Convergence of clock processes on infinite graphs and aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$

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Abstract. Using a method developed by Durrett and Resnick (1978) we establish general criteria for the convergence of properly rescaled clock processes of random dynamics in random environments on infinite graphs. This complements the results of Gayrard (2012), Bovier and Gayrard (2013), and Bovier et al. (2013): put together these results provide a unified framework for proving convergence of clock processes. As a first application we prove that Bouchaud's asymmetric trap model on $\mathbb{Z}^d$ exhibits a normal aging behavior for all $d \geq 2$. Namely, we show that certain two-time correlation functions, among which the classical probability to find the process at the same site at two time points, converge, as the age of the process diverges, to the distribution function of the arcsine law. As a byproduct we prove that the fractional kinetics process ages.

1. Introduction and main results

This introduction is made of three parts. In the first we describe the general setting and formulate the problems of interest. We state our abstract results in Section 1.2. Section 1.3 contains the application to Bouchaud's asymmetric trap model.

1.1. *Markov jump processes in random environments and clock processes.* Let $G = (\mathcal{V}, \mathcal{L})$ be a loop-free graph. The random environment is a collection of random variables, $\{\tau(x), x \in \mathcal{V}\}$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that are

only assumed to be positive. On $\mathcal{V}$ we consider a continuous time Markov jump process, X, with initial distribution μ, whose jump rates $(\lambda(x,y))_{x,y\in\mathcal{V}}$ satisfy

$$\tau(x)\lambda(x,y) = \tau(y)\lambda(y,x), \quad \forall (x,y) \in \mathcal{L}, \ x \neq y. \tag{1.1}$$

This implies that X is reversible with respect to the random measure on $\mathcal{V}$ that assigns to $x \in \mathcal{V}$ the mass $\tau(x)$.

Clock processes of X have recently been at the center of attention in connection with the study of aging and/or anomalous diffusions. Relevant questions on both topics can be formulated by writing X as a time change of another Markov process J,

$$X(t) = J(S^{\leftarrow}(t)), \qquad t \geq 0, \tag{1.2}$$

and making judicious choices of S, the so-called *clock process*. Here $S^{\leftarrow}$ denotes the generalized right continuous inverse of S. When studying aging the focus usually is on the total time elapsed along trajectories of X of a given length. This is given by the *discrete time clock process*

$$S(k) \equiv \sum_{i=0}^{k-1} \lambda^{-1}(J(i))e_i, \qquad k \geq 1, \tag{1.3}$$

where J is the discrete time chain with transition probabilities

$$p(x,y) \equiv \lambda(x,y)/\lambda(x) \quad \text{if } (x,y) \in \mathcal{L}, \tag{1.4}$$

and zero else,

$$\lambda(x) \equiv \sum_{y:(x,y)\in\mathcal{L}} \lambda(x,y), \quad x \in \mathcal{V}, \tag{1.5}$$

is the inverse of the mean holding time of X at x, and $\{e_i, \ i = 0,1,2,\ldots\}$ is an independent collection of i.i.d. mean one exponential random variables. Knowledge of the large k behavior of S combined with relation (1.2) then allows to deduce information on the long time behavior of the two-time correlation functions that are used to quantify aging in theoretical physics. When interested in scaling limits one looks at (1.2) from a different angle. One aims at expressing the process X as a time change of another continuous time process, J, for which the usual functional limit theorem holds. One is then naturally led to study the *continuous time clock process*

$$S(t) \equiv \int_0^t \lambda^{-1}(J(s))\widetilde{\lambda}(J(s))ds, \qquad t \geq 0, \tag{1.6}$$

where $\widetilde{\lambda}(x)$ denotes the inverse of the mean holding time of J at x.

It emerged from the bulk of works carried out in the past decade that the occurrence of stable subordinators as the limit of properly rescaled clock processes provides a basic mechanism for both aging and anomalous diffusive behaviors to set in in two main types of models. The first are phenomenological models – the so-called trap models of Bouchaud *et al.* Bouchaud (1992); Bouchaud and Dean (1995); Rinn et al. (2000, 2001). Introduced in theoretical physics to account for the phenomenon of aging then newly discovered in the physics of spin glasses, these are simple Markov jump processes that describe the dynamics of spin glasses on long time scales in terms of activated barrier crossing in landscapes made of random 'traps'. Another class of models stems from looking at the actual dynamics of microscopic spin glasses. Interesting such dynamics are Glauber dynamics on state spaces $\mathcal{V}_n = \{-1,1\}^n$ reversible with respect to the Gibbs measures associated to

random Hamiltonians of mean-field spin glasses, such as the REM and p-spin SK models.

The first connection between microscopic dynamics of spin systems and trap models was made in Ben Arous et al. (2003a,b, 2002) for a variant of the Glauber dynamics of the REM (the random hopping dynamics, hereafter RHD) on time scales close to equilibrium, and extended in Ben Arous and Černý (2008) to shorter time scales (but still exponential in n). There it is shown that the properly rescaled discrete time clock process (1.3) converges $\mathbb{P}$-a.s. to a stable subordinator. These results were partially extended to the p-spin SK models in Ben Arous et al. (2008), for all $p \geq 3$ and in a range of exponentially long time scales, whereas it was shown in Ben Arous and Gün (2012) that on sub-exponential times scales the clock process no longer converges to a stable subordinator but to an extremal process, and this for all $p \geq 2$; both these results were obtained in $\mathbb{P}$-law only.

The field gained new momentum with the paper Gayrard (2012). Based on a method developed by Durrett and Resnick (1978) in the late 70's to prove functional limit theorems for dependent random variables, a fresh view on the convergence of clock processes in random environment was proposed and general criteria for convergence of clock processes to subordinators were given. This allowed to improve all earlier results on aging of the RHD of the REM Gayrard (2010) and p-spin SK models Bovier and Gayrard (2013); Bovier et al. (2013), yielding $\mathbb{P}$-a.s. results for all $p > 4$ (in $\mathbb{P}$-probability else), and paved the way for new advances Gayrard (2015). In all the papers mentioned above clock processes are used to control suitable time-time correlation functions, and aging is deduced.

Meanwhile, in a different line of research, an important class of trap models on $\mathbb{Z}^d$ known as Bouchaud's asymmetric trap model (hereafter BATM) Rinn et al. (2000, 2001) was fully investigated both from the view point of aging and scaling limits, in different dimensions and for different values of the asymmetry parameter $\theta \in [0,1]$ (see Section 1.3 for the definition of BATM). In what follows we call BTM the 'symmetric' version of the model, obtained by setting $\theta = 0$. Aging was first proved in the seminal paper Fontes et al. (2002) for BTM on $\mathbb{Z}$, and extended to BATM on $\mathbb{Z}$ in Ben Arous and Černý (2005). Emphasis was first given to the discrete clock process of BTM in Ben Arous and Černý (2006), for $d = 2$, and later in Ben Arous and Černý (2007), for $d \geq 2$. In both these papers it is proved that for suitable scalings, the clock process converges to a stable subordinator. This is used in Ben Arous and Černý (2006) to study aging via correlation functions, and in Ben Arous and Černý (2007) to prove convergence of the properly normalized BTM to the so-called Fractional-Kinetics process (see (1.48)). More recently, Fontes and Mathieu (2014) established aging for transient variants of BTM on $\mathbb{Z}^d$ for all $d \geq 1$. The continuous time clock process (1.6) came into play later, in the study of BATM on $\mathbb{Z}^d$, $d \geq 2$, Barlow and Černý (2011a); Barlow and Zheng (2010); Černý (2011); Mourrat (2011). There, J is chosen as the so-called variable speed random walk (hereafter VSRW), that is to say, the continuous time Markov chain with rates $\widetilde{\lambda}(x, y) = \tau(x)\lambda(x, y)$. This is a central object in the literature on random conductance models and its scaling limit is well-understood (for the most recent and strongest results see Barlow and Deuschel (2010) and Andres et al. (2013)). Convergence of the rescaled clock process to a stable subordinator is established in Barlow and Černý (2011a); Černý (2011); Mourrat (2011) under various assumptions on d and using various techniques (see Section 1.3 for a detailed

discussion). Consequences for the scaling limit of BATM are drawn but not, to our knowledge, for correlation functions.

The question naturally arises as to whether the method put forward in Gayrard (2012) could allow to make progress on this issue. How to implement it however is not straightforward. The formulation of the general, abstract criteria for convergence of clock processes of Gayrard (2012) and Bovier and Gayrard (2013) was geared to the setting of sequences of finite graphs suited for dealing with mean field spin glasses. Furthermore, in all applications, explicit use is made of the fact that the discrete time chain J in (1.3) admits an invariant probability measure and is, moreover, sufficiently fast mixing. In contrast, the arena of BATM on $\mathbb{Z}^d$ is that of dynamics on infinite graphs that do not admit of an invariant probability measure.

In the present paper we address this question in the general setting of Markov jump processes on infinite graphs that satisfy (1.1). We formulate abstract sufficient conditions for properly rescaled clock processes of the form (1.2) (both continuous or discrete) to converge to stable subordinators. (It will be seen that the rôle of the invariant measure is now played by a certain 'mean empirical measure'.) We then apply this result to BATM for all $d \geq 2$. This, in turn, enables us to control several (classical or natural) correlation functions through which the aging behavior of the process can be characterized, and prove the existence of 'normal aging'.

1.2. *Main results.* In this paper, we consider continuous and discrete time clock processes in a unified setting and introduce notations that allow to handle them simultaneously. From now on let J be either a continuous or discrete time Markov chain having transition probabilities (1.4) and initial distribution μ. Continuous time chains are assumed to be non-explosive (see Chapter 3.5 in Norris (1998)). To a Markov chain J we associate a process $\ell = \{\ell_t(x), x \in \mathcal{V}, t \geq 0\}$ and a sequence $\widetilde{\Lambda} = \{\widetilde{\lambda}(x), x \in \mathcal{V}\}$ defined as follows. When J is a continuous time Markov chain $\widetilde{\lambda}(x)$ is the holding time parameter of J at x and $\ell_t(x)$ is the local time

$$\ell_t(x) \equiv \int_0^t \mathbb{1}_{J(s)=x} \, ds, \tag{1.7}$$

namely, the total time spent by J at x in the time interval $[0, t]$. When J is a discrete time Markov chain we set $\widetilde{\lambda}(x) \equiv 1$. In this case $\ell_t(x)$ is defined through

$$\ell_t(x) \equiv \sum_{i=0}^{\lfloor t \rfloor} e_i \mathbb{1}_{J(i)=x}, \tag{1.8}$$

where $\{e_i, i = 0, 1, 2, \ldots\}$ is a collection of i.i.d. mean one exponential random variables independent of everything else. Observe that this is the local time of a continuous time Markov chain whose mean holding times are identically one. The clock process is then given by

$$S^J(t) \equiv \sum_{x \in \mathcal{V}} \ell_t(x)\widetilde{\lambda}(x)\lambda^{-1}(x), \quad t \geq 0. \tag{1.9}$$

Notice that this definition is consistent with (1.3) and (1.6). In particular, one can check that the relation (1.2) between S^J, J, and X is satisfied. In the sequel we write P_μ for the law of J and $\mathcal{P}_\mu$ for the law of X with initial distribution μ. We also write $P_x \equiv P_{\delta_x}$ and $\mathcal{P}_x \equiv \mathcal{P}_{\delta_x}$. We denote expectations with respect to $\mathcal{P}_\mu$ and P_μ respectively by $\mathcal{E}_\mu$ and E_μ. Of course these are random measures on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let a_n and c_n be non-decreasing sequences. We think of c_n as the time scale on which the process X is observed, and of a_n as an auxiliary time scale for the

Markov chain J. The question of interest now becomes to find conditions for the re-scaled sequence

$$S_n^J(t) \equiv c_n^{-1} \sum_{x \in \mathcal{V}} \widetilde{\lambda}(x) \lambda^{-1}(x) \ell_{\lfloor a_n t \rfloor}(x), \quad t \geq 0, \tag{1.10}$$

to converge weakly, as a sequence of random elements in the space $D[0, \infty)$ of càdlàg functions on $[0, \infty)$, $\mathbb{P}$-almost surely in the random environment.

To answer this question we use a method developed by Durrett and Resnick Durrett and Resnick (1978) that yields criteria for sums of correlated random variables to converge that are particularly useful when applied to clock processes. Following Bovier and Gayrard (2013), we will not apply it to S_n^J directly but rather to a 'blocked' version of S_n^J. Namely, we introduce a new sequence, θ_n, chosen such that $\theta_n \ll a_n$, and use it to define the block variables

$$Z_{n,k}^J \equiv c_n^{-1} \sum_{x \in \mathcal{V}} \widetilde{\lambda}(x) \lambda^{-1}(x) \left(\ell_{\theta_n k}(x) - \ell_{\theta_n (k-1)}(x) \right), \quad k \geq 1. \tag{1.11}$$

The reason for this is that in many examples of interest, we do know that the jumps of the limiting clock process do not come from isolated jumps of S_n^J but from block variables, either because of strong local spatial correlations of the random environment (as in spin glasses) or because of strong local temporal correlations of the process J (as e.g. in BTM on $\mathbb{Z}^2$) or by a conjunction of these reasons. Set $k_n(t) \equiv \lfloor \lfloor a_n t \rfloor / \theta_n \rfloor$. The blocked clock process $S_n^{J,b}$ is then defined by

$$S_n^{J,b}(t) \equiv \sum_{k=0}^{k_n(t)-1} Z_{n,k+1}^J + Z_{n,0}^J, \quad t \geq 0. \tag{1.12}$$

where the sum over k is zero whenever $k_n(t) - 1 < 0$, and

$$Z_{n,0}^J \equiv \sum_{x \in \mathcal{V}} \widetilde{\lambda}(x) \lambda^{-1}(x) \ell_0(x)$$

(note that this is non-zero for discrete time chains J only).

The convergence criteria we obtain bear on a small number of quantities that we now introduce. For $x \in \mathcal{V}$ and $u > 0$ let

$$Q_n^u(x) \equiv \mathcal{P}_x \left(Z_{n,1}^J > u \right) \tag{1.13}$$

be the tail distribution of $Z_{n,1}^J$, starting in x. For each fixed $t > 0$, we construct a probability measure on $\mathcal{V}$ through

$$\pi_n^t(x) \equiv E_\mu \left((k_n(t))^{-1} \sum_{k=1}^{k_n(t)-1} \mathbb{1}_{J(k\theta_n)=x} \right), \quad x \in \mathcal{V}. \tag{1.14}$$

This is the empirical measure induced by the sequence $\{J(k\theta_n), k = 1, \ldots k_n(t)-1\}$, averaged over P_μ. Note that Q_n^u and π_n^t are not random in the chain J. Using these quantities, we define

$$\nu_n^t(u, \infty) \equiv k_n(t) \sum_{x \in \mathcal{V}} \pi_n^t(x) Q_n^u(x), \tag{1.15}$$

and

$$\sigma_n^t(u, \infty) \equiv k_n(t) \sum_{x \in \mathcal{V}} \pi_n^t(x) (Q_n^u(x))^2. \tag{1.16}$$

We are now ready to introduce the conditions of our main theorem. They are stated for given sequences a_n, c_n, θ_n, a given initial distribution μ, and fixed $\omega \in \Omega$.

(A-0) For all $u > 0$

$$\lim_{n \to \infty} \mathcal{P}_\mu \left(Z_{n,1}^J + Z_{n,0}^J > u \right) = 0. \tag{1.17}$$

(A-1) For all $t > 0$ there exists $c < \infty$ such that, uniformly in $x \in \mathcal{V}$,

$$\lim_{n \to \infty} \sum_{k=1}^{k_n(t)-1} P_\mu(J(k\theta_n) = x) = 0, \tag{1.18}$$

and

$$\lim_{n \to \infty} \sum_{k=1}^{k_n(t)-1} P_x(J(k\theta_n) = x) < c. \tag{1.19}$$

(A-2) There exists a sigma-finite measure ν on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x) d\nu(x) < \infty$ such that for all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$,

$$\lim_{n \to \infty} \nu_n^t(u, \infty) = t\nu(u, \infty). \tag{1.20}$$

(A-3) For all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$,

$$\lim_{n \to \infty} \sigma_n^t(u, \infty) = 0. \tag{1.21}$$

(A-4) For all $t > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} k_n(t) \sum_{x \in \mathcal{V}} \pi_n^t(x) \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \leq \varepsilon\}} = 0. \tag{1.22}$$

Theorem 1.1. *Assume that there exist sequences a_n, c_n, and θ_n and an initial distribution μ such that Conditions (A-0)-(A-4) are satisfied $\mathbb{P}$-a.s. Then, $\mathbb{P}$-a.s., as $n \to \infty$,*

$$S_n^{J,b} \Rightarrow V_\nu, \tag{1.23}$$

where V_ν is a subordinator with Lévy measure ν and zero drift. Convergence holds weakly in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology.

Let us emphasize that our statement is made for $S_n^{J,b}$ and holds in the strong J_1 topology, which immediately implies that S_n^J converges to the same limit in the weaker M_1 topology. As we just explained (see discussion below (1.11)), in many models of interest it is not true that S_n^J converges in the J_1 topology, but more information than contained in M_1 statements can be obtained by introducing a blocked clock process, $S_n^{J,b}$. (This is the case in the p-spin SK models Ben Arous et al. (2008), Bovier and Gayrard (2013), and BTM on $\mathbb{Z}^2$ Ben Arous and Černý (2006).) Of course in the case of continuous time clock processes, forming blocks is needed in order to make sense of writing convergence to subordinators statements in the J_1 topology. Let us finally stress that it is crucial for applications to correlation functions to make statements that are valid in the J_1 topology (see the discussion below (1.43)).

Let us comment on Conditions (A-0)-(A-4). Condition (A-0) is a condition on the initial distribution and ensures that the initial increment $S_n^{J,b}(0)$ converges to zero as $n \to \infty$. Conditions (A-2)-(A-4) have the same form as Conditions (A2-1)-(A3-1) in Bovier and Gayrard (2013) where sequences of finite state reversible Markov jump processes are studied. There it is assumed that J_n admits a unique invariant measure, π_n, and θ_n is chosen large compared to the mixing time of J_n

(see Condition (A1-1)). In the present setting, the empirical measure averaged over P_μ replaces the measure π_n, and Condition (A-1) plays the same rôle as Condition (A1-1). More precisely these conditions allow to replace J dependent, respectively J_n dependent, quantities by their average over P_μ, respectively P_{π_n}. We conclude this discussion with a lemma that sheds light on the complementarity of Theorem 1.1 and Theorem 1.3 in Bovier and Gayrard (2013); indeed the former can only be satisfied by J's that are transient or null-recurrent whereas the latter is designed for positive recurrent J's.

