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Clustering and coexistence in the one-dimensional vectorial Deffuant model

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Abstract. The vectorial Deffuant model is a simple stochastic process for the dynamics of opinions that also includes a confidence threshold. To understand the role of space in this type of social interactions, we study the process on the one-dimensional lattice where individuals are characterized by their opinion – in favor or against – about F different issues and where pairs of nearest neighbors potentially interact at rate one. Potential interactions indeed occur when the number of issues both neighbors disagree on does not exceed a certain confidence threshold, which results in one of the two neighbors updating her opinion on one of the issues both neighbors disagree on (if any). This paper gives sufficient conditions for clustering of the system and for coexistence due to fixation in a fragmented configuration, showing the existence of a phase transition between both regimes.

1. Introduction

In the voter model Clifford and Sudbury (1973); Holley and Liggett (1975), individuals are located on the vertex set of a graph and are characterized by one of two competing opinions. Individuals update their opinion independently at rate one by mimicking a random neighbor where the neighborhood is defined in an obvious manner from the edge set of the graph. This is the simplest model of opinion dynamics based on the framework of interacting particle systems. The model includes social influence, the tendency of individuals to become more similar when they interact. More recently, and particularly since the work of political scientist Axelrod Axelrod (1997), a number of variants of the voter model that also account for homophily, the tendency of individuals to interact more frequently with

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individuals who are more similar, have been introduced. These spatial processes are continuous-time Markov chains whose state at time t is a function that maps the vertex set V of a graph into a set of opinions:

$$\eta_t : V \longrightarrow \Gamma \ := \ \text{opinion set}.$$

The common modeling approach is to equip Γ with a metric, which allows one to define an opinion distance between neighbors, and to include homophily by assuming that neighbors interact at a rate which is a nonincreasing function of their opinion distance. This rate is often chosen to be the step function equal to zero if the opinion distance between the two neighbors is larger than a so-called confidence threshold and equal to one otherwise.

Model description – In the original version of the Deffuant model Deffuant et al. (2000), the opinion space is the unit interval equipped with the Euclidean distance. Neighbors interact at rate one if and only if the distance between their opinion does not exceed a certain confidence threshold, which results in a compromise strategy where both opinions get closer to each other by a fixed factor. The main conjecture about this opinion model is that, at least when the initial opinions are independent and uniformly distributed over the unit interval, the system reaches a consensus when the confidence threshold is larger than one half whereas disagreements persist in the long run when the confidence threshold is smaller than one half. This conjecture has been completely proved for the one-dimensional system in Häggström (2012); Lanchier (2012) using different techniques and we also refer to Häggström and Hirscher (2014) for additional results on the system in higher dimensions and/or starting from more general distributions. In contrast, in the vectorial version of the Deffuant model also introduced in Deffuant et al. (2000), the opinion space is the hypercube equipped with the Hamming distance:

$$\Gamma \ := \ \{0,1\}^F \quad \text{and} \quad H(u,v) \ := \ \text{card}\,\{i : u_i \neq v_i\} \quad \text{for all} \quad u,v \in \Gamma. \qquad (1.1)$$

As for the general class of opinion models described above, the system depends on a confidence threshold that we call θ from now on. To describe the dynamics, we also introduce the set

$$\Omega(x,y,\eta) \ := \ \{u \in \Gamma : H(u,\eta(y)) = H(\eta(x),\eta(y)) - 1\}$$

for each pair of neighbors x and y and each configuration η. The vectorial Deffuant model can then be formally defined as the Markov chain with generator

$$\begin{aligned} Lf(\eta) \ &= \ \textstyle\sum_x (\text{card}\,\{y : y \sim x\})^{-1} \sum_{y \sim x} (\text{card}\ \Omega(x,y,\eta))^{-1} \\ &\quad \textstyle\sum_{u \in \Omega(x,y,\eta)} \mathbf{1}\{1 \leq H(\eta(x),\eta(y)) \leq \theta\}\ [f(\eta_{x,u}) - f(\eta)] \end{aligned} \qquad (1.2)$$

where $y \sim x$ means that both vertices are nearest neighbors and where $\eta_{x,u}$ is the configuration obtained from configuration η by setting the opinion at x equal to u and leaving all the other opinions unchanged. In words, each individual looks at a random neighbor at rate one and updates her opinion by moving one unit towards the opinion of this neighbor along a random direction in the hypercube unless either the opinion distance between the two neighbors exceeds the confidence threshold or both neighbors already agree. These evolution rules, which are somewhat complicated thinking of each opinion as an element of the hypercube, have a very natural interpretation if one thinks of each opinion as a set of binary opinions – in favor or against – about F different issues. Using this point of view gives the following:

each individual looks at a random neighbor at rate one and imitates the opinion of this neighbor on an issue selected uniformly at random among the issues they disagree on (if any), which models social influence, unless the number of issues they disagree on exceeds the confidence threshold, which models homophily.

Main results – The results in Deffuant et al. (2000, section 4) are based on numerical simulations of the system on a complete graph where all the individuals are neighbors of each other, thus leaving out any spatial structure. In contrast, the main objective of this paper is to understand not only the role of the parameters but also the role of explicit space in the long-term behavior of the system. In particular, we specialize from now on in the one-dimensional system where each individual has exactly two nearest neighbors and set $V = \mathbb{Z}$. In this case, the process can exhibit two types of behavior: reach a consensus or get trapped in an absorbing state where the different opinions coexist. To define mathematically this dichotomy, we say that

- the system **clusters** whenever
$$\lim_{t \to \infty} P(\eta_t(x) = \eta_t(y)) = 1 \quad \text{for all} \quad x, y \in \mathbb{Z},$$
- the system **coexists** due to fixation whenever
$$P(\eta_t(x) = \eta_\infty(x) \text{ eventually in } t) \ = \ 1 \quad \text{for all} \quad x \in \mathbb{Z}$$
for some configuration η_∞ such that $P(\text{card}\,\{\eta_\infty(x) : x \in \mathbb{Z}\} = 2^F) = 1$.

The one-dimensional system has been studied numerically by Adamopoulos and Scarlatos (2012) who considered a percolation model starting from the uniform product measure and predicted a phase transition for the continuous-time model between clustering and coexistence at an approximate critical value $\theta_c \approx F/2$. We follow Adamopoulos and Scarlatos (2012) and study the system starting from the uniform product measure in which the opinions at different vertices are independent and equally likely, i.e.,

$$P(\eta_0(x) = u) \ = \ (1/2)^F \quad \text{for all} \quad x \in \mathbb{Z} \ \text{ and } \ u \in \Gamma. \tag{1.3}$$

The first key ingredient to prove both clustering and coexistence is to think of each opinion profile as a collection of F levels, each having two possible states, and put a particle between two neighbors at the levels they disagree on. This induces a coupling between the dynamics of opinions and a system of annihilating random walks similar to the one introduced in Lanchier and Schweinsberg (2012). The inclusion of a confidence threshold in the opinion model translates into the following in the system of random walks: particles jump at a positive rate except the ones that are part of a pile whose size exceeds the confidence threshold which do not move because they are carried by an edge connecting two individuals who disagree too much to interact. We call a particle either active or frozen depending on whether it jumps at a positive rate or cannot jump at all.

To begin with, we look at the process with $F \leq \theta$. In this case, regardless of the number of issues on which they disagree, neighbors can always interact so, at every issue-level, the configuration of opinions evolves according to a voter model where individuals interact at rates that vary across space and time. More precisely, whenever two neighbors disagree on a given issue, they come into an agreement on this issue at rate one over the total number of issues on which they disagree, their new common opinion being equally likely to be each of the two possible opinions.

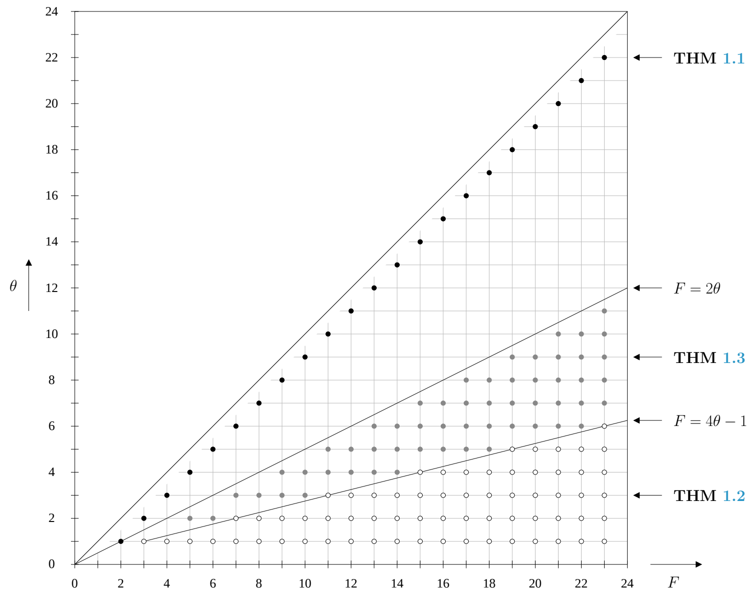

Figure 1.1. Phase diagram of the one-dimensional vectorial Deffuant model in the $F - \theta$ plane along with a summary of our theorems. The black dots correspond to the set of parameters for which clustering is proved whereas the white dots correspond to the set of parameters for which coexistence is proved.

