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Large deviations of the empirical current for the boundary driven Kawasaki process with long range interaction

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Abstract. We consider a lattice gas evolving in a bounded cylinder of length $2N+1$ and interacting via a Neuman Kac interaction of range N, in contact with particles reservoirs at different densities. We investigate the associated law of large numbers and large deviations of the empirical current and of the density. The hydrodynamic limit for the empirical density, obtained in the diffusive scaling, is given by a nonlocal, nonlinear evolution equation with Dirichlet boundary conditions.

1. Introduction

The large deviations principle is an inportant topic of interest for the study of macroscopic properties of non-equilibrium systems. In the last years, many papers have been devoted to the subject. We just quote a few of them where the issue is addressed in the context of lattice gas dynamics for which large deviation principles can be derived in the hydrodynamic scaling, Bertini et al. (2001); Bodineau and Derrida (2006); Derrida (2007); Bertini et al. (2006a) and references therein. Typical examples are systems in contact with two thermostats at different temperatures or with two reservoirs at different densities. A mathematical model for such systems is provided by reversible systems of hopping dynamics combined with the action of an external mechanism of creation and annihilation of particles, modeling the exchange reservoirs. The action of the reservoirs makes the full process non reversible. A principal generic feature of these systems is that they exhibit long range correlations in their steady state.

In this paper we consider a microscopic conservative system, with long range interaction with open boundaries. The system is contained in a cylinder $\Lambda_N =$

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$\{-N, \cdots, N\} \times \mathbb{T}_N^{d-1}$ of length $2N+1$ with axis in direction u_1, with $\mathbb{T}_N^{d-1}$ the $(d-1)$-dimensional microscopic torus of length $2N+1$ and N a scaling parameter, namely we impose periodic boundary conditions in all directions but u_1. In the bulk, particles evolve according to conservative dynamics (Kawasaki) perturbed by a modified version of Kac potential which we call *Neuman Kac potential*. The Kac potentials J_N are two-body interactions with range N and strength N^{-d}: $J_N(u) = N^{-d}J(u)$, $u \in \mathbb{R}^d$, where J is a smooth function with compact support. They have been introduced in Kac et al. (1963); Uhlenbeck et al. (1963); Hemmer et al. (1964), and then generalized in Lebowitz and Penrose (1966), to present a rigorous derivation of the van der Waals theory of a gas-liquid phase transition. There have been many interesting results on Kac Ising spin systems in equilibrium statistical mechanics. We refer for a survey to the book Presutti (2009). The so called *Neuman Kac potential*, $J_N^{\mathrm{neum}}(u) = N^{-d}J^{\mathrm{neum}}(u)$, $u \in \mathbb{R}^d$ (see (2.1) below) is the modification of the Kac potential that takes into account the fact that the particles are confined in a bounded domain.

Given $\beta \geq 0$ and a chemical potential $\lambda \in \mathbb{R}$, we consider the Hamiltonian

$$H_N^{\beta}(\eta) = -\beta \sum_{x,y\in\Lambda_N} J_N^{\mathrm{neum}}(x,y)\eta(x)\eta(y) \, + \, \lambda \sum_{x\in\Lambda_N} \eta(x)\,,$$

where $\eta = (\eta(x)\,,\ x \in \mathbb{Z})$, $\eta(x) \in 0,1$; $\eta(x) = 1$ if there is a particle at site x and $\eta(x) = 0$ if site x is empty. One can construct in a standard way an evolution conserving the total number of particles, the so-called Kawasaki dynamics, which can be described as follows. Particles attempt to jump to nearest neighbour sites at rates depending on the energy difference before and after the exchange, provided the nearest neighbour target sites are empty; attempted jumps to occupied sites are suppressed. The rates are chosen in such a way that the system satisfies a detailed balance condition with respect to a family of Gibbs measures, parametrized by the so-called chemical potential $\lambda \in \mathbb{R}$ and fixed β. To model the presence of the reservoirs, we superimpose at the boundary to the bulk dynamics a birth and death process. For a fixed smooth function $b(\cdot)$ defined on the boundary of the domain, the rates of this birth and death process are chosen so that a Bernoulli product measure of varying parameter b is reversible for it. This latter dynamics is of course not conservative and keeps the fixed value of the density equal to b at the boundary. This dynamics defines an *irreducible* Markov jump process on a finite state space; its stationary measure $\mu_N^{st,b}$ is unique. There is a flow of mass through the full system and $\mu_N^{st,b}$ encodes its long time behavior. The full dynamics is reversible only if $\beta = 0$ and b is constant. We introduce the empirical density π_t^N of particles and the integrated empirical current $\mathbf{W}_t^N$, which measures the total net flow of particles in the time interval $[0,t]$, associated to a trajectory $(\eta_\cdot)$.

We analyze here the behavior as $N \uparrow \infty$ of the system when the time is rescaled by N^2 (*diffusive limit*). Our purpose is to investigate the behavior of the current of particles. Problems of this kind have been studied in Bertini et al. (2006b) and in Bodineau and Lagouge (2012). In both documents the large deviations rate functionals are convex. The paper Bertini et al. (2006b), studied the simple exclusion process, in the torus with periodic conditions. The paper Bodineau and Lagouge (2012) is concerned by the reaction diffusion process, in a one-dimensional interval with two types of currents (conservative and non conservative); some conditions on the convexity on the functionals were imposed. Our goal is to extend these results to the d-dimentional boundary driven systems with long range interactions, for which the dynamical large deviations functionals are non-convex.

For important classes of models, the hydrodynamic limit and dynamical large deviations for the empirical density have been proven, see for example Kipnis et al. (1989); Quastel et al. (1999) for equilibrium dynamics and Bertini et al. (2006a); Bodineau and Derrida (2006); Bertini et al. (2009) in nonequilibrium dynamics. For Kawasaki dynamics with Kac potential, the law of large numbers for the empirical density has been proved on the torus with periodic boundary conditions in Giacomin and Lebowitz (1997), on the whole lattice in Marra and Mourragui (2000), and finally on a one-dimensional bounded interval (boundary driven) in Mourragui and Orlandi (2013). The hydrodynamic equation obtained for the boundary driven dynamics is the following nonlocal, nonlinear partial differential equation with Dirichlet conditions at the boundary Γ of the domain,

$$\begin{cases} \partial_t \rho_t \, = \, \nabla \cdot \Big\{ \nabla \rho_t - \beta \sigma(\rho_t) \nabla (J^{\rm neum} \star \rho_t) \Big\} \, = \, -\nabla \cdot \Big\{ \mathbf{J}^{\beta}(\rho_t) \Big\} \\ \rho_t \big|_{\Gamma} = \, b(\cdot) \quad \text{for} \quad 0 \le t \le T \, , \\ \rho_0(u) = \gamma(u) \, , \end{cases} \tag{1.1}$$

where $*$ stands for the spatial convolution and $\sigma(\rho) = 2\rho(1-\rho)$ is the mobility of the system. In the above formula $\mathbf{J}^{\beta}(\rho_t)$ is the instantaneous current at time t associated to the trajectory ρ:

$$\mathbf{J}^{\beta}(\rho_t) \, = \, -\nabla \rho_t + \beta \sigma(\rho_t) \nabla (J^{\rm neum} \star \rho_t) \, . \tag{1.2}$$

We shall denote by $\bar\rho$ the unique stationary solution of the hydrodynamic equation, i.e. $\bar\rho$ is the typical density profile for the stationary nonequilibrium state.

It follows from the hydrodynamic limit that the empirical current $\mathbf{W}^N_t$ converges weakly to the time integral of $\mathbf{J}^{\beta}(\rho_s)$ in the time interval $[0, t]$ (cf. Proposition 2.3). In addition to this we prove that when β is small enough, then the empirical particle density π^N obeys a law of large numbers with respect to the stationary measures (hydrostatic), i.e. it converges weakly under the unique stationary measure of the evolution process to the stationary solution $\bar\rho$, (see Proposition 2.2). This is obtained by deriving first the hydrodynamic behavior of the process (π^N_t) when η_0 is distributed according to the stationary measure. Then we exploit that for β small enough, the stationary solution $\bar\rho$ is unique and is a global attractor for the macroscopic evolution with a decay rate uniform with respect to the initial datum. This holds only for $\beta < \beta_0$ where $\beta_0 > 0$ depends on the diameter of the domain and on the chosen interaction J. Similar strategy for proving the hydrostatic is used in Farfan et al. (2011); Mourragui and Orlandi (2013). It results from the hydrostatics that if initially the particles are distributed according to the stationary state $\mu^{st,b}_N$, then for each $t > 0$, the mean empirical current $\mathbf{W}^N_t/t$ converges weakly to $\mathbf{J}^{\beta}(\bar\rho)$ as $N \uparrow \infty$ (see Proposition 2.4).

Further, we investigate the large deviations for the couple (current, density)= $(\mathbf{W}^N_t, \pi^N_t)$, that is we compute the asymptotic probability of observing an atypical macroscopic trajectory of the (current, density)= $(\mathbf{W}_t, \rho_t)$, when the number of particles tends to infinity. The result can be informally stated as follows. Given a trajectory $(\mathbf{W}_t, \rho_t)_{t\in[0,T]}$ on a fixed interval of time $[0, T]$, we have

$$\mathbb{P}^{\beta}_N \Big(\big(\mathbf{W}^N, \pi^N\big) \approx \big(\mathbf{W}, \rho\big) \Big) \, \sim \, \exp \Big\{ - N^d \mathcal{J}_T \big(\mathbf{W}, \rho\big) \Big\} \, ,$$

where $\mathbb{P}_N^\beta$ is the law of microscopic dynamics, $\sim$ denotes the logarithmic equivalence as $N \uparrow \infty$ and $(\mathbf{W}^N, \pi^N) \approx (\mathbf{W}, \rho)$ means that the trajectory $(\mathbf{W}^N, \pi^N)$ is in some neighborhood of $(\mathbf{W}, \rho)$ for an appropriate topology. The rate functional $\mathcal{J}_T$ is infinite in the set $\mathbf{E}^c$ of all paths $(\mathbf{W}, \rho)$ that do not satisfy the continuity equation $\partial_t \rho + \nabla \cdot \dot{\mathbf{W}}_t = 0$, and for which some suitable energy estimate does not holds (cf. (2.12)). Outside this set,

$$\mathcal{J}_T(\mathbf{W}, \rho) \;=\; \frac{1}{2} \int_0^T dt \left\langle \left[\dot{\mathbf{W}}_t - \mathbf{J}^\beta(\rho_t)\right], \frac{1}{\sigma(\rho_t)} \left[\dot{\mathbf{W}}_t - \mathbf{J}^\beta(\rho_t)\right] \right\rangle,$$

where $\dot{\mathbf{W}}_t$ is the instantaneous current at time t, $\langle \cdot, \cdot \rangle$ denotes integration with respect to the space variables and $\mathbf{J}^\beta(\cdot)$ is defined in (1.2).

Our proof relies on the method developed to study hydrodynamic large deviations for the density in Kipnis et al. (1989); Quastel et al. (1999); Bertini et al. (2009) and for the current Bertini et al. (2006b). The basic strategy of the proof of the lower bound consists of two steps, we first obtain this bound for smooth paths, then we extend it for general trajectories by showing that, for any given trajectory $(\mathbf{W}, \rho)$ with finite rate functional $\mathcal{J}_T(\mathbf{W}, \rho)$ one constructs a sequence of smooth paths $(\mathbf{W}^n, \rho^n)$ so that $(\mathbf{W}^n, \rho^n) \to (\mathbf{W}, \rho)$ in a suitable topology and $\mathcal{J}_T(\mathbf{W}^n, \rho^n) \to \mathcal{J}_T(\mathbf{W}, \rho)$. The proof in Bertini et al. (2006b) relies on the convexity of the rate functional. In the present case, because of the lack of convexity we modify the definition of the rate functional declaring it infinite in the set $\mathbf{E}^c$. The modified rate functional $\mathcal{J}_T$ makes the proof of the lower and upper bounds harder than the one in Bertini et al. (2006b).

The last result of this paper is the large deviations for the empirical density. In one dimension, it has been done in Mourragui and Orlandi (2013). In our context, one can achieve the proof either following the same scheme as in Mourragui and Orlandi (2013), or adapting the strategy of Bertini et al. (2006b), using the contraction principle.

The paper is organized as follows. In section 2, we introduce the model and state the main results. In Section 3, we introduce the perturbed model, we prove the law of large numbers for the current, and we collect some basic estimates needed along the paper. In Section 4, we state and prove some properties of the rate functionals. In sections 5 and 6, we derive the upper and lower bounds large deviations for the couple (current, density). Finally the density large deviations are recovered using the contraction principle in section 7.

2. Notation and Results

Fix a positive integer $d \geq 2$. Denote by Λ the open set $(-1, 1) \times \mathbb{T}^{d-1}$ and by $\overline{\Lambda} = [-1, 1] \times \mathbb{T}^{d-1}$ its closure, where $\mathbb{T}^k$ is the k-dimensional torus $[0, 1)^k$, and by $\Gamma = \partial\Lambda$ the boundary of Λ: $\Gamma = \{(u_1, \ldots, u_d) \in \overline{\Lambda} : u_1 = \pm 1\}$.

We introduce a smooth, symmetric, translational invariant probability kernel of range 1 on $\mathbb{S}_d = \mathbb{R} \times \mathbb{T}^{d-1}$, that is, a function $J : \mathbb{S}_d \times \mathbb{S}_d \to [0, 1]$ such that $J(u, v) = J(v, u) = J(0, v - u)$ for all $u, v \in \mathbb{S}_d$, $J(0, \cdot)$ is continuously differentiable, $J(0, u) = 0$, for all u such that $|u_1| > 1$, and $\int J(u, v) dv = 1$, for all $u \in \mathbb{S}_d$. This is the so called the Kac interaction on $\mathbb{S}_d$.

The Neuman Kac interaction J^{neum} is a symmetric probability kernel on Λ defined by imposing a reflection rule: when $(u, v) \in \overline{\Lambda} \times \overline{\Lambda}$, u interacts with v and with the reflected points of v where reflections are the ones with respect to the left and right boundaries of Λ. That is for all u and v in $\overline{\Lambda}$

$$J^{neum}(u,v) := J(u,v) + J(u, v + 2(1-v_1)e_1) + J(u, v - 2(1+v_1)e_1)\,, \tag{2.1}$$

where v_1 stands for the first cordinate of the vector $v = (v_1, \cdots, v_d)$ and $\{e_1, \ldots, e_d\}$ stands for the canonical basis of $\mathbb{R}^d$. By the assumption on J, $J^{neum}(u,v) = J^{neum}(v,u)$ and $\int J^{neum}(u,v)dv = 1$ for all $u \in \Lambda$, see Lemma 3.2. We defined the interaction (2.1) by boundary reflections only for convenience. It has the advantage to keep J^{neum} a symmetric probability kernel. This choice of the potential has been done already in De Masi et al. (2011); Mourragui and Orlandi (2013).

For an integer $N \geq 1$, denote by $\mathbb{T}_N^{d-1} = \{0, \ldots, N-1\}^{d-1}$, the discrete $(d-1)$-dimensional torus of length N. Let $\Lambda_N = \{-N, \ldots, N\} \times \mathbb{T}_N^{d-1}$ be the cylinder in $\mathbb{Z}^d$ of length $2N+1$ and basis $\mathbb{T}_N^{d-1}$ and let $\Gamma_N = \{(x_1, \ldots, x_d) \in \mathbb{Z} \times \mathbb{T}_N^{d-1} \,|\, x_1 = \pm N\}$ be the boundary of Λ_N. The elements of Λ_N are denoted by letters x, y and the elements of $\overline{\Lambda}$ by the letters u, v.

The configuration space is $\Sigma_N := \{0,1\}^{\Lambda_N}$; elements of Σ_N are denoted by η so that $\eta(x) = 1$, (resp. 0) if site x is occupied, (resp. empty) for the configuration η.

Fix a positive parameter $\beta \geq 0$, and a positive function $b : \Gamma \to \mathbb{R}_+$. Assume that there exists a neighbourhood V of $\overline{\Lambda}$ and a smooth function $\theta : V \to (0,1)$ in $\mathcal{C}^2(V)$ such that θ is bounded below by a strictly positive constant, bounded above by a constant smaller than 1 and such that the restriction of θ to Γ is equal to b. The boundary driven Kawasaki process with Neuman Kac interaction is the Markov process on Σ_N whose generator $\mathfrak{L}_N := \mathfrak{L}_{\beta,b,N}$ can be decomposed as

$$\mathfrak{L}_N \;:=\; N^2 \mathcal{L}_{\beta,N} \;+ N^2\, L_{b,N}\,. \tag{2.2}$$

The generator $\mathcal{L}_{\beta,N}$ describes the bulk dynamics which preserves the total number of particles. The pair interaction between x and y in Λ_N is given by

$$J_N(x,y) = N^{-d} J^{neum}(\frac{x}{N}, \frac{y}{N}).$$

The total interaction energy among particles is defined by the following Hamiltonian

$$H_N(\eta) = -\sum_{x,y \in \Lambda_N} J_N(x,y)\eta(x)\eta(y)\,. \tag{2.3}$$

The action of $\mathcal{L}_{\beta,N}$ on functions $f : \Sigma_N \to \mathbb{R}$ is then given by

$$\left(\mathcal{L}_{\beta,N} f\right)(\eta) = \sum_{i=1}^{d} \sum_{x, x+e_i \in \Lambda_N} C_N^\beta(x, x+e_i; \eta) \left[f(\eta^{x,x+e_i}) - f(\eta)\right]\,,$$

whith the rate of exchange occupancies C_N^β given by

$$C_N^\beta(x,y;\eta) = \exp\left\{-\frac{\beta}{2}[H_N(\eta^{x,y}) - H_N(\eta)]\right\}\,, \tag{2.4}$$

where $\eta^{x,y}$ is the configuration obtained from $\eta \in \Sigma_N$, by exchanging the occupation variables $\eta(x)$ and $\eta(y)$, i.e.

$$(\eta^{x,y})(z) := \begin{cases} \eta(y) & \text{if } \; z = x\,, \\ \eta(x) & \text{if } \; z = y\,, \\ \eta(z) & \text{if } \; z \neq x, y\,. \end{cases}$$

The generator $L_{b,N}$ models the particle reservoir at the boundary of Λ_N, it is defined by the infinitesimal generator of a birth and death process acting on Γ_N as

$$(L_{b,N} f)(\eta) \;=\; \sum_{x\in\Gamma_N} r_x\big(b(x/N),\eta\big)\big[f(\sigma^x\eta)-f(\eta)\big]\,,$$

where $\sigma^x\eta$ is the configuration obtained from η by flipping the configuration at x, i.e.

$$(\sigma^x\eta)(z) := \begin{cases} 1-\eta(x) & \text{if } \; z=x \\ \eta(z) & \text{if } \; z\neq x\,, \end{cases}$$

and for $x\in\Gamma_N$ and $\lambda\in(0,1)$ the rate $r_x\big(\lambda,\eta\big)$ is given by

$$r_x\big(\lambda,\eta\big) \;:=\; \lambda(1-\eta(x))+(1-\lambda)\eta(x)\,. \tag{2.5}$$

For any $\beta\geq 0$, the operator $\mathcal{L}_{\beta,N}$ is self-adjoint w.r.t. the Gibbs measures $\mu_N^{\beta,\lambda}$ associated to the Hamiltonian (2.3) and chemical potentials $\lambda\in\mathbb{R}$:

$$\mu_N^{\beta,\lambda}(\eta) = \frac{1}{Z_N^{\beta,\lambda}}\exp\{-\beta H_N(\eta)+\lambda\sum_{x\in\Lambda_N}\eta(x)\}\,, \qquad \eta\in\Sigma_N\,,$$

where $Z_N^{\beta,\lambda}$ is the normalization constant. This means that the rates of the bulk dynamics $\{C_N^\beta(x,y;\eta), \quad x,\, y\in\Lambda_N\}$, satisfies the detailed balance conditions:

$$C_N^\beta(x,y;\eta) = e^{-\beta[H_N(\eta^{x,y})-H_N(\eta)]}C_N^\beta(y,x;\eta^{x,y}).$$

For a smooth function $\rho:\bar\Lambda\to(0,1)$ and $x\in\Lambda_N$, let $\nu^N_{\rho(\cdot)}$ be the Bernoulli product measure on Σ_N with marginals given by

$$\nu^N_{\rho(\cdot)}(\eta(x)=1) = \rho(x/N)\;.$$

Let $\varphi(\alpha):=\log[\alpha/(1-\alpha)]$ be the chemical potential of the density α. It is easy to see that, $\nu^N_{\rho(\cdot)}$ can be rewritten as

$$\nu^N_{\rho(\cdot)}(\eta) = \prod_{x\in\Lambda_N}\frac{e^{\varphi(\rho(x/N))\,\eta(x)}}{1+e^{\varphi(\rho(x/N))}}\,, \tag{2.6}$$

and if $\rho(u)=b(u)$ for all $u\in\Gamma$, then $\nu^N_{\rho(\cdot)}$ is reversible for the process with generator $L_{b,N}$.