Lemma 1.2. *Let $x \in \mathcal{V}$. If x is transient then* (1.18) *and* (1.19) *are satisfied for any $\theta_n \gg 1$, whereas if x is positive recurrent they cannot be satisfied. In particular, (A-1) cannot hold if J admits an invariant probability measure.*

When J is random in the random environment the conditions of Theorem 1.1 may not be easy to handle. We now present an additional condition, (B-5), that enables us to replace π_n^t in (A-2)-(A-4) by a deterministic probability measure $\overline{\pi}_n^t$. In this way, all the dependence on the random environment in (A-2)-(A-4) is confined to the Q_n^u's. The following conditions, stated for given sequences a_n, c_n, θ_n, a given initial distribution μ, and for fixed $\omega \in \Omega$, imply the conditions of Theorem 1.1.

(B-5) Set $\mathcal{A}_n = \{(x,k): \ x \in \mathcal{V}, \ k \in [k_n(t) - 1]\}$, where $[m] \equiv \{0, \dots, m\}$. There exists a sequence of functions $h_n : \mathcal{V} \to [0,1]$ such that for all $t > 0$ and all $n \in \mathbb{N}$, the set $\mathcal{A}_n$ can be decomposed into the disjoint union of two sets, $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$, satisfying

$$\lim_{n\to\infty} \sup_{(x,k)\in\mathcal{A}_n^1} \frac{|P_\mu(J(k\theta_n) = x)) - h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} = 0, \qquad (1.24)$$

and

$$\lim_{n\to\infty} \sum_{(x,k)\in\mathcal{A}_n^2} \left|P_\mu(J(k\theta_n) = x) - h_{k\theta_n}(x)\right| Q_n^u(x) = 0, \qquad (1.25)$$

$$\lim_{n\to\infty} \sum_{(x,k)\in\mathcal{A}_n^2} \left|P_\mu(J(k\theta_n) = x) - h_{k\theta_n}(x)\right| \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \le \varepsilon\}} = 0. \qquad (1.26)$$

Observe that proving (1.24) corresponds to proving a uniform local central limit theorem for J.

For each $t > 0$ we define the measure $\overline{\pi}_n^t$, using h_n, through

$$\overline{\pi}_n^t(x) = (k_n(t))^{-1} \sum_{k=1}^{k_n(t)-1} h_{k\theta_n}(x), \quad x \in \mathcal{V}. \qquad (1.27)$$

By analogy with (1.15) and (1.16) we set for $t > 0$, $u > 0$

$$\overline{\nu}_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \overline{\pi}_n^t(x) Q_n^u(x), \qquad (1.28)$$

$$\overline{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \overline{\pi}_n^t(x) (Q_n^u(x))^2. \qquad (1.29)$$

The next conditions are nothing but Conditions (A-2)-(A-4) with π_n^t replaced by $\overline{\pi}_n^t$.

(B-2) There exists a sigma-finite measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge x) d\nu(x) < \infty$ such that for all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$,

$$\lim_{n\to\infty} \overline{\nu}_n^t(u,\infty) = t\nu(u,\infty). \qquad (1.30)$$

(B-3) For all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$,

$$\lim_{n \to \infty} \overline{\sigma}_n^t(u, \infty) = 0. \tag{1.31}$$

(B-4) For all $t > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} k_n(t) \sum_{x \in \mathcal{V}} \overline{\pi}_n^t(x) \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \leq \varepsilon\}} = 0. \tag{1.32}$$

Theorem 1.3. *Assume that there exist sequences $a_n, c_n,$ and θ_n and an initial distribution μ such that Conditions (A-0), (A-1), (B-2)-(B-5) are satisfied $\mathbb{P}$-a.s. Then, $\mathbb{P}$-a.s.,*

$$S_n^{J,b} \Rightarrow V_\nu, \tag{1.33}$$

where V_ν is a subordinator with Lévy measure ν and zero drift. Convergence holds weakly in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology.

The following lemma is instrumental in verifying the conditions of Theorem 1.1 or Theorem 1.3.

Lemma 1.4. *Let ν be a sigma finite measure on $(0, \infty)$. Suppose that for given a_n, c_n, θ_n, μ and fixed $t > 0$, $u > 0$ there exists $\Omega^\tau(u, t) \subseteq \Omega$ with $\mathbb{P}(\Omega^\tau(u, t)) = 1$ such that, on $\Omega^\tau(u, t)$, (A-0)-(A-4), respectively (B-2)-(B-5), are verified. Then, for these sequences and this initial distribution (A-0)-(A-4), respectively (B-2)-(B-5), are satisfied $\mathbb{P}$-a.s. for all $t > 0$, $u > 0$.*

1.3. *Application to Bouchaud's asymmetric trap model (BATM).* We now use Theorem 1.3 to prove aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$. Here $\mathcal{V} = \mathbb{Z}^d$, $d \geq 2$, $\mathcal{L}$ is the set of nearest neighbors on $\mathbb{Z}^d$, and $\mu \equiv \delta_0$. The random environment, $\{\tau(x), x \in \mathbb{Z}^d\}$, is a collection of i.i.d. random variables, with tail distribution given by

$$\mathbb{P}(\tau(0) > u) = \begin{cases} Cu^{-\alpha}(1 + L(u)), & u \in (\bar{c}, \infty), \\ 1, & u \in (0, \bar{c}], \end{cases} \tag{1.34}$$

where $\alpha \in (0, 1)$, $\bar{c}, C \in (0, \infty)$ are constants, and $L : (0, \infty) \to \mathbb{R}$ is a function that obeys $L(u) \to 0$ as $u \to \infty$. We write $x \sim y$ if x, y are nearest neighbors in $\mathbb{Z}^d$. The jump rates of X depend on a parameter, $\theta \in [0, 1]$, and are given by

$$\lambda(x, y) = (\tau(x))^{-1}(\tau(x)\tau(y))^\theta, \quad \text{if } x \sim y, \tag{1.35}$$

and zero else. Consider now the VSRW of this model, namely, the continuous time Markov chain, J, with jump rates

$$\widetilde{\lambda}(x, y) = (\tau(x)\tau(y))^\theta, \quad \text{if } x \sim y, \tag{1.36}$$

and zero else. Our interest is in the continuous time clock process S^J defined (as in (1.9)) through

$$S^J(t) = \sum_{x \in \mathcal{V}} \ell_t(x) \widetilde{\lambda}(x) \lambda^{-1}(x) = \sum_{x \in \mathcal{V}} \ell_t(x) \tau(x). \tag{1.37}$$

Our first theorem states convergence of the blocked clock process, $S_n^{J,b}$, for appropriate choices of block lengths, θ_n, in the J_1 topology.

Theorem 1.5. *Let $c_n = n$ and take*

$$\theta_n \quad = \quad n^{\alpha\gamma_2}\mathbb{1}_{d=2} + (\log n)^{\gamma_3}\mathbb{1}_{d\geq 3}, \tag{1.38}$$

$$a_n \quad = \quad n^{\alpha}(\log n)^{1-\alpha}\mathbb{1}_{d=2} + n^{\alpha}\mathbb{1}_{d\geq 3}, \tag{1.39}$$

where $\gamma_2 \in (0, 1/6), \gamma_3 > 12/(1-\alpha)$. Then, $\mathbb{P}$-a.s., as $n \to \infty$,

$$S_n^{J,b} \Rightarrow V_{\alpha}, \tag{1.40}$$

where V_{α} is a subordinator with Lévy measure $\nu(u, \infty) = \mathcal{K}u^{-\alpha}$, for $\mathcal{K} = \mathcal{K}(d, \alpha, \theta) > 0$, and zero drift. Convergence holds weakly in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology.

All earlier papers dealing with the clock process S_n^J focused on proving scaling limits for BATM. It was first proved in Barlow and Černý (2011a) that the properly rescaled process converges to a fractional kinetics process for $d \geq 3$. This was extended to $d = 2$ in Černý (2011). Shortly after Barlow and Černý (2011a), Mourrat (2011) gave an alternative proof of this result for $d \geq 5$. The method of Barlow and Černý (2011a); Černý (2011) relies on blocking with block length $\theta_n = \varepsilon n^{\alpha}$. In contrast, Mourrat (2011) proposed a method of proof that does not use blocking. Both approaches resulted in M_1 convergence for S_n^J. (Note that because S^J is a continuous time clock process, the method of Mourrat (2011) does not allow to obtain J_1 convergence statements for S^J.)

Let us comment on our choices of θ_n. Because J is recurrent when $d = 2$ and transient otherwise two cases must be distinguished. When $d = 2$ we first remark that (A-1) would be satisfied for any $\theta_n \gg \log n$. There, our constraint on θ_n comes from (A-2)-(A-4). In the course of verifying these conditions one sees that θ_n must be chosen in such a way that the mean values of local times in the time interval $[0, \theta_n]$ are of the order of $\log n$. Since these mean values are of order $\log \theta_n$ we take $\theta_n = n^{\alpha\gamma_2}$. When $d \geq 3$ Conditions (A-1)-(A-4) can a priori be verified for any diverging θ_n. Here, the constraint (1.38) on θ_n comes from using precise heat kernel estimates for J, taken from Barlow and Deuschel (2010), which are only valid for large enough time intervals (of course this was already the case in $d = 2$).

We now present our results on aging. Theorem 1.5 allows to control several correlation functions, which we now introduce. The first is the classical correlation function

$$\mathcal{C}_s^1(1, \rho) \equiv \mathcal{P}\left(X(s) = X(s(1+\rho))\right), \quad s > 0, \ \rho > 0, \tag{1.41}$$

which is the probability that at the beginning and the end of the time interval $(s, s(1+\rho))$ the process is in the same site. The second correlation function is the probability that during a certain time interval the process stays inside a ball of a certain radius. Specifically, writing $\theta_s \equiv \theta_{\lfloor s \rfloor}$,

$$\mathcal{C}_s^2(1, \rho) \equiv \mathcal{P}\left(\max_{v \in (s, s(1+\rho))} |X(s) - X(v)| \leq (\theta_s \log \theta_s)^{1/2}\right), \ s > 0, \ \rho > 0. \tag{1.42}$$

Notice that $\mathcal{C}_s^1$ and $\mathcal{C}_s^2$ clearly contain different information. Our third and last correlation function combines them both. For $s > 0, \rho > 0$ we define

$$\mathcal{C}_s^3(1, \rho) \equiv \mathcal{P}\left(X(s) = X(s(1+\rho)), \max_{v \in (s, s(1+\rho))} |X(s) - X(v)| \leq (\theta_s \log \theta_s)^{1/2}\right). \tag{1.43}$$

The proof of the next theorem relies on a well-known scheme, that goes back to Ben Arous and Černý (2008), that links aging to the arcsine law for subordinators

through the convergence of the clock process $S_n^{J,b}$. In this scheme, one aims at deducing convergence of correlation functions from convergence of the overshoot function of the blocked clock process, $\chi(S_n^{J,b})$. (The overshoot function of $Y \in D[0,\infty)$ is given by $\chi_u(Y) = Y(\mathcal{L}_u(Y)) - u$, $u > 0$, where $\mathcal{L}_u$ is the first passage time (see (5.2) for a definition).) For this, one needs χ to be continuous with respect to the topology in which $S_n^{J,b}$ converges, and because this is only true in the J_1 topology, it is all important that $S_n^{J,b}$ converges in that topology.

Let Asl_α denote the distribution function of the generalized arcsine law,

$$\mathrm{Asl}_\alpha(u) \equiv \tfrac{\sin\alpha\pi}{\pi} \int_0^u (1-x)^{-\alpha} x^{\alpha-1} dx, \quad u \in [0,1]. \tag{1.44}$$

Theorem 1.6. *Let $d \geq 2$. Under the assumptions of Theorem 1.5, for $i = 1, 2, 3$, $\mathbb{P}$-a.s.,*

$$\lim_{s \to \infty} \mathcal{C}_s^i(1,\rho) = \mathrm{Asl}_\alpha(1/(1+\rho)), \quad \rho > 0. \tag{1.45}$$

As pointed out below Theorem 1.5, it was proved that the rescaled process

$$X_s(t) \equiv a_s^{-1/2} X(st), \quad t \geq 0, \tag{1.46}$$

converges to the fractional kinetics process. Observe that the radius of the balls in (1.42) for which Theorem 1.6 holds is very small compared to the normalization of X_s, namely, $(\theta_s \log \theta_s)^{1/2} \ll a_s^{1/2}$. From this and Theorem 1.6 one readily deduces that the correlation function defined, for $\varepsilon > 0$, by

$$\mathcal{C}_s^\varepsilon(1,\rho) \equiv \mathcal{P}\left(\max_{v\in(1,1+\rho)} |X_s(1) - X_s(v)| \leq \varepsilon\right), \quad s > 0, \ \rho > 0, \tag{1.47}$$

converges to the arcsine distribution function. Interestingly, this, in turn, enables us to deduce results on the aging behavior of the fractional kinetics process itself. This is the content of Theorem 1.7 below. Recall that the fractional kinetics process is defined by

$$Z_{d,\alpha}(t) \equiv B_d(V_\alpha^\leftarrow(t)), \quad t \geq 0, \tag{1.48}$$

where B_d is a standard Brownian motion on $\mathbb{R}^d$ started in 0, V_α is an α-stable subordinator with zero drift that is independent of B_d, and $V_\alpha^\leftarrow(t) = \inf\{v : V_\alpha(v) > t\}$ its generalized right-continuous inverse. By analogy with (1.47) define

$$\mathcal{C}^\varepsilon(1,\rho) \equiv \mathcal{P}\left(\max_{v\in(1,1+\rho)} |Z_{d,\alpha}(1) - Z_{d,\alpha}(v)| \leq \varepsilon\right), \quad \rho > 0. \tag{1.49}$$

Theorem 1.7. *Let $d \geq 2$. Under the assumptions of Theorem 1.5, $\mathbb{P}$-a.s.,*

$$\lim_{\varepsilon \to 0} \lim_{s \to \infty} \mathcal{C}_s^\varepsilon(1,\rho) = \lim_{\varepsilon \to 0} \mathcal{C}^\varepsilon(1,\rho) = \mathrm{Asl}_\alpha(1/(1+\rho)), \quad \rho > 0. \tag{1.50}$$

Remark 1.8. As a final remark notice that our results are only valid for $d \geq 2$. It is known that the situation in $d = 1$ is completely different, see Fontes et al. (2002), Ben Arous and Černý (2005). The clock process converges to the integral of the local time of a Brownian motion on $\mathbb{R}$ with respect to the so-called random speed measure – a scaling limit of the random environment – and the scaling limit of X is a singular diffusion on $\mathbb{R}$; see e.g. Ben Arous and Černý (2007) and Černý (2011) for further discussions.

The remainder of the paper is structured as follows. Section 2 contains the proof of Theorem 1.1 and Theorem 1.3. In Section 3 we collect preparatory results for the proof of Theorem 1.5. The latter is carried out in Section 4. Finally, Section 5 contains the proofs of Theorem 1.6 and Theorem 1.7. Two lemmata are proven in the Appendix.

Acknowledgement. We thank Pierre Mathieu for pointing out that the proof of (4.24) in an earlier version was incomplete. We are grateful to an anonymous referee for identifying an error in our use of Theorem 3.1.

2. Proof of Theorem 1.1 and Theorem 1.3

We now come to the proofs of the abstract theorems of Section 1. We first prove Theorem 1.1. We then show that the conditions of Theorem 1.3 imply those of Theorem 1.1, thereby proving Theorem 1.3. Finally, we prove the lemmata of Section 1.

Proof of Theorem 1.1: As mentioned earlier, the proof is based on a result by Durrett and Resnick (1978) that gives conditions for partial sum processes of dependent random variables to converge. We use this result in a specialized form suitable for our application that we take from Gayrard (2012), namely Theorem 2.1 p. 7.

Throughout we fix a realization $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^{J,b}(t) \equiv S_n^{J,b}(t) - \left(Z_{n,1}^J + Z_{n,0}^J\right), \quad t > 0. \tag{2.1}$$

Condition (A-0) ensures that $\widehat{S}_n^{J,b} - S_n^{J,b}$ converges to zero, uniformly. Thus, we must show that under Conditions (A-1)-(A-4)

$$\widehat{S}_n^{J,b} \Rightarrow V_\nu. \tag{2.2}$$

This will follow if we can verify Conditions (D1)-(D3) of Theorem 2.1 in Gayrard (2012) for $\widehat{S}_n^{J,b}$.

For this, let $\{\mathcal{F}_{n,k}, k \geq 0\}$ be an array of sigma algebras, where for $k \geq 0$, $\mathcal{F}_{n,k}$ is generated by $\{\ell_s(x), \ s \leq \theta_n k, \ x \in \mathbb{Z}^d\}$. When J is continuous $\mathcal{F}_{n,k}$ is generated by $\{J(s), \ s \leq \theta_n k\}$, whereas when J is discrete $\mathcal{F}_{n,k}$ is generated by $\{J(i), e_i, \ i \leq \theta_n k\}$. Note that for $n, k \geq 1$, $Z_{n,k}^J$ is $\mathcal{F}_{n,k}$ measurable and $\mathcal{F}_{n,k-1} \subset \mathcal{F}_{n,k}$.