For the system of particles introduced above, this means that all the particles are active and evolve according to simple symmetric random walks that jump at rates that vary across space and time. Since such random walks are recurrent (because they are symmetric and jump at a positive rate) and since they annihilate by pair just like the interfaces of the one-dimensional voter model, the system of active particles goes extinct, which implies clustering of the opinion model.

Now, we look at the process with $F = \theta + 1$. In this case, only individuals who disagree on all issues do not interact. In particular, the only frozen piles are the ones with F particles, which results in a system of annihilating random walks similar enough to its counterpart for the Axelrod model so that the machinery in Lanchier and Schweinsberg (2012) also applies to our case. Using this machinery, it can be proved that each frozen particle will eventually become active or annihilate with an active particle. From this, it can be deduced that both frozen and active particles ultimately go extinct, which again implies clustering of the opinion model therefore we have the following result.

Theorem 1.1. – *Assume* (1.3) *and* $F = \theta + 1$. *Then, the system clusters.*

Even though the proof of this theorem is the same as the proof in Lanchier and Schweinsberg (2012), the main ingredients will be briefly explained in Section 3 for the sake of completeness.

To study the coexistence regime, we again use the coupling with annihilating random walks as well as a characterization of fixation due to Bramson and Griffeath (1989) based on certain spatial properties of so-called active paths that keep track of the offspring of the opinions initially present in the system. This characterization leads to a sufficient condition for survival of the frozen particles on a large interval, and therefore coexistence due to fixation, based on the initial number of active and frozen particles in this interval. Using estimates on the random number of active particles that annihilate with frozen particles to turn a pile of frozen particles into a smaller pile of active particles, a random weight is then attributed to each pile of particles at time zero. This, together with large deviation estimates for the cumulative weight in large intervals, implies that the system coexists whenever the expected value of the weight of a typical pile is positive. Relying on some symmetry property of the binomial random variable, we make explicit the set of parameters for which the expected value of the weight is positive, from which we deduce the following theorem.

Theorem 1.2. – *Assume* (1.3) *and* $F \geq 4\theta - 1$. *Then, the system coexists.*

Note that both theorems imply the existence of at least one phase transition between consensus and coexistence at some critical confidence threshold

$$\theta_c \in ((1/4)(F+1), F-1) \quad \text{for all} \quad F \geq 2$$

which gives a rigorous proof of part of the conjecture announced in Adamopoulos and Scarlatos (2012).

To gain some insight on the reason why the critical threshold might indeed be equal to $F/2$, we finally look at the system starting from a non-uniform product measure where two opposite designated opinion profiles start at high density whereas the other opinion profiles start at low density. More precisely, we now assume that the system starts from the product measure with

$$\begin{aligned} P\,(\eta_0(x) = u) \ &= \ 1/2 - (2^{F-1} - 1)\,\rho \quad \text{when} \quad u \in \{u_-, u_+\} \\ &= \ \rho \qquad\qquad\qquad\qquad\quad \text{when} \quad u \notin \{u_-, u_+\} \end{aligned} \tag{1.4}$$

where $\rho \in [0, 2^{-F})$ is a small parameter and where

$$u_- \ := \ (0, 0, \ldots, 0) \quad \text{and} \quad u_+ := (1, 1, \ldots, 1).$$

For the process starting from this initial distribution, the methodology developed to prove the previous theorem can be applied together with large deviation estimates for non-independent random variables to obtain the following result.

Theorem 1.3. – *Assume* (1.4) *and* $F > 2\theta$. *Then, there exists* $\rho_0 > 0$ *that depends on the two parameters such that the system coexists for all* $\rho \leq \rho_0$.

Even though the theorem only gives a sufficient condition for coexistence due to fixation, the proof somewhat suggests that this condition is also necessary. Our intuition relies on the fact that the largest blockades contain F frozen particles while the collision of $F - \theta$ active particles with such a blockade can create a total of θ active particles. In particular, when $F \leq 2\theta$, it is possible that the number of active particles created is at least equal to the number of active particles destroyed, which leads to a global extinction of all the particles and therefore clustering.

We refer the reader to Figure 1.1 for a summary of our results. The rest of this paper is devoted to proofs starting in the next section with the coupling with annihilating random walks which is then used to show our three theorems in the subsequent three sections.

2. Coupling with annihilating random walks

In this section, we follow the approach of Lanchier and Schweinsberg (2012) to define a coupling between the process and a collection of systems of symmetric annihilating random walks. The basic idea is to visualize each opinion profile, i.e., each vertex of the hypercube, using F levels each having two possible states and put particles between two neighbors at the levels they disagree on. For an illustration, we refer to Figure 2.2 where black and white dots represent the two possible opinions on each issue and the crosses indicate the position of the particles. To make this construction rigorous, we first identify the process with the spin system

$$\bar{\eta}_t : \mathbb{Z} \times \{1, 2, \ldots, F\} \ \to \ \{0, 1\} \quad \text{where} \quad \bar{\eta}_t(x, i) := i\text{th coordinate of } \eta_t(x).$$

This again defines a Markov process. To describe this system of particles, it is also convenient to identify the edges connecting neighbors with their midpoint

$$e \ := \ (x, x+1) \ \equiv \ x + 1/2 \quad \text{for all} \quad x \in \mathbb{Z}$$

and to define translations on this set of edges by setting

$$e + a := (x, x+1) + a \equiv x + 1/2 + a \quad \text{for all} \quad e \in \mathbb{Z} + 1/2 \ \text{ and } \ a \in \mathbb{R}.$$

The process that keeps track of the disagreements is then defined as

$$\xi_t(e, i) \ := \ \mathbf{1}\,\{\bar{\eta}_t(e - 1/2, \, i) \neq \bar{\eta}_t(e + 1/2, \, i)\} \tag{2.1}$$

and we put a particle on edge e at level i if and only if $\xi_t(e, i) = 1$ to visualize the corresponding configuration of interfaces. The reason for introducing this system of particles is that

- the limiting behavior of the process (1.2) can be easily translated into simple properties for the system of particles (2.1): clustering is equivalent to extinction of the particles whereas coexistence is equivalent to survival of the particles and
- the particles of (2.1) consist of a collection of systems of simple symmetric annihilating random walks somewhat easier to analyze than the vectorial Deffuant model itself.

The number of particles per edge, defined as

$$\zeta_t(e) \ := \ \xi_t(e, 1) + \xi_t(e, 2) + \cdots + \xi_t(e, F) \quad \text{for each edge } e,$$

is a key quantity to describe the dynamics of the system of particles since it is equal to the opinion distance between the two individuals connected by the edge:

$$\begin{aligned} \zeta_t(e) \ &= \ \operatorname{card}\{i : \xi_t(e, i) = 1\} \\ &= \ \operatorname{card}\{i : \bar{\eta}_t(e - 1/2, \, i) \neq \bar{\eta}_t(e + 1/2, \, i)\} \\ &= \ H(\eta_t(e - 1/2), \eta_t(e + 1/2)). \end{aligned}$$

Since the number of particles on the edge is equal to the opinion distance, an interaction along edge $e := (x, x+1)$ results in the following alternative:

(1) There are more than θ particles on the edge in which case nothing happens because the opinion distance between the two neighbors exceeds the confidence threshold.

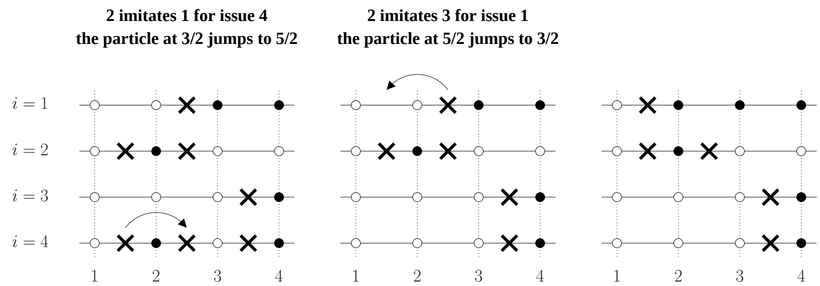

FIGURE 2.2. Illustration of the coupling between the vectorial Deffuant model and the system of simple symmetric annihilating random walks. Black and white dots represent the two possible states of the individuals' opinion on each issue while the crosses indicate the position of the particles. In our example, there are $2^4 = 16$ possible opinion profiles and the confidence threshold is $\theta \geq 2$. The two imitation events represented in this realization translate into two consecutive jumps of particles, with the first one resulting in the annihilation of two particles.

(2) There are at most θ particles on the edge in which case one of the issues for which the two neighbors disagree is chosen uniformly at random and the opinion of either vertex x or vertex $x+1$ at this level is switched. Note that the issues the two neighbors disagree on correspond to the levels which are occupied by a particle so, after the interaction, the particle at the chosen level disappears while the state of one of the two edges $e \pm 1$ at the same level switches from either empty to occupied or from occupied to empty.

Combining 1 and 2, we deduce that the system of particles (2.1) evolves at each level according to a system of simple symmetric annihilating random walks as illustrated in Figure 2.2. In addition, since the issues on which neighbors disagree are chosen for update uniformly at random, at each edge occupied by j particles, these particles jump individually at rate

$$\begin{aligned} r(j) &= j^{-1} \quad \text{when} \quad 0 < j \leq \theta \\ &= 0 \quad \text{when} \quad \theta < j \leq F \end{aligned} \tag{2.2}$$

making the F systems of symmetric annihilating random walks non-independent. Motivated by the transition rates in (2.2), we call an edge either a **live edge** or a **blockade** depending on whether they have at most or more than θ particles, respectively. Accordingly, we call the particles at this edge either **active** particles or **frozen** particles, respectively, and notice that active particles jump at a positive rate whereas frozen particles cannot jump at all.