Notice that in view of the diffusive scaling limit, the generator has been speeded up by N^2. We denote by (η_t) the Markov process on Σ_N with generator $\mathfrak{L}_N$. Since the Markov process (η_t) is irreducible, for each $N\geq 1$, $\beta\geq 0$, there exists a unique invariant measure $\mu_N^{st}=\mu_N^{st}(\beta,b)$ in which we drop the dependence on β and b from the notation. Moreover, if b is not constant then the invariant measure μ_N^{st} cannot be written in simple form.

For an integer $1\leq m\leq+\infty$ denote by $\mathcal{C}^m(\Lambda)$ the space of m-continuously differentiable real functions defined on $\overline{\Lambda}$. Let $\mathcal{C}_0^m(\Lambda)$ (resp. $\mathcal{C}_c^m(\Lambda)$), $1\leq m\leq+\infty$, be the subset of functions in $\mathcal{C}^m(\Lambda)$ which vanish at the boundary of Λ (resp. with compact support in Λ). We denote by $\mathcal{M}=\mathcal{M}(\Lambda)$ the space of finite signed measures on Λ, endowed with the weak topology. For a finite signed measure m and a continuous function $F\in\mathcal{C}^0(\Lambda)$, we let $\langle m,F\rangle$ be the integral of F with respect to m.

For each configuration η, denote by $\pi^N = \pi^N(\eta) \in \mathcal{M}$ the positive measure obtained by assigning mass N^{-d} to each particle of η :

$$\pi^N \,=\, N^{-d} \sum_{x\in\Lambda_N} \eta(x)\, \delta_{x/N}\ ,$$

where δ_u is the Dirac measure concentrated on u. Notice that for each $\eta \in \Sigma_N$, the total mass of the positive measure $\pi^N(\eta)$ is bounded by 3.

For $t \geq 0$ and two neighboring sites $x, y \in \Lambda_N$, denote by $\mathbf{N}_t^{x,y}$ the total number of particles that jumped from x to y in the macroscopic time interval $[0,t]$. For $1 \leq j \leq d$ and $x, x+e_j \in \Lambda_N$, we denote by $W_t^{x,x+e_j} = \mathbf{N}_t^{x,x+e_j} - \mathbf{N}_t^{x+e_j,x}$ the current through the edge $(x, x+e_j)$. We now define the current entering and leaving the system through the border points in the direction e_1. For $x \in \Gamma_N$, let $\mathbf{N}_t^{x,+}$ (resp. $\mathbf{N}_t^{x,-}$) be the number of particles created (resp. killed) at x due to the reservoir in the macroscopic time interval $[0,t]$, the current through Γ_N is then defined by $W_t^x = \mathbf{N}_t^{x,+} - \mathbf{N}_t^{x,-}$ if $x \in \Gamma_N^-$ and $W_t^x = \mathbf{N}_t^{x,-} - \mathbf{N}_t^{x,+}$ if $x \in \Gamma_N^+$, where Γ_N^-, resp. Γ_N^+ stands for the left, resp. right, boundary of Λ_N:

$$\Gamma_N^{\pm} = \{(x_1, \cdots, x_d) \in \Gamma_N : x_1 = \pm N\}\,.$$

For $t \geq 0$, we define the *empirical current* $\mathbf{W}_t^N = (W_{1,t}^N, \ldots, W_{d,t}^N) \in \mathcal{M}^d = \{\mathcal{M}(\Lambda)\}^d$ as the vector-valued finite signed measure on Λ induced by the net flow of particles in the time interval $[0,t]$:

$$\begin{cases} W_{1,t}^N \;=\; \dfrac{1}{N^{d+1}} \displaystyle\sum_{x,x+e_1\in\Lambda_N} W_t^{x,x+e_1}\delta_{x/N} \,+\, \dfrac{1}{N^{d+1}} \sum_{x\in\Gamma_N} W_t^x \delta_{x/N}\,, \\ W_{k,t}^N \;=\; \dfrac{1}{N^{d+1}} \displaystyle\sum_{x\in\Lambda_N} W_t^{x,x+e_k}\delta_{x/N} \quad \text{for} \qquad k = 2, \ldots, d\,. \end{cases} \tag{2.7}$$

For a continuous vector field $\mathbf{G} = (G_1, \ldots, G_d) \in (\mathcal{C}^0(\Lambda))^d$ the integral of $\mathbf{G}$ with respect to $\mathbf{W}_t^N$, also denoted by $\langle \mathbf{W}_t^N, \mathbf{G}\rangle$, is given by

$$\langle \mathbf{W}_t^N, \mathbf{G}\rangle = \sum_{k=1}^{d} \langle W_{k,t}^N\,,\, G_k\rangle\ , \tag{2.8}$$

where

$$\langle W_{1,t}^N\,,\, G_1\rangle \;=\; N^{-(d+1)} \Big\{ \sum_{x,x+e_1\in\Lambda_N} G_1(x/N)\, W_t^{x,x+e_1} \,+\, \sum_{x\in\Gamma_N} G_1(x/N)\, W_t^x \Big\}$$

and for $2 \leq k \leq d$,

$$\langle W_{k,t}^N\,,\, G_k\rangle \;=\; N^{-(d+1)} \sum_{x\in\Lambda_N} G_k(x/N)\, W_t^{x,x+e_k}\,.$$

The purpose of this article is to prove hydrodynamic limit and large deviations for the empirical current and for the density of particles. Fix $T > 0$. Let $\mathcal{F}^1$ be the subset of $\mathcal{M}$ of all absolutely continuous positive measures with respect to the Lebesgue measure with positive density bounded by 1:

$$\mathcal{F}^1 = \big\{\pi \in \mathcal{M} : \pi(du) = \rho(u)du \quad \text{and} \quad 0 \leq \rho(u) \leq 1 \;\text{ a.e.}\big\}\,.$$

For a metric space E ($E = \mathcal{M}, \mathcal{F}^1, \mathcal{M}^d, \Sigma_N, \cdots$), let $D([0,T], E)$ be the set of right continuous with left limits trajectories with values in E, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra. For a probability measure μ_N on Σ_N denote by $(\eta_t)_{t\in[0,T]}$ the Markov process with generator $\mathfrak{L}_N$

starting, at time $t=0$, by η_0 distributed according to μ_N. Denote by $\mathbb{P}^\beta_{\mu_N} := \mathbb{P}^{\beta,N}_{\mu_N}$ the probability measure on the path space $D([0,T],\Sigma_N)$ corresponding to the Markov process $(\eta_t)_{t\in[0,T]}$ and by $\mathbb{E}^\beta_{\mu_N}$ the expectation with respect to $\mathbb{P}^\beta_{\mu_N}$. When $\mu_N = \delta_{\eta^N}$ for some configuration $\eta^N \in \Sigma_N$, we write simply $\mathbb{P}^\beta_{\eta^N} = \mathbb{P}^{\beta,N}_{\delta_{\eta^N}}$ and $\mathbb{E}^\beta_{\eta^N} = \mathbb{E}^\beta_{\delta_{\eta^N}}$. We denote by π^N the map from $D([0,T],\Sigma_N)$ to $D([0,T],\mathcal{M})$ defined by $\pi^N(\eta_\cdot)_t = \pi^N(\eta_t)$ and by $Q^\beta_{\mu_N} = \mathbb{P}^\beta_{\mu_N}\circ(\pi^N)^{-1}$ the law of the process $\big(\pi^N(\eta_t)\big)_{t\in[0,T]}$.

2.1. *Hydrodynamics and hydrostatics.* The hydrodynamic and hydrostatic limits for the empirical measures π^N has been proved in one dimension in Mourragui and Orlandi (2013). The analysis in all dimension can be deducted from the same strategy. We shall therefore summarize the results omitting their proofs.

For integers n and m we denote by $C^{n,m}([0,T]\times\Lambda)$ the space of functions $F = F_t(u) : [0,T]\times\overline{\Lambda} \to \mathbb{R}$ with n derivatives in time and m derivatives in space which are continuous up to the boundary. We denote by $C^{n,m}_0([0,T]\times\Lambda)$ the subset of $C^{n,m}([0,T]\times\Lambda)$ of functions vanishing at the boundary of Λ, i.e. $F_t\big|_\Gamma \equiv 0$ for all $t\in[0,T]$. We finaly denote by $C^{n,m}_c([0,T]\times\Lambda)$ the subset of $C^{n,m}([0,T]\times\Lambda)$ of functions with compact support in $[0,T]\times\Lambda$.

Let $L^2(\Lambda)$ be the Hilbert space of functions $F:\Lambda\to\mathbb{R}$ such that $\int_\Lambda |F(u)|^2 du < \infty$ equipped with the inner product

$$\langle F, G\rangle = \int_\Lambda F(u)\, G(u)\, du\ .$$

The norm of $L^2(\Lambda)$ is denoted by $\|\cdot\|_{L_2(\Lambda)}$.

Let $H^1(\Lambda)$ be the Sobolev space of functions F with generalized derivatives $\nabla F = \big(\partial_1 F,\cdots,\partial_d F\big)$ in $L^2(\Lambda)$. $H^1(\Lambda)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{H^1}$, defined by

$$\langle F, G\rangle_{H^1} = \langle F, G\rangle + \sum_{i=1}^d \langle \partial_i F\,,\, \partial_i G\rangle\ ,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{H^1}$. Denote by $H^1_0(\Lambda)$ the closure of $C^\infty_c(\Lambda)$ in $H^1(\Lambda)$.

Denote by $\mathrm{Tr} : H^1(\Lambda) \to L^2(\Gamma)$ the continuous linear operator called trace operator, defined as the unique extension of the linear operator from $\mathcal{C}^0(\Lambda)$ to $L^2(\Gamma)$ which associates to any $F\in H^1(\Lambda)\cap\mathcal{C}^0(\Lambda)$ its boundary value: $\mathrm{Tr}(F) = F\big|_\Gamma$ (Zeidler (1990), Theorem 21.A.(e)). Recall that the space $H^1_0(\Lambda)$ is the space of functions F in $H^1(\Lambda)$ with zero trace:

$$H^1_0(\Lambda) = \big\{F\in H^1(\Lambda) :\ \mathrm{Tr}(F) = 0\big\}\ .$$

To state the hydrodynamic equation, we need some more notation. For a Banach space $(\mathbb{B},\|\cdot\|_{\mathbb{B}})$ we denote by $L^2([0,T],\mathbb{B})$ the Banach space of measurable functions $U:[0,T]\to\mathbb{B}$ for which

$$\|U\|^2_{L^2([0,T],\mathbb{B})}\ =\ \int_0^T \|U_t\|^2_{\mathbb{B}}\, dt\ <\ \infty$$

holds. For $m \in L_\infty(\Lambda)$ and $u \in \Lambda$, we set

$$(J^{\text{neum}} \star m)(u) = \int_\Lambda J^{\text{neum}}(u,v)m(v)dv\,,$$

and $\sigma(m) = 2m(1-m)$. For any smooth function F, let ΔF be the laplacian with respect to the space variables of a function F. For $F \in C_0^{1,2}([0,T]\times\Lambda)$, $\rho \in D([0,T],\mathcal{F}^1)$ denote

$$\begin{aligned}\ell_F^\beta(\rho) := \big\langle \rho_T, F_T\big\rangle - \langle \rho_0, F_0\rangle - \int_0^T dt\,\big\langle \rho_t, \partial_t F_t\big\rangle \\ - \int_0^T dt\,\big\langle \rho_t, \Delta F_t\big\rangle \;+\; \int_0^T dt \int_\Gamma b(r)\,\mathbf{n}_1(r)\,(\partial_1 F_t)(r)\,dS(r) \\ - \beta \int_0^T \langle \sigma(\rho_t), (\nabla F_t)\cdot\nabla(J^{\text{neum}}\star\rho_t)\rangle dt\,,\end{aligned}$$

where $\mathbf{n}$=$(\mathbf{n}_1,\ldots,\mathbf{n}_d)$ stands for the outward unit normal vector to the boundary surface Γ and dS for an element of surface on Γ. For $u,v\in\mathbb{R}^d$, $u{\cdot}v$ is the usual scalar product of u and v in $\mathbb{R}^d$, we denote by $|\cdot|$ the associated norm: $|u| = \sqrt{\sum_{i=1}^d |u_i|^2}$.

Denote by $\mathcal{A}_{[0,T]} \subset D\big([0,T];\mathcal{F}^1\big)$ the set of all weak solutions of the boundary value problem (1.1):

$$\mathcal{A}_{[0,T]} = \Big\{\rho \in L^2\big([0,T], H^1(\Lambda)\big)\;:\quad \forall F \in \mathcal{C}_0^{1,2}([0,T]\times\Lambda)\,,\; \ell_F^\beta(\rho) = 0\Big\}\,.$$

Proposition 2.1. *For any sequence of initial probability measures $(\mu_N)_{N\ge1}$, the sequence of probability measures $(Q^\beta_{\mu_N})_{N\ge1}$ is weakly relatively compact and all its converging subsequences converge to some limit $Q^{\beta,*}$ that is concentrated on absolutely continuous paths whose densities $\rho \in C([0,T],\mathcal{F}^1(\Lambda))$ are in $\mathcal{A}_{[0,T]}$. Moreover, if for any $\delta>0$ and for any function $F\in\mathcal{C}^0(\Lambda)$*

$$\lim_{N\to\infty} \mu^N\Big\{\Big|\langle \pi_N, F\rangle \;-\; \int_\Lambda \gamma(u)F(u)du\Big| \ge \delta\Big\} = 0\,, \tag{2.9}$$

for an initial profile $\gamma\in\mathcal{F}^1$, then the sequence of probability measures $(Q^\beta_{\mu_N})_{N\ge1}$ converges to the Dirac measure concentrated on the unique weak solution $\rho(\cdot,\cdot)$ of the boundary value problem (1.1). Accordingly, for any $t\in[0,T]$, any $\delta>0$ and any function $F\in\mathcal{C}^0(\Lambda)$

$$\lim_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big\{\Big|\langle \pi_N(\eta_t), F\rangle \;-\; \int_\Lambda \rho(t,u)F(u)du\Big| \ge \delta\Big\} = 0\,.$$

The proof of this Proposition is similar to the one of Theorem 2.1. in Mourragui and Orlandi (2013). Recall that the stationary measure μ_N^{st} depends on β and b. The asymptotic behavior of the empirical measure under the stationary state μ_N^{st} can be stated as follows.

Proposition 2.2. *There exists β_0 depending on Λ and J^{neum} so that, for any $\beta<\beta_0$, for any $F\in C^0(\Lambda)$, for any $\delta>0$,*

$$\lim_{N\to\infty} \mu_N^{st}\Big[\Big|\langle \pi_N(\eta), F\rangle \;-\; \int_\Lambda \bar\rho(u)F(u)du\Big| \ge \delta\Big] = 0\,,$$

where $\bar\rho$ is the unique weak solution of the following boundary value problem

$$\begin{cases} \Delta\rho(u) - \beta\nabla\cdot\Big\{\sigma(\rho(u))\nabla(J^{neum}\star\rho)(u)\Big\} = 0, \quad u\in\Lambda, \\ \rho(\cdot)\big|_\Gamma = \; b(\cdot)\,. \end{cases} \tag{2.10}$$

The proof of this Proposition is similar to the one of Theorem 2.3 in Mourragui and Orlandi (2013) and therefore is omitted. As noticed in the introduction, we need to show the uniqueness of the solution of the equation (2.10), and that the hydrodynamic equation (1.1) satisfy a comparison principle. For values of β larger than β_0, we are not able to show these two main ingredients used in Farfan et al. (2011); Mourragui and Orlandi (2013) to derive the hydrostatic limit.

Proposition 2.3. *Fix an initial profile $\gamma\in\mathcal{F}^1$ and consider a sequence of probability measures μ^N associated to γ in the sense of (2.9). Let ρ be the solution of the equation (1.1). Then, for each $T>0$, $\delta>0$ and $\mathbf{G}\in\big(C^1(\Lambda)\big)^d$,*

$$\lim_{N\to\infty}\mathbb{P}^\beta_{\mu_N}\Big[\,\big\langle\mathbf{W}^N_T,\mathbf{G}\big\rangle \;-\; \int_0^T dt\,\big\langle\big\{-\nabla\rho_t+\beta\sigma(\rho_t)\nabla(J^{neum}\star\rho_t)\big\}\,,\,\mathbf{G}\big\rangle\Big| > \delta\Big] \;=\; 0\,.$$

Next result concerns the asymptotic behavior of the mean empirical current $\mathbf{W}^N_T/T$ under the sequence of stationary measures $\{\mu^{st}_N \;:\; N\ge 1\}$.

Proposition 2.4. *There exists β_0 depending on Λ and J^{neum} so that, for any $\beta<\beta_0$, for any $T>0$, $\delta>0$ and $\mathbf{G}\in\big(C^1(\Lambda)\big)^d$,*

$$\lim_{N\to\infty}\mathbb{P}^\beta_{\mu^{st}_N}\Big[\,\big\langle\frac{1}{T}\mathbf{W}^N_T,\mathbf{G}\big\rangle \;-\; \big\langle\big\{-\nabla\bar\rho+\beta\sigma(\bar\rho)\nabla(J^{neum}\star\bar\rho)\big\}\,,\,\mathbf{G}\big\rangle\Big| > \delta\Big] \;=\; 0\,,$$

where $\bar\rho$ is the unique weak solution of the boundary value problem (2.10).

The proof of Proposition 2.3 is given for more general processes in section 3. We obtain then Proposition 2.4 as an immediate consequence from Proposition 2.2.

2.2. *Large deviations.* Fix a positive time $T>0$ and an initial profile $\gamma\in\mathcal{F}^1$. We are interested both on large deviations of the couple $(\mathbf{W}^N_t,\pi^N(\eta_t))_{t\in[0,T]}$ and on large deviations of the empirical measure $(\pi^N(\eta_t))_{t\in[0,T]}$ during the interval time $[0,T]$ and starting from the profile γ.