We first establish that Condition (D1) is satisfied. For $t > 0$ and $u > 0$ we define

$$\nu_n^{J,t}(u,\infty) \equiv \sum_{k=1}^{k_n(t)-1} \mathcal{P}_\mu\left(Z_{n,k+1}^J > u \mid \mathcal{F}_{n,k}\right). \tag{2.3}$$

This conditions then states that for all $u > 0$ such that $\nu(\{u\}) = 0$ and all $t > 0$ we have in $\mathcal{P}_\mu$-probability

$$\lim_{n \to \infty} \nu_n^{J,t}(u,\infty) = t\nu(u,\infty). \tag{2.4}$$

By the Markov property, $\nu_n^{J,t}(u,\infty)$ can be rewritten as

$$\nu_n^{J,t}(u,\infty) = \sum_{k=1}^{k_n(t)-1} \sum_{x \in \mathcal{V}} \mathbb{1}_{J(k\theta_n)=x} Q_n^u(x) = k_n(t) \sum_{x \in \mathcal{V}} \pi_n^{J,t}(x) Q_n^u(x), \tag{2.5}$$

where, for $x \in \mathcal{V}$,

$$\pi_n^{J,t}(x) \equiv (k_n(t))^{-1} \sum_{k=1}^{k_n(t)-1} \mathbb{1}_{J(k\theta_n)=x}, \tag{2.6}$$

denotes the empirical measure induced by the sequence $\{J(k\theta_n), k = 1, \ldots, k_n(t) - 1\}$. Taking the expectation with respect to $\mathcal{P}_\mu$, (2.5) yields

$$\mathcal{E}_\mu \nu_n^{J,t}(u,\infty) = k_n(t) \sum_{x \in \mathcal{V}} \mathcal{E}_\mu\left(\pi_n^{J,t}(x)\right) Q_n^u(x) = \nu_n^t(u,\infty). \tag{2.7}$$

Since (A-2) ensures that $\lim_{n \to \infty} \nu_n^t(u,\infty) = t\nu(u,\infty)$ it suffices to prove that

$$\lim_{n \to \infty} \mathcal{P}_\mu\left(\left|\nu_n^{J,t}(u,\infty) - \nu_n^t(u,\infty)\right| > \varepsilon\right) = 0, \qquad \forall \varepsilon > 0, \tag{2.8}$$

i.e. that we may replace $\pi_n^{J,t}$ by its mean value. We do this by means of a second order Chebyshev inequality. For $x, y \in \mathcal{V}$ and $k, j \in \mathbb{N}$ write

$$\bar{q}_{k,j}(x,y) \equiv P_\mu(J(k) = x, J(j) = y) \quad \text{and} \quad q_k(x,y) \equiv P_x(J(k) = y), \qquad (2.9)$$

with the convention that $q_k(y) \equiv P_\mu(J(k) = y)$. Then, on the one hand,

$$\mathcal{E}_\mu \left(\nu_n^{J,t}(u,\infty) \right)^2$$
$$= \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \left[k_n(t)\pi_n^t(x) + 2 \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} \bar{q}_{k\theta_n, j\theta_n}(x,x) \right] \quad (2.10)$$
$$+ \quad 2 \sum_{\substack{x,x' \in \mathcal{V} \\ x \neq x'}} Q_n^u(x)Q_n^u(x') \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} \bar{q}_{k\theta_n, j\theta_n}(x,x'),$$

and on the other hand,

$$\left(\mathcal{E}_\mu \nu_n^{J,t}(u,\infty) \right)^2$$
$$\geq \quad 2 \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} q_{k\theta_n}(x)q_{j\theta_n}(x) \qquad (2.11)$$
$$+ \quad 2 \sum_{\substack{x,x' \in \mathcal{V} \\ x \neq x'}} Q_n^u(x)Q_n^u(x') \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} q_{k\theta_n}(x)q_{j\theta_n}(x').$$

Combining (2.10) and (2.11), we obtain that

$$\mathcal{E}_\mu \left(\nu_n^{J,t}(u,\infty) \right)^2 - \left(\mathcal{E}_\mu \nu_n^{J,t}(u,\infty) \right)^2$$
$$\leq \quad \sigma_n^t(u,\infty)$$
$$+ \quad \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} [\bar{q}_{k\theta_n, j\theta_n}(x,x) - q_{k\theta_n}(x)q_{j\theta_n}(x)]$$
$$+ \quad \sum_{\substack{x,x' \in \mathcal{V} \\ x \neq x'}} Q_n^u(x)Q_n^u(x') \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} [\bar{q}_{k\theta_n, j\theta_n}(x,x') - q_{k\theta_n}(x)q_{j\theta_n}(x')]$$
$$\equiv \quad (I) + (II) + (III). \qquad (2.12)$$

By (A-3), (I) tends to zero as $n \to \infty$. To bound (II), we drop the terms involving $q_{k\theta_n}(x)q_{j\theta_n}(x)$, and use the Markov property to write

$$(II) \quad \leq \quad \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \sum_{k=1}^{k_n(t)-2} \sum_{j=k+1}^{k_n(t)-1} q_{k\theta_n}(x) P_x \left(J((j-k)\theta_n) = x \right)$$
$$\leq \quad \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \sum_{k=1}^{k_n(t)-2} q_{k\theta_n}(x) \sum_{j=1}^{k_n(t)-1} P_x \left(J(j\theta_n) = x \right)$$
$$\leq \quad k_n(t) \sum_{x \in \mathcal{V}} (Q_n^u(x))^2 \pi_n^t(x) \sum_{j=1}^{k_n(t)-1} P_x \left(J(j\theta_n) = x \right)$$
$$\leq \quad \sigma_n^t(u,\infty) \sup_{x \in \mathcal{V}} \sum_{j=1}^{k_n(t)-1} P_x \left(J(j\theta_n) = x \right). \qquad (2.13)$$

By (A-1) and (A-3), $(II) \to 0$ as $n \to \infty$.

Let us now show, using (A-1) and (A-2), that also (III) vanishes. Fix $x \in \mathcal{V}$, $k \geq 1$, and $j \geq k+1$. For every $x' \neq x$ we bound the term $Q_n^u(x')$ by 1. Now

$$\sum_{x': \, x' \neq x} \bar{q}_{k\theta_n, j\theta_n}(x,x') = P_\mu(J(k\theta_n) = x, J(j\theta_n) \neq x) \leq P_\mu(J(k\theta_n) = x), \quad (2.14)$$

and

$$\sum_{x': \, x' \neq x} q_{k\theta_n}(x)q_{j\theta_n}(x') = P_\mu(J(k\theta_n) = x)P_\mu(J(j\theta_n) \neq x), \qquad (2.15)$$

so that, combining (2.14) and (2.15),

$$(III) \quad \leq \quad \sum_{x \in \mathcal{V}} Q_n^u(x) \sum_{k=1}^{k_n(t)-2} P_\mu(J(k\theta_n) = x) \sum_{j=1}^{k_n(t)-1} P_\mu(J(j\theta_n) = x)$$
$$\leq \quad \nu_n^t(u,\infty) \sup_{x \in \mathcal{V}} \sum_{j=1}^{k_n(t)-1} P_\mu(J(j\theta_n) = x). \qquad (2.16)$$

By (A-2), $\nu_n^t(u, \infty)$ converges as $n \to \infty$ to $t\nu(u, \infty)$, which is a finite number. Thus invoking (A-1), $(II) \to 0$ as $n \to \infty$. Inserting our bounds in (2.12), the variance of $\nu_n^{J,t}(u, \infty)$ tends to zero as $n \to \infty$. The verification of Condition (D1) is complete.

Next we show that Condition (D2) of Theorem 2.1 in Gayrard (2012) is satisfied. For $t > 0$, $u > 0$ we define

$$\sigma_n^{J,t}(u, \infty) \equiv \sum_{k=1}^{k_n(t)-1} \left(\mathcal{P}_\mu \left(Z_{n,k+1}^J > u | \mathcal{F}_{n,k} \right) \right)^2. \tag{2.17}$$

This condition then states that for all $u > 0$ such that $\nu(\{u\}) = 0$ and all $t > 0$,

$$\lim_{n \to \infty} \mathcal{P}_\mu \left(\sigma_n^{J,t}(u, \infty) > \varepsilon \right) = 0, \qquad \forall \varepsilon > 0. \tag{2.18}$$

By the Markov property,

$$\sigma_n^{J,t}(u, \infty) = k_n(t) \sum_{x \in \mathcal{V}} \pi_n^{J,t}(x) \left(Q_n^u(x) \right)^2. \tag{2.19}$$

The expectation of $\sigma_n^{J,t}(u, \infty)$ with respect to $\mathcal{P}_\mu$ is equal to $\sigma_n^t(u, \infty)$ and tends by (A-3) to zero. Thus, Condition (D2) is satisfied.

It remains to verify Condition (D3) of Theorem 2.1 in Gayrard (2012). It is in particular satisfied if

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n(t)-1} \mathcal{E}_\mu Z_{n,k+1}^J \mathbb{1}_{Z_{n,k+1}^J \leq \varepsilon} = 0. \tag{2.20}$$

By the Markov property the left hand side of (2.20) is equal to the left hand side of (1.22) and vanishes by (A-4). This proving that Condition (D3) is satisfied. Therefore, the conditions of Theorem 2.1 in Gayrard (2012) are verified, and so $\widehat{S}_n^{J,b} \Rightarrow V_\nu$ where convergence holds weakly in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology and V_ν is a subordinator with Lévy measure ν and zero drift. $\qquad \square$

In the verification of Condition (D1) of Theorem 2.1 in Gayrard (2012), more precisely in the proof of the claim (II), $(III) \to 0$, one sees that Condition (A-1) is used to replace $\pi_n^{J,t}$ by its average over P_μ. This is to be contrasted with the setting of Bovier and Gayrard (2013) where (II) and (III) vanish because J is already in the invariant measure after θ_n steps, and hence for $x, x' \in \mathcal{V}$ and $j > k$ the event $\{J(k\theta_n) = x\}$ is essentially independent of $\{J(j\theta_n) = x'\}$.

Proof of Theorem 1.3: As in the proof of Theorem 1.1 we show that for given sequences a_n, c_n, θ_n, a given initial distribution μ and for fixed $\omega \in \Omega$ (B-2)-(B-5) $\Rightarrow$ (A-2)-(A-4). Since both Theorems require that the conditions are satisfied $\mathbb{P}$-a.s. for all $t > 0$ and all $u > 0$, it suffices to consider a fixed realization $\omega \in \Omega$ and fixed $u > 0$, $t > 0$. Let us first establish that, under the assumptions of Theorem 1.3,

$$\lim_{n \to \infty} \left| \nu_n^t(u, \infty) - \bar{\nu}_n^t(u, \infty) \right| = 0. \tag{2.21}$$

By (B-2), (2.21) implies (A-2). Next

$$\begin{aligned} & \left| \nu_n^t(u, \infty) - \bar{\nu}_n^t(u, \infty) \right| \\ \leq\ & \sum_{(x,k) \in \mathcal{A}_n^1} |P(J(k\theta_n) = x) - h_{k\theta_n}(x)| Q_n^u(x) \\ +\ & \sum_{(x,k) \in \mathcal{A}_n^2} |P(J(k\theta_n) = x) - h_{k\theta_n}(x)| Q_n^u(x). \end{aligned} \tag{2.22}$$

By (1.25) of (B-5) the second summand tends to zero. The first summand is smaller than

$$\sup_{(x,k)\in\mathcal{A}_n^2} \frac{|P(J(k\theta_n)=x)-h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} \sum_{(x,k)\in\mathcal{A}_n^2} h_{k\theta_n}(x)Q_n^u(x)$$

$$\leq \sup_{(x,k)\in\mathcal{A}_n^2} \frac{|P(J(k\theta_n)=x)-h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} \nu_n^t(u,\infty), \tag{2.23}$$

and (1.24) of (B-5) guarantees that it vanishes as $n \to \infty$, proving that (A-2) is satisfied. To establish that

$$\lim_{n\to\infty}\left|\sigma_n^t(u,\infty)-\overline{\sigma}_n^t(u,\infty)\right|=0, \tag{2.24}$$

we proceed as in (2.22). Bounding $Q_n^u(x) \leq 1$, the claim of (2.24) follows from (2.22)-(2.23) and (A-3) is satisfied as well. Condition (A-4) follows in a similar way. This finishes the proof of Theorem 1.3. $\qquad\square$

Proof of Lemma 1.2: Let us show that (1.18) and (1.19) are always satisfied for transient x and never for positive recurrent x. Since the ideas of proof are similar, we restrict ourselves to continuous time J's. Let $x \in \mathcal{V}$ be transient. Then, for $\mu' \in \{\delta_x, \mu\}$ and any $\theta_n \gg 1$,

$$\lim_{n\to\infty}\int_{\theta_n}^\infty P_{\mu'}(J(t)=x)dt=0. \tag{2.25}$$

Now, for all $s < t$, $P_{\mu'}(J(t)=x) \geq P_{\mu'}(J(t-s)=x)\exp(-s\widetilde{\lambda}^{-1}(x))$, and so

$$\sum_{k=1}^{k_n(t)-1} P_{\mu'}(J(k\theta_n)=x) = \sum_{k=1}^{k_n(t)-1}\int_{k\theta_n}^{k\theta_n+1} P_{\mu'}(J(k\theta_n)=x)dt$$

$$\leq e^{\widetilde{\lambda}^{-1}(x)}\int_{\theta_n}^\infty P_{\mu'}(J(t)=x)dt, \tag{2.26}$$

which by (2.25) tends to zero. This proves that (1.18) and (1.19) hold for transient $x \in \mathcal{V}$.

Since (1.19) can only be satisfied if $P_x(J(t)=x) \to 0$ and since by Theorem 1.8.3 in Norris (1998) $\lim_{t\to\infty} P_x(J(t)=x) > 0$ for positive recurrent $x \in \mathcal{V}$, (A-1) cannot hold for positive recurrent $x \in \mathcal{V}$. By Theorem 3.5.3 in Norris (1998) this also proves that (A-1) cannot hold for J that admit for an invariant probability measure. $\qquad\square$

Proof of Lemma 1.4: Since the proofs are the same, we only prove the claim for (A-0)-(A-4). Assume that (A-0)-(A-4) are satisfied $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$ and given a_n, c_n, θ_n, and μ. We construct a set $\Omega^\tau \subseteq \Omega$ of full measure on which (A-0)-(A-4) are satisfied for all $u > 0$, $t > 0$. The sums on the left hand sides of (1.18), (1.19), (1.22), and the quantities $\nu_n^t(u,\infty)$, and $\sigma_n^t(u,\infty)$ depend on t through $k_n(t)\pi_n^t(x)$, $x \in \mathcal{V}$, which is increasing in t. Moreover, as sums of tail distributions, the quantities $\mathcal{P}_\mu(S_n^{J,b}(0) > u)$, $\nu_n^t(u,\infty)$, and $\sigma_n^t(u,\infty)$ are decreasing in u. The right hand sides of (1.18)-(1.22) are continuous in t. The only right hand side that depends on u is that of (1.20) and in (A-2) we require that (1.20) holds for all continuity points of the mapping $u \mapsto \nu(u,\infty)$. Thus, $\Omega^\tau \equiv \bigcap_{u,t>0,t\in\mathbb{Q},u\in\mathbb{Q},\nu(\{u\})=0} \Omega^\tau(u,t) \subseteq \Omega$ is of full measure and (A-0)-(A-4) hold true for all $u > 0$ and all $t > 0$ on Ω^τ. The proof of Lemma 1.4 is finished. $\qquad\square$

3. Application to BATM

This section and the next are devoted to the proof of Theorem 1.5. In the present section we derive new conditions that imply (B-2)-(B-5) and are specific to BATM. We also show that (A-0) and (A-1) hold true for BATM. In Section 4 we prove that these new conditions are satisfied and give the conclusion of the proof.

3.1. *The VSRW.* We collect results for J that are used in the proof of Theorem 1.5. The VSRW is a well-studied Markov jump process in random environment (see Barlow and Deuschel (2010); Barlow and Černý (2011a); Černý (2011); Andres et al. (2013), and the references therein). The proof of Theorem 1.5 relies heavily on very precise results for J that can be found in Barlow and Deuschel (2010). The results that we are using repeatedly concern the heat kernel, which we now define. For $x, y \in \mathbb{Z}^d$ and $t > 0$ the heat kernel is given by

$$q_t(x, y) \equiv P_x(J(t) = y). \tag{3.1}$$

The bounds for $q_t(x, y)$ that are contained in Barlow and Deuschel (2010) allow us to control all hitting, local, and exit times of vertices and balls that we need for the proof of Theorem 1.5. Moreover, we use the local central limit theorem which can be found in Barlow and Deuschel (2010). Note that in virtue of Theorem 6.1 in Barlow and Deuschel (2010) and Lemma 9.1 in Barlow and Černý (2011a), these theorems apply in the present setting. We denote by $|\cdot|$ the Euclidean distance. For convenience, we restate Theorem 1.2 (a)-(c) (heat kernel bounds) and Theorem 5.14 (uniform local central limit theorem) from Barlow and Deuschel (2010).

Theorem 3.1. *There exists $c_1 \in (0, \infty)$ such that for all $x, y \in \mathbb{Z}^d$ and $t > 0$,*

$$q_t(x, y) \leq c_1 t^{-d/2}. \tag{3.2}$$

There exist identically distributed random variables $\{U_x\}_{x \in \mathbb{Z}^d}$ whose distribution satisfies

$$\mathbb{P}(U_x > v) \leq c_1 \exp(-c_2 v^{1/3}), \quad v > 0, \tag{3.3}$$

where $c_1, c_2 \in (0, \infty)$, and such that we have

$$q_t(x, y) \leq c_1 t^{-d/2} e^{-c_2 |x-y|\{1 \wedge |x-y|t^{-1}\}}, \quad if \ |x - y| \vee t^{1/2} \geq U_x, \tag{3.4}$$

$$q_t(x, y) \geq c_1 t^{-d/2} e^{-c_2 |x-y|^2 t^{-1}}, \quad if \ t \geq U_x^2 \vee |x - y|^{4/3}. \tag{3.5}$$

For $x \in \mathbb{R}^d$ write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor)$. There exists $c_v > 0$ such that, for $T > 0$,

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} \left| n^{d/2} q_{nt}(0, \lfloor n^{1/2} x \rfloor) - (2\pi c_v t)^{d/2} e^{-|x|^2/2c_v t} \right| = 0, \quad \mathbb{P}\text{-}a.s.. \tag{3.6}$$

For $x \in \mathbb{Z}^d$, define

$$A_n(x) \equiv \{\omega \in \Omega : \sup_{y:|x-y| \leq 2a_n} U_y \leq c_0 (\log a_n)^3\}; \tag{3.7}$$

by convention $A_n(0) \equiv A_n$. By (3.3), there exists $c_0 \in (0, \infty)$ such that $\mathbb{P}(A_n^c) \leq c_1 n^{-(5 \vee 2d)}$. Therefore, writing

$$A \equiv \bigcup_{n \geq 1} \bigcap_{m \geq n} A_m, \tag{3.8}$$

we have by Borel-Cantelli Lemma that $\mathbb{P}(A) = 1$. On the event A, we have for all but finitely many n that $\sup_{y:|y| \leq 2a_n} U_y \leq c_0 (\log a_n)^3$. We will make use of Theorem 3.1 on the events A_n and A. Whenever we do so, we check whether,

given x, y such that $|x|, |y| \leq a_n$ and $t > 0$, $|x - y| \wedge t^{1/2} \geq c_0(\log a_n)^3$ or $t^{1/2} \geq c_0(\log a_n)^3 \vee |x - y|^{2/3}$.

