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The limit distribution of ratios of jumps and sums of jumps of subordinators

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Abstract. Let V_t be a driftless subordinator, and let denote $m_t^{(1)} \geq m_t^{(2)} \geq \ldots$ its jump sequence on interval $[0,t]$. Put $V_t^{(k)} = V_t - m_t^{(1)} - \ldots - m_t^{(k)}$ for the k-trimmed subordinator. In this note we characterize under what conditions the limiting distribution of the ratios $V_t^{(k)}/m_t^{(k+1)}$ and $m_t^{(k+1)}/m_t^{(k)}$ exist, as $t \downarrow 0$ or $t \to \infty$.

1. Introduction and results

Let V_t, $t \geq 0$, be a subordinator with Lévy measure Λ and drift 0. Its Laplace transform is given by

$$\mathbf{E}\mathrm{e}^{-\lambda V_t} = \exp\left\{ -t \int_0^\infty \left(1 - \mathrm{e}^{-\lambda v}\right) \Lambda(\mathrm{d}v) \right\},$$

where the Lévy measure Λ satisfies

$$\int_0^\infty \min\{1, x\} \Lambda(\mathrm{d}x) < \infty. \tag{1.1}$$

Put $\overline{\Lambda}(x) = \Lambda((x,\infty))$. Then $\overline{\Lambda}(x)$ is nonincreasing and right continuous on $(0,\infty)$. Whenever we consider limit theorems, as $t \downarrow 0$, we also assume that $\overline{\Lambda}(0+) = \infty$,

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which is necessary and sufficient to assure that there is an infinite number of jumps up to time t, for any $t > 0$.

Denote $m_t^{(1)} \geq m_t^{(2)} \geq \ldots$ the ordered jumps of V_s up to time t, and for $k \geq 0$ consider the trimmed subordinator

$$V_t^{(k)} = V_t - \sum_{j=1}^{k} m_t^{(j)}.$$

We investigate the asymptotic distribution of jump sizes as $t \downarrow 0$ and $t \to \infty$. Specifically, we shall determine a necessary and sufficient condition in terms of the Lévy measure Λ for the convergence in distribution of the ratios $V_t^{(k)}/m_t^{(k+1)}$ and $m_t^{(k+1)}/m_t^{(k)}$. Observe in this notation that $V_t^{(0)} = V_t$ is the subordinator and $m_t^{(1)}$ is the largest jump.

An extended random variable W can take the value ∞ with positive probability, in which case W has a defective distribution function F, meaning that $F(\infty) < 1$. We shall call an extended random variable proper, if it is finite a.s. In this case its F is a probability distribution, i.e. $F(\infty) = 1$. Here we are using the language of the definition given on p. 127 of Feller (1966).

Theorem 1.1. *For any choice of $k \geq 0$ the ratio $V_t^{(k)}/m_t^{(k+1)}$ converges in distribution to an extended random variable W_k as $t \downarrow 0$ ($t \to \infty$) if and only if one of the following holds:*

(i) *$\overline{\Lambda}$ is regularly varying at 0 (∞) with parameter $-\alpha$, $\alpha \in (0,1)$, in which case W_k is a proper random variable with Laplace transform*

$$g_k(\lambda) = \frac{\mathrm{e}^{-\lambda}}{\left[1 + \alpha \int_0^1 \left(1 - \mathrm{e}^{-\lambda y}\right) y^{-\alpha-1} \mathrm{d}y\right]^{k+1}}; \tag{1.2}$$

(ii) *$\overline{\Lambda}$ is slowly varying at 0 (∞), in which case $W_k = 1$ a.s.;*

(iii) *the condition*

$$\frac{x\overline{\Lambda}(x)}{\int_0^x u\Lambda(\mathrm{d}u)} \longrightarrow 0 \quad \text{as } x \downarrow 0 \ (x \to \infty) \tag{1.3}$$

holds, in which case $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$, that is $W_k = \infty$ a.s.

Note that Theorem 1.1 says that the situation $0 < \mathbf{P}\{W_k = \infty\} < 1$ cannot happen.

The corresponding problem for nonnegative i.i.d. random variables was investigated by Darling (1952) and Breiman (1965), in the $k = 0$ case. In this case Darling proved the sufficiency parts corresponding to (i) and (ii) (Theorem 5.1 and Theorem 3.2 in Darling (1952)), in particular the limit W_0 has the same distribution as given by Darling in his Theorem 5.1, while Breiman proved the necessity parts corresponding to (i), (ii) and (iii) (Theorem 3 (p. 357), Theorem 2 and Theorem 4 in Breiman (1965)). A special case of Theorem 1 in Teugels (1981) gives the sufficiency analog of (i) in the case of i.i.d. nonnegative sums for any $k \geq 0$.