Both the vectorial Deffuant model and its coupled system of annihilating random walks starting from any initial configuration can be constructed from the same percolation structure using a standard argument due to Harris Harris (1972). This percolation structure consists of a random graph involving independent Poisson processes marking the times at which potential jumps or interactions occur and additional collections of independent Bernoulli random variables and uniform random variables to determine the outcome of each jump or interaction.

To make this construction rigorous, for each pair of individual-issue or vertex-level $(x, i) \in \mathbb{Z} \times \{1, 2, \ldots, F\}$,

- we let $(N_{x,i}(t) : t \geq 0)$ be a rate one Poisson process,
- we denote by $T_{x,i}(n)$ its nth arrival time: $T_{x,i}(n) := \inf\{t : N_{x,i}(t) = n\}$,
- we let $(B_{x,i}(n) : n \geq 1)$ be a collection of independent Bernoulli random variables with
$$P(B_{x,i}(n) = +1) \ = \ P(B_{x,i}(n) = -1) \ = \ 1/2,$$
- we let $(U_{x,i}(n) : n \geq 1)$ be a collection of independent Uniform $(0, 1)$ random variables.

Then, at each time $t := T_{x,i}(n)$, we draw an arrow
$$x \ \to \ y := x + B_{x,i}(n) \quad \text{with the label } i \tag{2.3}$$
and call this arrow an **active** arrow if and only if
$$\xi_{t-}(e, i) = 1 \ \text{ and } \ U_{x,i}(n) \leq r(\zeta_{t-}(e)) \quad \text{where} \quad e := x + (1/2)\, B_{x,i}(n). \tag{2.4}$$
The vectorial Deffuant model and the system of annihilating random walks can then be constructed from the resulting percolation structure by assuming that arrows which are not active have no effect on any of the two processes whereas if the i-arrow (2.3) is active (2.4) then

- at time t, the individual at y looks at the individual at x and imitates her opinion for the ith issue, therefore we set $\bar{\eta}_t(y, i) := \bar{\eta}_{t-}(x, i)$,
- the particle at $x + (1/2)\, B_{x,i}(n)$ at level i jumps to $x + (3/2)\, B_{x,i}(n)$.

To establish our results, it is also useful to identify the vertices where the opinions of some given space-time point originate from. This can be done looking at **active paths**: we say that there is an active i-path from (z, s) to (x, t) whenever there are sequences of times and vertices
$$s_0 \ = \ s \ < \ s_1 \ < \ \cdots \ < \ s_{n+1} \ = \ t \qquad \text{and} \qquad x_0 \ = \ z, x_1, \ldots, x_n \ = \ x$$
such that the following two conditions hold:

(1) For $j = 1, 2, \ldots, n$, there is an active i-arrow $x_{j-1} \to x_j$ at time s_j.

(2) For $j = 0, 1, \ldots, n$, there is no active i-arrow pointing at $\{x_j\} \times (s_j, s_{j+1})$.

We say that there is a **generalized active path** from (z, s) to (x, t) whenever

3. For $j = 1, 2, \ldots, n$, there is an active arrow $x_{j-1} \to x_j$ at time s_j.

Later, we will use the notations $\overset{i}{\rightsquigarrow}$ and $\rightsquigarrow$ to indicate the existence of an active i-path and a generalized active path. We also point out that for every space-time point (x, t) there is a unique space-time point $(z, 0)$ such that both points are connected by an active i-path which, using a simple induction, implies that the corresponding individuals at the corresponding times agree on the ith issue, i.e., z is the ancestor of (x, t) for the ith issue.

3. Proof of Theorem 1.1

In this section, we prove that, when $F = \theta + 1$, the opinion model clusters. In this case, the process is closely related to the two-state Axelrod model Axelrod (1997). Indeed, the one-dimensional construction in the previous section can be applied to the latter, which again results in a system of non-independent annihilating random walks. The only difference between the two models is that, in the system of random walks coupled with the Axelrod model, at each edge occupied by j particles, these particles jump at rate

$$r_{\mathrm{ax}}(j) \;\; = \;\; j^{-1}(1 - j/F) \quad \text{when} \quad j \neq 0. \tag{3.1}$$

In particular, when $F = \theta + 1$, it follows from (2.2) and (3.1) that, for both models, an edge is a blockade if and only if it has exactly F particles:

$$r(j) = 0 \qquad \text{if and only if} \qquad r_{\mathrm{ax}}(j) = 0 \qquad \text{if and only if} \qquad j = F.$$

Now, clustering of the two-state Axelrod model has been proved in Lanchier and Schweinsberg (2012) and, while it heavily relies on the fact that an edge is a blockade if and only if it has F particles, the proof is not sensitive to the exact rate at which active particles jump. In particular, Theorem 1.1 follows from the arguments introduced in Lanchier and Schweinsberg (2012) for the Axelrod model. In addition to the coupling with annihilating random walks, there are two key ingredients: each blockade breaks eventually with probability one and, as a consequence, the system of active and frozen particles goes extinct. We only give the idea of the proof and refer to Lanchier and Schweinsberg (2012, Sections 3–4) for more details.

Blockade destruction – The first step is to prove destruction of the blockades: assuming that a designated edge $e_\star$ is a blockade at some time t, we have

$$T \; := \; \inf\{s > t : \zeta_s(e_\star) \neq F\} \; < \; \infty \quad \text{with probability one.} \tag{3.2}$$

The proof of (3.2) will involve the probability

$p \;\; :=$ conditional probability that the leftmost edge of the system of random walks on $\mathbb{N}$ has F particles at all times given that it has F particles at time 0.

We will see that this probability is equal to zero but, seeking a contradiction, we will assume that it is strictly positive. The proof relies on two ingredients: parity preserving of the number of particles at each level and a symmetry argument due to Adelman Adelman (1976) to show site recurrence of systems of annihilating random walks. To briefly explain parity preserving, assume that

- edge $e^\star$ with $e^\star > e_\star$ also is a blockade at time t, i.e., $\zeta_t(e^\star) = F$, and
- between the two blockades $e_\star$ and $e^\star$, the number of particles at some level i and the number of particles at some other level j do not have the same parity, i.e., for some $i \neq j$, we have

$$\textstyle\sum_{e_\star \leq e \leq e^\star} \; \xi_t(e,i) \; \neq \; \sum_{e_\star \leq e \leq e^\star} \; \xi_t(e,j) \quad \mathrm{mod}\ 2. \tag{3.3}$$

Now, let τ be the first time one of these two blockades breaks. Since particles at the same level annihilate by pairs, the parity of the number of particles between the two blockades is preserved at each level and up to time τ. This, together with (3.3), implies that, up to time τ, there is at least one active particle between the two blockades so either this particle or another active particle outside the interval breaks one of the blockades after a finite time:

$$\tau := \inf\{s > t : \zeta_s(e_\star) \neq F \text{ or } \zeta_s(e^\star) \neq F\} < \infty \quad \text{with probability one.} \tag{3.4}$$

The property in (3.2) can be deduced from its analog (3.4) for two blockades also using some symmetry arguments through the following construction. To prove this statement, we first introduce

$$\begin{aligned} B_0 \ &:= \ \{e_\star, e_\star + 1, \ldots, e_*, e_* + 1, e_* + 2\} \quad \text{where} \\ e_* \ &:= \ \min \{e > e_\star : e \text{ and } e + 1 \text{ and } e + 2 \\ &\qquad\qquad \text{have not been updated by time } t\}. \end{aligned}$$

Next, we partition the half-line into intervals with the same length as B_0 by setting

$$B_n \ := \ B_0 + (e_* + 3 - e_\star)\, n \quad \text{for all} \quad n \geq 1$$

and let N be the smallest n such that

- the configurations of particles in B_0 and B_n at time t can be obtained from one another by translation or reflection and
- none of the edges in B_n has been updated by time t.

Then, letting $e^\star$ be the rightmost edge in B_N,

(a) the probability that the configurations of particles in B_0 and B_N at time t can be obtained from one another by reflection is $\geq 1/2$, and

(b) the probability that (3.3) holds, in which case (3.4) also holds, is $\geq 1/2$.

In case the events in (a) or (b) do not occur, we repeat the same construction starting from the same time t but replacing the interval B_0 with the interval

$$\begin{aligned} B_0' \ &:= \ \text{the smallest interval that contains } B_0 \text{ and } B_N \\ &= \ \{e_\star, e_\star + 1, \ldots, e^\star - 1, e^\star\} \end{aligned} \tag{3.5}$$

After at most a geometric number of steps, both (a) and (b) occur. To avoid cumbersome notation, we again let $e^\star$ be the rightmost edge of the rightmost interval in this construction. Because parity is preserved, one of the two blockades at $e_\star$ and $e^\star$ breaks after a finite time. The probability that, at this time, the rightmost blockade breaks because of the jump of an active particle that originates from its right is by definition at most $1 - p$. In particular,

(c) by symmetry due to reflection (a), the conditional probability given (b) that the blockade $e_\star$ breaks before the blockade $e^\star$ is at least $p/2$.