We now state two lemmata that are needed in the proof of Theorem 1.5. Their proofs are postponed to the appendix. The first concerns the distribution of the exit times of certain balls. We denote by $B_r(x)$ the ball of radius r centered at x; by convention $B_r \equiv B_r(0)$. We write $\eta(B_r(x))$ for the exit time of $B_r(x)$.

Lemma 3.2. *Let a_n be as in* (1.39). *There exists $c_4 \in (0, \infty)$ such that the following holds. For all sequences m_n, r_n such that $m_n \geq c_0^2 r_n (\log a_n)^6$ and $a_n \geq m_n$, on the event A_n,*

$$P_x(\eta(B_{r_n}(x)) \leq m_n) \leq e^{-c_4 r_n^2 m_n^{-1}}, \quad \forall x \in B_{a_n}. \tag{3.9}$$

For all sequences m_n, r_n such that $a_n \geq r_n \geq c_0(\log a_n)^3$ and $m_n \geq 3r_n^2$, on the event A_n,

$$P_x(\eta(B_{r_n}(x)) \geq m_n) \leq e^{-c_4 m_n^{1/2} r_n^{-1}}, \quad \forall x \in B_{a_n}. \tag{3.10}$$

The second lemma provides bounds on the expected number of different sites that J visits in certain time intervals. Given an increasing sequence of integers, m_n, we define the range of J in the time interval $[0, m_n]$ as

$$R_{m_n} \equiv \sum_{y \in \mathbb{Z}^d} \mathbb{1}_{\sigma(y) \leq m_n}, \tag{3.11}$$

where $\sigma(y) \equiv \inf\{t \geq 0 : J(t) = y\}$ is the hitting time of y.

Lemma 3.3. *Let m_n be such that $a_n \geq m_n \geq c_0^2(\log a_n)^6$. There exists $c_5 \in (0, \infty)$ such that the following holds for n large enough. For $d \geq 2$ and $k \in \{1, 2\}$,*

$$\mathbb{E}E_x R_{m_n}^k \leq c_5 \left(m_n(\log m_n)^{-k} \mathbb{1}_{d=2} + m_n^{1/k} \mathbb{1}_{d \geq 3} \right). \tag{3.12}$$

Moreover, for $d = 2$ there exists $f_{m_n} : (0, \infty) \to (0, \infty)$ such that, on the event A_n,

$$P(\sigma(x) \leq m_n) \leq f_{m_n}(|x|), \quad \text{for all } x \in B_{m_n}, \tag{3.13}$$

and f_{m_n} satisfies

$$\sum_{x \in B_{m_n}} \left(f_{m_n}(|x|) \right)^k \leq c_5 m_n(\log m_n)^{-k}, \quad k \in \{1, 2, 4\}. \tag{3.14}$$

Notice that by our choices of θ_n we may use Lemma 3.3 for $m_n \geq \theta_n^\delta$ for $\delta > 1/2$.

3.2. *Specializing Theorem 1.3 for BATM.* In this section we specialize Theorem 1.3 to the setting of BATM. More precisely, we will not study $S_n^{J,b}$ directly, but another process, $\overline{S}_n^{J,b}$, to which only those x contribute for which $\tau(x)$ is 'large enough'. For $x \in \mathbb{Z}^d$ we set

$$\gamma_n(x) \equiv c_n^{-1} \tau(x). \tag{3.15}$$

Let $\epsilon_n(d) \equiv (\log \theta_n)^{-6/(1-\alpha)} \mathbb{1}_{d=2} + \theta_n^{-1/3} \mathbb{1}_{d \geq 3}$ and denote the collection of 'large' traps by $T_n \equiv \{x \in \mathbb{Z}^d : \gamma_n(x) > \epsilon_n, \max_{y \sim x} \tau(y) \leq \epsilon_n^{-2/\alpha}\}$. Then,

$$\overline{S}_n^{J,b}(t) \equiv \sum_{k=0}^{k_n(t)-1} \sum_{x \in \mathbb{Z}^d} \gamma_n(x) \mathbb{1}_{x \in T_n} \left(\ell_{\theta_n(k+1)}(x) - \ell_{\theta_n k}(x) \right), \quad t > 0, \tag{3.16}$$

where $\ell_t(y) = \int_0^t \mathbb{1}_{J(s)=y} ds$.

Roughly speaking, the following lemma states that, $\mathbb{P}$-a.s., $\overline{S}_n^{J,b}$ is a good approximation for $S_n^{J,b}$. To simplify notation, we write $P \equiv P_0$, respectively $\mathcal{P} \equiv \mathcal{P}_0$.

Lemma 3.4. $\mathbb{P}$-*a.s.,* $\limsup_{n \to \infty} \mathcal{P}(\rho_\infty(S_n^{J,b}, \overline{S}_n^{J,b}) > \delta_n) = 0$, *where* $\delta_n \equiv \epsilon_n^{(1-\alpha)/2}$.

Proof: By definition of ρ_∞ it suffices to show this result with ρ_∞ replaced by ρ_r, Skorohod's J_1 metric on $D[0, r]$, for all $r > 0$. For convenience we take $r = 1$ and we get

$$
\begin{aligned}
\mathcal{P}\Big(S_n^{J,b}(1) - \overline{S}_n^{J,b}(1) > \delta_n\Big) &= \mathcal{P}\Big(\int_0^{a_n} \gamma_n(J(s))\mathbb{1}_{J(s) \notin T_n} > \delta_n\Big) \\
&\le \mathcal{P}\big(\int_0^{a_n} \gamma_n(J(s))\mathbb{1}_{J(s) \notin T_n, \gamma_n(J(s)) \le \epsilon_n^{-1}} > \delta_n\big) \\
&\quad + \mathcal{P}\big(\exists x \in \mathbb{Z}^d : \gamma_n(x) > \epsilon_n^{-1}, \ell_{a_n}(x) > 0\big).
\end{aligned}
\tag{3.17}
$$

To shorten notation, set $B_n \equiv T_n^c \cap \{y \in \mathbb{Z}^d : \gamma_n(y) \le \epsilon_n^{-1}\}$. Using a first order Chebyshev inequality to bound the first term in the right hand side of (3.17) and Boole's inequality for the second, we get that their sum is bounded above by

$$
\delta_n^{-1} \int_0^{a_n} \mathcal{E}\gamma_n(J(s))\mathbb{1}_{J(s) \in B_n} ds + \sum_{x \in \mathbb{Z}^d} P(\ell_{a_n}(x) > 0)\mathbb{1}_{\gamma_n(x) > \epsilon_n^{-1}}.
\tag{3.18}
$$

In order to establish that (3.18) tends $\mathbb{P}$-a.s. to zero, let us first consider subsequences of the form $c_{N,r} = \exp(N + r)$ for $r \in [0, 1]$ and establish that, uniformly in $r \in [0, 1)$, (3.18) tends $\mathbb{P}$-a.s. to zero as $N \to \infty$. Since $c_n = n = \exp(\lfloor \log n \rfloor + (\log n - \lfloor \log n \rfloor))$ this implies that (3.18) vanishes $\mathbb{P}$-a.s. as $n \to \infty$. To ease notation we write $a_{N,r} \equiv a_{\exp(N+r)}$ for $r \in [0, 1]$ and $N \in \mathbb{N}$, and use the same abbreviation for all n dependent quantities.

Now, the first summand in (3.18) satisfies

$$
\sup_{r \in [0,1)} \delta_{N,r}^{-1} \sum_{y \in B_{N,r}} \mathcal{E}\ell_{a_{N,r}}(y)\gamma_{N,r}(y) \le \delta_{N,1}^{-1} \sum_{y \in B_{N,1}} \mathcal{E}\ell_{a_{N,1}}(y)\gamma_{N,0}(y),
\tag{3.19}
$$

and the supremum over $r \in [0, 1)$ of the second is bounded above by

$$
\sum_{x \in \mathbb{Z}^d} P(\ell_{a_{N,1}}(x) > 0)\mathbb{1}_{\gamma_{N,0}(x) > \epsilon_{N,1}^{-1}} \le \sum_x P(\max_{y \sim x} \ell_{a_{N,1}}(y) > 0)\mathbb{1}_{\gamma_{N,0}(x) > \epsilon_{N,1}^{-1}}.
\tag{3.20}
$$

The lemma will be proven if we can show that the sum of the expectation of the right hand side of (3.19) and (3.20) with respect to the random environment, that is

$$
\begin{aligned}
&\sum_x \tfrac{1}{\delta_{N,1}}\mathbb{E}\big[\mathcal{E}\ell_{a_{N,1}}(x)\gamma_{N,0}(x)\mathbb{1}_{x \in B_{N,1}}\big] \\
&+ \sum_x \mathbb{E}\big[P(\ell_{a_{N,1}}(x) > 0)\big]\mathbb{P}(\gamma_{N,0}(0) > \epsilon_{N,1}^{-1}),
\end{aligned}
\tag{3.21}
$$

tends to zero fast enough. Notice that the second sum in (3.21) bounds (3.20) because $P(\max_{y \sim x} \ell_{a_{N,1}}(y) > 0)$ is independent of $\gamma_{N,0}(x)$. By (3.12) of Lemma 3.3, the second sum in (3.21) is bounded above by

$$
c_5 a_{N,1} c_{N,0}^{-\alpha} \epsilon_{N,1}^\alpha (1/\log a_{N,1} \mathbb{1}_{d=2} + \mathbb{1}_{d \ge 3}) \le \epsilon_{N,1}^\alpha,
\tag{3.22}
$$

which is summable in N. We now decompose the first sum in (3.21) into three sums according to the size of $|x|$. Namely, we set $D_1 \equiv B_{c_0(\log a_{N,1})^3}$, $D_2 \equiv B_{a_{N,1}^{1/2} \log\log a_{N,1}} \setminus D_1$, and $D_3 \equiv (B_{a_{N,1}^{1/2} \log\log a_{N,1}})^c$. When $x \in D_1$, we know by (3.2) of Theorem 3.1 that $\mathcal{E}\ell_{a_{N,1}}(x) \le c_1 \log a_{N,1}$, and so

$$
\tfrac{1}{\delta_{N,1}} \sum_{x \in D_1} \mathbb{E}\big[\mathcal{E}\ell_{a_{N,1}}(x)\gamma_{N,0}(x)\mathbb{1}_{x \in B_{N,1}}\big] \le \tfrac{c_1 |D_1| \log a_{N,1}}{\delta_{N,1}} \mathbb{E}\big[\gamma_{N,0}(0)\mathbb{1}_{0 \in B_{N,1}}\big].
\tag{3.23}
$$

One can check that $\mathbb{E}\gamma_{N,0}(0)\mathbb{1}_{0 \in B_{N,1}} \le c c_{N,1}^{-\alpha} \epsilon_{N,1}^{1-\alpha}$, and so (3.23) is smaller than, say, $c_{N,1}^{-\alpha/2}$ which is summable in N. For $x \in D_3$ we derive from (3.4) and (3.2)

of Theorem 3.1 that, $\mathcal{E}\ell_{a_{N,1}}(x) \leq e^{-c_2|x|^2/a_{N,1}} + \log a_{N,1} \mathbb{1}_{U_x > |x|}$. Thus, by (3.3) of Theorem 3.1,

$$
\begin{aligned}
& \tfrac{1}{\delta_{N,1}} \textstyle\sum_{x \in D_3} \mathbb{E}\left[\mathcal{E}\ell_{a_{N,1}}(x)\gamma_{N,0}(x)\mathbb{1}_{x \in B_{N,1}}\right] \\
& \leq \ \tfrac{c}{\delta_{N,1}} \textstyle\sum_{x \in D_3} \left\{ e^{-c_2/2|x|^2/a_{N,1}} c_{N,1}^{-\alpha} + \epsilon_{N,1}^{-1/\alpha} \log a_{N,1} \mathbb{P}(U_x > |x|) \right\} \qquad (3.24) \\
& \leq \ e^{-c'(\log\log a_{N,1})^2},
\end{aligned}
$$

which is summable in N. Finally, let $x \in D_2$. Let $A_{N,1} \equiv A_{\exp(N+1)}(0)$ be as in (3.7). In order to bound the contribution to the first sum in (3.21) coming from D_2 we distinguish between the events $A_{N,1}$ and $A_{N,1}^c$. By definition of $B_{N,1}$, we have that

$$
\begin{aligned}
& \tfrac{1}{\delta_{N,1}} \textstyle\sum_{x \in D_2} \mathbb{E}\left[\mathcal{E}\ell_{a_{N,1}}(x)\gamma_{N,0}(x)\mathbb{1}_{x \in B_{N,1}}\mathbb{1}_{A_{N,1}^c}\right] \\
& \leq \ \tfrac{\epsilon_{N,1}}{\delta_{N,1}} \mathbb{E}\left[\mathbb{1}_{A_{N,1}^c} \textstyle\sum_{x \in D_2} \mathcal{E}\ell_{a_{N,1}}(x)\right] \leq a_{N,1}\mathbb{P}(A_{N,1}^c) \leq a_{N,1}^{-4}. \qquad (3.25)
\end{aligned}
$$

On $A_{N,1}$ we have by (3.4) of Theorem 3.1 that $\mathcal{E}\ell_{a_{N,1}}(x) \leq c_3|x|^{2-d}$ if $d \geq 3$, and $\mathcal{E}\ell_{a_{N,1}}(x) \leq c_2 \log a_{N,1}$ if $d = 2$, and so,

$$
\begin{aligned}
& \tfrac{1}{\delta_{N,1}} \textstyle\sum_{x \in D_2} \mathbb{E}\left[\mathcal{E}\ell_{a_{N,1}}(x)\gamma_{N,0}(x)\mathbb{1}_{x \in B_{N,1}}\mathbb{1}_{A_{N,1}}\right] \\
& = \ \tfrac{c}{\delta_{N,1}} \textstyle\sum_{x \in D_2} c_{N,1}^{-\alpha}\epsilon_{N,1}^{1-\alpha} \left(|x|^{2-d}\mathbb{1}_{d \geq 3} + \log a_{N,1}\mathbb{1}_{d=2}\right) \\
& = \ \tfrac{c}{\delta_{N,1}} c_{N,1}^{-\alpha}\epsilon_{N,1}^{1-\alpha} \textstyle\sum_{k=1}^{a_{N,1}^{1/2} \log\log a_{N,1}} k \left(\mathbb{1}_{d \geq 3} + \log a_{N,1}\mathbb{1}_{d=2}\right) \\
& \leq \ c'\varepsilon^{-1}N^{-1-\alpha}\log N, \qquad (3.26)
\end{aligned}
$$

where $c, c' \in (0, \infty)$ and where the last line follows from (1.38), (1.39), and the construction of $c_{N,r}$. Collecting (3.22)-(3.26), the proof of Lemma 3.4 is complete. $\qquad\square$

Using Theorem 1.3, we can derive new conditions for the process $\overline{S}_n^{J,b}$ to converge. To present these conditions, we introduce the following quantities. For $x, y \in \mathbb{Z}^d$, $u > 0$, and $\varepsilon > 0$ we define

$$
\begin{aligned}
Q_n^u(x,y) & \equiv \ \mathcal{P}_x\left(\ell_{\theta_n}(y)\gamma_n(y) > u, \eta(B_{\theta_n}(x)) > \theta_n\right)\mathbb{1}_{y \in T_n}, \qquad (3.27) \\
M_n^\varepsilon(x,y) & \equiv \ \mathcal{E}_x\left(\ell_{\theta_n}(y)\gamma_n(y)\mathbb{1}_{\gamma_n(y)\ell_{\theta_n}(y) \leq \varepsilon}\mathbb{1}_{\eta(B_{\theta_n}(x)) > \theta_n}\right)\mathbb{1}_{y \in T_n}. \qquad (3.28)
\end{aligned}
$$

Note that $Q_n^u(x,y) = M_n^\varepsilon(x,y) = 0$ for $y \notin B_{\theta_n}(x)$. For $t > 0$ we set $d_n(t) \equiv \lfloor a_n t \rfloor^{1/2} \log\lfloor a_n t \rfloor$. For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ we take $h_n(x) = \mathbb{E}P(J(n) = x)$. Thus, $\overline{\pi}_n^t(x) = \mathbb{E}\pi_n^t(x)$. By analogy with (1.28) and (1.29) we write, for $u > 0$, $t > 0$,

$$
\widetilde{\nu}_n^t(u, \infty) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \overline{\pi}_n^t(x) \textstyle\sum_{y \in \mathbb{Z}^d} Q_n^u(x,y), \qquad (3.29)
$$

and

$$
\widetilde{\sigma}_n^t(u, \infty) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \overline{\pi}_n^t(x) \textstyle\sum_{y \in \mathbb{Z}^d} (Q_n^u(x,y))^2. \qquad (3.30)
$$

We also define for $\varepsilon > 0$, $t > 0$

$$
m_n^t(\varepsilon) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \overline{\pi}_n^t(x) \textstyle\sum_{y \in \mathbb{Z}^d} M_n^\varepsilon(x,y), \qquad (3.31)
$$

and finally we introduce for $\varepsilon > 0$ the set

$$
\mathcal{B}_n \equiv \{(x,k) \in B_{d_n(t)} \times [k_n(t) - 1] \setminus \{0\} : |x|^2 < \varepsilon k\theta_n, \ |x|^2 > k\theta_n/\varepsilon\}. \qquad (3.32)
$$

We are now ready to present our new conditions. They are stated for fixed $\omega \in \Omega$.