The necessary and sufficient condition in the cases (ii) and (iii), stated in the more general setup of Lévy processes without a normal component, is given by Buchmann et al. (2014), see their Theorem 3.1 and 5.1.

Next we shall investigate the asymptotic distribution of the ratio of two consecutive ordered jumps $m_t^{(k+1)}/m_t^{(k)}$, $k \geq 1$. We shall obtain the analog for subordinators of a special case of a result that Bingham and Teugels (1981) established for i.i.d. nonnegative random variables. This will follow from a general result on the asymptotic distribution of ratios of the form defined for $k \geq 1$ by

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)},\ t > 0,$$

where for each $k \geq 1$, $S_k = \omega_1 + \ldots + \omega_k$, with $\omega_1, \omega_2, \ldots$ being i.i.d. mean 1 exponential random variables and ψ is the nonincreasing and right continuous inverse function defined for $s > 0$ by

$$\psi(s) = \sup\{y : \overline{\Pi}(y) > s\},$$

with Π being a positive measure on $(0, \infty)$ such that $\overline{\Pi}(x) = \Pi((x, \infty)) \to 0$, as $x \to \infty$. Note that we do not require Π to be a Lévy measure. Also whenever we consider the asymptotic distribution of $r_k(t)$ as $t \downarrow 0$ we shall assume that $\overline{\Pi}(0+) = \infty$.

We call a function f *rapidly varying at 0* with index $-\infty$, $f \in \mathrm{RV}_0(-\infty)$, if

$$\lim_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} = \begin{cases} 0, & \text{for } \lambda > 1, \\ 1, & \text{for } \lambda = 1, \\ \infty, & \text{for } \lambda < 1. \end{cases}$$

Correspondingly, f is *rapidly varying at* ∞ with index $-\infty$, $f \in \mathrm{RV}_\infty(-\infty)$, if the same holds with $x \to \infty$.

Theorem 1.2. *For any choice of $k \geq 1$ the ratio $r_k(t)$ converges in distribution as $t \downarrow 0$ ($t \to \infty$) to a random variable Y_k if and only if one of the following holds:*

(i) $\overline{\Pi}$ *is regularly varying at 0 (∞) with parameter $-\alpha \in (-\infty, 0)$, in which case Y_k has the Beta$(k\alpha, 1)$ distribution, i.e.*

$$G_k(x) = \mathbf{P}\{Y_k \leq x\} = x^{k\alpha}, \quad x \in [0,1]; \tag{1.4}$$

(ii) $\overline{\Pi}$ *is slowly varying at 0 (∞), in which case $Y_k = 0$ a.s.*

(iii) $\overline{\Pi}$ *is rapidly varying at 0 (∞) with index $-\infty$, in which case $Y_k = 1$ a.s.*

Theorem 1.2 has some important applications to the asymptotic distribution of the ratio of two consecutive ordered jumps $m_t^{(k+1)}/m_t^{(k)}$, $k \geq 1$, of a Lévy process. Let X_t, $t \geq 0$, be a Lévy processes whose Lévy measure Λ is concentrated on $(0, \infty)$. Here in addition to $\overline{\Lambda}(x) \to 0$ as $x \to \infty$, we require that

$$\int_0^\infty \min\{1, x^2\} \Lambda(\mathrm{d}x) < \infty. \tag{1.5}$$

In this setup one has the distributional representation for $k \geq 1$

$$\left(m_t^{(k)}, m_t^{(k+1)}\right) \stackrel{\mathcal{D}}{=} \left(\varphi(S_k/t), \varphi(S_{k+1}/t)\right), \tag{1.6}$$

with φ defined for $s > 0$ to be

$$\varphi(s) = \sup\{y : \overline{\Lambda}(y) > s\}. \tag{1.7}$$

It is readily checked that φ is nonincreasing and right continuous. Moreover, whenever Λ is the Lévy measure of a subordinator V_t, condition (1.1) holds, which is equivalent to

$$\int_{\delta}^{\infty} \varphi(s)\mathrm{d}s < \infty, \text{ for any } \delta > 0. \tag{1.8}$$

The distributional representation in (1.6) follows from Proposition 1 in Kevei and Mason (2013), see the proof of Theorem 1.1 below. For general spectrally positive Lévy processes it can be deduced using the same methods that Maller and Mason (2010) derived the distributional representation for a Lévy process given in their Proposition 5.7.