In case the event in (c) does not occur, we repeat the same construction but starting from the time at which the blockade breaks and using the substitution (3.5). This construction implies the existence of an increasing sequence (t_j) of finite stopping times such that

$$P\,(A_k \,|\, (A_1 \cup A_2 \cup \cdots \cup A_{k-1})^c) \ \geq \ p/2 \quad \text{for all} \quad k > 0 \tag{3.6}$$

where A_j is the event that the blockade at $e_\star$ breaks between times t_{j-1} and t_j. Finally, seeking a contradiction, we now assume that (3.2) does not hold for the system of random walks where edges to the left of $e_\star$ have been removed. Since the probability that $e_\star$ has not been updated until time t is positive, the probability p must be positive so that (3.2) does not hold. But this, together with (3.6), implies that one of the A_k must occur so (3.2) holds, a contradiction. This implies the desired result for the process where edges to the left of $e_\star$ have been removed, from which the analog for the original process on the integers directly follows.

Extinction of the particles – To complete the proof of the theorem, it suffices to show extinction of the system of random walks, since this property is equivalent to clustering of the original opinion model. The proof deals with active particles and frozen particles separately. Seeking a contradiction, we first assume that the expected number of active particles at edge e, which does not depend on the choice of e due to translation invariance, does not converge:

$$\begin{aligned} &\limsup_{t\to\infty} \ E\,(\zeta_t(e)\,\mathbf{1}\{\zeta_t(e)\neq F\}) \\ &\quad \neq \ \liminf_{t\to\infty} \ E\,(\zeta_t(e)\,\mathbf{1}\{\zeta_t(e)\neq F\}). \end{aligned} \tag{3.7}$$

There are two types of events that reduce the number of active particles: annihilating events and active particles becoming frozen. In particular, the fact that the expected number of active particles per edge goes infinitely often from the lim sup to the lim inf in (3.7) implies that the expected number of annihilating events per edge or freezing events per edge is infinite. The former leads to an expected number of particles per edge decreasing to minus infinity while the latter leads to an expected number of frozen particles per edge increasing to infinity, i.e.,

$$\lim_{t\to\infty} \ E\,(\zeta_t(e)) = -\infty \quad \text{or} \quad \lim_{t\to\infty} \ E\,(\zeta_t(e)\,\mathbf{1}\{\zeta_t(e) = F\}) = \infty, \tag{3.8}$$

which is not possible. In particular, (3.7) is not true. We refer the reader to Lanchier and Schweinsberg (2012, Lemma 6) for more details. Seeking again a contradiction, we now assume that the expected number of active particles per edge converges to a positive limit:

$$\lim_{t\to\infty} \ E\,(\zeta_t(e)\,\mathbf{1}\{\zeta_t(e)\neq F\}) \ = \ \epsilon \ > \ 0. \tag{3.9}$$

Since one-dimensional symmetric random walks are recurrent, each active particle either gets annihilated or becomes frozen eventually with probability one therefore (3.9) implies that, at all times, the expected number of annihilating events per edge per unit of time or the expected number of freezing events per edge per unit of time is larger than some positive constant, which again leads to the impossible statement (3.8). It follows that (3.9) is not true. We refer the reader to Lanchier and Schweinsberg (2012, Lemma 7) for more details. To deal with the frozen particles, we first observe that, since the expected number of particles per edge can only decrease, it has a limit as time goes to infinity. This, together with the fact that the expected number of active particles per edge has a limit, implies that the expected number of frozen particles per edge has a limit as well. Seeking once more a contradiction, we assume that this limit is positive:

$$\lim_{t\to\infty} \ E\,(\zeta_t(e)\,\mathbf{1}\{\zeta_t(e) = F\}) \ = \ \epsilon \ > \ 0. \tag{3.10}$$

Since each blockade breaks eventually with probability one according to (3.2) and since each blockade destruction results in the annihilation of two particles, (3.10) implies that the expected number of annihilating events per edge per unit of time is larger than a positive constant, thus leading to the left-hand side of (3.8), again a contradiction. See Lanchier and Schweinsberg (2012, Lemma 8) for more details.

4. Proof of Theorem 1.2

This section is devoted to the study of the coexistence regime for the system starting from the uniform product measure. The first ingredient is a construction based on duality-like techniques due to Bramson and Griffeath Bramson and Griffeath (1989), which gives a sufficient condition (4.6) for fixation of an interacting particle system. Starting from this general condition, we then derive a more explicit condition for fixation using properties of the opinion model as well as large deviation estimates. To begin with, we state the following lemma, which is the analog of Bramson and Griffeath (1989, Lemma 2).

Lemma 4.1. – *For all* $(z, i) \in \mathbb{Z} \times \{1, 2, \ldots, F\}$*, let*

$$T(z, i) \ := \ \inf \{t : (z, 0) \overset{i}{\rightsquigarrow} (0, t)\}.$$

Then, the system fixates whenever

$$\lim_{N \to \infty} P\,(T(z, i) < \infty \ \textit{for some} \ z < -N \ \textit{and} \ i = 1, 2, \ldots, F) \ = \ 0. \qquad (4.1)$$

Proof: This follows the proof of Lanchier and Scarlatos (2013, Lemma 4). □

We now extend the construction in Bramson and Griffeath (1989, Section 3) using generalized active paths rather than active paths at a given level. In this construction, Lemma 4.1 is used to exhibit the connection between the initial configuration of the system, i.e., the initial number of active particles and the initial number of frozen particles, and the key event

$$H_N \ := \ \{T(z, i) < \infty \text{ for some } z < -N \text{ and some } i = 1, 2, \ldots, F\}$$

that appears in (4.1). The main objective is to construct a random interval such that all the blockades initially in this interval must have been destroyed by either active particles initially in this interval or active particles that result from the destruction of these blockades. Let

$$\tau \ = \ \inf \{T(z, i) : z \in (-\infty, -N) \text{ and } i = 1, 2, \ldots, F\}$$

be the first time that, for some issue $i = 1, 2, \ldots, F$, an active i-path that originates from the left of $-N$ hits the origin, and observe that H_N can be written as

$$H_N \ = \ \{\tau < \infty\}. \qquad (4.2)$$

Given H_N, we let $z^\star < -N$ be the initial position of this active path and

$$\begin{array}{rcl} z_- & := & \min \{z \in \mathbb{Z} : (z, 0) \rightsquigarrow (0, \tau)\} \ \leq \ z^\star \ < \ -N \\ z_+ & := & \max \{z \in \mathbb{Z} : (z, 0) \rightsquigarrow (0, \sigma) \text{ for some } \sigma < \tau\} \ \geq \ 0 \end{array} \qquad (4.3)$$

and define $I = (z_-, z_+)$. Since generalized active paths cannot cross edges which are occupied by a blockade, we have the following two properties:

- All the blockades initially in I must have been destroyed, i.e., turned into piles of θ active particles due to annihilating events, by time τ.
- None of the active paths can cross going from left to right the leftmost generalized active path in (4.3). Similarly, none of the active paths can cross from right to left the rightmost generalized active path. In particular, the active particles initially outside I cannot jump inside the space-time region delimited by the two generalized active paths.

This, together with (4.2), implies that, on the event H_N, all the blockades initially in the interval I must have been destroyed before time τ by either active particles initially in this interval or active particles that result from the destruction of these blockades. To quantify this last event, we use a collection of random variables, that we shall call **contributions** of the edges, which are measurable with respect to the initial configuration and the graphical representation of the process. More precisely, we let

$$\text{cont}\,(e) \ := \ \begin{array}{l} \text{number of active particles that either annihilate or} \\ \text{become frozen as the result of a jump onto } e \text{ before} \\ \text{the first jump of an active particle initially at } e \\ \text{minus the number of particles initially at } e \end{array} \qquad (4.4)$$

when e is initially a live edge, and

$$\text{cont}\,(e) \;:=\; \begin{array}{l}\text{number of active particles that either annihilate or} \\ \text{become frozenas the result of a jump onto } e \text{ before } e \\ \text{becomes a live edge minus the number of particles} \\ \text{initially at } e \text{ that ever become active}\end{array} \tag{4.5}$$

when e is initially a blockade. Basically, the contribution counts particles using different weights: particles initially active that become frozen are counted positively while particles initially frozen that become active are counted negatively. In addition, particles initially active that annihilate are counted positively if the annihilation is the result of their jump but negatively otherwise, so that overall pairs of active particles that annihilate do not contribute. Note that particles initially frozen that stay frozen are not counted because there are no such particles in the interval I. The fact that, on the event H_N, all the blockades initially in I must have been destroyed by active particles that are either initially in this interval or that result from the destruction of these blockades can then be written using these contributions as follows:

$$\begin{array}{rcl} H_N & = & \{\tau < \infty\} \\ & = & \{\tau < \infty \text{ and } \sum_{e \in I} \text{cont}\,(e) \leq 0\} \\ & \subset & \{\sum_{e \in (l,r)} \text{cont}\,(e) \leq 0 \text{ for some } l < -N \text{ and some } r \geq 0\}. \end{array} \tag{4.6}$$

We now briefly describe the structure of our proof to deduce fixation and coexistence. The first step is to find an explicit random function ϕ, that we shall call **weight**, defined on the edge set and which is stochastically smaller than the contribution. Then, proving large deviation estimates for the total weight of a large interval and using Lemma 4.1 and (4.6), we will deduce that fixation occurs whenever the expected value of the weight at a single edge is strictly positive. To complete the proof, we will invoke the symmetry of the probability mass function of the binomial random variable to study the sign of the expected weight from which fixation will follow for the parameter region described in the statement of the theorem with the exception of the three-feature system with threshold one. To study this last case, we will improve our stochastic bound for the contribution by also accounting for pairs of active particles forming blockades.

