

If $\mathbb{E}[Z_1] = 0$, we conclude that $(Y_{2n}/A_n)_n$ converges in distribution to U.

Assume now that $\mathbb{E}[Z_1] \neq 0$. Then $(W_n/\sqrt{n})_n$ is independent of $(U_n)_n$ and converges in distribution to some centered normal variable W (assumed to be independent of U).

Hence, if $1 < \beta < 2$, we conclude that $(Y_{2n}/A_n)_n$ converges in distribution to U.

If $\beta = 2$ and $\mathbb{E}[Z_1] \neq 0$, we can choose A_n such that U and W have the same distribution and we conclude that $(Y_{2n}/\sqrt{n + A_n^2})_n$ converges in distribution to a U.

Hence, for P-a.e. environment ω, we have

$$\sum_{n\in\mathbb{N}} P_\omega\big((X_{2n}, Y_{2n}) = (0,0)\big) = \sum_{n\in\mathbb{N}} P_\omega(X_{2n} = 0) \cdot P_\omega(Y_{2n} = 0) = \infty,$$

due to Theorem 1.1 applied with $\alpha > 1/\beta$ and due to the local limit theorem for $(Y_{2n})_n$. This implies the recurrence of $(X_n, Y_n)_n$ and so of $(M_n)_n$, for P-a.e. environment ω. □

Proposition 4.4. *If we replace Sinai's walk by the simple symmetric random walk on $\mathbb{Z}$ (i.e. if we replace ω_x by $1/2$) in the assumptions of Propositions 4.2 and 4.3, then the walk $(M_n)_n$ is recurrent if and only if $\sum_n \frac{1}{A_n\sqrt{n}} = \infty$.*

In particular it is transient as soon as $\beta < 2$.

Proof: We follow the proofs of Propositions 4.2 and 4.3 and we use the fact that $\mathbb{P}(X_{2n} = 0)$ is equivalent to $c/\sqrt{n}$ for some $c > 0$ as n goes to infinity. We have

$$\sum_n \mathbb{P}(M_n = 0) = \sum_n \mathbb{P}(X_{2n} = 0)\mathbb{P}\left(\exists K \in \{0, ..., \tau - 1\},\ Y_{2n} + \sum_{k=1}^K Z_k = 0\right),$$

where τ has the same distribution as τ_1 and where $(Z_k)_k$ is a sequence of i.i.d. random variables with distribution ν such that Y_{2n}, τ and $(Z_k)_k$ are independent. Now observe that

$$\mathbb{P}\left(\exists K \in \{0, ..., \tau - 1\},\ Y_{2n} + \sum_{k=1}^K Z_k = 0\right) \geq \mathbb{P}(Y_{2n} = 0) \sim \frac{C_1}{A_n}$$

for some $C_1 > 0$ due to the local limit theorem for $(Y_{2n})_n$ and that

$$\mathbb{P}\left(\exists K \in \{0, ..., \tau - 1\},\ Y_{2n} + \sum_{k=1}^K Z_k = 0\right) \leq \mathbb{P}\left(|Y_{2n}| \leq \sum_{k=1}^{\tau-1} |Z_k|\right)$$

$$\leq \sum_{m\geq 0} \mathbb{P}(|Y_{2n}| = m)\mathbb{P}\left(\sum_{k=1}^{\tau-1} |Z_k| \geq m\right)$$
$$\leq \frac{C_2}{A_n}\left(\mathbb{E}\left[\sum_{k=1}^{\tau-1} |Z_k|\right] + 1\right) \leq \frac{C_2}{A_n}(1 + \mathbb{E}[\tau]\mathbb{E}[|Z_1|]),$$

for $C_2 > 0$ using the uniform bound given by the local limit theorem. Hence

$$\mathbb{P}\left(\exists K \in \{0, ..., \tau - 1\},\ Y_{2n} + \sum_{k=1}^{K} Z_k = 0\right) \approx \frac{1}{A_n}.$$

■

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