When applying Theorem 1.2 to the asymptotic distribution of consecutive ordered jumps at 0 or ∞ of a Lévy process X_t whose Lévy measure Λ is concentrated on $(0,\infty)$, we have to keep in mind that (1.5) must always hold and (1.1) must be satisfied whenever X_t is a subordinator. For instance in the case of a subordinator V_t, whenever $m_t^{(k+1)}/m_t^{(k)}$ converges in distribution to a random variable Y_k as $t \downarrow 0$, Theorem 1.2 says that $\overline{\Lambda}$ is regularly varying at 0. Further since (1.1) must hold, the parameter $-\alpha$ is necessarily in $[-1,0]$, while there is no such restriction when considering convergence in distribution as $t \to \infty$. We note that in case of general Lévy processes for $k=1$ the sufficiency part corresponding to part (ii) in Theorem 1.2 is given in Theorem 3.1 in Buchmann et al. (2014).

In the special case when V_t is an α-stable subordinator, $\alpha \in (0,1)$, and $m^{(1)} > m^{(2)} > \ldots$ is its jump sequence on $[0,1]$, then $(m^{(1)}/V_1, m^{(2)}/V_1, \ldots)$ has the Poisson–Dirichlet law with parameter $(\alpha, 0)$ (PD$(\alpha,0)$), see Bertoin (2006) p. 90. The ratio of the $(k+1)^{\text{th}}$ and k^{th} element of a vector, which has the PD$(\alpha,0)$ law, has the Beta$(k\alpha,1)$ distribution (Proposition 2.6 in Bertoin (2006)).

2. Proofs

In the proofs we only consider the case when $t \downarrow 0$, as the $t \to \infty$ case is nearly identical.

2.1. *Proof of Theorem 1.1.* First we calculate the Laplace exponent of the ratio using the notation φ defined in (1.7). We see by the nonincreasing version of the change of variables formula stated in (4.9) Proposition of Revuz and Yor (1991), which is given in Lemma 1 in Kevei and Mason (2013),

$$\begin{aligned}
\mathbf{E}\mathrm{e}^{-\lambda V_t} &= \exp\left\{ -t \int_0^\infty \left(1 - \mathrm{e}^{-\lambda v}\right) \Lambda(\mathrm{d}v) \right\} \\
&= \exp\left\{ -t \int_0^\infty \left(1 - \mathrm{e}^{-\lambda \varphi(x)}\right) \mathrm{d}x \right\}.
\end{aligned}$$

The key ingredient of our proofs is a distributional representation of the subordinator V_t given in Proposition 1 in Kevei and Mason (2013), which follows from a general representation by Rosiński (2001). It states that for $t > 0$

$$V_t \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right). \tag{2.1}$$

From the proof of this result it is clear that $\varphi(S_i/t)$ corresponds to $m_t^{(i)}$, for $i \geq 1$. Therefore

$$\frac{V_t^{(k)}}{m_t^{(k+1)}} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}.$$

Conditioning on $S_{k+1} = s$ and using the independence we can write

$$\begin{aligned}\sum_{i=k+2}^{\infty} \varphi(S_i/t) &= \sum_{i=k+2}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i - s}{t}\right) \\ &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i}{t}\right) \\ &= \sum_{i=1}^{\infty} \varphi_{s/t}\left(S_i/t\right),\end{aligned}$$

where $\varphi_y(x) = \varphi(y+x)$. Note that the latter sum has the same form as in (2.1), therefore it is equal in distribution to a subordinator $V^{(s/t)}(t)$ with Laplace transform

$$\begin{aligned}\mathbf{E}\mathrm{e}^{-\lambda V_t^{(s/t)}} &= \exp\left\{-t\int_0^{\infty}\left(1-\mathrm{e}^{-\lambda\varphi_{s/t}(x)}\right)\mathrm{d}x\right\} \\ &= \exp\left\{-t\int_{s/t}^{\infty}(1-\mathrm{e}^{-\lambda\varphi(x)})\mathrm{d}x\right\}.\end{aligned} \tag{2.2}$$