Let $\mathfrak{A}_\gamma$ be the set of trajectories $(\mathbf{W},\pi)$ in $D([0,T],\mathcal{M}^{d+1})$ such that for any $t\in[0,T]$ and any $G\in\mathcal{C}^1_0(\Lambda)$

$$\langle\pi_t,G\rangle-\langle\gamma,G\rangle=\langle\mathbf{W}_t,\nabla G\rangle\,. \tag{2.11}$$

Define the energy functional $\mathcal{E}^\gamma=\mathcal{E}^{\gamma,T,\beta}:D([0,T],\mathcal{M}^{d+1})\to[0,\infty]$ by

$$\mathcal{E}^\gamma(\mathbf{W},\pi)=\begin{cases}\mathcal{Q}(\pi) & \text{if}\quad (\mathbf{W},\pi)\in\mathfrak{A}_\gamma\cap D([0,T],\mathcal{M}^d\times\mathcal{F}^1)\,,\\ +\infty & \text{otherwise,}\end{cases} \tag{2.12}$$

where the functional $\mathcal{Q}:D([0,T],\mathcal{F}^1)\to[0,\infty]$ is given for a trajectory $\pi\in D([0,T],\mathcal{F}^1)$ with $\pi_t=\rho_t(u)du\,,\; t\in[0,T]$ by the formula

$$\mathcal{Q}(\pi)=\sum_{k=1}^d\sup\Big\{\int_0^T dt\,\langle\rho_t,\partial_k H_t\rangle-2\int_0^T dt\langle\sigma(\rho_t)H_t,H_t\rangle\Big\}\,,$$

in which the supremum is carried over all $H \in C_c^\infty([0,T]\times\Lambda)$. It has been proved in Bertini et al. (2009); Farfan et al. (2011) that $\mathcal{Q}(\pi)$ is finite if and only if $\rho \in L^2\big([0,T],H^1(\Lambda)\big)$, and

$$\mathcal{Q}(\pi) \;=\; \frac{1}{8}\int_0^T dt\,\langle \frac{1}{\sigma(\rho_t)}\,,\,\nabla\rho_t\cdot\nabla\rho_t\rangle\,. \tag{2.13}$$

Notice that $\mathfrak{A}_\gamma \cap D([0,T],\mathcal{M}^d\times\mathcal{F}^1)$ is a closed and convex subset of $D([0,T],\mathcal{M}^{d+1})$. It follows immediately from the concavity of $\sigma(\cdot)$ that the functional $\mathcal{E}^\gamma$ is convex and lower semicontinuous.

We now define the large deviations functional for the pair $(\mathbf{W}^N,\pi^N)$ in the time interval $[0,T]$ with initial condition γ. For each $\mathbf{V}\in\big(C^{1,1}([0,T]\times\Lambda)\big)^d$, define the functional $\widehat{\mathbb{J}}^T_{\mathbf{V}} = \widehat{\mathbb{J}}^{T,\beta}_{\mathbf{V}} : D([0,T],\mathcal{M}^d\times\mathcal{F}^1)\to\mathbb{R}$ if $\pi_t=\rho_t(u)du\,,\ t\in[0,T]$ by

$$\widehat{\mathbb{J}}^T_{\mathbf{V}}(\mathbf{W},\pi) \;=\; \mathbf{L}^\beta_{\mathbf{V}}(\mathbf{W},\pi) \;-\; \frac{1}{2}\int_0^T dt\,\langle\sigma(\rho_t),\mathbf{V}_t\cdot\mathbf{V}_t\rangle\;, \tag{2.14}$$

where $\mathbf{L}^\beta_{\mathbf{V}}(\mathbf{W},\pi) := \mathbf{L}^\beta_{\mathbf{V},T}(\mathbf{W},\pi)$ is a linear function on $\mathbf{V}$:

$$\begin{aligned}\mathbf{L}^\beta_{\mathbf{V}}(\mathbf{W},\pi) \;=\; &\langle\mathbf{W}_T,\mathbf{V}_T\rangle \;-\; \int_0^T dt\,\langle\mathbf{W}_t,\partial_t\mathbf{V}_t\rangle\\ &-\int_0^T dt\,\langle\pi_t,\nabla\cdot\mathbf{V}_t\rangle \;+\; \int_0^T dt\int_\Gamma b(r)\,\mathbf{n}_1(r)\,V_1(t,r)\,dS(r)\\ &-\beta\int_0^T\langle\sigma(\rho_t),\mathbf{V}_t\cdot\nabla(J^{\rm neum}\star\rho_t)\rangle dt\,.\end{aligned}$$

The large deviations fuctional for $(\mathbf{W}^N,\pi^N)$ is finally defined from $D([0,T],\mathcal{M}^{d+1})$ to $[0,+\infty]$ by

$$\mathcal{J}^\gamma_T(\mathbf{W},\pi) \;=\; \begin{cases}\mathbb{J}_T(\mathbf{W},\pi) & \text{if } \mathcal{E}^\gamma(\mathbf{W},\pi)<\infty\,,\\ +\infty & \text{otherwise}\,,\end{cases} \tag{2.15}$$

where

$$\mathbb{J}_T(\mathbf{W},\pi) \;=\; \sup_{\mathbf{V}\in\left(C^{1,1}([0,T]\times\Lambda)\right)^d} \widehat{\mathbb{J}}^T_{\mathbf{V}}(\mathbf{W},\pi)\,.$$

It remains to define the rate functional for the empirical measure. Fix an initial profile $\gamma:\bar\Lambda\to[0,1]$, denote

$$\begin{aligned}\ell^\beta_F(\rho|\gamma) := &\big\langle\rho_T,F_T\big\rangle - \langle\gamma,F_0\rangle - \int_0^T dt\,\big\langle\rho_t,\partial_t F_t\big\rangle\\ &-\int_0^T dt\,\big\langle\rho_t,\Delta F_t\big\rangle \;+\; \int_0^T dt\int_\Gamma b(r)\,\mathbf{n}_1(r)\,(\partial_1 F_t)(r)\,dS(r)\\ &-\beta\int_0^T\langle\sigma(\rho_t),(\nabla F_t)\cdot\nabla(J^{\rm neum}\star\rho_t)\rangle dt\,,\end{aligned} \tag{2.16}$$

Denote by $\mathbb{I}^\gamma_T = \mathbb{I}^{\gamma,\beta}_T\colon D([0,T],\mathcal{F}^1)\longrightarrow[0,\infty]$ the functional given for a trajectory π with $\pi_t(du)=\rho_t(u)du\,,\ t\in[0,T]$ by

$$\mathbb{I}^\gamma_T(\pi) = \sup_{F\in C_0^{1,2}([0,T]\times\Lambda)} \widehat{\mathbb{I}}^{T,\gamma}_F(\pi)\,, \tag{2.17}$$

where for any function $F \in \mathcal{C}_0^{1,2}([0,T]\times\Lambda)$, $\widehat{\mathbb{I}}_F^{T,\gamma} = \widehat{\mathbb{I}}_F^{T,\gamma,\beta} \colon D([0,T],\mathcal{F}^1) \longrightarrow \mathbb{R}$ is given by

$$\widehat{\mathbb{I}}_F^{T,\gamma}(\pi) := \ell_F^{\beta}(\rho|\gamma) \; - \; \frac{1}{2}\int_0^T dt \left\langle \sigma(\rho_t), \nabla F_t \cdot \nabla F_t \right\rangle .$$

The rate functional $\mathcal{I}_T^{\gamma} : D([0,T],\mathcal{M}) \to [0,\infty]$ for the empirical measure is then given by

$$\mathcal{I}_T^{\gamma}(\pi) = \begin{cases} \mathbb{I}_T^{\gamma}(\pi) & \text{if} \quad \pi \in D([0,T],\mathcal{F}^1) \text{ and } \mathcal{Q}(\pi) < +\infty \, , \\ +\infty & \text{otherwise .} \end{cases} \tag{2.18}$$

We are now ready to state the large deviations results:

Theorem 2.5. *Fix $T > 0$ and an initial profile γ in $\mathcal{C}^0(\Lambda)$. Consider a sequence $\{\eta^N : N \geq 1\}$ of configurations associated to γ in the sense of (2.9). Then, for each closed set $\mathcal{C}$ and each open set $\mathcal{U}$ of $D([0,T],\mathcal{M}^{d+1})$, we have*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}^{\beta}_{\eta^N}\Big[(\mathbf{W}^N, \pi^N) \in \mathcal{C} \Big] \; \leq \; - \inf_{(\mathbf{W},\pi)\in\mathcal{C}} \mathcal{J}_T^{\gamma}(\mathbf{W},\pi) \, ,$$
$$\underline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}^{\beta}_{\eta^N}\Big[(\mathbf{W}^N, \pi^N) \in \mathcal{U} \Big] \; \geq \; - \inf_{(\mathbf{W},\pi)\in\mathcal{U}} \mathcal{J}_T^{\gamma}(\mathbf{W},\pi) \, .$$

The functional $\mathcal{J}_T^{\gamma}(\cdot,\cdot)$ is lower semi-continuous.

We prove this Theorem in sections 5 and 6. We have the following dynamical large deviation principle for the empirical measure.

Theorem 2.6. *Fix $T > 0$ and an initial profile γ in $\mathcal{C}^0(\Lambda)$. Consider a sequence $\{\eta^N : N \geq 1\}$ of configurations associated to γ in the sense of (2.9). Then, the sequence of probability measures $\{Q^{\beta}_{\eta^N} : N \geq 1\}$ on $D([0,T],\mathcal{M})$ satisfies a large deviation principle with speed N and rate function $\mathcal{I}_T^{\gamma}(\cdot)$, defined in (2.18):*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q^{\beta}_{\eta^N}\left(\pi^N \in C\right) \; \leq \; - \inf_{\pi\in C} \mathcal{I}_T^{\gamma}(\pi)$$
$$\underline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q^{\beta}_{\eta^N}\left(\pi^N \in U\right) \; \geq \; - \inf_{\pi\in U} \mathcal{I}_T^{\gamma}(\pi) \, ,$$

for any closed set $C \subset D([0,T],\mathcal{M})$ and open set $U \subset D([0,T],\mathcal{M})$. The functional $\mathcal{I}_T^{\gamma}(\cdot)$ is lower semi-continuous and has compact level sets.

The proof of this Theorem is given in Section 7. It relies on Theorem 2.5 and the contraction principle.

3. The perturbed dynamics and basic tools

In this section, we consider the perturbation of the original process (2.2), and we prove some results needed either to caracterize the behavior of the empirical current and the empirical density, either to prove large deviations principle.

3.1. *The modified process.* Fix $T > 0$, a time dependent vector-valued function $\mathbf{V} = (V_1, \ldots, V_d) \in \big(\mathcal{C}^{0,0}([0,T] \times \Lambda)\big)^d$ and a smooth function $H \in \mathcal{C}^{0,0}([0,T] \times \Gamma)$. Define at time t, $0 \le t \le T$, the following generators of a time inhomogeneous Markov process on Σ_N

$$\big(\mathcal{L}^{\mathbf{V}}_{\beta,N} f\big)(\eta) \;=\; \sum_{i=1}^{d} \sum_{x,x+e_i \in \Lambda_N} C^{\beta,V_i}_{N,t}(x, x+e_i; \eta) \left[f(\eta^{x,x+e_i}) - f(\eta)\right] ,$$

$$(L^H_{b,N} f)(\eta) \;=\; \sum_{x \in \Gamma_N} r^H_{x,t}\big(b(x/N), \eta\big)\left[f(\sigma^x \eta) - f(\eta)\right] ,$$

where the rate function $C^{\beta,V_i}_{N,t}(x, x+e_i; \eta)$ is defined through the rate C^β_N by

$$C^{\beta,V_i}_{N,t}(x, x+e_i; \eta) \;=\; C^\beta_N(x, x+e_i; \eta) e^{-[\eta(x+e_i)-\eta(x)]N^{-1} V_i(t,x/N)} , \tag{3.1}$$

and the rate at the boundary $r^H_{x,t}\big(b(x/N), \eta)\big)$ is defined through the rate r_x as

$$r^H_{x,t}\big(b(x/N), \eta\big) \;=\; r_x\big(b(x/N), \eta\big) e^{(2\eta(x)-1)N^{-1} H(t,x/N)} . \tag{3.2}$$

For a probability measure μ_N on Σ_N denote by $\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}$ the law of the inhomogeneous Markov process $(\eta_t)_{t\in[0,T]}$ on the path space $D\big([0,T], \Sigma_N\big)$ with generator $\mathfrak{L}^{\mathbf{V},H}_N = N^2 \mathcal{L}^{\mathbf{V}}_{\beta,N} + N^2 L^H_{b,N}$ and initial distribution μ_N. Let $Q^{\beta,\mathbf{V},H}_{\mu_N}$ be the measure of the process $(\pi^N_t)_{t\in[0,T]}$ on the state space $D\big([0,T], \mathcal{M}\big)$ induced from $\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}$.

Proposition 3.1. *Let μ_N be a sequence of probability measures on Σ_N corresponding to a macroscopic profile γ in the sense of (2.9). Then the sequence of probability measures $Q^{\beta,\mathbf{V},H}_{\mu_N}$ converges as $N \uparrow \infty$, to $Q^{\beta,\mathbf{V}}$. This limit point is concentrated on the unique weak solution $\rho^{\beta,\mathbf{V}}$ in $L^2([0,T], H^1(\Lambda))$ of the following boundary value problem*

$$\begin{cases} \partial_t \rho + \nabla \cdot \big\{\sigma(\rho)\big[\beta \nabla(J^{neum} \star \rho) \,+\, \mathbf{V}\big]\big\} = \Delta\rho \\ \rho(t,\cdot)\big|_\Gamma = \; b(\cdot) \quad \textit{for} \quad 0 \le t \le T \,, \\ \rho_0(u) = \gamma(u) \,. \end{cases} \tag{3.3}$$

Moreover, for each $t > 0$, $\delta > 0$ and $\mathbf{G} \in (\mathcal{C}^1(\Lambda))^d$, we have

$$\lim_{N\to\infty} \mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}\Big[\,\Big|\big\langle \mathbf{W}^N_t, \mathbf{G}\big\rangle \;-\; \int_0^t ds \,\big\langle \mathbf{J}(\rho^{\beta,\mathbf{V}}), \mathbf{G}\big\rangle\,\Big| > \delta\Big] \;=\; 0 \,, \tag{3.4}$$

where $\mathbf{J}(\rho^{\beta,\mathbf{V}})$ is is the instantaneous current associated to $\rho^{\beta,\mathbf{V}}$ and is given by

$$\mathbf{J}(\rho^{\beta,\mathbf{V}}) \;=\; -\nabla \rho^{\beta,\mathbf{V}} + \sigma(\rho^{\beta,\mathbf{V}})\big[\beta \nabla(J^{neum} \star \rho^{\beta,\mathbf{V}}) \,+\, \mathbf{V}\big] .$$

We postpone the derivation of this Proposition at the end of this section.

3.2. *Some useful tools.* In this subsection we collect some technical results which will be used in the proof both of the hydrodynamic limit and of the dynamical large deviation principle. We start by some properties of the potential $J^{\mathrm{neum}}(\cdot,\cdot)$ easily obtained by its definition.

Lemma 3.2. *The potential $J^{neum}(\cdot,\cdot)$ is a symmetric probability kernel. Moreover for any regular function $F : \Lambda \to \mathbb{R}$ and $1 \le k \le d$, we have the following:*

$$\Big|\partial_k\Big(\int_\Lambda J^{neum}(u,v) F(v) dv\Big)\Big| \le \int_\Lambda J^{neum}(u,v)\big|\partial_k F(v)\big| dv \,, \tag{3.5}$$

where for $1 \le k \le d$, $\partial_k F$ is the partial derivative in the direction e_k. In particular, if $|\cdot|_1$ stands for the l_1 norme of $\mathbb{R}^d$, then

$$\left|\nabla(J \star F)(u)\right|_1 \;\le\; \big(J \star \left|\nabla F\right|_1\big)(u)\,. \tag{3.6}$$

The proof of this Lemma is similar to the one of Lemma 3.1. in Mourragui and Orlandi (2013) and therefore is omitted.

Next, we show that for $t \ge 0$ and $\mathbf{V} = (V_1, \dots, V_d) \in \big(\mathcal{C}^{1,1}([0,T] \times \Lambda)\big)^d$, the rates $C_{N,t}^{\beta,V_i}$, $1 \le i \le d$ of the generator $\mathcal{L}_{\beta,N}^{\mathbf{V}}$ are a perturbation of the rates of the symmetric simple exclusion generator. For any $F \in \mathcal{C}^1(\Lambda)$, $u \in \Lambda$ and $1 \le k \le d$ denote by $\partial_k^N F(u)$ the discrete (space) derivative in the direction e_k:

$$\partial_k^N F(u) = N\big[F(u + e_k/N) - F(u)\big]\,, \text{ if } \quad u + e_k/N \in \Lambda\,. \tag{3.7}$$

Lemma 3.3. *Fix $t \ge 0$ and $\mathbf{V} = (V_1, \dots, V_d) \in \big(\mathcal{C}^{1,1}([0,T] \times \Lambda)\big)^d$. For any $1 \le k \le d$, $\eta \in \Sigma_N$ and any $x \in \Lambda_N$ with $x + e_k \in \Lambda_N$,*

$$C_{N,t}^{\beta,V_k}(x, x + e_k; \eta) = 1 - N^{-1}\big(\eta(x + e_k) - \eta(x)\big)\Upsilon_k^{\beta,\mathbf{V}}(\pi^N(\eta), t, x/N) + O(N^{-2})\,,$$

where

$$\Upsilon_k^{\beta,\mathbf{V}}(\pi^N(\eta), t, x/N) \;=\; \beta\partial_k^N(J^{neum} \star \pi^N(\eta))(x/N) \;+\; V_k(t, x/N)\,,$$

Proof: Recall from (3.1) that

$$C_{N,t}^{\beta,V_k}(x, x + e_k; \eta) = C_N^{\beta}(x, x + e_k; \eta) e^{-[\eta(x+e_k)-\eta(x)]N^{-1}V_k(t,x/N)}. \tag{3.8}$$

By definition of H_N, for all $x, y \in \Lambda_N$ and $\eta \in \Sigma_N$,

$$\begin{aligned} H_N(\eta^{x,y}) - H_N(\eta) &= \frac{1}{N^d}\big(\eta(x) - \eta(y)\big)^2 \big(J^{\mathrm{neum}}(\frac{x}{N}, \frac{y}{N}) - J^{\mathrm{neum}}(0,0)\big) \\ &\quad + \big(\eta(x) - \eta(y)\big)\frac{2}{N^d}\sum_{z \in \Lambda_N} \eta(z)\big[J^{\mathrm{neum}}(\frac{x}{N}, \frac{z}{N}) - J^{\mathrm{neum}}(\frac{y}{N}, \frac{z}{N})\big]\,. \end{aligned}$$

Thus, by Taylor expansion,

$$C_N^{\beta}(x, x+e_k; \eta) = 1 - \beta\big(\eta(x+e_k) - \eta(x)\big)N^{-1}\partial_k^N\big[\big(J^{\mathrm{neum}}\big)\star\pi^N(\eta)\big](x/N) + O(N^{-2})\,.$$

To conclude the proof of the Lemma, it remains to apply again Taylor expansion to the expression $e^{-[\eta(x+e_k)-\eta(x)]N^{-1}V_k(t,x/N)}$ in (3.8). □

It is well known that one of the main steps in the derivation of a large deviations principle for the empirical density is a superexponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime. Essentially, the problem consists in bounding expression such as $\langle Z, f\rangle_{\mu_N^{st}}$ in terms of Dirichlet form $N^2\langle -\mathfrak{L}_N\sqrt{f(\eta)}, \sqrt{f(\eta)}\rangle_{\mu_N^{st}}$, where Z is a local function and $\langle\cdot,\cdot\rangle_{\mu_N^{st}}$ represents the inner product with respect to stationary measure μ_N^{st}. In the context of boundary driven process, the fact that the invariant measure is not explicitly known introduces a technical difficulty. We fix as reference measure a product measure $\nu_{\theta(\cdot)}^N$, see (2.6), where θ is a smooth function with the only requirement that $\theta|_\Gamma \;=\; b$. There is therefore no reasons for $N^2\langle -\mathcal{L}_N\sqrt{f(\eta)}, \sqrt{f(\eta)}\rangle_{\nu_{\theta}^N(\cdot)}$ to be positive. Next lemma estimates this quantity.