(C-2) For all $u > 0$, $t > 0$

$$\lim_{n \to \infty} \tilde{\nu}_n^t(u, \infty) = t \mathcal{K} u^{-\alpha}. \tag{3.33}$$

(C-3) For all $u > 0$, $t > 0$

$$\lim_{n \to \infty} \tilde{\sigma}_n^t(u, \infty) = 0. \tag{3.34}$$

(C-4) For all $t > 0$ there exists $C(t) > 0$ such that for all $\varepsilon > 0$

$$\limsup_{n \to \infty} m_n^t(\varepsilon) \leq C(t)\varepsilon^{1-\alpha}. \tag{3.35}$$

(C-5) For all $u > 0$, $t > 0$, $\varepsilon > 0$ there exists $C(u,t) \in (0, \infty)$ and $N(\varepsilon)$ such that for $n \geq N(\varepsilon)$,

$$\sum_{(x,k) \in \mathcal{B}_n} \sum_y (k\theta_n)^{-d/2} e^{-c_2 |x|^2 / k\theta_n} Q_n^u(x,y) \leq C(u,t)\varepsilon, \tag{3.36}$$

$$\sum_{(x,k) \in \mathcal{B}_n} \sum_y (k\theta_n)^{-d/2} e^{-c_2 |x|^2 / k\theta_n} M_n^\varepsilon(x,y) \leq C(u,t)\varepsilon. \tag{3.37}$$

Proposition 3.5. *Assume that Conditions (C-2)-(C-5) are satisfied $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$, and $\varepsilon > 0$. Then, $\mathbb{P}$-a.s., $\overline{S}_n^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$, as $n \to \infty$, where $\overset{J_1}{\Longrightarrow}$ denotes weak convergence in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology.*

The next lemma gives us a very helpful bound for $\pi_n^t(x)$ which we will use in the proof of Proposition 3.5 and in the following sections.

Lemma 3.6. *There exists $c_3 \in (0, \infty)$ such that for all $t > 0$ and large enough n we have that if $d \geq 3$,*

$$\overline{\pi}_n^t(x) \leq (k_n(t)\theta_n)^{-1} \begin{cases} c_3(|x| \vee 1)^{2-d}, & \text{if } |x| \leq a_n^{1/2} \log \theta_n, \\ c_3 |x|^{2-d} e^{-1/2(\log \theta_n)^2}, & \text{else}, \end{cases} \tag{3.38}$$

and if $d = 2$,

$$\overline{\pi}_n^t(x) \leq (k_n(t)\theta_n)^{-1} \begin{cases} c_3(\log(a_n/|x|^2) \vee (\log \log a_n)), & \text{if } |x| \leq a_n^{1/2} \log \log a_n, \\ e^{-c_2/2(\log \log a_n)^2}, & \text{if } a_n^{1/2} \log \log a_n < |x|. \end{cases} \tag{3.39}$$

We first prove Proposition 3.5 assuming Lemma 3.6 and the lemma next.

Proof of Proposition 3.5: Let us apply Theorem 1.3 to $\overline{S}_n^{J,b}$. By Lemma 1.4 it suffices to prove that the conditions of Theorem 1.3 are verified $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$, and $\varepsilon > 0$.

We first prove that Condition (A-1) is satisfied. By (3.2) of Theorem 3.1 we have for all $\omega \in \Omega$, all $x, y \in \mathbb{Z}^d$, and all $t > 0$,

$$\sum_{k=1}^{k_n(t)-1} q_{k\theta_n}(x,y) \leq \sum_{k=1}^{k_n(t)-1} c_1 (\theta_n k)^{-d/2} \leq \theta_n^{-1} \sum_{k=1}^{k_n(t)-1} c_1 k^{-1} \leq 2c_1 \frac{\log a_n t}{\theta_n}. \tag{3.40}$$

By (1.38) and (1.39) this vanishes as $n \to \infty$, and hence (A-1) is satisfied $\mathbb{P}$-a.s.

To verify (A-0) and establish that (C-2)-(C-4) $\Rightarrow$ (B-2)-(B-4) we proceed as in the proof of Lemma 3.4 and consider subsequence $c_{N,r} = \exp(N + r)$ first (see the paragraph below (3.18)). Since $\overline{Z}_{N,r,1}^J \equiv \sum_{x \in \mathbb{Z}^d} \gamma_n(x) \mathbb{1}_{x \in T_n} \ell_{\theta_{N,r}}(x)$ is zero unless there is $y \in T_{N,r}$ for which $\ell_{\theta_{N,r}}(y) > 0$,

$$\sup_{r \in [0,1)} \mathcal{P}(\overline{Z}_{N,r,1}^J > u) \leq P(\eta(B_{\theta_{N,1}^{3/4}}) \leq \theta_{N,1}) + \sum_{y:|y| \leq \theta_{N,1}^{3/4}} \mathbb{1}_{y \in T_{N,0}}. \tag{3.41}$$

By definition of A (see (3.8)) and Lemma 3.2, the first term in the right hand side of (3.41) tends $\mathbb{P}$-a.s. to zero. The second term in the right hand side of (3.41) is bounded above, when taking expectation with respect to the random environment, by $\theta_{N,1}^{3d/4} c_{N,0}^{-\alpha} \epsilon_{N,0}^{-\alpha}$. This is summable in N and hence (A-0) is satisfied $\mathbb{P}$-a.s.

We establish now that (C-2)-(C-4) $\Rightarrow$ (B-2)-(B-4). First we prove that (C-2) $\Rightarrow$ (B-2), i.e. that $\sup_{r \in [0,1)} |\bar{\nu}_{N,r}^t(u,\infty) - \tilde{\nu}_{N,r}^t(u,\infty)|$ tends $\mathbb{P}$-a.s. to zero as $N \to \infty$. We have that

$$\sup_{r \in [0,1)} |\bar{\nu}_{N,r}^t(u,\infty) - \tilde{\nu}_{N,r}^t(u,\infty)| \le \Delta_N^1 + \Delta_N^2 + \Delta_N^3, \qquad (3.42)$$

where

$$\Delta_N^1 \equiv \sup_{r \in [0,1)} k_{N,r}(t) \sum_{x \notin B_{d_{N,r}(t)}} \bar{\pi}_{N,r}^t(x), \qquad (3.43)$$

$$\Delta_N^2 \equiv \sup_{r \in [0,1)} k_{N,r}(t) \sum_{x \in B_{d_{N,r}(t)}} \bar{\pi}_{N,r}^t(x) P_x(\eta(B_{\theta_{N,r}}(x)) \le \theta_{N,r}), \qquad (3.44)$$

$$\Delta_N^3 \equiv \sup_{r \in [0,1)} k_{N,r}(t) \sum_{x \in B_{d_{N,r}(t)}} \bar{\pi}_{N,r}^t(x) \sum_{y,y':|x-y|,|x-y'| \le \theta_n} \mathbb{1}_{y,y' \in T_{N,r}}. \qquad (3.45)$$

Let us now prove that, $\mathbb{P}$-a.s., Δ_N^i vanishes for $i = 1, 2, 3$ as $N \to \infty$. We have that

$$\begin{aligned}
\Delta_N^1 &= \sup_{r \in [0,1)} \sum_{k=1}^{k_{N,r}(t)-1} \sum_{x \notin B_{d_{N,r}(t)}} \mathbb{E} P(J(k\theta_{N,r}) = x) \\
&\le \sum_{k=1}^{k_{N,1}(t)} \mathbb{E} P(\eta(B_{d_{N,0}(t)}) \le k\theta_{N,1}) \\
&\le k_{N,1}(t) e^{-c_4 \theta_{N,0}} + k_{N,1}(t) \mathbb{P}(A_{N,1}^c), \qquad (3.46)
\end{aligned}$$

where we used (3.9) of Lemma 3.2 on $A_{N,1} = A_{\exp(N+1)}$ in the last step. By construction, $k_{N,1}(t)\mathbb{P}(A_{N,1}^c) \le \exp(-cN)$, and so (3.46) is bounded above by $\exp(-cN)$. Thus, $\mathbb{P}$-a.s., $\Delta_N^1 \to 0$. The same arguments yield that $\mathbb{P}$-a.s., $\Delta_N^2 \to 0$. Finally, writing

$$\bar{\pi}_N^t(x) \equiv \sup_{r \in [0,1)} \bar{\pi}_{N,r}^t(x), \qquad (3.47)$$

we get by a first order Chebyshev inequality that

$$\mathbb{P}\left(\Delta_N^3 > \varepsilon\right) \le \frac{k_{N,1}(t)}{\varepsilon} \sum_{x \in B_{d_{N,1}(t)}} \bar{\pi}_N^t(x) \sum_{y,y' \in B_{\theta_{N,1}}(0), y \ne y'} \mathbb{E}(\mathbb{1}_{y,y' \in T_{N,0}}). \qquad (3.48)$$

By Lemma 3.6 one can show that the sum over $x \in B_{d_{N,1}(t)}$ of $\bar{\pi}_N^t(x)$ is bounded above by $c_3 (\log \log a_{N,1})^3$. The sum over $y, y' \in B_{\theta_{N,1}}$ in (3.48) is equal to $\theta_{N,1}^d c_{N,0}^{-2\alpha} \epsilon_{N,0}^{-2\alpha}$. Thus, $\mathbb{P}\left(\Delta_N^3 > \varepsilon\right) \ll c_{N,0}^{-\alpha/4}$. This is summable in N, and so, $\mathbb{P}$-a.s, $\Delta_N^3 \to 0$. Therefore, (C-2) $\Rightarrow$ (B-2). In a similar way one can show that (C-3) $\Rightarrow$ (B-3). We now prove that (C-4) $\Rightarrow$ (B-4). Observe that for $r \in [0,1)$,

$$\begin{aligned}
&\sum_{x \in \mathbb{Z}^d} \bar{\pi}_{N,r}^t(x) \mathcal{E}_x \left[\overline{Z}_{N,r,1}^J \mathbb{1}_{\overline{Z}_{N,r,1}^J \le \varepsilon} \right] \\
&\le \Delta_N^1 + \Delta_N^2 + \sum_{x \in B_{d_{N,r}(t)}} \bar{\pi}_{N,r}^t(x) \mathcal{E}_x \left[\overline{Z}_{N,r,1}^J \mathbb{1}_{\overline{Z}_{N,r,1}^J \le \varepsilon} \mathbb{1}_{\eta(B_{\theta_{N,r}}(x)) > \theta_{N,r}} \right]. \qquad (3.49)
\end{aligned}$$

By (3.46), $\Delta_N^1, \Delta_N^2 \to 0$. It remains to establish that for all $r \in [0,1)$, $m_{N,r}^t(\varepsilon)$ is an upper bound for the last term in the right hand side of (3.49). This term is

equal to

$$\sum_{x\in B_{d_{N,r}(t)}} \overline{\pi}^t_{N,r}(x) \sum_{y\in \mathbb{Z}^d \cap T_{N,r}} \mathcal{E}_x\left[\gamma_n(y)\ell_{\theta_{N,r}}(y)\mathbb{1}_{\overline{Z}^J_{N,r,1}\leq\varepsilon}\mathbb{1}_{\eta(B_{\theta_{N,r}}(x))>\theta_{N,r}}\right].$$
(3.50)

Since $\{\overline{Z}^J_{N,r,1}\leq\varepsilon\}\subset\{\ell_{\theta_{N,r}}(y)\gamma_n(y)\leq\varepsilon\}$ for all $y\in\mathbb{Z}^d$ for which $\ell_{\theta_{N,r}}(y)>0$, $m^t_{N,r}(\varepsilon)$ bounds the last term in the right hand side of (3.49) from above. Thus, (C-4) $\Rightarrow$ (B-4).

Finally we prove that (C-5) $\Rightarrow$ (B-5). The local central limit theorem (3.6) of Theorem 3.1 implies that $\mathbb{P}$-a.s., for $\mathcal{A}^1_n = \{(x,k): k\geq 1, |x|^2\in(\varepsilon k\theta_n, k\theta_n/\varepsilon)\}$,

$$\lim_{n\to\infty}\sup_{(x,k)\in\mathcal{A}^1_n}|(k\theta_n)^{d/2}q_{k\theta_n}(x)-(2\pi c_v)^{d/2}e^{-|x|^2/(2c_v k\theta_n)}|=0, \qquad (3.51)$$

where $q_t(x)\equiv q_t(0,x)$. By (3.3) of Theorem 3.1, $(k\theta_n)^{d/2}q_{k\theta_n}(x)\leq c_1$ for all $x\in\mathbb{Z}^d$, $k\in\mathbb{N}$ and so, by bounded convergence,

$$\lim_{n\to\infty}\sup_{(x,k)\in\mathcal{A}^1_n}|(k\theta_n)^{d/2}h_{k\theta_n}(x)-(2\pi c_v)^{d/2}e^{-|x|^2/(2c_v k\theta_n T)}|=0, \qquad (3.52)$$

where $h_n(x)=\mathbb{E}q_n(x)$. Thus, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\sup_{(x,k)\in\mathcal{A}^1_n}\tfrac{|q_{k\theta_n}(x)-h_{k\theta_n}(x)|}{h_{k\theta_n}(x)}=0, \qquad (3.53)$$

proving that (1.24) is satisfied for $\mathcal{A}^1_n$. Hence, it suffices to verify (1.25) and (1.26) for the set $\mathcal{A}^2_n\equiv\mathbb{Z}^d\times[k_n(t)-1]\setminus\mathcal{A}^1_n$. The set $\mathcal{A}^2_n$ is the disjoint union of $[k_n(t)-1]\times\mathbb{Z}^d\setminus B_{d_n(t)}$ and $\mathcal{B}_n$. Let us now verify (1.25) and (1.26) for each of these sets separately. Since $\Delta^1_n\to 0$, $\mathbb{P}$-a.s., we know that $[k_n(t)-1]\times\mathbb{Z}^d\setminus B_{d_n(t)}$ satisfies (1.25) and (1.26). By (3.4) of Theorem 3.1 we have that on the event A_n $q_{k\theta_n}(x)\leq c_1(k/\theta_n)^{-d/2}e^{-c_2|x|^2/k\theta_n}$ for all $x\in B_{d_n(t)}$ and all $k\geq 1$. By construction, $\mathbb{P}(A^c_n)|B_{d_n(t)}|\ll a_n^{d/2}n^{-(4\vee d)}\ll n^{-2}$ which is summable in n. Thus, by Borel-Cantelli Lemma, (C-5) implies (1.25) and (1.26) for $\mathcal{B}_n$. Thus, (C-5) $\Rightarrow$ (B-5). The proof of Proposition 3.5 is complete. $\square$

Proof of Lemma 3.6: Let us construct a bound on $\overline{\pi}^t_n$. By (3.4) of Theorem 3.1 we have

$$\mathbb{E}(\pi^t_n(x)\mathbb{1}_{U_x\leq\theta_n})$$
$$\leq \tfrac{1}{k_n(t)}\left(\sum_{k=\lfloor|x|^2/\theta_n\rfloor\vee 1}^{k_n(t)-1}(k\theta_n)^{-d/2}e^{-c_2|x|/(k\theta_n)}+\sum_{k=1}^{\lfloor|x|/\theta_n\rfloor\wedge 1}e^{-c_2|x|}\right)$$
$$\leq c_1(k_n(t)\theta_n)^{-1}\int_{1/2|x|\vee\theta_n}^{a_n t}s^{-d/2}e^{-c_2|x|^2 s^{-1}}ds, \qquad (3.54)$$

with the convention that $\sum_{k=1}^0=0$. Let $d\geq 3$ first. We substitute $u=c_2|x|^2 s^{-1}$ and get

$$\mathbb{E}(\pi^t_n(x)\mathbb{1}_{U_x\leq\theta_n})\leq c''(k_n(t)\theta_n)^{-1}|x|^{2-d}\Gamma(d/2-2,|x|^2/a_n), \qquad (3.55)$$

where $c''\in(0,\infty)$. By (3.3), $\mathbb{E}\pi^t_n(x)\mathbb{1}_{U_x>\theta_n}\leq c_1 e^{-c_2\theta_n^{1/3}}$, and so, taking c_3 large enough we get that (3.38) holds. Now let $d=2$. An asymptotic analysis reveals in (3.54) that for $|x|\leq\sqrt{a_n/\log a_n}$ we have for some $c'\in(0,\infty)$ that

$$\mathbb{E}(\pi^t_n(x)\mathbb{1}_{U_x\leq\theta_n})\leq c'(k_n(t)\theta_n)^{-1}\log(a_n/|x|^2). \qquad (3.56)$$

Moreover, when $|x|\geq\sqrt{a_n}\log\log a_n$, $\mathbb{E}(\pi^t_n(x)\mathbb{1}_{U_x\leq\theta_n})\leq e^{-c_2/2|x|^2/a_n}$. Since (3.54) is a decreasing function of $|x|$, we may bound $\overline{\mathbb{E}}(\pi^t_n(x)\mathbb{1}_{U_x\leq\theta_n})\leq\log\log a_n$ for $|x|\in[\sqrt{a_n/\log a_n},\sqrt{a_n}\log\log a_n]$. Since $\mathbb{E}\pi^t_n(x)\mathbb{1}_{U_x>\theta_n}\leq c_1 e^{-c_2/2\theta_n^{1/3}}$, this proves (3.39). This finishes the proof of Lemma 3.6. $\square$

4. Verification of Conditions (C-2)-(C-5)

In this section we show that (C-2)-(C-5) are satisfied. Let $u > 0$, $t > 0$ and $\varepsilon > 0$ be fixed. In Section 4.1 we establish that $\lim_{n\to\infty} \mathbb{E}\widetilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$. Next, in Section 4.2, we prove that $\mathbb{P}$-a.s., $\lim_{n\to\infty} \widetilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$. In Section 4.3 we prove that $\lim_{n\to\infty} \mathbb{E}\widetilde{\sigma}_n^t(u,\infty) = 0$ and show that, $\mathbb{P}$-a.s. $\widetilde{\sigma}_n^t(u,\infty)$ tends to zero. In Section 4.4 we establish that $\mathbb{E}m_n^t(\varepsilon) \leq C(t)\varepsilon^{1-\alpha}$ and show that $m_n^t(\varepsilon)$ concentrates $\mathbb{P}$-a.s. around its mean value. We verify Condition (C-5) in Section 4.4. Finally, we conclude the proof of Theorem 1.3 in Section 4.5.

4.1. *Convergence of* $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. This section is devoted to the proof of convergence of $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$, which is the most demanding part of the proof of Theorem 1.3.

Lemma 4.1. *For all* $u > 0$, $t > 0$, $\lim_{n\to\infty} \mathbb{E}\widetilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$.