Now we can compute the Laplace transform of the ratio $V_t^{(k)}/m_t^{(k+1)}$. Since S_{k+1} has Gamma$(k+1,1)$ distribution, the law of total probability and (2.2) give

$$\begin{aligned}&\mathbf{E}\exp\left\{-\lambda\frac{V_t^{(k)}}{m_t^{(k+1)}}\right\} = \mathbf{E}\exp\left\{-\lambda\frac{\sum_{i=k+1}^{\infty}\varphi(S_i/t)}{\varphi(S_{k+1}/t)}\right\} \\ &= \int_0^{\infty}\frac{s^k}{k!}\mathrm{e}^{-s}\left[\mathrm{e}^{-\lambda}\,\mathbf{E}\exp\left\{-\frac{\lambda}{\varphi(s/t)}\sum_{i=1}^{\infty}\varphi_{s/t}(S_i/t)\right\}\right]\mathrm{d}s \\ &= \mathrm{e}^{-\lambda}\int_0^{\infty}\frac{s^k}{k!}\mathrm{e}^{-s}\exp\left\{-t\int_{s/t}^{\infty}\left[1-\mathrm{e}^{-\frac{\lambda}{\varphi(s/t)}\varphi(x)}\right]\mathrm{d}x\right\}\mathrm{d}s \\ &= \frac{t^{k+1}}{k!}\mathrm{e}^{-\lambda}\int_0^{\infty}u^k\exp\left\{-t\left(u+\int_u^{\infty}\left[1-\mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right]\mathrm{d}x\right)\right\}\mathrm{d}u \\ &= \frac{t^{k+1}}{k!}\mathrm{e}^{-\lambda}\int_0^{\infty}u^k\mathrm{e}^{-t\Psi(u,\lambda)}\mathrm{d}u,\end{aligned} \tag{2.3}$$

where

$$\Psi(u,\lambda) = u + \int_u^{\infty}\left[1-\mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right]\mathrm{d}x. \tag{2.4}$$

Since φ is right continuous on $(0,\infty)$, $\Psi(\cdot,\lambda)$ is also right continuous on $(0,\infty)$. Further a short calculation shows that this function is strictly increasing for any $\lambda > 0$, moreover for $u_1 > u_2$

$$\Psi(u_1,\lambda) - \Psi(u_2,\lambda) \geq \mathrm{e}^{-\lambda}(u_1 - u_2).$$

Clearly $\Psi(\infty,\lambda) = \infty$ and therefore

$$\Psi_k(u,\lambda) := \Psi\left(((k+1)u)^{1/(k+1)},\lambda\right)$$

has a right continuous increasing inverse function given by

$$Q_\lambda(s) = \inf\{v : \Psi_k(v,\lambda) > s\}, \text{ for } s \geq 0,$$

such that $Q_\lambda(0) = 0$ and $\lim_{x\to\infty} Q_\lambda(x) = \infty$. (For the right continuity part see (4.8) Lemma in Revuz and Yor (1991).)

Necessity. Assuming that $V_t^{(k)}/m_t^{(k+1)}$ converges in distribution as $t \to 0$ to some extended random variable W_k, we can apply Theorem 2a on p. 210 of Feller (1966) to conclude that its Laplace transform also converges, i.e.

$$\int_0^\infty u^k \mathrm{e}^{-t\Psi(u,\lambda)}\mathrm{d}u = \int_0^\infty \mathrm{e}^{-t\Psi_k(v,\lambda)}\mathrm{d}v$$

$$= \int_0^\infty \mathrm{e}^{-ty}\mathrm{d}Q_\lambda(y) \sim \frac{\mathrm{e}^\lambda g_k(\lambda) k!}{t^{k+1}}, \text{ as } t \to 0,$$

where $g_k(\lambda) = \mathbf{E}\mathrm{e}^{-\lambda W_k}$, and W_k can possibly have a defective distribution, i.e. possibly $\mathbf{P}\{W_k = \infty\} > 0$. (Here we used the change of variables formula given in (4.9) Proposition in Revuz and Yor (1991).) By Karamata's Tauberian theorem (Theorem 1.7.1 in Bingham et al. (1989))

$$Q_\lambda(y) \sim \frac{y^{k+1}}{k+1}\mathrm{e}^\lambda g_k(\lambda), \quad \text{as } y \to \infty,$$

and thus by Theorem 1.5.12 in Bingham et al. (1989)

$$\Psi_k(v,\lambda) \sim \left(\frac{(k+1)v}{\mathrm{e}^\lambda g_k(\lambda)}\right)^{1/(k+1)}, \quad \text{as } v \to \infty,$$

and hence

$$\Psi(u,\lambda) \sim u\left[\mathrm{e}^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}}, \quad \text{as } u \to \infty.$$