For each probability measure ν on Σ_N and each function $f \in L^2(\nu)$, define the following functionals

$$\begin{aligned} \mathcal{D}_{0,N}\big(f,\nu\big) &= \frac{1}{2}\sum_{i=1}^{d}\sum_{x,x+e_i\in\Lambda_N}\int \big(f(\eta^{x,x+e_i})-f(\eta)\big)^2\,d\nu(\eta)\,, \\ \mathcal{D}_{b,N}\big(f,\nu\big) &= \frac{1}{2}\sum_{x\in\Gamma_N}\int r_x\big(b(x/N),\eta\big)\,\big(f(\sigma^x\eta)-f(\eta)\big)^2\,d\nu(\eta)\,. \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let $\theta : \overline{\Lambda} \to (0,1)$ be a smooth function such that $\theta(\cdot)\big|_{\Gamma} = b(\cdot)$. There exist two positive constants $C_0 \equiv C_0(\|\nabla\theta\|_\infty, J^{neum}, \mathbf{V})$, $C_0' \equiv C_0'(b,H)$ so that for any $a>0$ and for $f \in L^2\big(\nu^N_{\theta(\cdot)}\big)$,*

$$\begin{aligned} \langle f\,,\,\mathcal{L}^{\mathbf{V}}_{\beta,N}f\rangle_{\nu^N_{\theta(\cdot)}} &\le -\big(1-a\big)\mathcal{D}_{0,N}\big(f,\nu^N_{\theta(\cdot)}\big)+\frac{C_0}{a}N^{-2+d}\|f\|^2_{L^2(\nu^N_{\theta(\cdot)})}, \\ \langle f\,,\,L^H_{b,N}f\rangle_{\nu^N_{\theta(\cdot)}} &= -\big(1-a\big)\mathcal{D}_{b,N}\big(f,\nu^N_{\theta(\cdot)}\big)+\frac{C_0'}{a}N^{-2+d}\|f\|^2_{L^2(\nu^N_{\theta(\cdot)})}\,. \end{aligned} \tag{3.10}$$

The proof of this lemma is similar to the one of Lemma 3.3 in Mourragui and Orlandi (2013) and is thus omitted.

This lemma permits us to prove the superexponential estimate. For a cylinder function Ψ denote the expectation of Ψ with respect to the Bernoulli product measure ν^N_α by $\widetilde{\Psi}(\alpha)$:

$$\widetilde{\Psi}(\alpha) = E^{\nu^N_\alpha}[\Psi]\,.$$

For a positive integer l and $x \in \Lambda_N$, denote the empirical mean density on a box of size $2l+1$ centered at x by $\eta^l(x)$:

$$\eta^l(x) = \frac{1}{|\Lambda_l(x)|}\sum_{y\in\Lambda_l(x)}\eta(y)\,,$$

where

$$\Lambda_l(x) = \Lambda_{N,l}(x) = \{y\in\Lambda_N : |y-x|\le l\}\,. \tag{3.11}$$

For $1\le j\le d$, define the cylinder function $\Psi_j = [\eta(e_j)-\eta(0)]^2$, For each $\mathbf{V} = (V_1,\cdots,V_d)$, $\mathbf{G} = (G_1,\cdots,G_d)$ in $(\mathcal{C}^{0,1}([0,T]\times\Lambda))^d$, and each $\varepsilon>0$, let

$$\begin{aligned} \mathcal{G}^{\mathbf{G},\mathbf{V},\beta}_{N,\varepsilon}(s,\eta) \;=\; &\frac{1}{N^d}\sum_{j=1}^{d}\sum_{x,x+e_j\in\Lambda_N}G_j(s,x/N) \\ &\times\Upsilon^{\beta,\mathbf{V}}_j(\pi^N(\eta),s,x/N)\,\Big[\tau_x\Psi_j(\eta)-\widetilde{\Psi}_j(\eta^{\varepsilon N}(x))\Big]\,. \end{aligned} \tag{3.12}$$

For a continuous function $H:[0,T]\times\Gamma\to\mathbb{R}$, let

$$\mathcal{H}^H_N(s,\eta) \;=\; \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}H(s,x/N)\big[\eta(x)-b(x/N)\big]\,. \tag{3.13}$$

Proposition 3.5. *Fix $\mathbf{G},\mathbf{V}\in(\mathcal{C}^{0,0}([0,T]\times\Lambda))^d$, H in $\mathcal{C}^{0,0}([0,T]\times\Gamma)$ and $\beta\ge 0$. For any sequence of initial measures μ_N and every $\delta>0$,*

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}\Big[\Big|\int_0^T\mathcal{G}^{\mathbf{G},\mathbf{V},\beta}_{N,\varepsilon}(s,\eta_s)\,ds\Big|>\delta\Big] \;=\; -\infty\,, \\ &\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}\Big[\Big|\int_0^T\mathcal{H}^H_N(s,\eta_s)\,ds\Big|>\delta\Big] \;=\; -\infty\,. \end{aligned}$$

We conclude this section by the Girsanov formula needed in the proof of the large deviations. Indeed, in order to compare the original dynamics to a perturbed dynamics with regular drifts $\mathbf{V}, H$ (3.1) and (3.2), we have to compute the Radon-Nikodym derivative of the modified process with respect to the original one (see Kipnis and Landim (1999), Appendix 1, Proposition 7.3). Fix a vector-valued function $\mathbf{V} \in (\mathcal{C}^{0,0}([0,T]\times\Lambda))^d$ and a function $H \in \mathcal{C}^{0,0}([0,T]\times\Gamma)$. For any initial measure μ_N and any positive time $t>0$, the Radon-Nikodym derivative of $\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}$ with respect to $\mathbb{P}^{\beta}_{\mu_N}$ restricted to the time interval $[0,t]$ is gives by

$$\frac{d\mathbb{P}^{\beta,\mathbf{V},H}_{\mu_N}}{d\mathbb{P}^{\beta}_{\mu_N}}\big((\eta_s)_{s\in[0,t]}\big) \;=\; \mathbb{M}^{\beta,\mathbf{V}}_t \,\times\, \mathbb{B}^{b,H}_t\,, \tag{3.14}$$

where $\mathbb{M}^{\beta,\mathbf{V}}_t$ and $\mathbb{B}^{b,H}_t$ are two exponential martingales given by,

$$\begin{aligned}
\mathbb{M}^{\beta,\mathbf{V}}_t = \exp\Big(\sum_{k=1}^{d}\sum_{x,x+e_k\in\Lambda_N}\Big\{\int_0^t \frac{1}{N}V_k(s,x/N)\,dW^{x,x+e_k}_s \\
- N^2\int_0^t \big[\eta_s(x)+\eta_s(x+e_j)\big]C^{\beta}_N\big(x,x+e_k;\eta_s\big) \\
\big[e^{-[\nabla^{x,x+e_k}\eta_s(x)]\frac{1}{N}V_k(s,x/N)}\,-\,1\big]ds\Big\}\Big)\,,\\
\mathbb{B}^{b,H}_t = \exp\Big(\sum_{x\in\Gamma_N}\Big\{\int_0^t \frac{1}{N}H(s,x/N)\,dW^{x}_s \\
- N^2\int_0^t r_x\big(b(x/N),\eta_s(x)\big)\big[e^{[2\eta_s(x)-1]\frac{1}{N}H(s,x/N)}\,-\,1\big]ds\Big\}\Big)\,,
\end{aligned}$$

where the rate $r_x(\cdot,\cdot)$ is given by (2.5) and for any function $g:\Sigma_N\to\mathbb{R}$ and $x,y\in\Lambda_N$, we have denoted $\nabla^{x,y}g(\eta)=[g(\eta^{x,y})-g(\eta)]$.

3.3. *Proof of Prposition 3.1.* The identification of the limit for the empirical density $(\pi^N(\eta_t))_{t\in[0,T]}$ is similar to the one of Mourragui and Orlandi (2013). We therefore switch to the limit (3.4). Following the same steps as in Bertini et al. (2006b), we consider the family of jump martingales

$$\begin{aligned}
\widetilde{W}^{x,y}_t &= W^{x,y}_t - N^2\int_0^t \big[\eta(x)-\eta(y)\big]C^{\beta,V_i}_{N,t}(x,y;\eta_s)ds \text{ for } y=x+e_i\,,\ x,y\in\Lambda_N\,,\\
\widetilde{W}^{y}_t &= W^{y}_t - N^2\int_0^t \Big\{\eta_s(y)(1-b(y/N))e^{N^{-1}H(s,y/N)} \\
&\qquad - (1-\eta_s(y))b(y/N)e^{-N^{-1}H(s,y/N)}\Big\}ds\,, \qquad y\in\Gamma_N^+\,,\\
\widetilde{W}^{y}_t &= W^{y}_t - N^2\int_0^t \Big\{(1-\eta_s(y))b(y/N)e^{-N^{-1}H(s,y/N)} \\
&\qquad - \eta_s(y)(1-b(y/N))e^{N^{-1}H(s,y/N)}\Big\}ds\,, \qquad y\in\Gamma_N^-\,.
\end{aligned}$$

Recall from (2.7) the definition of the empirical measures $(W^N_{j,t})_{t\geq 0}$, $1\leq j\leq d$. Fix a smooth vector field $\mathbf{G}=(G_1,\cdots,G_d)\in\big(\mathcal{C}^{1,1}([0,T]\times\Lambda)\big)^d$, and consider the

$\mathbb{P}^{\beta,\mathbf{V},H}_{\mu^N}$-martingale $\widetilde{\mathbf{W}}^{\mathbf{G},\mathbf{V},H}_t \equiv \widetilde{\mathbf{W}}^{\mathbf{G},\mathbf{V},H,N,\beta}_t$, $t \in [0,T]$, defined by

$$
\begin{aligned}
\widetilde{\mathbf{W}}^{\mathbf{G},\mathbf{V},H}_t \;=\; & \sum_{k=1}^d \Big\{ \big\langle W^N_{k,t}\,,\, G_k \big\rangle \\
& - \frac{N^2}{N^{d+1}} \sum_{x,x+e_k\in\Lambda_N} G_k(x/N) \int_0^t \big[\eta_s(x)-\eta_s(x+e_k)\big] C^{\beta,V_k}_{N,t}\big(x,x+e_k;\eta_s\big) ds \Big\} \\
& - \frac{N^2}{N^{d+1}} \sum_{x\in\Gamma_N} G_1(x/N)\mathbf{n}_1(x/N) \int_0^t \Big\{ \eta_s(x)(1-b(x/N))e^{N^{-1}H(s,x/N)} \\
& \qquad\qquad - (1-\eta_s(x))b(x/N)e^{-N^{-1}H(s,x/N)} \Big\} ds\,.
\end{aligned}
$$

From Lemma 3.3 and Taylor expansion the integral term of the last expression is equal to

$$
\begin{aligned}
& - \frac{N^2}{N^{d+1}} \sum_{k=1}^d \sum_{x,x+e_k\in\Lambda_N} G_k(x/N) \int_0^t \big[\eta_s(x)-\eta_s(x+e_k)\big]\, ds \\
& - \frac{1}{N^d} \sum_{k=1}^d \sum_{x,x+e_k\in\Lambda_N} G_k(x/N) \int_0^t \big[\eta_s(x)-\eta_s(x+e_k)\big]^2 \Upsilon^{\beta,\mathbf{V}}_k(\pi^N(\eta_s),s,x/N)\, ds \\
& - \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N} G_1(x/N)\mathbf{n}_1(x/N) \int_0^t \big[\eta_s(x)-b(x/N)\big] ds \;+\; O_{\mathbf{G},\beta,\mathbf{V},H}\big(N^{-1}\big)\,,
\end{aligned}
$$

where for $1\le k\le d$, $\eta\in\Sigma_N$, $s\ge 0$ and $x\in\Lambda_N$,

$$
\Upsilon^{\beta,\mathbf{V}}_k(\pi^N(\eta),s,x/N) \;=\; \beta\partial^N_k(J^{\text{neum}}\star\pi^N(\eta_s))(x/N) \;+\; V_k(s,x/N)\,,
$$

for any smooth function G, $\partial^N_j G$ is defined in (3.7), and $O_{\mathbf{G},\beta,\mathbf{V},H}\big(N^{-1}\big)$ is an expression whose absolute value is bounded by CN^{-1} for some constant depending on $\mathbf{G}$, β, J^{neum}, $\mathbf{V}$ and H. A summation by parts and Taylor expansion permit to rewrite the martingale $\widetilde{\mathbf{W}}^{\mathbf{G},\mathbf{V},H}_t$ as

$$
\begin{aligned}
\widetilde{\mathbf{W}}^{\mathbf{G},\mathbf{V},H}_t \;=\; & \big\langle \mathbf{W}^N_t\,,\,\mathbf{G}\big\rangle \;-\; \frac{1}{N^d}\sum_{k=1}^d \sum_{x\in\Lambda_N\setminus\Gamma_N} \int_0^t ds\, \big(\partial_k G_k\big)(x/N)\eta_s(x) \\
& - \frac{1}{N^d}\sum_{k=1}^d \sum_{x\in\Lambda_N\setminus\Gamma_N} \int_0^t ds\, G_k(x/N)\big[\eta_s(x)-\eta(x+e_k)\big]^2 \Upsilon^{\beta,\mathbf{V}}_k(\pi^N(\eta_s),s,x/N)\big] \\
& + \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N} G_1(x/N)\mathbf{n}_1(x/N) \int_0^t \eta_s(x) ds \\
& - \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N} G_1(x/N)\mathbf{n}_1(x/N) \int_0^t \big[\eta_s(x)-b(x/N)\big] ds \;+\; O_{\mathbf{G},\beta,\mathbf{V},H}\big(N^{-1}\big)\,.
\end{aligned}
$$

Here, Γ^-_N, resp. Γ^+_N, stands for the left, resp. right, boundary of Λ_N:

$$
\Gamma^\pm_N = \{(x_1,\cdots,x_d)\in\Gamma_N : x_1 = \pm N\}
$$

Next, we use the replacement lemma stated in Proposition 3.5. We obtain that the martingal $\widetilde{\mathbf{W}}_t^{\mathbf{G},\mathbf{V},H}$ can be replaced by

$$\big\langle \mathbf{W}_t^N\,,\,\mathbf{G}\big\rangle - \int_0^t ds \Big\{ \sum_{k=1}^d < \pi^N(\eta_s), \partial_k G_k > \Big\} + \int_0^t ds \int_\Gamma G_1(r)\, b(r)\, \mathbf{n}_1(r)\, dS(r)$$
$$- \int_0^t ds \Big\{ \frac{1}{N^d} \sum_{k=1}^d \sum_{x\in\Lambda_N\setminus\Gamma_N} G_k(x/N) \sigma\left(\eta_s^{\varepsilon N}(x)\right) \Upsilon_k^{\beta,\mathbf{V}}(\pi^N(\eta_s), s, x/N) \Big\}\,.$$

On the other hand, a simple computation shows that the expectation of the quadratic variation of the martingale $\widetilde{\mathbf{W}}_t^{\mathbf{G},\mathbf{V},H}$ vanishes as $N \uparrow +\infty$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{\beta,\mathbf{V},H}\Big[\sup_{0\le t\le T} |\widetilde{\mathbf{W}}_t^{\mathbf{G},\mathbf{V},H}| > \delta \Big] \;=\; 0\,. \tag{3.15}$$

Finally, recall that by the first part of the proposition, the empirical density converges to the solution of the equation (3.3). This concludes the proof. □

Remark 3.6. The hydrodynamic equation (3.3) of the perturbed process does not depend on the function H. This follows from Lemma 3.5 and Lemma 3.4, where it is shown that the density at the boundary can be replaced by the function b when $N \uparrow +\infty$.

4. Properties of the rate functionals

In this section, we prove representation results for the rates $\mathcal{J}_T^\gamma(\cdot)$ and $\mathcal{I}_T^\gamma(\cdot)$, see Lemma 4.5, the lower semicontinuity and the compactness of the level sets , see Proposition 4.2.

4.1. *Lower semicontinuity.* We first prove that the functional $\mathcal{J}_T^\gamma$ is larger than $\mathcal{I}_T^\gamma$:

Lemma 4.1. *For any* $(\mathbf{W},\pi) \in D([0,T],\mathcal{M}^{d+1})$,

$$\mathcal{I}_T^\gamma(\pi) \;\le\; \mathcal{J}_T^\gamma(\mathbf{W},\pi)\,.$$

Proof: When $\mathcal{J}_T^\gamma(\mathbf{W},\pi) = +\infty$, the inequality is trivially verified. Suppose then that $\mathcal{J}_T^\gamma(\mathbf{W},\pi) < +\infty$. This implies that $\pi \in D\big([0,T],\mathcal{F}^1\big)$, $(\mathbf{W},\pi) \in \mathfrak{A}_\gamma$, $\mathcal{Q}(\pi) < +\infty$ and $\mathcal{J}_T^\gamma(\mathbf{W},\pi) = \mathbb{J}_T(\mathbf{W},\pi)$. Furthermore, by definition, since $\pi \in D\big([0,T],\mathcal{F}^1\big)$ and $\mathcal{Q}(\pi) < +\infty$, we have $\mathcal{I}_T^\gamma(\mu) = \mathbb{I}_T^\gamma(\mu)$.

Let $F \in \mathcal{C}_0^{1,2}([0,T]\times\Lambda)$, since $(\mathbf{W},\pi) \in \mathfrak{A}_\gamma$, we have

$$\widehat{\mathbb{I}}_F^{T,\gamma}(\pi) \;=\; \widehat{\mathbb{J}}_{\nabla F}^T(\mathbf{W},\pi) \;\le\; \mathcal{J}_T^\gamma(\mathbf{W},\pi)\,.$$

To conclude the proof, it is enough to take the supremum over all $F \in \mathcal{C}_0^{1,2}([0,T]\times\Lambda)$, on the left hand side of the last inequality. □

The main result of this subsection is stated in the following proposition.

Proposition 4.2. *For every profile $\gamma \in \mathcal{F}^1$, the functional $\mathcal{J}_T^\gamma$, resp. $\mathcal{I}_T^\gamma$ defined in* (2.15)*, resp.* (2.18) *is lower semicontinuous for the topology of the space $D([0,T],\mathcal{M}^{d+1})$, resp. $D([0,T],\mathcal{M})$. Moreoever the functional $\mathcal{I}_T^\gamma$ has compact level sets in $D([0,T],\mathcal{M})$.*

The proof is split in several lemmata. We follow the general scheme used in Quastel et al. (1999); Bertini et al. (2009). Denote

$$\mathcal{B}_\gamma^b = \{(\pi_t(du))_{t\in[0,T]} = (\rho_t(u)du)_{t\in[0,T]} \ : \ \rho \in L^2([0,T], H^1(\Lambda))\,, \\ \rho_0(\cdot) = \gamma(\cdot); \quad \mathrm{Tr}(\rho_t)(\cdot) = b(\cdot), \ \text{ for a.e. } t \in (0,T]\}.$$

Lemma 4.3. *Let π be a trajectory in $D([0,T],\mathcal{M})$ such that $\mathcal{I}_T^\gamma(\pi) < \infty$. Then π belongs to $\mathcal{B}_\gamma^b \cap C([0,T],\mathcal{F}^1)$. Furthermore, there exists a positive constant $C_0 = C_0(\beta, J^{neum})$ such that*

$$\mathcal{Q}(\pi) \ \le \ C_0\big\{1 + \mathcal{I}_T^\gamma(\pi)\big\}\,. \tag{4.1}$$

Proof: The proof of the first statement of this Lemma is similar to the one of Lemma 4.1 in Farfan et al. (2011) and is therefore omitted. One can prove (4.1) by using the same arguments as in the proof of Proposition 4.3. Quastel et al. (1999) or Lemma 4.9. in Bertini et al. (2009). □

The proof of the lower-semicontinuity of the rate function $\mathcal{I}_T^\gamma$ is based on compactness arguments; its basic tools is given by the next Proposition. We refer to Bertini et al. (2009); Farfan et al. (2011) for the proof.

Proposition 4.4. *Let $\{\pi^n : n \ge 1\}$ be a sequence of functions in $D([0,T],\mathcal{M})$ such that*

$$\sup_{n\in\mathbb{N}} \big\{\mathcal{I}_T^\gamma(\pi^n)\big\} < \infty$$

with $\pi^n(t,du) = \rho^n(t,u)du$, for $t \in [0,T]$ and $n \in \mathbb{N}$. Suppose that the sequence ρ^n converges weakly in $L^2([0,T]\times\Lambda)$ to some ρ. Then, ρ^n converges strongly in $L^2([0,T]\times\Lambda)$ to ρ.