Proof of Lemma 4.1: Since the τ's are identically distributed we have

$$\sum_{y\in\mathbb{Z}^d} \mathbb{E}Q_n^u(x,y) = \sum_{y\in\mathbb{Z}^d} \mathbb{E}Q_n^u(0,y), \quad \forall x \in B_{d_n}(t). \tag{4.1}$$

The statement of the lemma is thus equivalent to

$$\lim_{n\to\infty} k_n(t) \sum_{y\in\mathbb{Z}^d} \mathbb{E}Q_n^u(0,y) = t\nu(u,\infty), \quad \forall u > 0, \; \forall t > 0. \tag{4.2}$$

In view of (3.27), the sum in (4.2) is over $y \in B_{\theta_n}$. In fact, we can restrict it to $y \in B_{\theta_n} \setminus \{0\}$ because $\mathbb{E}Q_n(0,0) \leq \mathbb{E}(\mathbb{1}_{0\in T_n}) = c_n^{-\alpha}\epsilon_n^{-\alpha} \ll k_n(t)$. Also, we have that

$$\mathbb{P}(A_n)\mathbb{E}(\widetilde{\nu}_n^t(u,\infty)|A_n) \leq \mathbb{E}\widetilde{\nu}_n^t(u,\infty) \leq \mathbb{P}(A_n)\mathbb{E}(\widetilde{\nu}_n^t(u,\infty)|A_n) + k_n(t)\theta_n^d\mathbb{P}(A_n^c), \tag{4.3}$$

where A_n is as in (3.7) and satisfies $k_n(t)\theta_n^d\mathbb{P}(A_n^c) \leq c_1 a_n \theta_n^{d-1} n^{-5}$. Therefore, it suffices to calculate $\mathbb{E}(\widetilde{\nu}_n^t(u,\infty)|A_n)$. Let us also distinguish two cases depending on whether $d \geq 3$ or $d = 2$.

Case 1. Let $d \geq 3$ and take $y \in B_{\theta_n} \setminus \{0\}$. Set $k(\theta_n) = \theta_n(\log\theta_n)^{-1}$ and $h(\theta_n) = \theta_n - k(\theta_n)$. By the Markov property, writing $f_{\sigma(y)}$ for the density function of the hitting time, $\sigma(y)$, of y, we have on A_n

$$
\begin{aligned}
&\mathcal{P}(\ell_{\theta_n}(y)\gamma_n(y) > u, \eta(B_{\theta_n}) > \theta_n) \\
\geq\;& \int_0^{\theta_n} f_{\sigma(y)}(t)\mathcal{P}_y(\ell_{\theta_n-t}(y)\gamma_n(y) > u)dt - P(\eta(B_{\theta_n}) \leq \theta_n) \\
\geq\;& \int_0^{h(\theta_n)} f_{\sigma(y)}(t)dt\, \mathcal{P}_y(\ell_{k(\theta_n)}(y)\gamma_n(y) > u) - e^{-c_4\theta_n^{1/2}} \\
=\;& P(\sigma(y) \leq h(\theta_n))\, \mathcal{P}_y(\ell_{k(\theta_n)}(y)\gamma_n(y) > u) - e^{-c_4\theta_n^{1/2}},
\end{aligned}
\tag{4.4}
$$

where we used (3.9) of Lemma 3.2 in the second step. We first deal with the second probability in (4.4). Setting $B_n^1 \equiv B_{\sqrt{k(\theta_n)(\log\theta_n)^{-2}}}(y)$ we have,

$$
\begin{aligned}
\mathcal{P}_y(\ell_{k(\theta_n)}(y)\gamma_n(y) > u) \geq\;& \mathcal{P}_y(\ell_{\eta(B_n^1)}(y)\gamma_n(y) > u, \eta(B_n^1) < k(\theta_n)) \\
\geq\;& \mathcal{P}_y(\ell_{\eta(B_n^1)}(y)\gamma_n(y) > u) - \mathcal{P}_y(\eta(B_n^1) \geq k(\theta_n)).
\end{aligned}
\tag{4.5}
$$

By (3.10) of Lemma 3.2, on A_n, the second term in (4.5) is smaller than $e^{-c_4(\log\theta_n)^2}$. To bound the first term in (4.5) we use the well-know fact that when J starts in y, $\ell_{\eta(B_n^1)}(y)$ is exponentially distributed. Let

$$g_{B_n^1}(y) \equiv E_y\left[\int_0^{\eta(B_n^1(y))} \mathbb{1}_{J(s)=y}ds\right] \tag{4.6}$$

denote the mean value of $\ell_{\eta(B_n^1)}(y)$. Eq. (4.4) then becomes

$$\mathcal{P}_y(\ell_{k(\theta_n)}(y)\gamma_n(y) > u) \geq e^{-u\left(\gamma_n(y)g_{B_n^1}(y)\right)^{-1}} - e^{-c_4(\log\theta_n)^2}, \tag{4.7}$$

and we get

$$Q_n^u(0,y) \geq P(\sigma(y) \leq h(\theta_n))\left(e^{-u(\gamma_n(y)g_{B_n^1}(y))^{-1}} - e^{-c_4(\log\theta_n)^2}\right)\mathbb{1}_{y\in T_n}. \tag{4.8}$$

To get an upper bound we write (using the Markov property)

$$
\begin{aligned}
\mathcal{P}(\ell_{\theta_n}(y)\gamma_n(y) > u, \eta(B_{\theta_n}) > \theta_n) &\leq& \mathcal{P}(\ell_{\theta_n}(y)\gamma_n(y) > u) \\
&=& \int_0^{\theta_n} f_{\sigma(y)}(t)\mathcal{P}_y(\ell_{\theta_n-t}(y)\gamma_n(y) > u)dt \\
&\leq& P(\sigma(y) \leq \theta_n)\,\mathcal{P}_y(\ell_{\theta_n}(y)\gamma_n(y) > u). \quad (4.9)
\end{aligned}
$$

Set $B_n^2 \equiv B_{\sqrt{\theta_n}\log\theta_n}(y)$. By (3.9) of Lemma 3.2 we know on A_n that J exits B_n^2 before time θ_n with a probability smaller than $e^{-c_4(\log\theta_n)^2}$. Thus, proceeding as in (4.5)

$$Q_n^u(0,y) \leq P(\sigma(y) \leq \theta_n)\left(e^{-u(\gamma_n(y)g_{B_n^2}(y))^{-1}} + e^{-c_4(\log\theta_n)^2}\right)\mathbb{1}_{y\in T_n} \tag{4.10}$$

The contribution to $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ coming from the error terms $\exp(-c_4(\log\theta_n)^2)$ in (4.8) and (4.10) is negligible because

$$k_n(t)\sum_{|y|\leq\theta_n}\mathbb{E}\left[\mathbb{1}_{y\in T_n}e^{-c_4(\log\theta_n)^2}\right] \ll e^{-c_4/2(\log\theta_n)^2}. \tag{4.11}$$

To calculate $\mathbb{E}(Q_n^u(0,y)|A_n)$, we distinguish whether $\theta > 0$ or $\theta = 0$. In the first case several objects depend on the random environment: the distribution of $\sigma(y)$, the mean local time $g_{B_n^i}(y)$, and $\gamma_n(y)$. Thus we first seek upper and lower bounds on the distribution of $\sigma(y)$ and on $g_{B_n^i}(y)$ that are independent of $\gamma_n(y)$. Moreover, we look for upper and lower bounds for $g_{B_n^i}(y)$ that are independent of n.

Let us begin with bounds for $P(\sigma(y) \leq \theta_n)$. We show now that we may approximate the distribution of $\sigma(y)$ by that of $\min_{y'\sim y}\sigma(y')$, which is independent of $\gamma_n(y)$. Since $y \neq 0$ we know that $\min_{y'\sim y}\sigma(y') \leq \sigma(y)$, implying that

$$P(\sigma(y) \leq \theta_n) \leq P(\min_{y'\sim y}\sigma(y') \leq \theta_n). \tag{4.12}$$

By the definition of T_n we know that all the traps in the neighborhood $y \in T_n$ have size smaller than $\epsilon_n^{-2/\alpha}$. This implies that, as soon as J visits a neighbor y' of y, it jumps to y with probability larger than $1 - 2d(\epsilon_n^{-2/\alpha}c_n^{-1})^\theta$. This term goes to 1 when $\theta > 0$ and we get, for all $\varepsilon > 0$ and $y' \sim y$, that

$$P_{y'}(\varepsilon < \sigma(y) \leq h(\theta_n)) \leq \epsilon_n^{-2\theta/\alpha}c_n^{-\theta} \ll c_n^{-\theta/2}. \tag{4.13}$$

Thus,

$$P(\sigma(y) \leq h(\theta_n))\,\mathbb{1}_{y\in T_n} \geq \left(P(\min_{y'\sim y}\sigma(y') \leq h(\theta_n)) - c_n^{-\theta/2}\right)\mathbb{1}_{y\in T_n}. \tag{4.14}$$

As in (4.11), we see that the contribution of the error $c_n^{-\theta/2}$ to $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is of order $o(1)$.

Let us now approximate $g_{B_n^i}(y)$ by random variables, $\widetilde{g}_\infty(y)$, that are independent of $\gamma_n(y)$. This approximation follows closely the ideas of Barlow and Černý (2011a). For $i = 1,2$ we use the classical variational representation (see e.g. Chapter 3 in Bovier (2009)) to write

$$(g_{B_n^i}(y))^{-1} = \inf\left\{\tfrac{1}{2}\sum_{x\sim z}\widetilde{\lambda}(x,z)(f(x)-f(z))^2 : f|_y = 1, f|_{B_n^i} = 0\right\}, \tag{4.15}$$

and define, setting $N(y) \equiv \{y' : y' \sim y\} \cup \{y\}$,

$$(\widetilde{g}_{B_n^i}(y))^{-1} \equiv \inf\left\{ \tfrac{1}{2} \sum_{x \sim z} \widetilde{\lambda}(x,z)(f(x)-f(z))^2 : f|_{N(y)} = 1, f|_{B_n^i} = 0 \right\}. \quad (4.16)$$

As in the proof of Lemma 6.2 in Barlow and Černý (2011a) one can show on A_n that for all $\varepsilon > 0$ there exists $N(\varepsilon)$ uniform in the realization of the random environment, such that for $n \geq N(\varepsilon)$,

$$\widetilde{g}_{B_n^i}(y) \leq g_{B_n^i}(y) \leq (1+\varepsilon)\widetilde{g}_{B_n^i}(y), \quad \forall y \in T_n \cap B_{\theta_n}. \quad (4.17)$$

Combining (4.14), (4.17), (4.8), and (4.3) we get that $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is bounded below by

$$k_n(t) \sum_{|y| \leq \theta_n} \mathbb{E}\left[P(\min_{y' \sim y} \sigma(y') \leq h(\theta_n)) e^{-u(\gamma_n(y)\widetilde{g}_{B_n^1}(y))^{-1}} \mathbb{1}_{y \in T_n} | A_n \right] - o(1). \quad (4.18)$$

Similarly, we obtain by (4.14), (4.17), and (4.10) that $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is smaller than

$$k_n(t) \sum_{|y| \leq \theta_n} \mathbb{E}\left[P(\min_{y' \sim y} \sigma(y') \leq \theta_n) e^{-u(1-\varepsilon_n)(\gamma_n(y)\widetilde{g}_{B_n^2}(y))^{-1}} \mathbb{1}_{y \in T_n} | A_n \right] + o(1). \quad (4.19)$$

Let $g_\infty(y) = \lim_{n \to \infty} g_{B_n^1}(y) = \lim_{n \to \infty} g_{B_n^2}(y)$. As in the proof of Lemma 3.5 in Barlow and Černý (2011a), one can show that, on the event A_n, for all $\varepsilon' > 0$, there exists $N(\varepsilon')$, uniform in the random environment, such that for $n \geq N(\varepsilon')$ we have

$$(1 - \varepsilon')g_\infty(y) \leq g_{B_n^i}(y) \leq g_\infty(y), \qquad \forall y \in B_{d_n(t)}. \quad (4.20)$$

This with (4.17) implies that for all $\varepsilon'' > 0$ there exists $N(\varepsilon'')$ such that for $n \geq N(\varepsilon'')$, for all $y \in T_n \cap B_{\theta_n}$, $(1 - \varepsilon'')\widetilde{g}_\infty(y) \leq g_{B_n^i}(y) \leq (1+\varepsilon'')\widetilde{g}_\infty(y)$, where $\widetilde{g}_\infty(y) = \lim_{n \to \infty} \widetilde{g}_{B_n^1}(y) = \lim_{n \to \infty} \widetilde{g}_{B_n^2}(y)$. Equipped with (4.18) and (4.3) we take expectation with respect to $\gamma_n(y)$ and obtain that $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is bounded below by

$$\tfrac{t(1-\varepsilon'')^\alpha \Gamma(1+\alpha,\epsilon_n)}{u^\alpha \theta_n} \sum_{|y| \leq \theta_n} \mathbb{E}\left[\widetilde{g}_\infty^\alpha(y) P(\min_{y' \sim y} \sigma(y') \leq h(\theta_n))(1 - \mathbb{1}_{T_n(y)^c}) \right] - o(1), \quad (4.21)$$

where $T_n(y)^c = \{\max_{y' \sim y} \tau(y') > \epsilon_n^{-2/\alpha}\}$. The contribution to (4.21) coming from $T_n(y)^c$ is of order $o(1)$: using (4.17), and (3.2) of Theorem 3.1, and proceeding as in (4.12), it is by Lemma 3.3 smaller than $c_1 c_5 4 d^2 \epsilon_n^2$. Also, as in the proof of Lemma 3.3 one sees that adding $|y| > \theta_n$ in (4.21) produces at most an error of the order of $e^{-c_4/2\theta_n}$, and so

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty) \geq \tfrac{t(1-\varepsilon'')^\alpha \Gamma(1+\alpha,\epsilon_n)}{u^\alpha \theta_n} \sum_{y \in \mathbb{Z}^d} \mathbb{E}\left[\widetilde{g}_\infty^\alpha(y) P(\min_{y' \sim y} \sigma(y') \leq h(\theta_n)) \right] - o(1). \quad (4.22)$$

Similarly,

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty) \leq \frac{t\Gamma(1+\alpha)}{u^\alpha \theta_n} \sum_{y \in \mathbb{Z}^d} \mathbb{E}\left[\widetilde{g}_\infty^\alpha(y) P(\min_{y' \sim y} \sigma(y') \leq \theta_n) \right] + o(1). \quad (4.23)$$

Since $\epsilon_n \to 0$, $\Gamma(1+\alpha,\epsilon_n) \to \Gamma(1+\alpha)$. It remains to establish that

$$\lim_{n \to \infty} \theta_n^{-1} \sum_{y \in \mathbb{Z}^d} \mathbb{E}\left[\widetilde{g}_\infty^\alpha(y) P(\min_{y' \sim y} \sigma(y') \leq h_i(\theta_n)) \right] = K, \quad \text{for } i = 1, 2, \quad (4.24)$$

where $h_1(\theta_n) = \theta_n$ and $h_2(\theta_n) = h(\theta_n)$. Since $h_2 = h_1 - o(h_1)$, we only present the proof for h_1. For $\beta \in [0,1]$, set $f_n^\beta(x) \equiv \sum_{y \in \mathbb{Z}^d} \mathbb{E}\left[(\widetilde{g}_\infty(y)/c_6)^\beta P(\min_{y' \sim y} \sigma(y') \leq \right.$

$n)\big]$, where $c_6 \in (0,\infty)$ is such that $c_6 \geq g_\infty(y) \geq \widetilde{g}_\infty(y)$ for all $y \in \mathbb{Z}^d$. Using a 'quasi' sub-additivity argument (see (4.26) below), we now establish that $\lim_{n\to\infty} f_n^\beta/n = K'$ where $K' = \inf\{f_n^\beta/n : n \in \mathbb{N}\} \in (0,\infty)$. First note that by Lemma 3.3 $f_n^\beta/n \leq f_n^0/n \leq 2dc_5$, and so, $K' < \infty$. To see that $K' > 0$, we use that $f_m^\beta \geq f_m^1$ and bound f_m^1 from below:

$$\sum_{y\in\mathbb{Z}^d} g_\infty(y)P(\sigma(y) \leq m) \geq \sum_{y\in\mathbb{Z}^d} E\ell_m(y) \geq m \qquad (4.25)$$

As in the proof of (4.17) one can show that $\widetilde{g}_\infty(y) \geq (2d)^2 g_\infty(y)$, and hence $f_m^1 \geq (2d)^{-2}m$, which proves that $K' > 0$. Let us now assume that for all $\varepsilon > 0$ there exists N large enough such that for all $n, m \geq N$,

$$f_{n+m}^\beta \leq (1+\varepsilon)f_m^\beta + f_n^\beta. \qquad (4.26)$$

Then convergence to K' follows. Indeed, by construction of K' there exists M such that $f_M^\beta/M < K'+\varepsilon/2$. Now, let $N^\star = N'M$, $N' \geq N$, be such that $f_{2M}^\beta/N^\star < \varepsilon/2$. For $n \geq N^\star$ write $n = sM + r \geq N^\star$ where $s \geq N$, $r \leq M$. Then, by (4.26),

$$f_n^\beta/n \leq (1+\varepsilon)\tfrac{s-1}{n}f_M^\beta + f_{M+r}^\beta/n \leq (1+\varepsilon)\tfrac{s-1}{s+2}f_M^\beta/M + f_{2M}^\beta/N^\star \leq (1+2\varepsilon)K' - \varepsilon. \qquad (4.27)$$

Thus, f_n^β/n converges to K' because by construction, $f_n^\beta/n \geq K'$. It remains to establish the claim of (4.26). The difference $f_{n+m}^\beta - f_n^\beta$ is equal to

$$\sum_{y\in\mathbb{Z}^d} \mathbb{E}\big[(\tfrac{\widetilde{g}_\infty(y)}{c_6})^\beta P(\min_{y'\sim y} \sigma(y') \in (n, n+m))\big]$$
$$\leq \sum_{z,y} \mathbb{E}\big[(q_n(z) - \mathbb{E}q_n(z))(\tfrac{\widetilde{g}_\infty(y)}{c_6})^\beta P_z(\min_{y'\sim y} \sigma(y') \leq m)\big] + f_m^\beta. \qquad (4.28)$$