Lemma 4.2. – *The contribution* cont (e) *is stochastically larger than*

$$\begin{array}{rcll} \phi(e) & := & -j & \textit{when} \quad \zeta_0(e) = j \leq \theta \\ & := & j + 2\,(X_j - \theta) & \textit{when} \quad \zeta_0(e) = j > \theta \end{array}$$

where $X_j := \text{Bernoulli}\,(1 - j/F)$.

Proof: Let j be the initial number of particles at e and assume first that $j \leq \theta$. In this case, the edge is initially a live edge therefore (4.4) implies that

$$\text{cont}\,(e) \;\geq\; \text{minus the number of particles initially at } e \;=\; -j$$

almost surely, which proves the first part of the lemma. Now, assume that $j > \theta$, implying that the edge is initially a blockade. Then, observe that $j - \theta$ active particles must annihilate with some of the frozen particles of the blockade to break the blockade, and that this results in a total of exactly θ frozen particles initially at edge e becoming active. This together with (4.5) gives the following lower bound for the contribution:

$$\mathrm{cont}\,(e) \quad \geq \quad (j-\theta)-\theta \ = \ j-2\theta \quad \text{almost surely.}$$

The last step to improve the bound as indicated in the statement of the lemma is to also estimate the number of active particles that become frozen as the result of a jump onto e before the blockade becomes a live edge. More precisely, we look at the probability that the first jump of an active particle onto the blockade results in an annihilating event, which can be computed explicitly using the following symmetry argument: since both the initial distribution and the dynamics of the model are invariant by permutation of the levels, and since the configuration of particles outside e is independent of the distribution of particles at e by the time of the first jump of an active particle onto e, this first jump occurs with equal probability at each level. In particular, the first jump of an active particle onto the blockade e results in either

- an annihilating event with probability j/F, in which case the number of active particles required to break the blockade is the same as before or
- a blockade increase with probability $1-j/F$, in which case one active particle becomes frozen and one additional active particle is required to eventually break the blockade.

Since two additional active particles are eliminated in the event of a blockade increase, we deduce that the contribution of e is stochastically larger than

$$(1-X_j)(j-2\theta)+X_j\,(j-2\theta+2) \ = \ j+2\,(X_j-\theta)$$

where $X_j = \mathrm{Bernoulli}\,(1-j/F)$. This completes the proof. $\square$

In the next lemma, we prove large deviation estimates for the weight in a large interval, from which we deduce, in the subsequent lemma, that the system fixates whenever the expected value of the weight function is strictly positive.

Lemma 4.3. – *There exist $C_1<\infty$ and $c_1>0$ such that, for all $\epsilon>0$,*

$$P\,(\textstyle\sum_{e\in(-N,0)}\phi(e)\leq N(E\phi(e)-\epsilon)) \ \leq \ C_1\exp(-c_1N\epsilon^2).$$

Proof: The idea is to prove that the number of j-edges in a given interval is a binomial random variable and then apply the large deviation estimates

$$\begin{array}{rcl} P\,(Z\leq N(p-\epsilon)) & \leq & \exp(-(1/2)\,Np^{-1}\epsilon^2) \\ & \leq & \exp(-(1/2)\,N\epsilon^2) \\ P\,(Z\geq N(p+\epsilon)) & \leq & \exp(-(1/2)\,N(1-p)^{-1}\epsilon^2) \\ & \leq & \exp(-(1/2)\,N\epsilon^2) \end{array} \tag{4.7}$$

where $Z=\mathrm{Binomial}\,(N,p)$. First, we note that, starting from the uniform product measure, the opinions at two adjacent vertices at a given level are initially either equal or different with probability one half, independently of the rest of the initial configuration. This implies that the initial number of particles at any given edge is a binomial random variable:

$$p_j \ := \ P\,(\zeta_0(e)=j) \ = \ \binom{F}{j}(1/2)^F \quad \text{for each edge } e. \tag{4.8}$$

Recalling the expression of the bound $\phi(e)$, we thus obtain

$$\begin{aligned} E\phi(e) &= \textstyle\sum_{j\leq\theta}(-j)\,p_j + \sum_{j>\theta}(j+2\,(1-j/F-\theta))\,p_j \\ &= \textstyle\sum_{j\leq\theta}(-j)\,p_j + \sum_{j>\theta}(j-2\theta)\,p_j + \sum_{j>\theta}2\,(1-j/F)\,p_j. \end{aligned} \tag{4.9}$$

To also have an explicit expression for the weight of $(-N,0)$, we let

$$\begin{aligned} \Omega_j &:= \{e\in(-N,0):\zeta_0(e)=j\} \\ e_N(j) &:= \operatorname{card}\Omega_j \quad\text{for}\quad j=0,1,\ldots,F, \end{aligned}$$

denote the set of and the number of j-edges in $(-N,0)$, respectively. Using again that initially the different pairs edge-level are independently empty or occupied by a particle with equal probability, a simple extension of the symmetry argument from Lemma 4.2 implies that the Bernoulli random variables that determine the outcome of the first jump onto the blockades are independent. It follows that the random weight of the interval $(-N,0)$ can be expressed as

$$\begin{aligned} \textstyle\sum_{e\in(-N,0)}\phi(e) &= \textstyle\sum_{j\leq\theta}(-j)\,e_N(j) \\ &\quad+\textstyle\sum_{j>\theta}(j+2\,(X_{e,j}-\theta))\,e_N(j) \\ &= \textstyle\sum_{j\leq\theta}(-j)\,e_N(j) \\ &\quad+\textstyle\sum_{j>\theta}(j-2\theta)\,e_N(j)+\sum_{j>\theta}\sum_{e\in\Omega_j}2X_{e,j} \end{aligned} \tag{4.10}$$

where the random variables $X_{e,j}$ are independent Bernoulli random variables with the same success probability $1-j/F$. Combining the expressions (4.9) and (4.10), we deduce that

$$\begin{aligned} &P\left(\textstyle\sum_{e\in(-N,0)}\phi(e)\leq N(E\phi(e)-\epsilon)\right) \\ &\quad\leq \textstyle\sum_{j\leq\theta}P\left((-j)(e_N(j)-Np_j)\leq -N\epsilon/2F\right) \\ &\quad+ \textstyle\sum_{j>\theta}P\left((j-2\theta)(e_N(j)-Np_j)\leq -N\epsilon/2F\right) \\ &\quad+ \textstyle\sum_{j>\theta}P\left(\sum_{e\in\Omega_j}2X_{e,j}-2\,(1-j/F)Np_j\leq -N\epsilon/2F\right). \end{aligned} \tag{4.11}$$

To bound the first two terms, we first use (4.8) and independence to deduce

$$e_N(j) \;=\; \operatorname{Binomial}(N,p_j) \quad\text{for all}\quad j=0,1,\ldots,F. \tag{4.12}$$

Then, using (4.7) and (4.12), we get

$$\begin{aligned} &P\left((-j)(e_N(j)-Np_j)\leq -N\epsilon/2F\right) \\ &\qquad\leq\; P\left(e_N(j)-Np_j\geq N\epsilon/2F^2\right) \;\leq\; \exp(-N\epsilon^2/8F^4) \end{aligned} \tag{4.13}$$

for $j=1,2,\ldots,\theta$, and

$$\begin{aligned} &P\left((j-2\theta)(e_N(j)-Np_j)\leq -N\epsilon/2F\right) \\ &\qquad\leq\; P\left(e_N(j)-Np_j\notin(-N\epsilon/2F^2,N\epsilon/2F^2)\right) \\ &\qquad\leq\; 2\,\exp(-N\epsilon^2/8F^4) \end{aligned} \tag{4.14}$$

for $j = \theta + 1, \ldots, F$. Finally, using again the second inequality in (4.7) together with the fact that the random variables $X_{e,j}$ are independent, we get

$$\begin{aligned} & P\,(\textstyle\sum_{e \in \Omega_j} 2X_{e,j} - 2\,(1 - j/F) N p_j \\ & \qquad\qquad \leq -N\epsilon/2F \,|\, e_N(j) < N\,(p_j + \epsilon/16F)) \\ & \quad \leq \; P\,(\textstyle\sum_{e \in \Omega_j} X_{e,j} - (1 - j/F) N p_j \\ & \qquad\qquad \leq -N\epsilon/4F \,|\, \mathrm{card}\,\Omega_j = N\,(p_j + \epsilon/16F)) \\ & \quad \leq \; P\,(\mathrm{Binomial}\,(N(p_j + \epsilon/16F), 1 - j/F) \\ & \qquad\qquad\qquad \leq N((1 - j/F)\,p_j) - \epsilon/4F) \\ & \quad \leq \; P\,(\mathrm{Binomial}\,(N(p_j + \epsilon/16F), 1 - j/F) \\ & \qquad\qquad\qquad \leq N(p_j + \epsilon/16F)(1 - j/F - \epsilon/16F)) \\ & \quad \leq \; \exp(-(1/2)\,N\,(p_j + \epsilon/16F)(\epsilon/16F)^2) \end{aligned}$$

from which we deduce that

$$\begin{aligned} & P\,(\textstyle\sum_{e \in \Omega_j} 2X_{e,j} - 2\,(1 - j/F) N p_j \leq -N\epsilon/2F) \\ & \quad \leq \; P\,(\textstyle\sum_{e \in \Omega_j} 2X_{e,j} - 2\,(1 - j/F) N p_j \\ & \qquad\qquad \leq -N\epsilon/2F \,|\, e_N(j) < N\,(p_j + \epsilon/16F)) \\ & \qquad + \; P\,(e_N(j) \geq N\,(p_j + \epsilon/16F)) \\ & \quad \leq \; \exp(-(1/2)\,N\,(p_j + \epsilon/16F)(\epsilon/16F)^2) \\ & \qquad\qquad\qquad + \; \exp(-(1/2)\,N\,(\epsilon/16F)^2). \end{aligned} \tag{4.15}$$