Proof of Proposition 4.2. The proof for the functional $\mathcal{I}_T^\gamma$ is omitted since it's the same as for the one dimensional boundary driven Kawasaki process with Neuman Kac interaction Mourragui and Orlandi (2013).

To prove the lower semicontinuity of the functional $\mathcal{J}_T^\gamma$, we have to show that for all $a \ge 0$ the set

$$\mathbf{E}_a \ = \ \Big\{(\mathbf{W},\pi) \in D([0,T],\mathcal{M}^{d+1}) \ : \ \mathcal{J}_T^\gamma(\mathbf{W},\pi) \le a\Big\}$$

is closed in $D([0,T],\mathcal{M}^{d+1})$. Fix $a \ge 0$ and consider a sequence $\{(\mathbf{W}^n,\pi^n) : n \ge 1\}$ in $\mathbf{E}_a$ converging to some $(\mathbf{W},\pi)$ in $D([0,T],\mathcal{M}^{d+1})$, and denote by $\pi_t^n(du) = \rho_t^n(u)du$. Then for all $\mathbf{V}$ in $(\mathcal{C}([0,T]\times\Lambda))^d$ and F in $\mathcal{C}([0,T]\times\Lambda)$,

$$\begin{aligned} \lim_{n\to\infty} \int_0^T dt\, \langle \mathbf{W}_t^n, \mathbf{V}_t\rangle &= \int_0^T dt\, \langle \mathbf{W}_t, \mathbf{V}_t\rangle\,, \\ \lim_{n\to\infty} \int_0^T dt\, \langle \pi_t^n, F_t\rangle &= \int_0^T dt\, \langle \pi_t, F_t\rangle. \end{aligned} \tag{4.2}$$

We claim that $\mathcal{E}^\gamma(\mathbf{W},\pi) < +\infty$. Indeed, from the lower semicontinuity of $\mathcal{I}_T^\gamma$, Lemma 4.1 and Lemma 4.3, π belongs to $\mathcal{B}_\gamma^b$ and $\mathcal{Q}(\pi) \ \le \ C_a$ for some positive constant C_a. Moreover, for any $F \in \mathcal{C}_0^1(\Lambda)$

$$\begin{aligned} 0 \ &= \lim_{n\to\infty} \sup_{t\in[0,T]} \Big\{\langle \pi_t^n, F\rangle - \langle \gamma, F\rangle - \langle \mathbf{W}_t^n, \nabla F\rangle\Big\} \\ &= \sup_{t\in[0,T]} \Big\{\langle \pi_t, F\rangle - \langle \gamma, F\rangle - \langle \mathbf{W}_t, \nabla F\rangle\Big\}\,, \end{aligned}$$

proving that $(\mathbf{W},\pi)\in\mathfrak{A}_\gamma$ and then $\mathcal{E}^\gamma(\mathbf{W},\pi)<+\infty$, so that $\mathcal{J}_T^\gamma(\mathbf{W},\pi)=\mathbb{J}_T(\mathbf{W},\pi)$.

Denote by ρ the density of π: $\pi_t(du)=\rho_t(u)du$. Since ρ^n converges weakly to ρ in $L^2([0,T]\times\Lambda)$ (cf. (4.2)), by Proposition 4.4, ρ_n converges strongly to ρ in $L^2([0,T]\times\Lambda)$, hence for any $\mathbf{V}$ in $(\mathcal{C}^{1,1}([0,T]\times\Lambda))^d$

$$\lim_{n\to\infty}\Big\{\mathbf{L}^\beta_{\mathbf{V}}(\mathbf{W}^n,\pi^n)\;-\;\frac{1}{2}\int_0^T dt\,\langle\sigma(\rho^n_t),\mathbf{V}_t\cdot\mathbf{V}_t\rangle\Big\}$$
$$=\mathbf{L}^\beta_{\mathbf{V}}(\mathbf{W},\pi)\;-\;\frac{1}{2}\int_0^T dt\,\langle\sigma(\rho_t),\mathbf{V}_t\cdot\mathbf{V}_t\rangle\,.$$

Since $(\mathbf{W}^n,\pi^n)$ belongs to $\mathbf{E}_a$, the left hand side is bounded by a. Taking the supremum over $\mathbf{V}$ in $(\mathcal{C}^{1,1}([0,T]\times\Lambda))^d$ we obtain that $\mathbb{J}_T(\mathbf{W},\pi)\le a$ and conclude the proof of the lower semicontinuity of $\mathcal{J}_T^\gamma$. □

4.2. *Representation theorem.* Given a path $\pi\in D([0,T];\mathcal{F}^1)$ with $\pi(t,du)=\rho(t,u)du$, we denote by $\mathbb{L}^2(\sigma(\pi))$ the Hilbert space of (equivalence classes of) measurable vector-valued functions $\{\mathbf{G}:[0,T]\times\Lambda\to\mathbf{R}^d:\int_0^T\langle\sigma(\rho(t,u)),\mathbf{G}(t,u)\cdot\mathbf{G}(t,u)\rangle dt<\infty\}$ endowed with the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\sigma(\pi)}$ induced by

$$\langle\!\langle\mathbf{V},\mathbf{G}\rangle\!\rangle_{\sigma(\pi)}\;=\;\int_0^T dt\int_\Lambda du\,\sigma(\pi(t,u))\,\mathbf{V}(t,u)\cdot\mathbf{G}(t,u)\,.$$

The norm of $\mathbb{L}^2(\sigma(\pi))$ is denoted by $\|\cdot\|_{\mathbb{L}^2(\sigma(\pi))}$.

Denote by $H^1_0(\sigma(\pi))$ the Hilbert space obtained by quotienting and completing $C^{1,2}_0([0,T]\times\Lambda)$ with respect to the pre-inner product defined by

$$\langle F,H\rangle_{1,\sigma(\pi)}=\langle\!\langle\nabla F,\nabla H\rangle\!\rangle_{\sigma(\pi)}\,.$$

The norm of $H^1_0(\sigma(\pi))$ is denoted by $\|\cdot\|_{H^1_0(\sigma(\pi))}$.

Lemma 4.5. *Let $(\mathbf{W},\pi)\in D([0,T],\mathcal{M}^{d+1})$ such that $\mathcal{J}_T^\gamma(\mathbf{W},\pi)<\infty$. There exists a function $\mathbf{U}$ in $\mathbb{L}^2(\sigma(\pi))$ such that $(\mathbf{W},\pi)$ is the weak solution of the equation*

$$\partial_t\mathbf{W}_t\;=\;-\nabla\rho_t+\sigma(\rho_t)\big[\beta\nabla(J^{neum}\star\rho_t)\;+\;\mathbf{U}\big]\,,\quad\mathbf{W}_0=0\,,\tag{4.3}$$

in the following sense : for any $\mathbf{G}\in\big(C^{1,1}([0,T]\times\Lambda)\big)^d$,

$$\mathbf{L}^\beta_{\mathbf{G}}(\mathbf{W},\pi)\;=\;\langle\!\langle\mathbf{G},\mathbf{U}\rangle\!\rangle_{\sigma(\pi)}\;=\;\int_0^T dt\,\langle\sigma(\pi_t),\mathbf{G}_t\cdot\mathbf{U}_t\rangle\,,$$

where the linear function $\mathbf{G}\mapsto\mathbf{L}^\beta_{\mathbf{G}}(\mathbf{W},\pi)$ is defined by (2.14).

Furthermore, there exists a function $F\in H^1_0(\sigma(\pi))$ such that $\rho(\cdot,\cdot)$ solves the equation (3.3) and $\nabla\cdot\big(\sigma(\rho)(\mathbf{U}-\nabla F)\big)=0$ in the weak sense described by (4.6). Moreover,

$$\mathcal{J}_T^\gamma(\mathbf{W},\pi)\;=\frac{1}{2}\,\|\mathbf{U}\|_{\mathbf{L}^2(\sigma(\pi))}\;=\;\frac{1}{2}\int_0^T dt\,\langle\sigma(\rho_t),\mathbf{U}_t\cdot\mathbf{U}_t\rangle\tag{4.4}$$

and

$$\mathcal{I}_T^\gamma(\pi)\;=\;\frac{1}{2}\,\|F\|_{H^1_0(\sigma(\pi))}\;=\;\frac{1}{2}\int_0^T dt\,\langle\sigma(\rho_t),\nabla F_t\cdot\nabla F_t\rangle\,.\tag{4.5}$$

Proof: Assume that $\mathcal{J}_T^\gamma(\mathbf{W},\pi) < \infty$, then $\mathcal{E}^\gamma(\mathbf{W},\pi) < \infty$ and $\mathbb{J}_T(\mathbf{W},\pi) < \infty$. Following the arguments in Kipnis and Landim (1999, §10.5), from Riesz representation theorem, we derive the existence of a function $\mathbf{U}$ in $\mathbb{L}^2(\sigma(\pi))$ satisfying (4.4) and (4.3).

On the other hand, from Lemma 4.1, we have $\mathcal{I}_T^\gamma(\pi) < \infty$. Using again the Riesz representation theorem (cf. Kipnis and Landim (1999, §10.5)), we derive the existence of a function F in $H_0^1(\sigma(\pi))$ such that ρ is the weak solution of the boundary value problem (3.3), with $\mathbf{V} = \nabla F$. Then, the representation (4.5) for the functional $\mathcal{I}_T^\gamma$ follows immediately. Finally, equation (4.3) and the fact that $(\mathbf{W},\pi) \in \mathfrak{A}_\gamma$ yield,

$$\langle\!\langle (\mathbf{U} - \nabla F), \nabla G \rangle\!\rangle_{\sigma(\rho)} = 0, \tag{4.6}$$

for all $G \in C_0^{1,2}([0,T] \times \Lambda)$. □

5. large deviations upper bound for the empirical current

In this section, we prove the large deviations upper bounds stated in Theorem 2.5 and in Theorem 2.6. In view of the definitions of the energy functional $\mathcal{E}^\gamma$ and the rate functional for the large deviations, we need to exclude in the large deviations regime, paths $(\mathbf{W}_t, \pi_t)_{t\in[0,T]}$ which do not belong to $\mathfrak{A}_\gamma$, and with infinite energy $\mathcal{Q}(\pi) = +\infty$.

5.1. *The set* $\mathfrak{A}_\gamma$. Fix a positive profile γ and let $\widetilde{\mathfrak{A}}_\gamma$ be the set of trajectories $(\mathbf{W},\pi)$ in $D([0,T],\mathcal{M}^{d+1})$ such that for any $G \in \mathcal{C}_0^2(\Lambda)$ and any $\varphi \in \mathcal{C}^1([0,T])$

$$\sup_{0\le t\le T} \mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W},\pi) = 0,$$

where for $(G,\varphi) \in \mathcal{C}_0^2(\Lambda) \times \mathcal{C}^1([0,T])$ and $0 \le t \le T$,

$$\begin{aligned} \mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W},\pi) =& \langle \pi_t, G\rangle\varphi(t) - \langle \gamma, G\rangle \varphi(0) - \int_0^t ds \langle \pi_s, G\rangle \varphi'(s) \\ & - \langle \mathbf{W}_t, \nabla G\rangle \varphi(t) + \int_0^t ds \langle \mathbf{W}_s, \nabla G\rangle \varphi'(s)\,. \end{aligned} \tag{5.1}$$

Here φ' stands for the time derivative of φ.

Lemma 5.1. *Fix* $(\mathbf{W},\pi)$ *in* $D([0,T],\mathcal{M}^{d+1})$ *such that*

$$\sup_{(G,\varphi)} \sup_{0\le t\le T} \left\{ \mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W},\pi) \right\} < \infty\,,$$

where the supremum is taken over all $(G,\varphi) \in \mathcal{C}_0^2(\Lambda) \times \mathcal{C}^1([0,T])$. *Then* $(\mathbf{W},\pi)$ *belongs to* $\mathfrak{A}_\gamma$.

Proof: Let $M > 0$ be such that $\mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W},\pi) \le M$, for all $(G,\varphi) \in \mathcal{C}_0^2(\Lambda) \times \mathcal{C}^1([0,T])$, and $0 \le t \le T$. Fix a function $G \in \mathcal{C}_0^2(\Lambda)$ and $0 \le t_1 < t_2 \le T$, we have

$$\left\{ \langle \pi_{t_1}, G\rangle - \langle \pi_{t_2}, G\rangle \right\} - \left\{ \langle \mathbf{W}_{t_1}, \nabla G\rangle - \langle \mathbf{W}_{t_2}, \nabla G\rangle \right\} \le M\ .$$

Applying this last inequality to the functions $-G$ and then to AG for positive number $A > 0$, we get,

$$\left| \left\{ \langle \pi_t, G\rangle - \langle \pi_s, G\rangle \right\} - \left\{ \langle \mathbf{W}_t, \nabla G\rangle - \langle \mathbf{W}_s, \nabla G\rangle \right\} \right| \le \frac{M}{A}\,,$$

for all $A > 0$. It remains to let $A \uparrow +\infty$. $\square$

The following lemma is needed for the proof of the upper bound of the large deviations, which consists in two steps. We shall prove at first an upper bound with an auxiliary rate functional $\mathfrak{F}_a$ for small $a > 0$ (Proposition 5.6). This new rate functional will allow to take the large deviations rate functional equal to $+\infty$ on the set of paths (W, π), which do not belong to $\widetilde{\mathfrak{A}}_\gamma$.

Lemma 5.2. *Fix a sequence $\{\eta^N \in \Sigma_N : N \geq 1\}$ of configurations. For any $(G, \varphi) \in \mathcal{C}_0^2(\Lambda) \times \mathcal{C}^1([0,T])$ and any $a > 0$, we have*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{E}^\beta_{\eta^N} \Big[\exp \Big(a\, N^d \sup_{0 \leq t \leq T} \mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W}^N, \pi^N) \Big) \Big] \;\leq\; 0\,.$$

Proof: The proof follows the general scheme used in Bertini et al. (2006b). Notice however that in our context there are some additional difficulties due to the boundary terms. Fix $(G, \varphi) \in \mathcal{C}_0^2(\Lambda) \times \mathcal{C}^1([0,T])$. For any time $s \in [0,T]$, we have the following microscopic relation

$$\eta_s(x) = \eta_0(x) + \sum_{j=2}^d \big(W_s^{x-e_j,x} - W_s^{x,x+e_j} \big) + \begin{cases} W_s^{x-e_1,x} - W_s^{x,x+e_1} & \text{if } x \in \Lambda_N \setminus \Gamma_N\,, \\ W_s^{x} - W_s^{x,x+e_1} & \text{if } x \in \Gamma_N^-\,, \\ W_s^{x-e_1,x} - W_s^{x} & \text{if } x \in \Gamma_N^+\,. \end{cases}$$

Since G vanishes at the boundary Γ, the classical spatial summations by parts and integrations by parts in time, permit to rewrite the two terms of $\mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W}^N, \pi^N)$ as

$$\begin{aligned} &\langle \pi_t, G\rangle \varphi(t) - \langle \pi_0, G\rangle \varphi(0) - \int_0^t ds \langle \pi_s, G\rangle \varphi'(s) \\ &\qquad = \frac{1}{N^{d+1}} \sum_{j=1}^d \sum_{x \in \Lambda_N \setminus \Gamma_N^+} \int_0^t \partial_j^N G(x/N) \varphi(s) dW_s^{x,x+e_j}\,, \\ &\langle \mathbf{W}_t, \nabla G\rangle \varphi(t) - \int_0^t ds \langle \mathbf{W}_s, \nabla G\rangle \varphi'(s) \\ &\qquad = \frac{1}{N^{d+1}} \sum_{j=1}^d \sum_{x \in \Lambda_N \setminus \Gamma_N^+} \int_0^t \partial_j G(x/N) \varphi(s) dW_s^{x,x+e_j} \\ &\qquad\quad + \frac{1}{N^{d+1}} \sum_{x \in \Gamma_N} \int_0^t \partial_1 G(x/N) \mathbf{n}_1(x/N) \varphi(s) dW_s^x\,, \end{aligned}$$

where $\partial_j^N G(x/N)$ is the discrete derivative defined in (3.7) and $\partial_j G$ is the partial derivative of the function G in the direction e_j. Let $H : \Gamma \to \mathbb{R}$ be the function given by $H(s,u) = -\partial_1 G(u) \mathbf{n}_1(u) \varphi(s)$ and for $1 \leq j \leq d$ and $N > 1$, denote by $\mathbf{V}^N = (V_1^N, \cdots, V_d^N)$ the time dependent vector valued function defined by $\mathbf{V}_j^N(s,u) = N\big[\partial_j^N G(u) - \partial_j G(u)\big]\varphi(s)$, we obtain

$$\begin{aligned} aN^d \mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W}^N, \pi^N) &= \frac{a}{N^2} \sum_{j=1}^d \sum_{x \in \Lambda_N \setminus \Gamma_N^+} \int_0^t V_j^N(s, x/N) dW_s^{x,x+e_j} \\ &\quad + \frac{a}{N} \sum_{x \in \Gamma_N} \int_0^t H(s, x/N) dW_s^x\,. \end{aligned}$$

Thus by Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{1}{N^d}\log \mathbb{E}^{\beta}_{\eta^N}\Big[\exp\Big(aN^d \sup_{0\le t\le T}\mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W}^N,\pi^N)\Big)\Big]\\
&\quad\le \frac{1}{2N^d}\log \mathbb{E}^{\beta}_{\eta^N}\Big[\exp\Big(\frac{2a}{N^2}\sup_{0\le t\le T}\sum_{j=1}^d\sum_{x\in\Lambda_N\setminus\Gamma_N^+}\int_0^t V_j^N(s,x/N)dW_s^{x,x+e_j}\Big)\Big] \qquad (5.2)\\
&\quad+\frac{1}{2N^d}\log \mathbb{E}^{\beta}_{\eta^N}\Big[\exp\Big(\frac{2a}{N}\sup_{0\le t\le T}\sum_{x\in\Gamma_N}\int_0^t H(s,x/N)dW_s^{x}\Big)\Big].
\end{aligned}$$

Next, we control separately the two terms of the right hand side of (5.2) using the mean one exponential martingales $\mathbb{M}_t^{\beta,\frac{2a}{N}\mathbf{V}}$ and $\mathbb{B}_t^{b,2aH}$ defined in the Girsanov formula (3.14):

$$\begin{aligned}
\mathbb{M}_t^{\beta,\frac{2a}{N}\mathbf{V}} &= \exp\Big(\frac{2a}{N^2}\sum_{j=1}^d\sum_{x\in\Lambda_N\setminus\Gamma_N^+}\int_0^t V_j^N(s,x/N)dW_s^{x,x+e_j} - R_{0,t}^{\frac{2a}{N}\mathbf{V}}\Big),\\
\mathbb{B}_t^{b,2aH} &= \exp\Big(\frac{2a}{N}\sum_{x\in\Gamma_N}\int_0^t H(s,x/N)dW_s^{x} - R_{b,t}^{2aH}\Big),
\end{aligned} \qquad (5.3)$$

where

$$\begin{aligned}
R_{0,t}^{\frac{2a}{N}\mathbf{V}} &= N^2\sum_{j=1}^d\sum_{x\in\Lambda_N\setminus\Gamma_N^+}\int_0^t\Big\{[\eta_s(x)+\eta_s(x+e_j)]C_N^{\beta}\big(x,x+e_j;\eta_s\big)\times\\
&\qquad\qquad\Big(e^{-[\nabla^{x,x+e_j}\eta_s(x)]\frac{2a}{N^2}V_j^N(s,x/N)}-1\Big)\Big\}ds\\
R_{b,t}^{2aH} &= N^2\sum_{x\in\Gamma_N}\int_0^t r_x\big(b(x/N),\eta_s(x)\big)\Big(e^{[2\eta_s(x)-1]\frac{2a}{N}H(s,x/N)}-1\Big)ds\,.
\end{aligned}$$

We start by the boundary term which differs from the proof of Bertini et al. (2006b). Recall from (3.13) the definition of $\mathcal{H}_N^H(s,\eta)$. Let $\delta>0$, and define the set

$$E_{N,\delta}^H = \Big\{\eta_\cdot\in D([0,T],\Sigma_N):\Big|\int_0^T\mathcal{H}_N^H(t,\eta_t)dt\Big|\le\delta\Big\}.$$