Substituting back into (2.4) we obtain for any $\lambda > 0$

$$\lim_{u\to\infty} \frac{1}{u}\int_u^\infty \left(1 - \mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right)\mathrm{d}x = \left[\mathrm{e}^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1. \tag{2.5}$$

Note that the limit W_k is ≥ 1, with probability 1, and so $g_k(\lambda) \leq \mathrm{e}^{-\lambda}$. Thus for any λ

$$\left[\mathrm{e}^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1 \geq 0.$$

For any $x \geq 0$ we have $1 - \mathrm{e}^{-x} \leq x$. Therefore by (2.5) we obtain for any $\lambda > 0$

$$\liminf_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \geq \frac{1}{\lambda}\left(\left[\mathrm{e}^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right). \tag{2.6}$$

On the other hand, by monotonicity $\varphi(x)/\varphi(u) \leq 1$ for $u \leq x$. Therefore for any $0 < \varepsilon < 1$ there exists a $\lambda_\varepsilon > 0$, such that for all $0 < \lambda < \lambda_\varepsilon$

$$1 - \mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}} \geq (1-\varepsilon)\frac{\lambda\varphi(x)}{\varphi(u)}, \text{ for } x \geq u.$$

Using again (2.5) and keeping (1.8) in mind, this implies that for such λ

$$\limsup_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \leq \frac{1}{1-\varepsilon}\frac{1}{\lambda}\left(\left[\mathrm{e}^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right). \tag{2.7}$$

In particular, we obtain that, whenever $g_k(\lambda) \not\equiv 0$ (i.e. $\mathbf{P}\{W_k < \infty\} > 0$)

$$0 \leq \liminf_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \leq \limsup_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x < \infty.$$

Note that in (2.6) the greatest lower bound is 0 for all $\lambda > 0$ if and only if $g_k(\lambda) = e^{-\lambda}$, in which case $W_k = 1$. Then the upper bound for the limsup in (2.7) is 0, thus

$$\lim_{u\to\infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x)\mathrm{d}x = 0,$$

which by Proposition 2.6.10 in Bingham et al. (1989) applied to the function $f(x) = x\varphi(x)$ implies that $\varphi \in \mathrm{RV}_\infty(-\infty)$, and so, by Theorem 2.4.7 in Bingham et al. (1989), $\overline{\Lambda}$ is slowly varying at 0. We have proved that $W_k = 1$ if and only if $\overline{\Lambda}$ is slowly varying at 0.

In the following we assume that $\mathbf{P}\{W_k > 1\} > 0$, therefore the liminf in (2.6) is strictly positive. Let

$$a = \liminf_{\lambda\downarrow 0} \frac{1}{\lambda}\left(\left[e^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right) \le \limsup_{\lambda\downarrow 0} \frac{1}{\lambda}\left(\left[e^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right) = b.$$

By (2.7) and (2.6), $a > 0$ and $b < \infty$. Moreover

$$b \le \liminf_{u\to\infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x)\mathrm{d}x \le \limsup_{u\to\infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x)\mathrm{d}x \le a,$$

which forces

$$a = b = \lim_{u\to\infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x)\mathrm{d}x = \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\left(\left[e^\lambda g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right).$$

By Karamata's theorem (Theorem 1.6.1 (ii) in Bingham et al. (1989)) we obtain that φ is regularly varying at infinity with parameter $-a^{-1} - 1 =: -\alpha^{-1}$, so Λ is regularly varying with parameter $-\alpha$ at zero with $\alpha \in (0,1)$.

Let us consider the case when $W_k = \infty$ a.s., that is $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$. All the previous computations are valid, with $g_k(\lambda) = \mathbf{E}e^{-\lambda\infty} \equiv 0$. Thus, from (2.6) we have

$$\lim_{u\to\infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x)\mathrm{d}x = \infty.$$

From this, through the change of variables formula we obtain (1.3).

Sufficiency and the limit. Consider first the special case when $\varphi(x) = x^{-\frac{1}{\alpha}}$, $\alpha \in (0,1)$. Then a quick calculation gives

$$\frac{1}{u}\int_u^\infty \left(1 - e^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right)\mathrm{d}x = \alpha\int_0^1 \left(1 - e^{-\lambda y}\right) y^{-\alpha-1}\mathrm{d}y.$$

By formula (2.5) for the Laplace transform of the limit we obtain (1.2).