The lemma then follows from (4.11) and (4.13)–(4.15). □

Lemma 4.4. – *The system fixates whenever $E\phi(e) > 0$.*

Proof: Let $\epsilon := E\phi(e) > 0$. Then, according to Lemma 4.3,

$$\begin{aligned} P\,(\textstyle\sum_{e \in (-N,0)} \phi(e) \leq 0) \;\; &= \;\; P\,(\textstyle\sum_{e \in (-N,0)} \phi(e) \leq N(E\phi(e) - \epsilon)) \\ &\leq \;\; C_1 \exp(-c_1 N \epsilon^2). \end{aligned}$$

This, together with (4.6) and Lemma 4.2, implies that

$$\begin{aligned} & \lim_{N \to \infty} P\,(H_N) \; = \; \lim_{N \to \infty} P\,(\tau < \infty \text{ and } \textstyle\sum_{e \in I} \mathrm{cont}\,(e) \leq 0) \\ & \quad \leq \; \lim_{N \to \infty} P\,(\textstyle\sum_{e \in (l,r)} \mathrm{cont}\,(e) \leq 0 \text{ for some } l < -N \text{ and some } r \geq 0) \\ & \quad \leq \; \lim_{N \to \infty} P\,(\textstyle\sum_{e \in (l,r)} \phi(e) \leq 0 \text{ for some } l < -N \text{ and some } r \geq 0) \\ & \quad \leq \; \lim_{N \to \infty} \textstyle\sum_{l < -N} \sum_{r \geq 0} P\,(\sum_{e \in (l,r)} \phi(e) \leq 0) \\ & \quad \leq \; \lim_{N \to \infty} \textstyle\sum_{l < -N} \sum_{r \geq 0} C_1 \exp(-c_1\,(r - l)\epsilon^2) \; = \; 0. \end{aligned}$$

In particular, Lemma 4.1 implies that the system fixates. □

Having Lemma 4.4 in hand, the last step is to exhibit the set of parameters for which the expected value of the weight is positive. This can be done by making the expression of the expected value more explicit but this leads to messy calculations. As previously mentioned, we use instead the symmetry of the probability mass function of the binomial random variable when the success probability is equal to one half. Using this approach, the expected value of the weight can be expressed as a sum of positive values when $F \geq 4\theta$, which is done in the next lemma.

Lemma 4.5. – *Assume (1.3) and $F \geq 4\theta$. Then, $E\phi(e) > 0$.*

Proof: To begin with, we introduce

$$\begin{array}{rcl} K_- & := & \text{the largest integer smaller than or equal to } (1/2)(F-1) \\ K_+ & := & \text{the smallest integer larger than or equal to } (1/2)(F+1) \end{array}$$

and observe that $K_- + K_+ = F$ and

$$\begin{array}{rcl} K_+ - K_- & = & 1 \quad \text{when } F \text{ is odd} \\ & = & 2 \quad \text{when } F \text{ is even.} \end{array} \tag{4.16}$$

Letting $q_j(\theta, F) := 2\,(1-\theta) + (1 - 2/F)\,j$, we also have

$$\begin{array}{rcl} q_{F-j}(\theta, F) & = & 2\,(1-\theta) + (1-2/F)(F-j) \\ & = & F - 2\theta - (1-2/F)\,j \\ q_j(\theta, F) + q_{F-j}(\theta, F) & = & 2\,(1-\theta) + F - 2\theta \;=\; F - 4\theta + 2 \\ q_{F/2}(\theta, F) & = & F/2 - 2\theta + 1 \;\geq\; 1 \quad \text{for } F \geq 4\theta \text{ and even.} \end{array}$$

In particular, considering the intervals

$$\begin{array}{rcl} J_1 & := & [0, \theta] \\ J_2 & := & [\theta + 1, K_-] \\ J_3 & := & [K_+, F - (\theta + 1)] \\ J_4 & := & [F - \theta, F] \end{array} \tag{4.17}$$

recalling (4.9) and using the symmetry $p_j = p_{F-j}$, we obtain

$$\begin{array}{rcl} E\phi(e) & = & \sum_{j \leq \theta} (-j)\,p_j + \sum_{j > \theta} (j + 2\,(1 - j/F - \theta))\,p_j \\ & \geq & \sum_{j \in J_1} (-j)\,p_j + \sum_{k=2,3,4} \sum_{j \in J_k} q_j(\theta, F)\,p_j \\ & = & \sum_{j \in J_1} ((-j) + q_{F-j}(\theta, F))\,p_j \\ & & \quad + \sum_{j \in J_2} (q_j(\theta, F) + q_{F-j}(\theta, F))\,p_j \\ & = & \sum_{j \in J_1} (F - 2\theta - 2\,(1 - 1/F)\,j)\,p_j \\ & & \quad + \sum_{j \in J_2} (F - 4\theta + 2)\,p_j \\ & \geq & \sum_{j \in J_1} (F - 2\,(\theta + j))\,p_j + \sum_{j \in J_2} (F - 4\theta + 2)\,p_j \;>\; 0 \end{array} \tag{4.18}$$

for all $F \geq 4\theta$ since in this case all the terms in the previous two sums are nonnegative with also some terms that are strictly positive. □

The approach of the previous proof does not extend to the case $F = 4\theta - 1$ because, for this set of parameters, the first sum in the last line of (4.18) contains a negative term. To deal with this case also avoiding messy calculations, we find a lower bound using the binomial random variable and then invoke standard large deviation estimates for this distribution.

Lemma 4.6. – *Assume (1.3) and $F = 4\theta - 1$ with $\theta \geq 2$. Then, $E\phi(e) > 0$.*

Proof: Let $F = 4\theta - 1$ and observe that

$$\begin{array}{rcl} F - 2\,(\theta + j) & \geq & 4\theta - 1 - 2\,(\theta + \theta) \;=\; -1 \quad \text{for all} \quad j \in J_1 \\ F - 4\theta + 2 & = & 4\theta - 1 - 4\theta + 2 \;=\; 1 \quad \text{for all} \quad j \in J_2. \end{array}$$

In particular, recalling (4.18), we obtain

$$\begin{aligned} E\phi(e) &\geq \textstyle\sum_{j \in J_1} (F - 2(\theta + j))\, p_j + \sum_{j \in J_2} (F - 4\theta + 2)\, p_j \\ &\geq \textstyle\sum_{j \in J_1} (-p_j) + \sum_{j \in J_2} p_j \\ &= P(Z \in J_2) - P(Z \in J_1) \end{aligned}$$

where $Z = \text{Binomial}\,(F, 1/2)$. Using that, according to (4.16), the intervals in (4.17) form a partition of the range of Z when F is odd, we obtain

$$\begin{aligned} E\phi(e) &\geq (1/2)(P(Z \in J_2) + P(Z \in J_3) - P(Z \in J_1) - P(Z \in J_4)) \\ &\geq (1/2)(1 - 2\,P(Z \in J_1) - 2\,P(Z \in J_4)) \;=\; 1/2 - 2\,P(Z \in J_1). \end{aligned}$$

Then, using the standard large deviation estimate

$$P(Z \leq F(1/2 - \epsilon)) \;\leq\; \exp(-F\epsilon^2) \quad \text{for all} \quad \epsilon \in (0, 1/2)$$

and taking $\epsilon = 13/54$, we deduce that, for all $\theta \geq 7$,

$$\begin{aligned} E\phi(e) &\geq 1/2 - 2\,P(Z \leq \theta) \\ &\geq 1/2 - 2\,P(Z \leq (4\theta - 1)(1/2 - 13/54)) \\ &\geq 1/2 - 2\,\exp(-(13/54)^2(4\theta - 1)) \;>\; 0. \end{aligned} \tag{4.19}$$

In addition, explicit calculations for $2 \leq \theta \leq 6$ show that

$$E\phi(e) \;\geq\; \left(3\binom{7}{0} + \left(\frac{9}{7}\right)\binom{7}{1} - \left(\frac{3}{7}\right)\binom{7}{2} + \binom{7}{3}\right)\left(\frac{1}{2}\right)^7 \;=\; 19/64. \tag{4.20}$$

The lemma follows from combining (4.19)–(4.20). $\square$

Putting together Lemmas 4.4–4.6, we obtain the theorem except for the three-issue threshold one system in which case a direct calculation gives

$$E\phi(e) \;=\; -\binom{3}{1}(1/2)^3 + \left(\frac{2}{3}\right)\binom{3}{2}(1/2)^3 + \binom{3}{3}(1/2)^3 \;=\; 0.$$

To also prove fixation when $\theta = 1$ and $F = 3$, the idea is to slightly improve the definition of our weight function to make its expected value strictly positive by also accounting for pairs of initially active particles that form a blockade before jumping onto a blockade.