According to the definition of $\mathbb{B}_t^{b,2aH}$ and using the following inequality,

$$\varlimsup_{N\to\infty}\frac{1}{N^d}\log(a_N+b_N)\le\max\Big\{\varlimsup_{N\to\infty}\frac{1}{N^d}\log a_N,\ \varlimsup_{N\to\infty}\frac{1}{N^d}\log b_N\Big\}, \qquad (5.4)$$

we reduce the control of the second term of the right hand side of (5.2) to the following claims. For any $\delta>0$,

$$\varlimsup_{N\to\infty}\frac{1}{2N^d}\log\mathbb{E}^{\beta}_{\eta^N}\Big[\sup_{0\le t\le T}\Big\{\mathbb{B}_t^{b,2aH}\times\exp\big(R_{b,t}^{2aH}\big)\Big\}1_{(E_{N,\delta}^H)^c}\Big]=-\infty\,. \qquad (5.5)$$

and

$$\varlimsup_{\delta\to0}\varlimsup_{N\to\infty}\frac{1}{2N^d}\log\mathbb{E}^{\beta}_{\eta^N}\Big[\sup_{0\le t\le T}\Big\{\mathbb{B}_t^{b,2aH}\times\exp\big(R_{b,t}^{2aH}\big)\Big\}1_{E_{N,\delta}^H}\Big]\le 0\,. \qquad (5.6)$$

By Schwartz inequality, the expression in the first limit is bounded above by

$$\varlimsup_{N\to\infty}\frac{1}{4N^d}\log\mathbb{E}^{\beta}_{\eta^N}\Big[\sup_{0\le t\le T}\Big(\mathbb{B}_t^{b,2aH}\times\exp\big(R_{b,t}^{2aH}\big)\Big)^2\Big]+\varlimsup_{N\to\infty}\frac{1}{4N^d}\log\mathbb{P}^{\beta}_{\eta^N}\Big[(E_{N,\delta}^H)^c\Big].$$

From Lemma 3.5, for any $\delta > 0$, the second term in the last expression is equal to $-\infty$. Consider the first term. Since $G \in \mathcal{C}_0^2(\Lambda)$, a Taylor expansion shows that $\sup_{0\le t\le T} |R_{b,t}^{2aH}| \le N^d a(1+\frac{a}{N})C(H,T)$ for some constant $C(H,T)$ depending on H and T. Moreover, we can write the martingale $\mathbb{B}_t^{b,2aH}$ as

$$\begin{aligned}\mathbb{B}_t^{b,2aH} \;&=\; \Big(\mathbb{B}_t^{b,aH}\Big)^2 \exp\Big(2R_{b,t}^{aH} - R_{b,t}^{2aH}\Big)\\ &\le \Big(\mathbb{B}_t^{b,aH}\Big)^2 \exp\big\{a(1+\frac{a}{N})C(H,T)\big\}\,.\end{aligned} \tag{5.7}$$

Here and below $C(H,T)$ is a bounded constant depending on H and T whose value may change from line to line. Therefore,

$$\begin{aligned}&\frac{1}{4N^d}\log \mathbb{E}_{\eta^N}^{\beta}\Big[\sup_{0\le t\le T}\Big(\mathbb{B}_t^{b,2aH}\times \exp\big(R_{b,t}^{2aH}\big)\Big)^2\Big]\\ &\quad\le a(1+\frac{a}{N})C(H,T)\;+\;\frac{1}{4N^d}\log \mathbb{E}_{\eta^N}^{\beta}\Big[\sup_{0\le t\le T}\Big(\mathbb{B}_t^{b,2aH}\Big)^2\Big]\,.\end{aligned}$$

Since $\big(\mathbb{B}_t^{b,2aH}\big)_{t\in[0,T]}$ is a positive martingale equal to 1 at time 0, by Doob's inequality (cf. Proposition 2.16. in Ethier and Kurtz (1986)), the last expression in bounded above by

$$\begin{aligned}&a(1+\frac{a}{N})C(H,T)\;+\;\frac{1}{4N^d}\log \mathbb{E}_{\eta^N}^{\beta}\Big[4\Big(\mathbb{B}_T^{b,2aH}\Big)^2\Big]\\ &\quad\le a(1+\frac{a}{N})C(H,T)\;+\;\frac{1}{4N^d}\log \mathbb{E}_{\eta^N}^{\beta}\Big[\mathbb{B}_T^{b,4aH}\Big]\;=\;a(1+\frac{a}{N})C(H,T)\,,\end{aligned} \tag{5.8}$$

where we have used again the identity (5.7). This concludes the proof of (5.5).

On the other hand, a Taylor expansion shows that on the set $E_{N,\delta}^H$, for any $0 \le t \le T$, we have

$$|R_{b,t}^{2aH}| \le N^d a\big(\delta + \frac{a}{N}C(H)\big)\,,$$

for some positive constant $C(H)$. We then check the limit (5.6) by using again the same arguments as in (5.7), (5.8) and letting $N\uparrow\infty$ then $\delta\downarrow 0$.

We now consider the first term of the right hand side of (5.2). Since $G \in \mathcal{C}_0^2(\Lambda)$, Lemma 3.3, a Taylor expansion and a summation by parts allow to show that for any $0 \le t \le T$,

$$\begin{aligned}R_{0,t}^{\beta,\frac{2a}{N}\mathbf{V}} &\le a o_{\mathbf{V}}(1)\sum_{j=1}^{d}\sum_{x\in\Lambda_N}\int_0^T dt\,\eta_t(x) + a\beta T N^{d-1}C(\mathbf{V}) + ta^2N^{d-2}C(\mathbf{V},\beta)\\ &\le\; a\Big\{o_{\mathbf{V}}(1) + \frac{\beta}{N}C(\mathbf{V}) + \frac{a}{N^2}C(\mathbf{V},\beta)\Big\}N^dT\,,\end{aligned}$$

where $o_{\mathbf{V}}(1)$ is an expression depending on $\mathbf{V}$ which vanishes as $N\uparrow\infty$. It remains to apply again the same arguments as in (5.7), (5.8) for the martingale $\mathbb{M}_t^{\beta,\frac{2a}{N}\mathbf{V}}$:

$$\begin{aligned}\mathbb{M}_t^{\beta,\frac{2a}{N}\mathbf{V}} \;&=\; \Big(\mathbb{M}_t^{\beta,\frac{a}{N}\mathbf{V}}\Big)^2 \exp\Big(2R_t^{\beta,\frac{a}{N}\mathbf{V}} - R_t^{\beta,\frac{2a}{N}\mathbf{V}}\Big)\\ &\le \Big(\mathbb{M}_t^{\beta,\frac{a}{N}\mathbf{V}}\Big)^2 e^{N^d r_N(\mathbf{V},a,T)}\,,\end{aligned}$$

where $r_N(\mathbf{V},a,T)$ stands for an expression depending on $\mathbf{V}$, a and T which vanishes as $N\uparrow\infty$. $\square$

5.2. *The energy estimate $\mathcal{Q}$.* In this subsection, we state an energy estimate which is one of the main ingredients in the proof of large deviations and also in the proof of hydrodynamic limit. For $\pi \in D\big([0,T],\mathcal{F}^1\big)$, with $\pi_t(du)=\rho_t(u)du$, $0\le t\le T$, $\delta>0$, $1\le i\le d$, and $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$ define

$$\widetilde{\mathcal{Q}}^\delta_{i,H}(\pi) \;=\; \int_0^T dt\langle\pi_t,\partial_i H_t\rangle - \delta\int_0^T dt\langle\sigma(\rho_t)H_t,H_t\rangle\,, \tag{5.9}$$
$$\widetilde{\mathcal{Q}}^\delta_{i}(\pi) \;=\; \sup_{H\in C_c^\infty([0,T]\times\Lambda)}\Big\{\widetilde{\mathcal{Q}}^\delta_{i,H}(\pi)\Big\}\,,$$

Notice that

$$\mathcal{Q}(\pi) \;=\; \sum_{i=1}^d \widetilde{\mathcal{Q}}^2_i(\pi)\,,$$

where $\mathcal{Q}(\cdot)$ is defined in (2.12). We shall denote $\mathcal{Q}_i=\widetilde{\mathcal{Q}}^2_i$, so that $\mathcal{Q}=\sum_{i=1}^d\mathcal{Q}_i$.

For each $\varepsilon>0$ and π in $\mathcal{M}$, denote by π^ε the absolutely continuous measure obtained by smoothing the measure π:

$$\pi^\varepsilon(du) \;=\; \frac{1}{\kappa_\varepsilon}\frac{\pi(\mathbf{\Lambda}_\varepsilon(u))}{|\mathbf{\Lambda}_\varepsilon(u)|}\;du\;,$$

where $\mathbf{\Lambda}_\varepsilon(u)=\{v\in\Lambda:|v-u|\le\varepsilon\}$, $|A|$ stands for the Lebesgue measure of the set A, and $\{\kappa_\varepsilon:\varepsilon>0\}$ is a strictly decreasing sequence converging to 1. Denote

$$\pi^{N,\varepsilon} \;=\; \big(\pi^N\big)^\varepsilon\;,$$

and notice that for N sufficiently large $\pi^{N,\varepsilon}$ belongs to $\mathcal{F}^1$ because $\kappa_\varepsilon>1$. Moreover, for any $G\in\mathcal{C}^0(\Lambda)$,

$$\langle\pi^{N,\varepsilon},G\rangle \;=\; \frac{1}{N^d}\sum_{x\in\Lambda_N}G(x/N)\eta^{\varepsilon N}(x)\;+\;O(N,\varepsilon)\;,$$

where $O(N,\varepsilon)$ is absolutely bounded by $C\{N^{-1}+\varepsilon\}$ for some finite constant C depending only on G.

Lemma 5.3. *Fix a sequence $\{\eta^N\in\Sigma_N:N\ge1\}$ of configurations and $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$. There exists a positive constant C_1 depending only on b and β so that for any given $\delta_0>0$, for any δ, $0\le\delta\le\delta_0$ and any $1\le i\le d$, we have*

$$\overline{\lim_{\varepsilon\to0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(\delta\,N^d\,\widetilde{\mathcal{Q}}^{\delta_0}_{i,H}(\pi^{N,\varepsilon})\Big)\Big]\;\le\;C_1(T+1)\,.$$

The proof of this Lemma is similar to the one of Lemma 3.8. in Mourragui and Orlandi (2013), and therefore is omitted.

Corollary 5.4. *Fix a sequence $\{\eta^N\in\Sigma_N:N\ge1\}$ of configurations and $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$. There exists a positive constant C_1 depending only on b and β so that for any given $\delta_0>0$, for any δ, $0\le\delta\le\delta_0$,*

$$\overline{\lim_{\varepsilon\to0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(\delta\,N^d\sup_{1\le i\le d}\widetilde{\mathcal{Q}}^{\delta_0}_{i,H}(\pi^{N,\varepsilon})\Big)\Big]\;\le\;C_1(T+1)\,. \tag{5.10}$$

Proof: From the inequality (5.4) the limit in (5.10) is bounded above by

$$\max_{1\le i\le d}\Big\{\overline{\lim_{\varepsilon\to0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(\delta\,N^d\widetilde{\mathcal{Q}}^{\delta_0}_{i,G}(\pi^{N,\varepsilon})\Big)\Big].$$

The thesis follows By Lemma 5.3. □

5.3. *The functional $\mathcal{E}^\gamma$.* For $(G,\varphi)\in\mathcal{C}_0^2(\Lambda)\times\mathcal{C}^1([0,T])$ and $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$, denote by $\mathfrak{E}^{\gamma,H}_{(G,\varphi)}$ the functional

$$\mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W},\pi)=\sup_{1\le i\le d}\big\{\mathcal{Q}_{i,H}(\pi)\big\}+\sup_{0\le t\le T}\big\{\mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W},\pi)\big\}\,, \tag{5.11}$$

where $\mathcal{Q}_{i,H}(\pi)=\tilde{\mathcal{Q}}^2_{i,H}(\pi)$ with $\delta=2$, and $\mathbb{V}^{t,\gamma}_{(G,\varphi)}$ are defined in (5.9) and (5.1).

Lemma 5.5. *Fix a sequence $\{\eta^N\in\Sigma_N:N\ge1\}$ of configurations, $(G,\varphi)\in\mathcal{C}_0^2(\Lambda)\times\mathcal{C}^1([0,T])$ and $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$. There exists a positive constant C_2 depending only on b and β so that, for any $0\le\delta\le1$ and any $1\le i\le d$,*

$$\overline{\lim_{\varepsilon\to0}}\;\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(\delta\,N^d\mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W}^N,\pi^{N,\varepsilon})\Big)\Big]\;\le\;C_2(T+1)\,.$$

Proof: By Schwarz inequality,

$$\begin{aligned}&\frac{1}{N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(\delta\,N^d\mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W}^N,\pi^{N,\varepsilon})\Big)\Big]\\&\quad\le\;\frac{1}{2N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(2\delta\,N^d\sup_{0\le t\le T}\big\{\mathbb{V}^{t,\gamma}_{(G,\varphi)}(\mathbf{W}^N,\pi^{N,\varepsilon}\big)\Big)\Big]\\&\qquad+\;\frac{1}{2N^d}\log\mathbb{E}^\beta_{\eta^N}\Big[\exp\Big(2\delta\,N^d\sup_{1\le i\le d}\big\{\mathcal{Q}_{i,H}(\pi^{N,\varepsilon})\big\}\Big)\Big]\,.\end{aligned}$$

The result is an imediate consequence of Lemma 5.2 and of Corollary 5.4. □

5.4. *Upper bound.* In this section we investigate the upper bound of the large deviations principle for compact sets and then for closed sets of the couple $(\mathbf{W}^N,\pi^N)$ on the topological space $D([0,T),\mathcal{M}^{d+1})$. We follow the strategy of Mourragui and Orlandi (2013), relying on some properties of the rate function that we proved in the last subsections. Notice however that in the present case the proof is slightly more demanding due to the definition of the energy functional $\mathcal{E}^\gamma$. We first prove an upper bound with an auxiliary rate functional.

Recall from (5.11) the definition of $\mathfrak{E}^{\gamma,H}_{(G,\varphi)}$. We introduce the functional $\mathfrak{E}^\gamma:D([0,T],\mathcal{M}^d\times\mathcal{F}^1)\to[0,+\infty]$ defined by

$$\mathfrak{E}^\gamma(\mathbf{W},\pi)\;=\;\sup_{G,\varphi,H}\Big\{\mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W},\pi)\Big\}\,, \tag{5.12}$$

where the supremum is carried over all $(G,\varphi,H)\in\mathcal{C}_0^2(\Lambda)\times\mathcal{C}^1([0,T])\times\mathcal{C}_c^\infty([0,T]\times\Lambda)$. Notice that $\mathfrak{E}^\gamma(\mathbf{W},\pi)<+\infty$ if and only if $\mathcal{E}^\gamma(\mathbf{W},\pi)<+\infty$.

For each $0\le a\le1$, let $\mathfrak{F}_a:D([0,T],\mathcal{M}^{d+1})\to[0,+\infty]$ be the functional given by

$$\mathfrak{F}_a(\mathbf{W},\pi)=\begin{cases}\mathbb{J}_T(\mathbf{W},\pi)+a\,\mathfrak{E}^\gamma(\mathbf{W},\pi) & \text{if}\quad D([0,T],\mathcal{M}^d\times\mathcal{F}^1)\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

Proposition 5.6. *Let $\mathcal{K}$ be a compact set of $D([0,T],\mathcal{M}^{d+1})$. There exists a positive constants C_2, such that for any $0<a\le1$,*

$$\overline{\lim_{N\to\infty}}\;\frac{1}{N^d}\log Q^\beta_{\eta^N}(\mathcal{K})\;\le\;-\frac{1}{1+a}\inf_{(W,\pi)\in\mathcal{K}}\mathfrak{F}_a(\mathbf{W},\pi)\;+\;\frac{a}{1+a}C_2(T+1)\,.$$

Proof: Fix a compact set $\mathcal{K}$ of $D([0,T],\mathcal{M}^{d+1})$ and functions $(G,\varphi)\in\mathcal{C}_0^2(\Lambda)\times\mathcal{C}^1([0,T])$, $H\in\mathcal{C}_c^\infty([0,T]\times\Lambda)$, $\mathbf{V}=(V_1,\cdots,V_d)\in(\mathcal{C}^{1,1}([0,T]\times\Lambda))^d$. Denote by $\mathbf{O}$ the vector-valued function $(0,\cdots,0)$, where each component is the zero function and recall from (3.12) and (3.13), the definition of $\mathcal{G}_{N,\varepsilon}^{\mathbf{V},\mathbf{0},\beta}$ and $\mathcal{H}_N^{\partial_1 V_1}$. For $\delta>0$, let $B_{N,\varepsilon,\delta}^{\mathbf{V},\mathbf{0},\beta}$, $E_{N,\delta}^{\partial_1 V_1}$ be the sets of trajectories $(\eta_t)_{t\in[0,T]}$ defined by

$$\begin{aligned}B_{N,\varepsilon,\delta}^{\mathbf{V},\mathbf{0},\beta} &= \Big\{\eta_\cdot\in D([0,T],\Sigma_N):\Big|\int_0^T\mathcal{G}_{N,\varepsilon}^{\mathbf{V},\mathbf{0},\beta}(t,\eta_t)dt\Big|\le\delta\Big\}\,,\\ E_{N,\delta}^{\partial_1 V_1} &= \Big\{\eta_\cdot\in D([0,T],\Sigma_N):\Big|\int_0^T\mathcal{H}_N^{\partial_1 V_1}(t,\eta_t)dt\Big|\le\delta\Big\}\end{aligned}$$

and set

$$A_{N,\varepsilon,\delta}^{\mathbf{V},\beta} \;=\; B_{N,\varepsilon,\delta}^{\mathbf{V},\mathbf{0},\beta}\cap E_{N,\delta}^{\partial_1 V_1}\,.$$

By (5.4) and the superexponential estimates stated in Proposition 3.5, for any $\delta>0$

$$\varlimsup_{\varepsilon\to0}\varlimsup_{N\to\infty}\frac{1}{N^d}\log Q_{\eta^N}^\beta\Big(\mathcal{K}\cap\big(A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}\big)^c\Big) \;=\; -\infty\,, \tag{5.13}$$

where $\big(A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}\big)^c$ stands for the complementary of the set $A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}$.