The sufficiency can be proved by standard arguments for regularly varying functions. Using Potter bounds (Theorem 1.5.6 in Bingham et al. (1989)) one can show that for $\alpha \in (0,1)$

$$\lim_{u\to\infty} \frac{1}{u}\Psi(u,\lambda) = 1 + \alpha\int_0^1 \left(1 - e^{-\lambda y}\right) y^{-\alpha-1}\mathrm{d}y,$$

from which, through formula (2.3), the convergence readily follows. As already mentioned, cases (ii) and (iii) are treated in Buchmann et al. (2014).

2.2. *Proof of Theorem 1.2*. Using that $\psi(y) \le x$ if and only if $\overline{\Pi}(x) \le y$, for the distribution function of the ratio we have for $x \in (0,1)$

$$\begin{aligned} \mathbf{P}\left\{ r_k(t) \le x \right\} &= \mathbf{P}\left\{ \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \le x \right\} \\ &= \int_0^\infty \frac{s^{k-1}}{(k-1)!} \mathrm{e}^{-s}\, \mathbf{P}\left\{ \psi\left(\frac{s+S_1}{t} \right) \le x\psi\left(\frac{s}{t}\right) \right\} \mathrm{d}s \\ &= \int_0^\infty \frac{s^{k-1}}{(k-1)!} \mathrm{e}^{-s}\, \mathrm{e}^{-[t\overline{\Pi}(x\psi(s/t))-s]} \mathrm{d}s \\ &= \frac{t^k}{(k-1)!} \int_0^\infty u^{k-1}\, \mathrm{e}^{-t\overline{\Pi}(x\psi(u))} \mathrm{d}u. \end{aligned} \tag{2.8}$$

Necessity. Assume that the limit distribution function G_k exists. Write

$$\frac{t^k}{(k-1)!} \int_0^\infty u^{k-1} \mathrm{e}^{-t\overline{\Pi}(x\psi(u))} \mathrm{d}u = \frac{t^k}{(k-1)!} \int_0^\infty \mathrm{e}^{-t\Phi_k(v,x)} \mathrm{d}v, \tag{2.9}$$

where $\Phi_k(v,x) = \overline{\Pi}\left(x\psi((kv)^{1/k}) \right)$. Note that for each $x \in (0,1)$ the function $\Phi_k(\cdot, x)$ is monotone nondecreasing, since $\overline{\Pi}$ and ψ are both monotone nonincreasing. Let

$$\mathcal{G}_k = \{x : x \text{ is a continuity point of } G_k \text{ in } (0,1) \text{ such that } G_k(x) > 0\}.$$

First assume that $\mathbf{P}\{Y_k < 1\} > 0$. Clearly we can now proceed as in the proof of Theorem 1 to apply Karamata's Tauberian theorem (Theorem 1.7.1 in Bingham et al. (1989)) to give that for any $x \in \mathcal{G}_k$,

$$\lim_{u\to\infty} \frac{\overline{\Pi}(x\psi(u))}{u} = [G_k(x)]^{-\frac{1}{k}}. \tag{2.10}$$

In fact, there is a small difference here compared to the proof of Theorem 1.1. We have to be more cautious, as $\Phi_k(v,x)$ is not necessarily right-continuous as a function of $v > 0$. To use the machinery from the proof of Theorem 1.1 we need to consider the right-continuous version $\widetilde{\Phi}_k(v,x) := \Phi_k(v+,x)$. Since, in (2.9) we integrate with respect to the Lebesgue measure and Φ_k and $\widetilde{\Phi}_k$ are equal almost everywhere, substituting Φ_k with $\widetilde{\Phi}_k$ leaves the integral unchanged. Therefore, proceeding as before we obtain that

$$\widetilde{\Phi}_k(v,x) \sim \left(\frac{kv}{G_k(x)} \right)^{1/k}, \quad \text{as } v \to \infty,$$

and since the right-hand function is continuous, we also get that

$$\Phi_k(v,x) \sim \left(\frac{kv}{G_k(x)} \right)^{1/k}, \quad \text{as } v \to \infty,$$

form which now (2.10) does indeed follow.

We claim that (2.10) implies the regular variation of $\overline{\Pi}$. When $\overline{\Pi}$ is continuous and strictly decreasing we get by changing variables to $\psi(u) = t$, $u = \overline{\Pi}(t)$, that we have for any $x \in \mathcal{G}_k$

$$\lim_{t\downarrow 0} \frac{\overline{\Pi}(tx)}{\overline{\Pi}(t)} = [G_k(x)]^{-\frac{1}{k}},$$

which by an easy application of Proposition 1.10.5 in Bingham et al. (1989) implies that $\overline{\Pi}$ is regularly varying.