Lemma 4.7. – *The system with $\theta = 1$ and $F = 3$ fixates.*

Proof: We define the weight of a blockade as before by setting

$$\begin{aligned} \phi(e) &:= 2 + 2\,(X_{e,2} - 1) \;=\; 2\,X_{e,2} \quad &\text{when} \quad \zeta_0(e) = 2 \\ &:= 3 + 2\,(X_{e,3} - 1) \;=\; 2\,X_{e,3} + 1 \quad &\text{when} \quad \zeta_0(e) = 3 \end{aligned} \tag{4.21}$$

where the random variables $X_{e,j} = \text{Bernoulli}\,(1 - j/3)$ are again independent. To improve our estimate for the weight of an edge initially occupied by an active particle, we take into account the possibility that, before it jumps, this active particle forms a blockade of size two with another active particle. Let A_e be such an event for an active particle initially at e. To compute the probability of this event, we introduce the following two events:

B_e^- := there is initially an active particle at $e-1$ which is not at the same level as the active particle initially at edge e

B_e^+ := there is initially an active particle at $e+1$ which is not at the same level as the active particle initially at edge e.

Observe that, on the event $B_e^{\pm}$, the event A_e occurs whenever the first jump of an active particle either directed to or starting from one of the two edges e and $e \pm 1$ is a jump $e \pm 1 \to e$. Since all the active particles jump at the same rate,

$$\begin{aligned} P(A_e) \ &\geq \ P(A_e \cap (B_e^- \setminus B_e^+)) \\ &\qquad + P(A_e \cap (B_e^+ \setminus B_e^-)) + P(A_e \cap (B_e^- \cap B_e^+)) \\ &\geq \ (1/6)\, P(B_e^- \setminus B_e^+) \\ &\qquad + (1/6)\, P(B_e^+ \setminus B_e^-) + (2/8)\, P(B_e^- \cap B_e^+). \end{aligned} \tag{4.22}$$

Independence and basic counting also imply that

$$\begin{aligned} P(B_e^- \setminus B_e^+) \ &= \ P(B_e^-)(1 - P(B_e^+)) \\ &= \ (2/8) \times (6/8) \ = \ 3/16 \\ P(B_e^+ \setminus B_e^-) \ &= \ P(B_e^+)(1 - P(B_e^-)) \\ &= \ (2/8) \times (6/8) \ = \ 3/16 \\ P(B_e^- \cap B_e^+) \ &= \ P(B_e^-)\, P(B_e^+) \\ &= \ (2/8) \times (2/8) \ = \ 1/16. \end{aligned} \tag{4.23}$$

Combining (4.22)–(4.23), we deduce that

$$P(A_e) \ \geq \ (1/6) \times (3/16) + (1/6) \times (3/16) + (2/8) \times (1/16) \ = \ 5/64.$$

Note also that the events in (4.22) for different edges are independent whenever the two edges are at least distance four apart therefore our previous stochastic lower bound for the contribution of an active particle can be improved by setting

$$\begin{aligned} \phi(e) \ &:= \ -1 \qquad \text{when} \quad \zeta_0(e) = 1 \ \text{ and } \ e + 1/2 \neq 0 \mod 4 \\ &:= \ 2X_{e,1} - 1 \quad \text{when} \quad \zeta_0(e) = 1 \ \text{ and } \ e + 1/2 = 0 \mod 4 \end{aligned} \tag{4.24}$$

where the random variables $X_{e,1} = \text{Bernoulli}\,(5/64)$ are independent. Using the independence of these random variables, our proof of the large deviation estimates in Lemma 4.3 easily extends to the weight function defined in (4.21) and (4.24) and we get: for all $\epsilon > 0$,

$$P\left(\textstyle\sum_{e \in (-N,0)} \phi(e) \leq N(m - \epsilon)\right) \ \leq \ C_1 \exp(-c_1 N)$$

for suitable $C_1 < \infty$ and $c_1 > 0$, where

$$\begin{aligned} m \ &:= \ ((-3/4) + (1/4)\, E(2X_{e,1} - 1))\, P(\zeta_0(e) = 1) \\ &\qquad + E(2X_{e,2})\, P(\zeta_0(e) = 2) \ + \ E(2X_{e,3} + 1)\, P(\zeta_0(e) = 3) \\ &= \ ((-3/4) + (1/4)(10/64 - 1))(3/8) \\ &\qquad + 2(1 - 2/3)(3/8) + 1/8 \ = \ 15/1024 \ > \ 0. \end{aligned}$$

In particular, fixation follows from the argument in the proof of Lemma 4.4. □

The previous two lemmas imply fixation under the assumptions of Theorem 1.2. Our proof implies more generally that each edge initially occupied by a blockade has a positive probability of never being updated. Since in addition the process starts from a product measure and is constructed from a collection of independent Poisson processes, the ergodic theorem implies that

$$\begin{array}{l} \lim_{N \to \infty} (2N)^{-1} \ \mathrm{card}\,\{e \in (-N, N) : \zeta_t (e) = j \ \text{for all} \ t \geq 0\} \\ \qquad = \ P\,(\zeta_t (e) = j \ \text{for all} \ t \geq 0) \ > \ 0 \quad \text{for all} \quad j > \theta. \end{array} \tag{4.25}$$

Now, each time the opinion of an individual is updated, the two edges incident to the corresponding vertex are updated as well, so whenever an edge is never updated the two nearest neighbors on both sides of this edge are never updated. This, together with (4.25), implies that

$$\begin{array}{l} \lim_{N \to \infty} \ \mathrm{card}\,\{e \in (-N, N) : \eta_t (e - 1/2) = \eta_0 (e - 1/2) \\ \qquad \qquad \neq \eta_0 (e + 1/2) = \eta_t (e + 1/2) \ \text{for all} \ t \geq 0\} \\ \quad \geq \ \sum_{j > \theta} \lim_{N \to \infty} \ \mathrm{card}\,\{e \in (-N, N) : \zeta_t (e) = j \ \text{for all} \ t \geq 0\} \ = \ \infty \end{array}$$

so there are infinitely many pairs of neighbors who disagree and are never updated. Since in addition each of the opinion profiles is equally likely to appear on both sides of a blockade at time 0 due to the obvious symmetry of the process, we deduce that the system fixates in a configuration where all the opinion profiles are present: the system coexists due to fixation.

5. Proof of Theorem 1.3

We now assume that the system starts from the product measure (1.4). Our approach to study fixation in this case is similar to the one in the previous section, the only additional difficulty being to extend the large deviation estimates in Lemma 4.3 to non-uniform initial distributions where the number of particles at adjacent edges are no longer independent. In particular, the number of edges in a given interval and with a given initial number of particles is no longer a binomial random variable. In order to simplify the calculations, we define the weight function in the worst case scenario assuming that all the particles initially active never become frozen, i.e., we set all the random variables X_j introduced in Lemma 4.2 equal to zero so that

$$\begin{array}{rcll} \phi(e) & := & -j & \text{when} \quad \zeta_0 (e) = j \leq \theta \\ & := & j - 2\theta & \text{when} \quad \zeta_0 (e) = j > \theta. \end{array} \tag{5.1}$$

Denote the initial densities of opinion as

$$\rho(u) \ := \ P\,(\eta_0 (x) = u) \quad \text{for all} \quad u \in \Gamma.$$

To extend Lemma 4.3 to such product measures, we first study

$$e_N (u, v) \ := \ \mathrm{card}\,\{x \in [-N, 0] : \eta_0 (x) = u \ \text{and} \ \eta_0 (x + 1) = v\}$$

the number of edges connecting individuals with opinion u and v, respectively. The next lemma gives large deviation estimates for the number of such edges which itself relies on large deviation estimates for the number of changeovers in a sequence of independent Bernoulli trials.

Lemma 5.1. – *There exist* $C_2 < \infty$ *and* $c_2 > 0$ *such that, for all* $\epsilon > 0$ *small,*

$$P\left(e_N(u,v) - N\rho(u)\rho(v) \notin (-\epsilon N, \epsilon N)\right) \ \leq \ C_2 \exp(-c_2 N \epsilon^2) \quad \textit{for} \quad u \neq v.$$

Proof: Let $X_0, X_1, \ldots, X_N$ be independent Bernoulli trials and let Z_N be the number of changeovers, that is, the number of pairs of consecutive trials resulting in different outcomes

$$Z_N \ := \ \mathrm{card}\,\{j = 0, 1, \ldots, N-1 : X_j \neq X_{j+1}\}.$$

According to Lemma 7 in Lanchier and Scarlatos (2014), there exist $C_3 < \infty$ and $c_3 > 0$ such that, for all $\epsilon > 0$,

$$\begin{aligned} & P\left(Z_N - 2N\,(1-p)\,p \notin (-\epsilon N/2, \epsilon N/2)\right) \\ & \quad = \ P\left(Z_N - EZ_N \notin (-\epsilon N/2, \epsilon N/2)\right) \ \leq \ C_3 \exp(-c_3 N \epsilon^2) \end{aligned} \tag{5.2}$$

where p is the common success probability of all the Bernoulli trials. Now, for any u, the number of edges connecting an individual with initial opinion u to an individual with a different opinion is equal in distribution to the number of changeovers when $p = \rho(u)$. In particular, the large deviation estimate in (5.2) implies that, for every opinion profile $u \in \Gamma$,

$$\begin{aligned} & P\left(\textstyle\sum_{v \neq u} e_N(u,v) - N\rho(u)\,(1-\rho(u)) \notin (-\epsilon N/2, \epsilon N/2)\right) \\ & \qquad \leq \ C_3 \exp(-c_3 N \epsilon^2). \end{aligned} \tag{5.3}$$