Recall from (5.11) the definition of $\mathfrak{E}_{(G,\varphi)}^{\gamma,H}$. To short notation we denote by $\widehat{\mathcal{K}}_{N,\varepsilon,\delta}^{\mathbf{V},\beta}=\mathcal{K}\cap A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}$, and write

$$\begin{aligned}&\frac{1}{N^d}\log Q_{\eta^N}^\beta\Big(\mathcal{K}\cap A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}\Big) \;=\\ &\quad\frac{1}{N^d}\log\mathbb{E}_{\eta^N}^\beta\Big[\mathbb{1}\{\widehat{\mathcal{K}}_{N,\varepsilon,\delta}^{\mathbf{V},\beta}\}e^{-\frac{a}{1+a}N^d\mathfrak{E}_{(G,\varphi)}^{\gamma,H}(\mathbf{W}^N,\pi^{N,\varepsilon})}e^{\frac{a}{1+a}N^d\mathfrak{E}_{(G,\varphi)}^{\gamma,H}(\mathbf{W}^N,\pi^{N,\varepsilon})}\Big]\,.\end{aligned}$$

By Hölder inequality the right hand side of the last equality is bounded above by

$$\begin{aligned}&\frac{1}{1+a}\frac{1}{N^d}\log\mathbb{E}_{\eta^N}^\beta\Big[\mathbb{1}\{\widehat{\mathcal{K}}_{N,\varepsilon,\delta}^{\mathbf{V},\beta}\}e^{-aN^d\mathfrak{E}_{(G,\varphi)}^{\gamma,H}(\mathbf{W}^N,\pi^{N,\varepsilon})}\Big]\\ &\qquad+\;\frac{a}{1+a}\frac{1}{N^d}\log\mathbb{E}_{\eta^N}^\beta\Big[e^{N^d\mathfrak{E}_{(G,\varphi)}^{\gamma,H}(\mathbf{W}^N,\pi^{N,\varepsilon})}\Big]\,.\end{aligned} \tag{5.14}$$

From Lemma 5.5, the limsup when $N\uparrow\infty$ and $\varepsilon\downarrow0$ of the second term of this inequality is bounded by $\dfrac{a}{1+a}C_2(T+1)$, while the first term can be rewriten as the expectation with respect to the perturbed process introduced in Subsection 3.1 whose law is given by $\mathbb{P}_{\eta^N}^{\beta,\mathbf{V}}$, that is

$$\frac{1}{1+a}\frac{1}{N^d}\log\mathbb{E}_{\eta^N}^{\beta,\mathbf{V}}\Big[\frac{d\mathbb{P}_{\eta_N}^\beta}{d\mathbb{P}_{\eta_N}^{\beta,\mathbf{V}}}\,\mathbb{1}\{\widehat{\mathcal{K}}_{N,\varepsilon,c}^{F,\mathbf{V},\beta}\}e^{-aN^d\mathfrak{E}_{(G,\varphi)}^{\gamma,H}(\mathbf{W}^N,\pi^{N,\varepsilon})}\Big]\,. \tag{5.15}$$

By (3.14), the Radon-Nikodym derivative of $\mathbb{P}_{\eta_N}^\beta$ with respect to the probability $\mathbb{P}_{\eta_N}^{\beta,\mathbf{V}}$ defined by the Girsanov formula satisfies on the set $A_{N,\varepsilon,\delta}^{\mathbf{V},\beta}$

$$\frac{d\mathbb{P}_{\eta_N}^\beta}{d\mathbb{P}_{\eta_N}^{\beta,\mathbf{V}}} \;=\; \exp N^d\Big\{-\widehat{\mathbb{J}}_{\mathbf{V}}^T(\mathbf{W}^N,\pi^{N,\varepsilon})\;+\;r(N,\varepsilon,\delta,\mathbf{V})\Big\}\,,$$

where $\widehat{\mathbb{J}}_{\mathbf{V}}^T(\cdot)$ is the functional defined in (2.14), and $r(N,\varepsilon,c,\mathbf{V})$ is a quanity satisfying

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\lim_{N\to\infty}r(N,\varepsilon,\delta,\mathbf{V})=0\,.$$

We now exclude paths whose densities are not absolutely continuous with respect to the Lebesgue measure. Fix a sequence $\{f_k : k \geq 1\}$ of smooth nonnegative functions dense in $\mathcal{C}^0(\Lambda)$ for the uniform topology. For $k \geq 1$ and $\varrho > 0$, let

$$\mathbf{D}_{k,\varrho} = \Big\{(\mathbf{W}, \pi) \in D([0,T], \mathcal{M}^{d+1}) : \\ 0 \leq < \pi_t, f_k > \leq \int_\Lambda f_k(x)\, dx \,+\, C_k \varrho \,,\;\; 0 \leq t \leq T\Big\}\,,$$

where $C_k = C(\|\nabla f_k\|_\infty)$ is positive constants depending on the gradient ∇f_k of f_k. The sets $\mathbf{D}_{k,\varrho}$, $k \geq 1$, $\varrho > 0$ are closed subsets of $\in D([0,T], \mathcal{M}^{d+1})$, as well as

$$\mathfrak{D}_{m,\varrho} \;=\; \bigcap_{k=1}^{m} \mathbf{D}_{k,\varrho}\;, \quad m \geq 1\,.$$

Note that the empirical measure π^N belongs to $\mathfrak{D}_{m,\varrho}$ for N sufficiently large. We have that

$$D([0,T], \mathcal{M}^d \times \mathcal{F}^1) = \cap_{n\geq 1} \cap_{m \geq 1} \mathfrak{D}_{m,1/n}. \tag{5.16}$$

For $m, n \in \mathbb{Z}_+$, let $\widehat{\mathcal{E}}^{\gamma,\varepsilon,m,n}_{(G,\varphi),H} : D([0,T], \mathcal{M}^{d+1}) \to \mathbb{R} \cup \{\infty\}$ be the functional given by

$$\widehat{\mathcal{E}}^{\gamma,\varepsilon,m,n}_{(G,\varphi),H}(\mathbf{W}, \pi) = \begin{cases} \mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W}, \pi^\varepsilon) & \text{if} \quad \pi \in \mathfrak{D}_{m,\frac{1}{n}}\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{5.17}$$

It is lower semicontinuous because so is $(\mathbf{W}, \pi) \mapsto \mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W}, \pi^\varepsilon)$, and because $\mathfrak{D}_{m,1/n}$ is closed.

Recollecting all previous estimates. Using the inequality (5.4), optimizing over π in $\mathcal{K}$ and letting $N \uparrow \infty$, we obtain that, for any $m, n \in \mathbb{Z}_+$, $0 < a \leq 1$, $\delta > 0$ and ε small enough

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log Q^\beta_{\eta^N}\Big(\mathcal{K}\Big) \;\leq\; \frac{1}{1+a} \sup_{(\mathbf{W},\pi)\in\mathcal{K}} \widehat{\mathfrak{S}}^{a,\delta,\varepsilon,m,n}_{\mathbf{V},H,G,\varphi}(\mathbf{W}, \pi)\,. \tag{5.18}$$

Here, we have denoted

$$\widehat{\mathfrak{S}}^{a,\delta,\varepsilon,m,n}_{\mathbf{V},H,G,\varphi}(\mathbf{W}, \pi) \;=\; \max\Big\{\Big(-\widehat{\mathbb{J}}^T_{\mathbf{V}}(\mathbf{W}, \pi^\varepsilon) + a\mathfrak{R}^{a,\delta,\varepsilon,m,n}_{H,G,\varphi}(\mathbf{W}, \pi)\Big)\,,\, U_{0,a}(\mathbf{V}, \varepsilon)\Big\}\,,$$

where

$$\mathfrak{R}^{a,\delta,\varepsilon,m,n}_{H,G,\varphi}(\mathbf{W}, \pi) = -\widehat{\mathcal{E}}^{\gamma,\varepsilon,m,n}_{(G,\varphi),H}(\mathbf{W}, \pi) + U_{1,a}(G, \varphi, H, \varepsilon) + r(N, \varepsilon, \delta, \mathbf{V})\,,$$

$$U_{1,a}(G, \varphi, H, \varepsilon) = \varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}^\beta_{\eta^N}\Big[e^{N^d \mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W}^N, \pi^{N,\varepsilon})}\Big]\,,$$

$$U_{0,a}(\mathbf{V}, \varepsilon) = (1+a) \varlimsup_{N\to\infty} \frac{1}{N^d} \log Q^\beta_{\eta^N}\Big(\mathcal{K} \cap \big(A^{\mathbf{V},\beta}_{N,\varepsilon,\delta}\big)^c\Big)\,.$$

Note that, for each $m, n \in \mathbb{Z}_+$, $0 < a \leq 1$, $\delta > 0$ and $\varepsilon > 0$, the functional $\widehat{\mathfrak{S}}^{a,\delta,\varepsilon,m,n}_{\mathbf{V},H,G,\varphi}$ is lower semicontinuous. Minimizing the right hand side of the inequality (5.18) over $m, n \in \mathbb{Z}_+$, $\delta > 0$ and $0 < \varepsilon < 1$, and using Lemma A2.3.3 in Kipnis and Landim (1999) for our compact $\mathcal{K}$, we get

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log Q^\beta_{\eta^N}\Big(\mathcal{K}\Big) \;\leq\; \frac{1}{1+a} \sup_{(\mathbf{W},\pi)\in\mathcal{K}} \inf_{\delta,\varepsilon,m,n} \widehat{\mathfrak{S}}^{a,\delta,\varepsilon,m,n}_{\mathbf{V},H,G,\varphi}(\mathbf{W}, \pi)\,.$$

By (5.13), (5.4), (5.16) and Lemma 5.5

$$\varlimsup_{\delta\to 0} \varlimsup_{\varepsilon\to 0} \varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \widehat{\mathfrak{S}}^{a,\delta,\varepsilon,m,n}_{\mathbf{V},H,G,\varphi}(\mathbf{W}, \pi) \leq -\mathfrak{F}^{(G,\varphi),a}_{\mathbf{V},H}(\mathbf{W}, \pi) + aC_2(T+1)\,,$$

where

$$\mathfrak{F}^{(G,\varphi),a}_{\mathbf{V},H}(\mathbf{W},\pi) = \begin{cases} \widehat{\mathbb{J}}^T_{\mathbf{V}}(\mathbf{W},\pi) + a\mathfrak{E}^{\gamma,H}_{(G,\varphi)}(\mathbf{W},\pi) & \text{if} \quad \pi \in D([0,T],\mathcal{M}^d \times \mathcal{F}^1)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

This result and the last inequality imply,

$$\begin{aligned} &\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q^{\beta}_{\eta^N}\Big(\mathcal{K}\Big) \\ &\qquad \le -\frac{1}{1+a} \inf_{(\mathbf{W},\pi)\in\mathcal{K}} \Big\{\mathfrak{F}^{(G,\varphi),a}_{\mathbf{V},H}(\mathbf{W},\pi)\Big\} + \frac{a}{1+a} C_2(T+1)\,, \end{aligned}$$

for any $\mathbf{V}, H, G, \varphi$. To conclude the proof of the proposition, it remains to Minimize the last inequality over $\mathbf{V}, H, G, \varphi$, and to use again Lemma A2.3.3 in Kipnis and Landim (1999) for the compact $\mathcal{K}$. □

Proof of the upper bound. Denote by $\widehat{\mathcal{E}}^\gamma : D([0,T],\mathcal{M}^{d+1})$ the lower semicontinuous functional

$$\widehat{\mathcal{E}}^\gamma(\mathbf{W},\pi) = \begin{cases} \mathfrak{E}^\gamma(\mathbf{W},\pi) & \text{if} \quad (\mathbf{W},\pi) \in D([0,T],\mathcal{M}^d \times \mathcal{F}^1)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Let $\mathcal{K}$ be a compact set of $D([0,T],\mathcal{M}^{d+1})$. If for all $(\mathbf{W},\pi) \in \mathcal{K}$, $\widehat{\mathcal{E}}^\gamma(\mathbf{W},\pi) = +\infty$ then the upper bound is trivially satisfied. Suppose that $\inf\limits_{(\mathbf{W},\pi)\in\mathcal{K}} \big\{\widehat{\mathcal{E}}^\gamma(\mathbf{W},\pi)\big\} < \infty$, from Proposition 5.6, for any $0 < a \le 1$,

$$\begin{aligned} \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q^{\beta}_{\eta^N}\big(\mathcal{K}\big) \;&\le\; -\frac{1}{1+a} \inf_{\substack{(\mathbf{W},\pi)\in\mathcal{K} \\ \widehat{\mathcal{E}}_\gamma(\mathbf{W},\pi)<\infty}} \mathfrak{F}_a(\mathbf{W},\pi) \;+\; \frac{a}{1+a} C_2(T+1) \\ &= -\frac{1}{1+a} \inf_{(\mathbf{W},\pi)\in\mathcal{K}} \Big\{\mathcal{J}^\gamma_T(\mathbf{W},\pi) \;+\; a\widehat{\mathcal{E}}_\gamma(\mathbf{W},\pi)\Big\} \;+\; \frac{a}{1+a} C_2(T+1) \\ &\le -\frac{1}{1+a} \inf_{(\mathbf{W},\pi)\in\mathcal{K}} \mathcal{J}_T{}^\gamma(\mathbf{W},\pi) - \frac{a}{1+a} \inf_{(\mathbf{W},\pi)\in\mathcal{K}} \widehat{\mathcal{E}}_\gamma(\mathbf{W},\pi) + \frac{a}{1+a} C_2(T+1)\,. \end{aligned}$$

To conclude the proof of the upper bound for compact sets, it remains to let $a \downarrow 0$.

To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\big\{Q^{\beta}_{\eta^N}\,,\ N \ge 1\big\}$. The proof presented in Bertini et al. (2006b); Bodineau and Lagouge (2012) is easily adapted to our context. □

6. large deviations lower bound for the empirical current

The strategy of the proof of the lower bound consists of two steps. We first get a lower bound for neighbourhoods of regular trajectories. Then we extend the lower bound for all open set by showing in Theorem 6.3 that the set of all regualar trajectories is $\mathcal{J}^\gamma_T$-dense in the following sens:

Definition 6.1. A subset $\mathcal{A}$ of $D([0,T],\mathcal{M}^{d+1})$ is said to be $\mathcal{J}^\gamma_T$-dense if for every $(\mathbf{W},\pi)$ in $D([0,T],\mathcal{M}^{d+1})$ such that $\mathcal{J}^\gamma_T(\mathbf{W},\pi) < \infty$, there exists a sequence $\{(\mathbf{W}^n,\pi^n) : n \ge 1\}$ in $\mathcal{A}$ such that $(\mathbf{W}^n,\pi^n)$ converge to $(\mathbf{W},\pi)$ in $D([0,T],\mathcal{M}^{d+1})$ and $\lim\limits_{n\to\infty} \mathcal{J}^\gamma_T(\mathbf{W}^n,\pi^n) \;=\; \mathcal{J}^\gamma_T(\mathbf{W},\pi)$.

To clarify the meaning of regular trajectory, we consider the heat equation given by the boundary value problem (1.1) for $\beta = 0$:

$$\begin{cases} \partial_t \rho & = \Delta\rho \quad \text{in} \quad \Lambda \times (0,T), \\ \rho_0(\cdot) & = \gamma(\cdot) \quad \text{in} \quad \Lambda, \\ \rho_t|_\Gamma & = b \quad \text{for} \quad 0 \le t \le T\,. \end{cases} \tag{6.1}$$

Denote by $\rho^{(0)}$ its unique weak solution, and set $\pi_t^{(0)}(du) = \rho_t^{(0)}(u)du$. Let $(\mathbf{W}_t^{(0)})_{t\in[0,T]}$ be the solution of the equation

$$\partial_t \rho^0 + \nabla \cdot \partial_t \mathbf{W}_t = 0\,,$$

given by linear forms

$$\langle \mathbf{W}_t, \mathbf{V} \rangle = \int_0^t \Big\langle 1, -\nabla\rho_s^0 \cdot \mathbf{V} \Big\rangle ds\,, \quad \mathbf{V} \in \big(\mathcal{C}^1(\Lambda)\big)^d, \;\; t\in[0,T]\,.$$

Notice that, and an approximation of $\frac{\nabla\rho^{(0)}}{\sigma(\rho^{(0)})}$ by smooth functions shows that $\mathcal{Q}(\rho^{(0)}) < \infty$, (see Bertini et al. (2009), (5.1)). Moreover, by construction $(\mathbf{W}^0, \pi^{(0)}) \in \mathfrak{A}_\gamma$, and

$$\mathcal{J}^\gamma(\mathbf{W}^0, \pi^0) \le \frac{\beta^2}{4} \int_0^T dt \int_\Lambda \big|\nabla \rho_t^{(0)}\big|^2 < \infty\,.$$

(see Mourragui and Orlandi (2013), Lemma 5.8.).

Definition 6.2. A trajectory $(\mathbf{W}, \pi) \in D([0,T], \mathcal{M}^{d+1})$ is said to be regular if

(i) $\mathcal{J}_T^\gamma(\mathbf{W}, \pi) < \infty$, $\pi(t, du) = \rho_t(u)du$.
(ii) There exists $c > 0$ such that $(\mathbf{W}, \pi) = (\mathbf{W}^0, \pi^0)$ in the time interval $[0, c]$.
(iii) For all $0 < \delta \le T$, there exists $\varepsilon > 0$ such that $\varepsilon \le \rho_t(u) \le 1 - \varepsilon$ for $(t,u) \in [\delta, T] \times \Lambda$.
(iv) There exists $\mathbf{V} \in \big(\mathcal{C}^{1,1}([0,T] \times \Lambda)\big)^d$ such that ρ is the solution of the boundary value problem (3.3).

We denote by $\mathcal{A}^0$ the class of all regular trajectories.

The proof of the lower bound for regular trajectories is similar to the one in the convex periodic case. We show that for any path $(\mathbf{W}, \pi)$ in $\mathcal{A}^0$, for each neighborhood $\mathcal{N}_{(\mathbf{W},\pi)}$ of $(\mathbf{W}, \pi)$,

$$\varliminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}^\beta \left\{ \mathcal{N}_{(\mathbf{W},\pi)} \right\} \ge -\mathcal{J}_T^\beta(\mathbf{W}, \pi)\,. \tag{6.2}$$

We refer to Kipnis and Landim (1999), section 10.5 or Mourragui and Orlandi (2013), section 6.4 for the proof of (6.2) for $(\mathbf{W}, \pi)$ in $\mathcal{A}^0$.

As mentioned at the beginning of this section, the lower bound of the large deviations principle is then accomplished for general trajectotries using the next result.

Theorem 6.3. *The class $\mathcal{A}^0$ is $\mathcal{J}_T^\gamma$-dense.*

The proof of this theorem is an adaptation of the I-density presented in Bertini et al. (2009); Farfan et al. (2011); Mourragui and Orlandi (2013) for the couple $(\mathbf{W}, \pi)$. We therefore provide only a presentation of its main steps, with an outline of the proofs.