Note that the jumps of $\overline{\Pi}$ correspond to constant parts of ψ, and vice versa. Put $\mathcal{J} = \{z : \overline{\Pi}(z-) > \overline{\Pi}(z)\}$ for the jump points of $\overline{\Pi}$. For $z \in \mathcal{J}$ and $y \in [\overline{\Pi}(z), \overline{\Pi}(z-))$ we have $\psi(y) = z$. Substituting into (2.10) we have

$$\lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \quad \text{and} \quad \lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}. \tag{2.11}$$

To see how the second limit holds in (2.11) note that for any $0 < \varepsilon < 1$ and $z \in \mathcal{J}$, we have $\psi\left(\varepsilon\overline{\Pi}(z) + (1-\varepsilon)\overline{\Pi}(z-)\right) = z$ and thus

$$\lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\varepsilon\overline{\Pi}(z) + (1-\varepsilon)\overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}.$$

Since $0 < \varepsilon < 1$ can be chosen arbitrarily close to 0 this implies the validity of the second limit in (2.11). Therefore by choosing any $x \in \mathcal{G}_k$ we get

$$\lim_{z\downarrow 0} \frac{\overline{\Pi}(z-)}{\overline{\Pi}(z)} = 1. \tag{2.12}$$

Let

$$\mathcal{A} = \{z > 0 : \overline{\Pi}(z-\varepsilon) > \overline{\Pi}(z) \ \text{ for all } z > \varepsilon > 0\}.$$

This set contains exactly those points z for which $\psi(\overline{\Pi}(z)) = z$. With this notation formula (2.10) can be written as

$$\lim_{z\downarrow 0, z\in\mathcal{A}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \ \text{for } x \in \mathcal{G}_k. \tag{2.13}$$

This together with (2.12) will allow us to apply Proposition 1.10.5 in Bingham et al. (1989) to conclude that $\overline{\Pi}$ is regularly varying. We shall need the following technical lemma.

Lemma 2.1. *Whenever (2.12) holds, there exists a strictly decreasing sequence $z_n \in \mathcal{A}$ such that $z_n \to 0$ and*

$$\lim_{n\to\infty} \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)} = 1. \tag{2.14}$$

Proof. Choose $z_1 \in \mathcal{A}$ such that $\overline{\Pi}(z_1) > 0$, and define for each $n \geq 1$

$$z_{n+1} = \sup\left\{z > 0 : \overline{\Pi}(z) > \left(1 + \frac{1}{n}\right)\overline{\Pi}(z_n-)\right\}.$$

Notice that the sequence $\{z_n\}$ is well-defined, since $\overline{\Pi}(0+) = \infty$ and it is decreasing. Further we have

$$\overline{\Pi}(z_{n+1}-) \geq \left(1 + \frac{1}{n}\right)\overline{\Pi}(z_n-) \text{ and } \overline{\Pi}(z_{n+1}) \leq \left(1 + \frac{1}{n}\right)\overline{\Pi}(z_n-),$$

where the second inequality follows by right continuity of $\overline{\Pi}$. Also note that $z_{n+1} < z_n$, since otherwise if $z_{n+1} = z_n$, then

$$\overline{\Pi}(z_{n+1}-) = \overline{\Pi}(z_n-) \geq \left(1 + \frac{1}{n}\right)\overline{\Pi}(z_n-),$$

which is impossible. Observe that each z_{n+1} is in $\mathcal{A}$ since by the definition of z_{n+1} for all $0 < \varepsilon < z_{n+1}$

$$\overline{\Pi}(z_{n+1} - \varepsilon) > \left(1 + \frac{1}{n}\right)\overline{\Pi}(z_n-) \geq \overline{\Pi}(z_{n+1}).$$

Clearly since $\{z_n\}$ is a decreasing and positive sequence, $\lim_{n\to\infty} z_n = z^*$ exists and is ≥ 0. By construction

$$\overline{\Pi}(z_{n+1}-) \geq \left(1+\frac{1}{n}\right)\overline{\Pi}(z_n-) \ \geq \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\overline{\Pi}(z_1-).$$

The infinite product $\prod_{n=1}^{\infty}(1+1/n) = \infty$ forces $z^* = 0$. Also by construction we have

$$1 \leq \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n-)} = \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)}\left(\frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)}\right) \leq 1+\frac{1}{n}.$$