The first summand in the right hand side of (4.28) is smaller than εf_m^β if

$$\varepsilon_m^\beta \equiv \sum_z \sum_y \mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)|P_z(\sigma(y) \leq m)\big] \leq \varepsilon m. \qquad (4.29)$$

We divide the sum into $z \in B_{n^{1/2}/\varepsilon'}$ and $z \notin B_{n^{1/2}/\varepsilon'}$. Let $z \in B_{n^{1/2}/\varepsilon'}$. From the proof of Lemma 3.3 we know that there exists $c'' \in (0,\infty)$ such that

$$\sum_{y\in\mathbb{Z}^d} P_z(\sigma(y) \leq m)) \leq c'' \sum_{|z-y|\leq m^{1/2}\log m} |y|^{2-d}e^{-c_4/2|y|^2/m}(g_\infty(y))^{-1}$$
$$+ m^{(d+1)/2}\mathbb{1}_{A_m^c}(z) + \sum_{|z-y|>m^{1/2}\log m} \mathbb{1}_{A_{|z-y|^2}^c}(z), \qquad (4.30)$$

where $A_m(z)$ is as in (3.7). Let us first control the contribution to ε_m^β coming from the first sum in (4.30). We bound $1/g_\infty(y) \leq 1/\varepsilon' + (g_\infty(y))^{-1}\mathbb{1}_{g_\infty(y)<\varepsilon'}$ and call (I), respectively (II) the contribution to ε_m^β coming from $1/\varepsilon'$, respectively $(g_\infty(y))^{-1}\mathbb{1}_{g_\infty(y)<\varepsilon'}$. Now,

$$(I) = (c''m)/\varepsilon' \sum_{|z|\leq n^{1/2}/\varepsilon'} \mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)|/\mathbb{E}q_n(z)\big]\mathbb{E}q_n(z). \qquad (4.31)$$

Since $q_n(z) \leq c_1 n^{-d/2}$, the contribution to (I) from $z \in B_{\varepsilon'n^{1/2}}$ is smaller than $(\varepsilon')^{d-1}m$. By (3.6) of Theorem 3.1 for $z \in B_{n^{1/2}/\varepsilon'} \setminus B_{\varepsilon'n^{1/2}}$, $\mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)|/\mathbb{E}q_n(z)\big]$ tends uniformly to zero, and so (I) is bounded above by εm. Also,

$$(II) \leq c_1 m(\varepsilon')^{-d}c\mathbb{E}\big[(g_\infty(0))^{-1}\mathbb{1}_{g_\infty(0)<1/\varepsilon'}\big] \leq c_1 m(\varepsilon')^{-d}\exp(-c/\varepsilon'^{1/3}) < \varepsilon m, \qquad (4.32)$$

where we used $g_\infty(0) \geq U_0^{1/(2-d)}$ and (3.3). We now bound the contribution to ε_m^β coming from the second line in (4.30). By (3.2) of Theorem 3.1, $|q_n(z) - \mathbb{E}q_n(z)| \leq n^{-d/2}$ for all $z \in B_{n^{1/2}/\varepsilon'}$. By the identical distribution of the U's and since $|B_{n^{1/2}/\varepsilon'}|n^{-d/2} \leq \varepsilon'^{-d/2}$, it suffices to prove that $m^{(d+1)/2}\mathbb{P}(A_m^c) +$

$\sum_{|y|>m^{1/2}\log m}\mathbb{P}(A_{|y|^2}^c)$ vanishes as $m\to\infty$. This follows by definition of A_n and (3.3). Finally, let $z\notin B_{n^{1/2}/\varepsilon'}$. By Cauchy-Schwarz inequality, the contribution to ε_m^β coming from $z\notin B_{n^{1/2}/\varepsilon'}$ is bounded above by

$$\sum_{|z|>n^{1/2}/\varepsilon'}(\mathbb{E}|q_n(z)|^2)^{1/2}\sum_y(\mathbb{E}(P(\sigma(y)\le m))^2)^{1/2}. \tag{4.33}$$

By (3.3) and (3.4) of Theorem (3.1), $\mathbb{E}|q_n(z)|^2\le n^{-d}e^{-2c_4|z|^2/n^2}$. One can check that the sum over $z\notin B_{n^{1/2}/\varepsilon'}$ of $n^{-d/2}e^{-c_4|z|^2/n^2}$ is bounded above by $e^{-c_4/\varepsilon'}$. It remains to establish that the sum over y in (4.33) is of smaller order than m. This follows from (4.30) and (3.3). This proves (4.26). Thus (4.24) holds, and so,

$$u^{-\alpha}tK\Gamma(1+\alpha)(1+\varepsilon)^\alpha+o(1)\ge\mathbb{E}\widetilde{\nu}_n^t(u,\infty)\ge u^{-\alpha}tK\Gamma(1+\alpha,\epsilon_n)(1-\varepsilon)^\alpha-o(1). \tag{4.34}$$

Since $\varepsilon>0$ is arbitrary we see that $\lim_{n\to\infty}\mathbb{E}\widetilde{\nu}_n^t(u,\infty)=u^{-\alpha}t\mathcal{K}$, where $\mathcal{K}\equiv K\Gamma(1+\alpha)$. This concludes the proof of Lemma 4.1 for $d\ge3$ and $\theta>0$. When $\theta=0$, the proof simplifies because J is independent of the random environment. More precisely, it suffices to use Lemma 3.5 in Barlow and Černý (2011a) to replace $g_{B_n^i}(y)$ by $g_\infty(y)$ to get

$$\frac{t\Gamma(1+\alpha,\epsilon_n)}{u^\alpha\theta_n}\sum_{y\in\mathbb{Z}^d}g_\infty(y)^\alpha P\left(\sigma(y)\le h(\theta_n)\right)-o(1)$$
$$\le\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$$
$$\le\frac{t\Gamma(1+\alpha)(1+\varepsilon')^\alpha}{u^\alpha\theta_n}\sum_{y\in\mathbb{Z}^d}g_\infty^\alpha(y)P(\sigma(y)\le\theta_n)+o(1). \tag{4.35}$$

By the same arguments as for $\theta>0$, we can show that both bounds converge to $u^{-\alpha}t\mathcal{K}$ as $n\to\infty$. This finishes the proof of Lemma 4.1 for $d\ge3$.

Case 2. Let $d=2$. The pattern of proof is similar to that of Case 1 and relies on (4.4) and (4.9). The difference lies in the behavior $g_{B_n^i}(y)$. By definition in (4.6), on A_n,

$$g_{B_n^i}(y)=\int_0^\infty P_y(J(t)=y,\eta(B_n^1)>t)dt\ge\int_{\sqrt{\theta_n}}^{\theta_n}P_y(J(t)=y)dt-e^{-c_4(\log\theta_n)^2}, \tag{4.36}$$

where we used (3.9) of Lemma 3.2. By (3.5) of Theorem 3.1, the integral in the right hand side of (4.36) is larger than $c_1/2\log\theta_n$, showing that $g_{B_n^i}(y)$ diverges as $n\to\infty$. Thus, instead of substituting $\widetilde{g}_\infty(y)$ for $g_{B_n^i}(y)$ we use (4.17) to approximate $g_{B_n^i}(y)$ by $\widetilde{g}_{B_n^i}(y)$ for n large enough. A number of results from Černý (2011) will allow us to deal with $\widetilde{g}_{B_n^i}(y)$.

Let us begin with the construction of a lower bound on $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. We deduce from (4.14), that bounding $P(\sigma(y)\le h(\theta_n))\ge P(\min_{y\sim y'}\sigma(y')\le h(\theta_n))$ for $y\in T_n$, produces in $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ an error of the order $c_n^{-\varepsilon}$ for $\varepsilon>0$. We use (4.17) to substitute $\widetilde{g}_{B_n^1}(y)$ for $g_{B_n^1}(y)$. Since $P(\min_{y'\sim y}\sigma(y')\le h(\theta_n))$ and $\widetilde{g}_{B_n^1}(y)$ are independent of $\gamma_n(y)$, we can proceed as in Case 1 and take expectation with respect to $\gamma_n(y)$. Doing this yields

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty)\ge\frac{t\log\theta_n\Gamma(1+\alpha,\epsilon_n)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\le\theta_n}\mathbb{E}\left[\widetilde{g}_{B_n^1}^\alpha(y)P\left(\min_{y'\sim y}\sigma(y')\le h(\theta_n)\right)|A_n\right]-o(1)$$
$$\ge\frac{t\log\theta_n\Gamma(1+\alpha,\epsilon_n)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\le\theta_n}\mathbb{E}\left[\widetilde{g}_{B_n^1}^\alpha(y)P(\sigma(y)\le h(\theta_n))|A_n\right]-o(1), \tag{4.37}$$

since $\min_{y'\sim y}\sigma(y)\leq\sigma(y)$. We now construct an upper bound on $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. Again, by (4.17) and since $\min_{y'\sim y}\sigma(y)\leq\sigma(y)$, $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is bounded above by

$$\frac{t\log\theta_n\Gamma(1+\alpha)(1+\varepsilon)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}\big[g_{B_n^2}^\alpha(y)P(\min_{y'\sim y}\sigma(y')\leq\theta_n)|A_n\big]+o(1). \qquad (4.38)$$

We show now that, up to a negligible error term, we may substitute $P(\sigma(y)\leq\theta_n)$ for $P(\min_{y'\sim y}\sigma(y')\leq\theta_n)$ for all $y\in B_{\theta_n}$. To see this note for $y\in B_{\theta_n}$

$$P(\min_{y'\sim y}\sigma(y')\leq\theta_n)\leq P(\sigma(y)\leq\theta_n)+\sum_{y'\sim y}P(\sigma(y')\leq\theta_n<\sigma(y)). \qquad (4.39)$$

By the Markov property, for all $y'\sim y$, $P(\sigma(y')\leq\theta_n<\sigma(y))$ is bounded above by

$$P(\sigma(y')\leq h(\theta_n))P_{y'}(\sigma(y)>k(\theta_n))+P(\sigma(y')\in(h(\theta_n),\theta_n))$$
$$=\quad \delta_n^1(y')+\delta_n^2(y'). \qquad (4.40)$$

By (4.3) it thus suffices to establish that

$$\theta_n^{-1}(\log\theta_n)^{1-\alpha}\sum_{|y|\leq\theta_n}\sum_{y'\sim y}\mathbb{E}\big[\widetilde{g}_{B_n^1}^\alpha(y)\big(\delta_n^1(y')+\delta_n^2(y')\big)|A_n\big]=o(1). \qquad (4.41)$$

As in Lemma 3.3 in Černý (2011) one can show on A_n that there exists $c_9\in(0,\infty)$ such that, for all $y\in B_{d_n(t)}$, $\widetilde{g}_{B_n^i}(y)\leq c_9\log\theta_n$ for $i=1,2$. By (3.14) of Lemma 3.3, $\sum_{y,y'\sim y}\delta_n^2(y')\leq c_5\theta_n/(\log\theta_n)^2$. Hence the contribution to the left hand side of (4.41) coming from δ_n^2 is of order $o(1)$. To see that the same is true for δ_n^1, we use (3.14) of Lemma 3.3 to bound

$$\mathbb{E}[P(\sigma(y')\leq h(\theta_n))P_{y'}(\sigma(y)>k(\theta_n))|A_n]\leq f_{h(\theta_n)}(|y'|)\mathbb{E}[P(\sigma(x)>k(\theta_n))|A_n], \qquad (4.42)$$

where $|x|=1$. By recurrence and irreducibility, $\mathbb{E}[P(\sigma(x)>k(\theta_n))|A_n]\leq\varepsilon$ for n large enough and (4.42) implies that $\sum_y\delta_n^1(y)\leq c_5\varepsilon$. This concludes the proof of (4.41). Finally, combining (4.38)-(4.42),

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty)\leq\frac{t\log\theta_n\Gamma(1+\alpha)(1+\varepsilon)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}\big[g_{B_n^2}^\alpha(y)P(\sigma(y)\leq\theta_n)|A_n\big]+o(1). \qquad (4.43)$$

We now show that (4.37) and (4.43) tend to the same limit $\mathcal{K}tu^{-\alpha}$. By Proposition 3.1 in Černý (2011) we know that there exists $\bar{K}$ such that, as $r\to\infty$, $(\bar{K}\log r)^{-1}\widetilde{g}_{B_r^{1/2}(0)}(0)$ converges $\mathbb{P}$-a.s. to one for $i=1,2$. Thus, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}(\bar{K}\log\theta_n)^{-1}\widetilde{g}_{B_n^1}(0)=\lim_{n\to\infty}(\bar{K}\log\theta_n)^{-1}\widetilde{g}_{B_n^2}(0)=1. \qquad (4.44)$$

For $\varepsilon>0$ define $\mathcal{B}_n(y)\equiv\{|(\bar{K}\log\theta_n)^{-1}g_{B_n^2}(y)-1|\leq\varepsilon\}$. Then,

$$(4.43)\leq\frac{t\log\theta_n(1+\varepsilon)^\alpha\mathcal{K}'}{u^\alpha\theta_n}\sum_{y\in B_{\theta_n}}\mathbb{E}\big[P(\sigma(y)\leq\theta_n)((1+\varepsilon)+c_9^\alpha/\mathcal{K}'\mathbb{1}_{\mathcal{B}_n^c(y)})|A_n\big], \qquad (4.45)$$

where $\mathcal{K}'\equiv\Gamma(1+\alpha)\bar{K}^\alpha$ and where we used that $\widetilde{g}_{B_n^1}(y)\leq c_9\log\theta_n$. As in Lemma 3.3 in Černý (2011), on A_n, we have that $\widetilde{g}_{B_n^1}(y)\geq c_8\log\theta_n$, for all $y\in B_{\theta_n}$. Hence, we can bound (4.37) from below in a similar way. Thus, convergence of (4.37) and (4.43) follows if we can establish that

$$\lim_{n\to\infty}\theta_n(\log\theta_n)^{-1}\mathbb{E}[ER_{\theta_n}|A_n]=\bar{K}^{-1}, \qquad (4.46)$$
$$\lim_{n\to\infty}\theta_n(\log\theta_n)^{-1}\sum_{y\in B_{\theta_n}}\mathbb{E}\big[P(\sigma(y)\leq\theta_n)\mathbb{1}_{\mathcal{B}_n^c(y)}|A_n\big]=0, \qquad (4.47)$$

where R_{θ_n} is defined in (3.11). Let us first prove (4.47). By (3.14) of Lemma 3.3 and (4.3)

$$\sum_y\mathbb{E}\big[P(\sigma(y)\leq\theta_n)\mathbb{1}_{\mathcal{B}_n^c(y)}|A_n\big]\leq\sum_y f_{\theta_n}(|y|)\mathbb{P}\big(\mathcal{B}_n^c(y)\big)+\varepsilon\leq\frac{\theta_n}{\log\theta_n}\mathbb{P}\big(\mathcal{B}_n^c\big), \qquad (4.48)$$

where we used that the τ's are i.i.d. By Proposition 3.1 in Černý (2011), $\mathbb{P}(\mathcal{B}_n^c) \to 0$, and so (4.47) holds. Let us now construct upper and lower bounds for $\theta_n(\log\theta_n)^{-1}\mathbb{E}[ER_{\theta_n}|A_n]$ that coincide in the limit. We begin with the lower bound. By the Markov property,

$$\theta_n = \sum_{y\in B_{\theta_n}} E\ell_{\theta_n}(y) \le \sum_{y\in B_{\theta_n}} P(\sigma(y) \le \theta_n)E_y\ell_{\theta_n}(y). \qquad (4.49)$$

To bound $\mathbb{E}[ER_{\theta_n}|A_n]$ from below it suffices to construct an upper bound on $E_y\ell_{\theta_n}(y)$. By Theorem 3.2 one can show on A_n that for all $y \in B_{d_n(t)}$, $E_y\ell_{\theta_n}(y) \in (c_8 \log\theta_n, c_9 \log\theta_n)$, yielding

$$\theta_n \le \log\theta_n \sum_{|y|\le\theta_n} \mathbb{E}\big[P(\sigma(y) \le \theta_n > 0)\big(\bar{K}(1+\varepsilon) + c_8\mathbb{1}_{\mathcal{B}_n^c(y)}\big)|A_n\big] + o(1). \quad (4.50)$$

Together with (4.47),

$$1 \le \theta_n^{-1}\log\theta_n\bar{K}(1+\varepsilon)\mathbb{E}[ER_{\theta_n}|A_n] + c_8\mathbb{P}(\mathcal{B}_n^c), \qquad (4.51)$$

i.e. $\lim_{n\to\infty}\theta_n^{-1}\log\theta_n\mathbb{E}[ER_{\theta_n}|A_n]$ is bounded below by $\bar{K}^{-1}$. For the upper bound we again use the Markov property and get that

$$\theta_n + k(\theta_n) = \sum_{y\in B_{\theta_n}} E\ell_{\theta_n+k(\theta_n)}(y) \ge \sum_{y\in B_{\theta_n}} P(\ell_{\theta_n}(y) > 0)E_y\ell_{k(\theta_n)}(y). \quad (4.52)$$

Since $k(\theta_n)\log\theta_n/\theta_n \to 0$, we can show that the upper bound coincides with the lower bound. The claim of (4.46) is proved. Finally, using (4.46) and (4.47) in (4.37) and (4.43),

$$\mathcal{K}u^{-\alpha}t(1-\varepsilon)^{1+\alpha} \le \lim_{n\to\infty}\mathbb{E}\tilde{\nu}_n^t(u,\infty) \le \mathcal{K}u^{-\alpha}t(1+\varepsilon)^{1+\alpha}, \qquad (4.53)$$

where $\mathcal{K} = \mathcal{K}'\bar{K}^{-1}$. Since $\varepsilon > 0$ is arbitrary this proves the convergence of $\mathbb{E}\tilde{\nu}_n^t(u,\infty)$. This finishes the proof of Lemma 4.1 for $d = 2$. $\qquad\square$