Lemma 6.4. *The set of all trajectories satisfying (i) and (ii) is $\mathcal{J}_T^\gamma$-dense.*

Proof: Fix a path $(\mathbf{W},\pi)$ such that $\mathcal{J}_T^\gamma(\mathbf{W},\pi)<\infty$. For $\varepsilon>0$, define $(\mathbf{W}^\varepsilon,\pi^\varepsilon)$ as

$$\big(\mathbf{W}_t^\varepsilon,\pi_t^\varepsilon\big) = \begin{cases} \big(\mathbf{W}_t^{(0)},\pi_t^{(0)}\big) & \text{for } 0\le t\le\varepsilon,\\ \big(\mathbf{W}_{2\varepsilon-t}^{(0)},\pi_{2\varepsilon-t}^{(0)}\big) & \text{for } \varepsilon\le t\le 2\varepsilon,\\ \big(\mathbf{W}_{t-2\varepsilon},\pi_{t-2\varepsilon}\big) & \text{for } 2\varepsilon\le t\le T.\end{cases}$$

Clearly, $\lim_{\varepsilon\to0}\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big)=\big(\mathbf{W},\pi\big)$ in $D([0,T],\mathcal{M}^{d+1})$. The same strategy as in Lemma 5.4, Bertini et al. (2009) or Lemma 5.11, Mourragui and Orlandi (2013), yields $\mathcal{J}_T^\gamma\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big)<\infty$, for all $\varepsilon>0$, and $\lim\limits_{\varepsilon\to0}\mathcal{J}_T^\gamma\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big) = \mathcal{J}_T^\gamma\big(\mathbf{W},\pi\big)$. This concludes the proof. □

Lemma 6.5. *The set of all trajectories satisfying (i), (ii) and (iii) is $\mathcal{J}_T^\gamma$-dense.*

Proof: Denote by $\mathcal{A}^1$ the set of all trajectories $(\mathbf{W},\pi)$ satisfying (i), (ii) and (iii). By the previous lemma, it is enough to show that each trajectory $(\mathbf{W},\pi)$ satisfying (i) and (ii) can be approximated by trajectories in $\mathcal{A}^1$. Fix such trajectory $(\mathbf{W},\pi)$. For each $0<\varepsilon\le1$, let $(\mathbf{W}^\varepsilon,\pi^\varepsilon)$ given by

$$\mathbf{W}^\varepsilon=(1-\varepsilon)\mathbf{W}+\varepsilon\mathbf{W}^{(0)}\,,\qquad \pi^\varepsilon=(1-\varepsilon)\pi+\varepsilon\pi^{(0)}\,.$$

Repeating the arguments presented in Mourragui and Orlandi (2013, Lemma 5.12.), one can prove that $\lim_{\varepsilon\to0}\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big)=\big(\mathbf{W},\pi\big)$ in $D([0,T],\mathcal{M}^{d+1})$, $\mathcal{J}_T^\gamma\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big)<\infty$, for all $\varepsilon>0$ and $\lim\limits_{\varepsilon\to0}\mathcal{J}_T^\gamma\big(\mathbf{W}^\varepsilon,\pi^\varepsilon\big) = \mathcal{J}_T^\gamma\big(\mathbf{W},\pi\big)$. □

Proof of Theorem 6.3. Recall that $\mathcal{A}^1$ stands for the set of all trajectories $(\mathbf{W},\pi)$ satisfying (i), (ii) and (iii). From the previous lemmata, it is enough to show that each trajectory $(\mathbf{W},\pi)$ in $\mathcal{A}^1$ can be approximated by trajectories of $\mathcal{A}^1$ satisfying (iv). Fix $(\mathbf{W},\pi)\in\mathcal{A}^1$ and denote $\rho_t(\cdot)$ the density of π_t for $0\le t\le T$. By Lemma 4.5, there exist $\mathbf{U}=(U_1,\cdots,U_d)\in\mathbb{L}^2(\sigma(\pi))$ and $F\in H_0^1(\sigma(\pi))$ such that ρ solves the equation (3.3) with $\mathbf{V}=\nabla F$ and $\mathbf{W}$ solves the equation (4.3). We claim that $\mathbf{U}\in\big(L^2([0,T]\times\Lambda)\big)^d$ and $F\in L^2([0,T],H^1(\Lambda))$. Indeed, from condition (ii), ρ is the weak solution of (6.1) in some time interval $[0,2\delta]$ for some $\delta>0$. In particular, $\rho_t=\rho_t^{(0)}$, $\mathbf{W}_t=-\displaystyle\int_0^t\nabla\rho_s^{(0)}ds$ for $0\le t\le 2\delta$, which implies that $\mathbf{U}_t=\nabla F_t=-\beta\nabla(J^{\rm neum}\star\rho_t)$ a.e in $[0,2\delta]\times\Omega$. On the other hand, from condition (iii), there exists $\varepsilon>0$ such that $\varepsilon\le\rho_t(\cdot)\le1-\varepsilon$ for $\delta\le t\le T$. Hence, by Lemma 4.1,

$$\begin{aligned}\int_0^T dt\int_\Lambda\big|\mathbf{U}(t,u)\big|^2du &\le \int_0^\delta dt\int_\Lambda\beta^2\big|\nabla(J^{\rm neum}\star\rho_t)(u)\big|^2du+\frac{1}{\sigma(\varepsilon)}\|\mathbf{U}\|^2_{\mathbb{L}^2(\sigma(\pi))}\\ &\le\beta^2\int_0^T dt\int_\Lambda\big|\nabla\rho_t(u)\big|^2du+\frac{2}{\sigma(\varepsilon)}\mathcal{J}_T^\gamma(\mathbf{W},\pi)<\infty\,,\\ \int_0^T dt\int_\Lambda\big|\nabla F_t(u)\big|^2du &\le \int_0^\delta dt\int_\Lambda\beta^2\big|\nabla(J^{\rm neum}\star\rho_t)(u)\big|^2du+\frac{1}{\sigma(\varepsilon)}\|F\|^2_{H_0^1(\sigma(\pi))}\\ &\le\beta^2\int_0^T dt\int_\Lambda\big|\nabla\rho_t(u)\big|^2du+\frac{2}{\sigma(\varepsilon)}\mathcal{I}_T^\gamma(\pi)<\infty\,.\end{aligned}$$

Let $\big\{\mathbf{U}^n=(U_1^n,\cdots,U_d^n),\ \ n\ge1\big\}\subset(C^{1,1}([0,T]\times\Lambda))^d$ and $\big\{F^n,\ n\ge1\big\}\subset C^{1,2}([0,T]\times\Lambda)$ be two sequences of functions such that $\lim\limits_{n\to+\infty}\mathbf{U}^n=$

$\mathbf{U}$ in $\big(L^2\big([0,T]\times\Lambda\big)\big)^d$, and $\lim_{n\to+\infty} F^n = F$ in $L^2\big([0,T], H^1(\Lambda)\big)$. For each integer $n>0$, let $\mathbf{W}^n$ be the weak solution of the equation

$$\partial_t \mathbf{W}_t \;=\; -\nabla\rho^n_t + \sigma(\rho^n_t)\big[\beta\nabla(J^{\rm neum}\star\rho^n_t) \;+\; \mathbf{U}^n_t\big]\,, \qquad \mathbf{W}_0 = 0\,, \tag{6.3}$$

where ρ^n is the weak solution of (3.3) with ∇F^n in place of $\mathbf{V}$. We set $\pi^n(t,du) = \rho^n(t,u)du$.

We examine in this paragraph the energy $\mathcal{E}^\gamma(\mathbf{W}^n,\pi^n)$. Recall from Lemme 4.5 that $\nabla\cdot\sigma(\rho^n)\big(U^n - \nabla F^n\big) = 0$ in the weak sense. Since $\mathbf{W}^n$ solves the equation (6.3), and ρ^n solves the equation (3.3), we have for any $G\in\mathcal{C}^1_0(\Lambda)$, $t\in[0,T]$,

$$\begin{aligned}
\langle \mathbf{W}^n_t, \nabla G\rangle \;&=\; \int_0^t \big\langle 1, \big[-\nabla\rho^n_s + \beta\sigma(\rho^n_s)\nabla(J^{\rm neum}\star\rho^n_s)\big]\cdot\nabla G\big\rangle ds\\
&\quad + \int_0^t \big\langle\sigma(\rho^n_s), \mathbf{U}^n_s\cdot\nabla G\big\rangle ds\\
&= \langle\pi^n_t, G\rangle - \langle\pi^n_0, G\rangle + \int_0^t ds\langle\sigma(\rho^n_s), (-\nabla F^n_s + \mathbf{U}^n_s)\cdot\nabla G\rangle\\
&= \langle\pi^n_t, G\rangle - \langle\pi^n_0, G\rangle\,.
\end{aligned}$$

This proves that $(\mathbf{W}^n,\pi^n)\in\mathfrak{A}_\gamma$. On the other hand, since $\sigma(\pi^n)$ is bounded above by $1/2$, from Lemma 4.1

$$\begin{aligned}
\mathcal{I}^\gamma_T(\pi^n) &= \frac{1}{2}\int_0^T\; dt\;\langle\sigma(\rho^n_t), \nabla F^n_t\cdot\nabla F^n_t\rangle\\
&\le \mathcal{J}^\gamma(\mathbf{W}^n,\pi^n) = \frac{1}{2}\int_0^T\; dt\;\langle\sigma(\rho^n_t), \mathbf{U}^n_t\cdot\mathbf{U}^n_t\rangle\\
&\le \frac{1}{4}\int_0^T dt\int_\Lambda \big|\mathbf{U}^n_t(u)\big|^2 du\,.
\end{aligned}$$

In particular, $\{\mathcal{I}^\gamma_T(\pi^n),\ n\ge 1\}$ and $\{\mathcal{J}^\gamma_T(\mathbf{W}^n,\pi^n),\ n\ge 1\}$ are uniformly bounded. Thus, Lemma 4.3, implies the uniform boundedness of the sequence $\{\mathcal{Q}(\pi^n),\ n\ge 1\}$.

In order to extract a converging subsequence from the sequence $\big\{(\mathbf{W}^n,\pi^n),\ n\ge 1\big\}$, we need to show the relative compactness of the set $\big\{(\mathbf{W}^n,\pi^n),\ n\ge 1\big\}$ in the topological space $D([0,T],\mathcal{M}^{d+1})$. By construction, for any $s,t\in[0,T]$, any $\mathbf{V}\in\big(\mathcal{C}^1(\Lambda)\big)^d$ and any $G\in\mathcal{C}^2_0(\Lambda)$,

$$\begin{aligned}
&\big|\langle\mathbf{W}^n_t,\mathbf{V}\rangle + \langle\pi^n_t, G\rangle - \langle\mathbf{W}^n_s,\mathbf{V}\rangle - \langle\pi^n_s, G\rangle\big|\\
&\quad= \Big|\int_s^t \big\langle 1, \big[-\nabla\rho^n_\tau + \sigma(\rho^n_\tau)\beta\nabla(J^{\rm neum}\star\rho^n_\tau)\big]\cdot\big(\mathbf{V}+\nabla G\big)\big\rangle d\tau\\
&\qquad + \int_s^t \big\langle\sigma(\rho^n_\tau), \big[\mathbf{U}^n_\tau\cdot\mathbf{V} \;+\; \nabla F^n_\tau\cdot\nabla G\big]\big\rangle d\tau\Big|\,.
\end{aligned}$$

For shortness of notation, we shall denote $\Lambda_T = [0,T]\times\Lambda$ and for a vector valued measurable function $\mathbf{H}\in\big(L^2(\Lambda_T)\big)^d$ (resp. $\mathbf{V}\in\big(L^2(\Lambda)\big)^d$), we shall denote $\|\mathbf{H}\|_{L^2(\Lambda_T)} = \int_0^T dt\int_\Lambda |\mathbf{H}_t(u)|^2 du$ (resp. $\|\mathbf{V}\|_{L^2(\Lambda)} = \int_\Lambda |\mathbf{V}(u)|^2 du$). Since $\sigma(\cdot)$ is bounded by $1/2$, by Schwartz inequality, the right hand side of the last equality is bounded above by

$$\sqrt{|t-s|}\Big(\big\|\mathbf{V}\big\|_{L^2(\Lambda)}+\big\|\nabla G\big\|_{L^2(\Lambda)}\Big)\Big\{\|\nabla\rho^n\|_{L^2(\Lambda_T)}+\frac{\beta}{2}\|\nabla(J^{\text{neum}}\star\rho^n)\|_{L^2(\Lambda_T)}\Big\}$$
$$+\sqrt{|t-s|}\frac{1}{2}\Big\{\big\|\mathbf{V}\big\|_{L^2(\Lambda)}\|\mathbf{U}^n\|_{L^2(\Lambda_T)}+\big\|\nabla G\big\|_{L^2(\Lambda)}\|\nabla F^n\|_{L^2(\Lambda_T)}\Big\}\,.$$

Recall that for each u, $J^{\text{neum}}(u,v)dv$ is a probability density, by Lemma 3.2 , Jensen inequality and Fubini's Theorem,

$$\|\nabla(J^{\text{neum}}\star\rho^n)\|^2_{L^2(\Lambda_T)}\le\int_0^T dt\int_\Lambda J^{\text{neum}}\star|\nabla\rho^n_t|^2\;=\;\|\nabla\rho^n\|^2_{L^2(\Lambda_T)}\,.$$

Hence, for any $s,t\in[0,T]$,

$$\begin{aligned}&\big|\langle\mathbf{W}^n_t,\mathbf{V}\rangle+\langle\pi^n_t,G\rangle-\langle\mathbf{W}^n_s,\mathbf{V}\rangle-\langle\pi^n_s,G\rangle\big|\\&\qquad\le\sqrt{|t-s|}\,M\Big\{\big\|\mathbf{V}\big\|_{L^2(\Lambda)}+\big\|G\big\|_{L^2(\Lambda)}\Big\}\,,\end{aligned}\tag{6.4}$$

where the constant $M=C(\rho,\mathbf{U},F,\beta)$ is such that

$$\sup_{n\ge1}\Big\{\big(1+\frac{\beta}{2}\big)\|\nabla\rho^n\|_{L^2(\Lambda_T)}+\frac{1}{2}\|\mathbf{U}^n\|_{L^2(\Lambda_T)}+\frac{1}{2}\|\nabla F^n\|_{L^2(\Lambda_T)}\Big\}\le M\,.$$

Analogously, we obtain

$$\begin{aligned}&\sup_{n\ge1}\sup_{0\le t\le T}\big(\|\mathbf{W}^n_t\|+\|\pi^n_t\|\big)\\&\qquad\le1+M'\sqrt{T}\sup_{n\ge1}\Big\{\|\nabla\rho^n\|_{L^2(\Lambda_T)}\;+\;\|\mathbf{U}^n\|_{L^2(\Lambda_T)}\Big\}\;<\;\infty\end{aligned}\tag{6.5}$$

for some positive constant $M'=M'(|\Lambda|,\beta)$, where for each $n\ge1$, $\|\mathbf{W}^n\|$ (resp. $\|\pi^n\|$) stands for the total variation of the signed measure $\mathbf{W}^n$ (resp. of the measure π^n).

The relative compactness for the set $\big\{(\mathbf{W}^n,\pi^n),\ n\ge1\big\}$, follows from (6.4),(6.5) and the compactness criterium for the Skorohod topology (see Ethier and Kurtz (1986) Theorem 6.3 page 123).

Let $\{(\mathbf{W}^{n_k},\pi^{n_k}):\,k\ge1\}$ be a subsequence of $\{(\mathbf{W}^n,\pi^n):\,n\ge1\}$ converging to some $(\mathbf{W}^*,\pi^*)$ in $D([0,T],\mathcal{M}^{d+1})$ and denote by ρ^* the density of π^*. We claim that $(\mathbf{W}^*,\pi^*)=(\mathbf{W},\pi)$ and $\lim_{k\to\infty}\mathcal{J}^\gamma_T\big(\mathbf{W}^{n_k},\pi^{n_k}\big)=\mathcal{J}^\gamma_T\big(\mathbf{W},\pi\big)$. On the one hand, $\{\rho^{n_k}:\,k\ge1\}$ converges weakly to ρ^* in $L^2\big(\Lambda_T\big)$. Since $\mathcal{J}^\gamma_T(\mathbf{W}^n,\pi^n)$ is uniformly bounded, by Proposition 4.4 and Lemma 4.1, ρ^{n_k} converges to ρ^* strongly in $L^2(\Lambda_T)$. For every G in $\mathcal{C}^{1,2}_0(\Lambda_T)$, we have

$$\begin{aligned}\langle\pi^{n_k}_T,G_T\rangle-\langle\gamma,G_0\rangle&=\int_0^T dt\,\langle\pi^{n_k}_t,\partial_tG_t\rangle\\&\quad+\int_0^T dt\,\langle\pi^{n_k}_t,\Delta G_t\rangle\;-\;\int_0^T dt\int_\Gamma b(r)\,\mathbf{n}_1(r)\,(\partial_1F_t)(r)\,dS(r)\\&\quad+\int_0^T\langle\nabla G_t,\sigma(\rho^{n_k}_t)\big[\beta\nabla(J^{\text{neum}}\star\rho^{n_k}_t)+\nabla F^{n_k}_t\big]\rangle\,dt.\end{aligned}$$

Letting $k\to\infty$, we obtain that ρ^* is a weak solution of equation (3.3) with $\mathbf{V}=\nabla F$. Thus, by uniqueness of weak solutions of (3.3), $\pi^*=\pi$. On the other hand, for every $\mathbf{V}$ in $(\mathcal{C}^{1,1}(\Lambda_T))^d$ and any $k \geq 1$, we have

$$\begin{aligned}\langle \mathbf{W}_T^{n_k}, \mathbf{V}_T \rangle \;=\; & \int_0^T dt\, \langle \mathbf{W}_t^{n_k}, \partial_t \mathbf{V}_t \rangle \\ & + \int_0^T dt\, \langle \pi_t^{n_k}, \nabla \cdot \mathbf{V}_t \rangle \;-\; \int_0^T dt \int_\Gamma b(r)\, \mathbf{n}_1(r)\, V_1(t,r)\, dS(r) \\ & + \beta \int_0^T \langle \sigma(\rho_t^{n_k}), \mathbf{V}_t \cdot \big[\nabla (J^{\text{neum}} \star \rho_t^{n_k}) + \mathbf{U}^{n_k}\big] \rangle dt\,.\end{aligned}$$

Since $\{\mathbf{W}^{n_k} : k \geq 1\}$ converges weakly to $\mathbf{W}^*$ in $L^2\big(\Lambda_T\big)$ and ρ^{n_k} converges to ρ strongly in $L^2(\Lambda_T)$, letting $k \to \infty$, we obtain that $\mathbf{W}^*$ is a weak solution of the equation (4.3) (associated to ρ and $\mathbf{U}$). This proves the first part of the claim. To conclude the proof it remains to prove that $\lim_{k\to\infty} \mathcal{J}_T^\gamma(\mathbf{W}^{n_k}, \pi^{n_k}) = \mathcal{J}_T^\gamma(\mathbf{W}, \pi)$. The sequence $(\rho^{n_k})_{k>0}$ converges to ρ strongly in $L^2(\Lambda_T)$ and the sequence $(\mathbf{U}^{n_k})_{k>0}$ converges to $\mathbf{U}$ in $L^2(\Lambda_T)$. Taking into account that ρ is bounded and σ is Lipschitz, we obtain

$$\begin{aligned}\lim_{k\to\infty} \mathcal{J}_T^\gamma(\mathbf{W}^{n_k}, \pi^{n_k}) &= \lim_{k\to\infty} \frac{1}{2} \int_0^T dt\, \langle \sigma(\rho_t^{n_k}), \mathbf{U}_t^{n_k} \cdot \mathbf{U}_t^{n_k} \rangle \\ &= \frac{1}{2} \int_0^T dt\, \langle \sigma(\rho_t), \mathbf{U}_t \cdot \mathbf{U}_t \rangle \;=\; \mathcal{J}_T^\gamma(\mathbf{W}, \pi)\,.\end{aligned}$$

This concludes the proof. □

7. large deviations for the empirical density

In this section we prove Theorem 2.6. As we mentioned in the introduction, the large deviations principle for the empirical density can be recovered from the one for the current. Indeed, it follows from Theorem 2.5 and the contraction principle, that the rate function $\widetilde{\mathcal{I}}_T^\gamma$ for the empirical density is given by the variational formula

$$\widetilde{\mathcal{I}}_T^\gamma(\pi) \;=\; \inf_{\mathbf{W}\,:(\mathbf{W},\pi)\in\mathfrak{A}_\gamma} \mathcal{J}_T^\gamma(\mathbf{W}, \pi)\;, \tag{7.1}$$

where $\mathfrak{A}_\gamma$ is defined by (2.11). To conclude the proof of Theorem 2.6, we then need to show that the functional $\mathcal{I}_T^\gamma$ in (2.18) coincides with the functional $\widetilde{\mathcal{I}}_T^\gamma$ on the whole space $D([0,T],\mathcal{M})$.

Fix $\pi \in D([0,T],\mathcal{M})$. From Lemma 4.1, we have

$$\mathcal{I}_T^\gamma(\pi) \;\leq\; \widetilde{\mathcal{I}}_T^\gamma(\pi)\,. \tag{7.2}$$

Conversely, suppose that $\mathcal{I}_T^\gamma(\pi) < \infty$, then by Lemma 4.5, there exists $F \in H_0^1(\sigma(\pi))$ such that $\pi(t,du) = \rho(t,u)du$ and ρ solves the equation (3.3) with $\mathbf{V} = \nabla F$. Let $\mathbf{W}^F$ the weak solution of the equation (4.3) with $\mathbf{U} = \nabla F$, it is easy to check that $(\mathbf{W}^F, \pi) \in \mathfrak{A}_\gamma$ and

$$\widetilde{\mathcal{I}}_T^\gamma(\pi) \;\leq\; \mathcal{J}_T^\gamma(\mathbf{W}^F, \pi) \;=\; \mathcal{I}_T^\gamma(\pi)\,. \tag{7.3}$$

We deduce from (7.2) and (7.3), that for each $\pi \in D([0,T],\mathcal{M})$, $\mathcal{I}_T^\gamma(\pi) < +\infty$ if and only if $\widetilde{\mathcal{I}}_T^\gamma(\pi) < +\infty$ and then $\widetilde{\mathcal{I}}_T^\gamma(\pi) = \mathcal{I}_T^\gamma(\pi)$ which concludes the proof of (7.1). □

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