By (2.12) we have

$$\lim_{n\to\infty}\frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)} = 1.$$

Therefore we get (2.14). □

According to Proposition 1.10.5 in Bingham et al. (1989) to establish that $\overline{\Pi}$ is regularly varying at zero it suffices to produce λ_1 and λ_2 in $(0,1)$ such that for $i = 1,2$

$$\frac{\overline{\Pi}(\lambda_i z_n)}{\overline{\Pi}(z_n)} \to d_i \in (0,\infty)\,,\text{ as } n\to\infty,$$

where $(\log\lambda_1)\,/\,(\log\lambda_2)$ is finite and irrational. This can clearly be done using (2.13) and $\mathbf{P}\{Y_k < 1\} > 0$. Necessarily $\overline{\Pi}$ has index of regular variation parameter $-\alpha \in (-\infty, 0]$. For $\alpha \in (0,\infty)$ the limiting distribution function has the form (1.4). In the case $\alpha = 0$, $\overline{\Pi}$ is slowly varying at 0 and we get that $G_k(x) = 1$ for $x \in (0,1)$, i.e. $Y_k = 0$ a.s.

Now consider the case when $\mathbf{P}\{Y_k = 1\} = 1$, i.e. $G_k(x) = 0$ for any $x \in (0,1)$. We once more use Theorem 1.7.1 in Bingham et al. (1989), with $c = 0$ this time, and as an analog of (2.10) we obtain

$$\lim_{u\to\infty}\frac{\overline{\Pi}(x\psi(u))}{u} = \infty.$$

This readily implies that

$$\lim_{z\downarrow 0, z\in\mathcal{A}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = \infty.$$

Moreover, the analogs of formula (2.11) also hold, i.e.

$$\lim_{z\downarrow 0, z\in\mathcal{J}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = \infty, \quad \text{and} \quad \lim_{z\downarrow 0, z\in\mathcal{J}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z-)} = \infty.$$

(Note, however, that this does not imply (2.12).) Let $z \notin \mathcal{A}$, and define $z' = \inf\{v : v \in \mathcal{A}, v > z\}$. Clearly, $z' \downarrow 0$ as $z \downarrow 0$. If $z' \in \mathcal{A}$ then necessarily it is a jump point, $z' \in \mathcal{J}$, and $\overline{\Pi}(z'-) = \overline{\Pi}(z)$. Then

$$\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z'-)} \geq \frac{\overline{\Pi}(xz')}{\overline{\Pi}(z'-)},$$

and the latter tends to ∞ as $z \downarrow 0$. On the other hand, when $z' \notin \mathcal{A}$ it is simple to see that $\overline{\Pi}(z') = \overline{\Pi}(z)$ and $\overline{\Pi}(z'+\varepsilon) < \overline{\Pi}(z')$ for any $\varepsilon > 0$. Moreover, we can find

$z < z'' \in \mathcal{A}$, such that $\overline{\Pi}(z) \leq \overline{\Pi}(z'') + 1 \leq 2\overline{\Pi}(z'')$ (we tacitly assumed that z is small enough). Thus

$$\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} \geq \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z'') + 1} \geq \frac{\overline{\Pi}(xz'')}{2\overline{\Pi}(z'')},$$

and the lower bound goes to ∞ as $z \downarrow 0$. Summarizing, we have proved that

$$\lim_{z \downarrow 0} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = \infty,$$

for any $x \in (0,1)$, that is, $\overline{\Pi}$ is rapidly varying at 0 with index $-\infty$.

Sufficiency. Assume that $\overline{\Pi}$ is regularly varying at 0 with index $-\alpha \in (-\infty, 0)$. Then its inverse function ψ is regularly varying at ∞ with index $-1/\alpha$, therefore simply

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to \left(\frac{S_k}{S_{k+1}} \right)^{1/\alpha} \quad \text{a.s., as } t \downarrow 0,$$

which has the distribution G_k in (1.4). Assume now that $\overline{\Pi}$ is slowly varying at 0. Then $\psi \in \mathrm{RV}_\infty(-\infty)$, therefore

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to 0 \quad \text{a.s., as } t \downarrow 0.$$

Finally, if $\overline{\Pi} \in \mathrm{RV}_0(-\infty)$ then ψ is slowly varying at infinity, so

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to 1 \quad \text{a.s., as } t \downarrow 0,$$

and the theorem is completely proved.

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