In addition, since each individual with initial opinion u preceding a changeover is independently followed by any of the remaining $2^F - 1$ opinions, the conditional distribution of the number of edges

$$\begin{aligned} & e_N(u,v) \quad \text{given} \quad \textstyle\sum_{w \neq u} e_N(u,w) = K \quad \text{is} \\ & \qquad\qquad \mathrm{Binomial}\,(K, \rho(v)(1-\rho(u))^{-1}). \end{aligned} \tag{5.4}$$

Letting $K_+ := N\rho(u)(1-\rho(u)) + \epsilon N/2$, observing that, for $\epsilon > 0$ small,

$$K_+\left(\rho(v)(1-\rho(u))^{-1} + (1/4)\,\rho(u)^{-1}(1-\rho(u))^{-1}\,\epsilon\right) \ \leq \ N\left(\rho(u)\rho(v) + \epsilon\right)$$

and combining (5.3)–(5.4) with the large deviation estimates (4.7), we get

$$\begin{aligned} & P\left(e_N(u,v) - N\rho(u)\rho(v) \geq \epsilon N\right) \\ & \quad \leq \ P\left(\textstyle\sum_{w \neq u} e_N(u,w) - N\rho(u)(1-\rho(u)) \geq \epsilon N/2\right) \\ & \qquad + \ P\left(e_N(u,v) - N\rho(u)\rho(v) \geq \epsilon N \mid \right. \\ & \qquad\qquad \left. \textstyle\sum_{w \neq u} e_N(u,w) - N\rho(u)(1-\rho(u)) < \epsilon N/2\right) \\ & \quad \leq \ C_3 \exp(-c_3 N \epsilon^2) \\ & \qquad + \ P\left(\mathrm{Binomial}\,(K_+, \rho(v)(1-\rho(u))^{-1}) \geq N\left(\rho(u)\rho(v) + \epsilon\right)\right) \\ & \quad \leq \ C_3 \exp(-c_3 N \epsilon^2) + \exp(-(1/32)\,\rho(u)^{-2}(1-\rho(u))^{-2}\,K_+\,\epsilon^2). \end{aligned} \tag{5.5}$$

Similarly, letting $K_- := N\rho(u)(1-\rho(u)) - \epsilon N/2$, we have

$$K_-\left(\rho(v)(1-\rho(u))^{-1} - (1/4)\,\rho(u)^{-1}(1-\rho(u))^{-1}\,\epsilon\right) \ \geq \ N\left(\rho(u)\rho(v) - \epsilon\right)$$

and the same reasoning as in (5.5) gives

$$\begin{aligned} P\,(e_N(u,v) - N\rho(u)\rho(v) \leq -\epsilon N) \\ \leq\; C_3 \exp(-c_3 N \epsilon^2) \\ + P\,(\mathrm{Binomial}\,(K_-, \rho(v)(1-\rho(u))^{-1}) \leq N\,(\rho(u)\rho(v) - \epsilon)) \\ \leq\; C_3 \exp(-c_3 N\epsilon^2) + \exp(-(1/32)\,\rho(u)^{-2}(1-\rho(u))^{-2}\,K_-\,\epsilon^2). \end{aligned} \tag{5.6}$$

The lemma follows from combining (5.5)–(5.6). $\square$

Lemma 5.2. – *The system fixates whenever $E\phi(e) > 0$.*

Proof: For all $u, v \in \Gamma$, we set

$$\begin{aligned} h(u,v) \;&:=\; -H(u,v) \quad &\text{when}\quad H(u,v) \leq \theta \\ &:=\; H(u,v) - 2\theta \quad &\text{when}\quad H(u,v) > \theta \end{aligned}$$

where $H(u,v)$ is the Hamming distance (1.1), and observe that

$$\begin{aligned} \textstyle\sum_{e \in (-N,0)} (\phi(e) - E\phi(e)) \;=\; \sum_{e \in (-N,0)} \phi(e) - NE\phi(e) \\ = \textstyle\sum_{u \neq v} h(u,v)\, e_N(u,v) \\ - N \textstyle\sum_{u \neq v} h(u,v)\, P\,(\eta_0(x) = u \text{ and } \eta_0(x+1) = v) \\ = \textstyle\sum_{u \neq v} h(u,v)\,(e_N(u,v) - N\rho(u)\rho(v)). \end{aligned}$$

Then, letting

$$m \;:=\; \max_{u,v} |h(u,v)| \;=\; \max(\theta, F - 2\theta)$$

and applying Lemma 5.1, we get

$$\begin{aligned} P\,(\textstyle\sum_{e \in (-N,0)} (\phi(e) - E\phi(e)) \notin (-\epsilon N, \epsilon N)) \\ =\; P\,(\textstyle\sum_{u \neq v} h(u,v)\,(e_N(u,v) - N\rho(u)\rho(v)) \notin (-\epsilon N, \epsilon N)) \\ \leq\; P\,(e_N(u,v) - N\rho(u)\rho(v) \\ \notin (-\epsilon N/mF^2, \epsilon N/mF^2) \text{ for some } u \neq v) \\ \leq\; C_2\, F^2\, \exp(-c_2 N \epsilon^2/m^2 F^4) \end{aligned} \tag{5.7}$$

for all $\epsilon > 0$ small. Finally, we fix $\epsilon \in (0, E\phi(e))$ small enough so that (5.7) holds and follow the same reasoning as in Lemma 4.4 to deduce that

$$\begin{aligned} \lim_{N\to\infty} P\,(H_N) \;\leq\; \lim_{N\to\infty} \textstyle\sum_{l < -N} \sum_{r > 0} P\,(\sum_{e \in (l,r)} \phi(e) \leq 0) \\ \leq\; \lim_{N\to\infty} \textstyle\sum_{l<-N} \sum_{r>0} P\,(\sum_{e \in (l,r)} (\phi(e) - E\phi(e)) \leq -\epsilon(r-l)) \\ \leq\; \lim_{N\to\infty} \textstyle\sum_{l<-N} \sum_{r>0} C_2\, F^2\, \exp(-c_2 (r-l)\epsilon^2/m^2 F^4) \;=\; 0. \end{aligned}$$

As in Lemma 4.4, we deduce fixation from Lemma 4.1. $\square$

In view of Lemma 5.2, the last step to complete the proof of the theorem is to show the positivity of the expected value of the weight function when $F > 2\theta$ and the system starts from the product measure (1.4) with $\rho > 0$ small. This is done in the next lemma.

Lemma 5.3. – *Assume (1.4) and $F > 2\theta$. Then, there exists $\rho_0 > 0$ such that*

$$E\phi(e) > 0 \quad \textit{for all} \quad \rho \leq \rho_0.$$

Proof: To begin with, we observe that

$$P(\zeta_0(e) = j) \; = \; \textstyle\sum_{H(u,v)=j} P(\eta_0(x) = u \text{ and } \eta_0(x+1) = v) \tag{5.8}$$

and that, under the assumption (1.4),

$$\begin{aligned} N_j \; &:= \; \text{card}\,\{(u,v) \in \Gamma^2 : H(u,v) = j\} \\ &= \; 2^F \,\text{card}\,\{v \in \Gamma : H(u_-, v) = j\} \; = \; 2^F \binom{F}{j}. \end{aligned} \tag{5.9}$$

Now, among the pairs with $H(u,v) = F$,

- exactly two pairs include both u_- and u_+,
- the remaining $N_F - 2$ pairs do not include any of these two opinions.

This, together with (5.8)–(5.9), implies that

$$P(\zeta_0(e) = F) \; = \; 2\,\rho(u_-)^2 + (2^F - 2)\,\rho^2. \tag{5.10}$$

In addition, among the pairs with $H(u,v) = j < F$,

- exactly $4\binom{F}{j}$ pairs include either u_- or u_+,
- the remaining $N_j - 4\binom{F}{j}$ pairs do not include any of these two opinions.

This, together with (5.8)–(5.9), implies that

$$P(\zeta_0(e) = j) \; = \; 4\binom{F}{j}\rho(u_-)\rho + (2^F - 4)\binom{F}{j}\rho^2. \tag{5.11}$$

Recalling (1.4) and (5.1) and combining (5.10)–(5.11), we deduce

$$\begin{aligned} E\phi(e) \;\; = \;\; & \sum_{j=0}^{\theta} (-j)\left(4\binom{F}{j}\left(\frac{1}{2} - (2^{F-1} - 1)\,\rho\right)\rho + (2^F - 4)\binom{F}{j}\rho^2\right) \\ & + \sum_{j=\theta+1}^{F-1} (j - 2\theta)\left(4\binom{F}{j}\left(\frac{1}{2} - (2^{F-1} - 1)\,\rho\right)\rho + (2^F - 4)\binom{F}{j}\rho^2\right) \\ & + (F - 2\theta)\left(2\left(\frac{1}{2} - (2^{F-1} - 1)\,\rho\right)^2 + (2^F - 2)\,\rho^2\right). \end{aligned}$$

In particular, as a function of ρ, the expected weight is a degree two polynomial with constant term $(1/2)(F - 2\theta) > 0$. Therefore, by continuity, there exists $\rho_0 > 0$ such that the expected value of the weight is positive for all $\rho \leq \rho_0$. $\square$

Fixation under the assumptions of Theorem 1.3 directly follows from Lemma 4.1 and the previous two lemmas. To deduce that the one-dimensional system coexists, we use again the argument following the proof of Lemma 4.7.

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