4.2. *Convergence of $\tilde{\nu}_n^t(u,\infty)$.* Let $u > 0$, $t > 0$. In this section we prove that $\mathbb{P}$-a.s., $\lim_{n\to\infty}\tilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$. Once more, let us consider subsequences $c_{N,r} = \exp(N + r)$, $r \in [0,1)$ (see the paragraph below (3.18)). For $r,s \in [0,1)$, we define

$$Q_{N,r,s}(x) \equiv \sum_y Q_{N,r,s}(x,y), \qquad (4.54)$$

where we write, for $x,y \in \mathbb{Z}^d$,

$$Q_{N,r,s}(x,y) \equiv \mathcal{P}_x(\ell_{\theta_{N,s}}(y)\gamma_{N,r}(y) > u, \eta(B_{\theta_{N,r}}) \le \theta_{N,s})\mathbb{1}_{y\in T_{N,r}}, \qquad (4.55)$$

where $T_{N,r}$ is defined above(3.16). For $i = 0,\ldots,N$, we set $r_i \equiv i/N$ and define

$$\nu_{N,i}^1 \equiv \sum_{x\in B_{d_{N,r_i}}} Q_{N,r_{i+1},r_i}(x)\inf_{r\in\mathcal{I}_i}\overline{\pi}_{N,r_i}^t(x), \qquad (4.56)$$

$$\nu_{N,i}^2 \equiv \sum_{x\in B_{d_{N,r_{i+1}}}} Q_{N,r_i,r_{i+1}}(x)\sup_{r\in\mathcal{I}_i}\overline{\pi}_{N,r_i}^t(x). \qquad (4.57)$$

Then we have that, for $r \in \mathcal{I}_i \equiv [r_i, r_{i+1})$,

$$\nu_{N,i}^1 \le \tilde{\nu}_{N,r}^t(u,\infty) \le \nu_{N,i}^2. \qquad (4.58)$$

To prove that $\lim_{n\to\infty}\tilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$ $\mathbb{P}$-a.s., we use (4.58) as follows. First, we will derive from Lemma 4.1 that, for $j = 1,2$, $\mathbb{E}\nu_{N,i}^j$ converges to $t\nu(u,\infty)$. Then, we will prove in Lemma 4.2 that, for $d \ge 3$, respectively, $d = 2$, the second

and fourth moments of $\nu_{N,i}^j - \mathbb{E}\nu_{N,i}^j$ are bounded above by N^{-3}. Together with (4.58), we thus get by Chebyshev's inequality that

$$\mathbb{P}(\sup_{r \in [0,1)} |\widetilde{\nu}_{N,r}^t(u, \infty) - t\nu(u, \infty)| > \varepsilon)$$

$$\leq \sum_{i=0}^{N-1} \left\{ \mathbb{P}(|\nu_{N,i}^1 - \mathbb{E}\nu_{N,i}^1| > \varepsilon/2) + \mathbb{P}(|\nu_{N,i}^2 - \mathbb{E}\nu_{N,i}^2| > \varepsilon/2) \right\} \leq 32\varepsilon^{-4}N^{-2}, \quad (4.59)$$

which is summable in N and therefore proves that $\mathbb{P}$-a.s., $\lim_{n \to \infty} \widetilde{\nu}_n^t(u, \infty) = t\nu(u, \infty)$.

Fix $j \in \{1, 2\}$. Let us derive from Lemma 4.1 that $\lim_{N \to \infty} \mathbb{E}\nu_{N,i}^j = t\nu(u, \infty)$. By definition of r_i this lemma implies that $k_{N,r_i}(t)\mathbb{E}(Q_{N,r_{i+1},r_i},(0))$ and $k_{N,r_{i+1}}(t)\mathbb{E}(Q_{N,r_i,r_{i+1}}(0))$ both converge to $t\nu(u, \infty)$. Convergence of $\mathbb{E}\nu_{N,i}^j$ to the same quantity will follow if we can establish that the following vanishes as $N \to \infty$

$$\sup_{k=1,\dots,k_{N,r_i}(t)} \sum_{x \in B_{d_{N,r_{i+1}}}} \sup_{r,s \in \mathcal{I}_i} \mathbb{E}|P(J(k\theta_{N,r}) = x) - P(J(k\theta_{N,s}) = x)|. \quad (4.60)$$

Fix $k \in 1, \dots, k_{N,r_i}(t)$. We divide $B_{d_{N,r_{i+1}}}$ into $B_{(\varepsilon k\theta_{N,1})^{1/2}}$, $B_{d_{N,r_{i+1}}} \setminus B_{(k\theta_{N,0}/\varepsilon)^{1/2}}$, and $B_{(k\theta_{N,0}/\varepsilon)^{1/2}} \setminus B_{(\varepsilon k\theta_{N,1})^{1/2}}$ and construct bounds that are uniform in k. By (3.2) of Theorem 3.1, the contribution to (4.60) coming from $B_{(\varepsilon k\theta_{N,1})^{1/2}}$ is bounded above by ε^d. By (3.3) and (3.4) of Theorem 3.1, the contribution to (4.60) coming from $B_{d_{N,r_{i+1}}} \setminus B_{(k\theta_{N,0}/\varepsilon)^{1/2}}$ is smaller than $e^{-c_4/\varepsilon}$. By definition of r_i, dominated convergence, and the uniform local central limit theorem ((3.6) of Theorem 3.1), the contribution to (4.60) coming from $B_{(k\theta_{N,0}/\varepsilon)^{1/2}} \setminus B_{(\varepsilon k\theta_{N,1})^{1/2}}$ tends to zero. This proves that (4.60) vanishes as $N \to \infty$.

The following lemma bounds the second and fourth moment of $\nu_{N,i}^j - \mathbb{E}\nu_{N,i}^j$ for $d \geq 3$, respectively, $d = 2$.

Lemma 4.2. *Let* $i \in \{0, \dots, N - 1\}$, $j \in \{1, 2\}$. *For* $d \geq 3$, *for all* $u > 0$, $t > 0$, *for* N *large enough,*

$$\mathbb{E}(\nu_{N,i}^j - \mathbb{E}\nu_{N,i}^j)^2 \leq K(u, t)(\log \theta_{N,1})^2 \theta_{N,1}^{-1/2}, \quad (4.61)$$

and for $d = 2$, *for all* $u > 0$, $t > 0$, *for* N *large enough,*

$$\mathbb{E}(\nu_{N,i}^j - \mathbb{E}\nu_{N,i}^j)^4 \leq K(u, t)u^{-\alpha}(\log \log a_{N,1})^6 (\log \theta_{N,0})^{-4}, \quad (4.62)$$

where $K(u, t) \in (0, \infty)$.

The proof of Lemma 4.2 relies on Lemma 4.3 below. We first prove Lemma 4.3 and Lemma 4.2 next.

Lemma 4.3. *For all* $u > 0$, *for* $k \in \{2, 4\}$, *for* $r, s \in [0, 1)$, *for* N *large enough,*

$$\mathbb{E}(Q_{N,r,s}(0))^k \leq \sum_x \mathbb{E}(Q_{N,0,1}(0, x))^k + c_{N,0}^{-3\alpha/2} \leq \rho_N(d, k) \equiv Ku^{-\alpha}\rho_N(d, k), \quad (4.63)$$

where $K \in (0, \infty)$ *and where we set*

$$\rho_N(d, k) \equiv \theta_{N,1}c_{N,0}^{-\alpha}(\log \theta_{N,0})^{-k+\alpha}\mathbb{1}_{d=2} + \theta_{N,1}^{1/2}c_{N,0}^{-\alpha}\mathbb{1}_{d \geq 3}. \quad (4.64)$$

Proof of Lemma 4.3: Let $k \in \{2, 4\}$ and $r, s \in [0, 1)$. Since $Q_{N,r,s}(x) \leq Q_{N,0,1}(x)$, we have by (4.55) and (4.54) that

$$\mathbb{E}(Q_{N,r,s}(0))^k \leq \sum_y \mathbb{E}(Q_{N,0,1}(0, y))^k + \sum_{\substack{y' \neq y \\ |y'-y| \leq \theta_{N,1}}} \mathbb{E}(\mathbb{1}_{y',y \in T_{N,0}}). \quad (4.65)$$

The double sum in (4.65) is bounded above by

$$\sum_{\substack{y'\neq y \\ |y'-y|\leq \theta_{N,1}}} \mathbb{E}\mathbb{1}_{y,y'\in T_{N,0}} \leq \theta_{N,1}^d (c_{N,0}\epsilon_{N,1})^{-2\alpha} \leq c_{N,0}^{-3\alpha/2}, \tag{4.66}$$

where we used (1.38) and (1.39). It remains to bound the first term in the right hand side of (4.65). Since $\mathbb{P}(A_{N,1}^c)\theta_{N,1}^d \leq \rho_N(d,k)$, it suffices to bound $\mathbb{E}(Q_{N,0,1}(0,y)\mathbb{1}_{A_{N,1}})^k$. For $y \in B_{\theta_{N,1}}$ we know by (4.9) and (4.10) that $\mathbb{E}(Q_{N,0,1}(0,y)\mathbb{1}_{A_{N,1}})^k$ is smaller than

$$\mathbb{E}\big(\mathbb{1}_{y\in T_{N,0}} P_y\big(\ell_{\theta_{N,1}}(y)\gamma_{N,0}(y) > u\big) P(\sigma(y) \leq \theta_{N,1})\mathbb{1}_{A_{N,1}}\big)^k$$
$$\leq \mathbb{E}\big(\mathbb{1}_{y\in T_{N,0}} P_y\big(\ell_{B_{N,1}^2}(y)\gamma_{N,0}(y) > u\big) P(\sigma(y) \leq \theta_{N,1})\mathbb{1}_{A_{N,1}}\big)^k$$
$$+ e^{-c_4(\log\theta_{N,0})^2}\mathbb{P}(y\in T_{N,0}). \tag{4.67}$$

The contribution to $\mathbb{E}(Q_{N,0,1}(0,y)\mathbb{1}_{A_{N,1}})^k$ of the second term in the right hand side of (4.67) is negligible because $\theta_{N,1}^d c_{N,0}^{-\alpha}\epsilon_{N,0}^{-\alpha} \ll \rho_N(d)$. It remains to bound the first summand in (4.67). Recall from the proof of Lemma 4.1 that $\ell_{B_{N,1}^2}(y)$ has exponential distribution with mean value $g_{B_{N,1}^2}(y)$, which on $A_{N,1}$ is bounded above by $c(\log\theta_{N,1}\mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3})$, for all $y \in B_{\theta_{N,1}}$. Also, recall that $P(\sigma(y) \leq \theta_{N,1}) \leq P(\min_{y'\sim y}\sigma(y) \leq \theta_{N,1})$. Thus, for all $y \in B_{\theta_{N,1}}$

$$\mathbb{E}\big(\mathbb{1}_{A_{N,1}}\mathcal{P}(\ell_{B_{N,1}^2}(y)\gamma_{N,0}(y) > u)\mathbb{1}_{y\in T_{N,1}}\big)^k$$
$$\leq \mathbb{E}\Big(\mathbb{1}_{A_{N,1}} P(\min_{y'\sim y}\sigma(y') \leq \theta_{N,1})\exp(-u(\gamma_{N,0}(y)c(\mathbb{1}_{d\geq 3} + \log\theta_{N,1}\mathbb{1}_{d=2}))^{-1}))\Big)^k$$
$$\leq \mathbb{E}\exp(-ku(\gamma_{N,0}(0)c(\mathbb{1}_{d\geq 3} + \log\theta_{N,1}\mathbb{1}_{d=2}))^{-1})\mathbb{E}\big(P(\min_{y'\sim y}\sigma(y') \leq \theta_{N,1})\big)^k. \tag{4.68}$$

We bound the terms in (4.68) separately. The expectation with respect to $\gamma_{N,0}(0)$ is bounded above by $Cu^{-\alpha}c_{N,0}^{-\alpha}(\mathbb{1}_{d\geq 3} + \log\theta_{N,1}\mathbb{1}_{d=2})$, for some $C \in (0,\infty)$. Moreover,

$$\sum_{y\in B_{\theta_{N,1}}} \mathbb{E}\big(P(\min_{y'\sim y}\sigma(y') \leq \theta_n)\big)^k \leq \sum_y c\mathbb{E}(\mathbb{1}_{A_{N,1}}P(\sigma(y) \leq \theta_{N,1}))^k + e^{-c'N}, \tag{4.69}$$

where $c, c' \in (0,\infty)$ and where we used the definition of $A_{N,1}$. By Lemma 3.3, (4.69) is bounded above by $c_5\theta_{N,1}(\log\theta_{N,0})^{-k}$, for $d = 2$. For $d \geq 3$ and $k = 2$, the same lemma implies that (4.69) is smaller than $c_5\theta_{N,1}^{1/2}$. Since the right hand side of (4.69) is decreasing in k the same is true for $k = 4$. Collecting (4.68)-(4.69) yields

$$\sum_{y\in B_{\theta_{N,1}}} \mathbb{E}\big(\mathcal{P}(\ell_{B_{N,1}^2}(y)\gamma_{N,0}(y) > u)\mathbb{1}_{y\in T_{N,0}}\big)^k \leq Ku^{-\alpha}\rho_N(d,k), \tag{4.70}$$

for some $K \in (0,\infty)$. This finishes the proof of Lemma 4.3. $\qquad\square$

We are now ready to present the proof of Lemma 4.2.

Proof of Lemma 4.2: Without loss of generality, we prove Lemma 4.2 for $j = 1$ only. We distinguish whether $d = 2$ or $d \geq 3$. We begin with $d \geq 3$. By (3.47), the variance of $\nu_{N,i}^1$ is bounded above by

$$k_{N,1}^2(t)\sum_{x\in B_{d_{N,1}(t)}} (\overline{\pi}_N^t(x))^2\mathbb{E}\big(Q_{N,0,1}(x)\big)^2 \tag{4.71}$$

$$+ 2k_{N,1}^2(t)\sum_{x\neq x', |x-x'|\leq 2\theta_{N,1}} \overline{\pi}_N^t(x)\overline{\pi}_n^t(x')\mathbb{E}\big(Q_{N,0,1}(x)Q_{N,0,1}(x')\big), \tag{4.72}$$

where we used that $Q_{N,0,1}(x)$ only depends on $\tau(x)$ for $x \in B_{\theta_{N,1}}(x)$. By Lemma 4.3, (4.71) is bounded above by

$$\leq \sum_{x \in B_{d_{N,1}(t)}} (k_{N,1}(t)\overline{\pi}_N^t(x))^2 \rho_N(d) \ll K(u,t)(\log \theta_{N,1})^4 \theta_{N,1}^{-1/2}, \qquad (4.73)$$

for $K(u,t) \in (0,\infty)$. Since this satisfies (4.61), it suffices to control (4.72).

Fix $x, x' \in B_{d_{N,1}(t)}$. To bound $\mathbb{E}Q_{N,0,1}(x)Q_{N,0,1}(x')$ we distinguish two cases, depending on the size of $|x-x'|$. We define the sets $D_1(x) \equiv B_{\theta_{N,1}} \setminus B_{\sqrt{\theta_{N,1}(x)}\log \theta_{N,1}}(x)$ and $D_2(x) \equiv B_{\sqrt{\theta_{N,1}(x)}\log \theta_{N,1}}(x)$. Let $x' \in D_1(x)$ first. We use the same arguments as in the proof of Lemma 4.3 to bound $\mathbb{E}(Q_{N,0,1}(x)Q_{N,0,1}(x'))$. By analogy with (4.66)-(4.70),

$$\mathbb{E}(Q_{N,0,1}(x)Q_{N,0,1}(x')\mathbb{1}_{A_{N,1}}) \leq K'2du^{-\alpha}c_{N,0}^{-\alpha}\mathbb{E}\mathbb{1}_{A_{N,1}}I_{\theta_{N,1}}(x,x') + c_{N,0}^{-3\alpha/2}, \quad (4.74)$$

for some $K' \in (0,\infty)$ and where $I_{\theta_{N,1}}(x,x')$ is the expected intersection range of J starting in x and an independent copy J' starting in x',

$$I_{\theta_{N,1}}(x,x') \equiv \sum_{y:|x-y|\wedge|x'-y|\leq \theta_{N,1}} E_x E'_{x'} \mathbb{1}_{\sigma(y)\leq \theta_{N,1}} \mathbb{1}_{\sigma'(y)\leq \theta_{N,1}}. \qquad (4.75)$$

We bound $I_{\theta_{N,1}}(x,x')$ by

$$I_{\theta_{N,1}}(x,x') \leq P_x P'_{x'}\left(\max_{y\in\mathbb{Z}^d}(\sigma(y)\vee\sigma'(y))\leq \theta_{N,1}\right)\left(E_x R_{\theta_{N,1}} \vee E_{x'} R_{\theta_{N,1}}\right). \quad (4.76)$$

Since $x' \in D_1(x)$, the probability in (4.76) is smaller than the probability that, during the time interval $[0,\theta_n]$, J (or J') visits a point that is at distance at least $\frac{1}{2}\sqrt{\theta_n}\log \theta_n$ from its starting point. By (3.9) of Lemma 3.2, on $A_{N,1}$, this is bounded above by $e^{-c_4/4(\log \theta_{N,0})^2}$. Thus, by Lemma 3.3,

$$\mathbb{E}I_{\theta_{N,1}}(x,x')\mathbb{1}_{A_{N,1}} \leq c_5\theta_{N,1}\exp(-c'(\log \theta_{N,0})^2), \qquad (4.77)$$

where $c' = c_4/4$. We use (4.77) and get that, for $x \in B_{d_{N,1}(t)}, x' \in D_1(x)$,

$$\mathbb{E}(Q_{N,0,1}(x)Q_{N,0,1}(x')\mathbb{1}_{A_{N,1}}) \leq c_5(\rho_N(d)\theta_{N,1}e^{-c'(\log \theta_{N,0})^2} + c_{N,0}^{-3\alpha/2}). \qquad (4.78)$$

By (3.38) of Lemma 3.6 we have for any ball $B_r(y)$ with $r \leq d_{N,1}(t)$ that

$$k_{N,1}(t)\sum_{x\in B_r(y)}\overline{\pi}_N^t(x) \leq \frac{\log\log a_{N,1}}{\theta_{N,1}}\min(r^2, a_{N,1}). \qquad (4.79)$$

By (4.79),

$$(k_{N,1}(t))^2 \sum_{x\in B_{d_{N,1}(t)}, x'\in D_1(x)} \overline{\pi}_N^t(x)\overline{\pi}_N^t(x') \leq c_3 k_{N,1}(t)(\log\log a_{N,1})^2. \qquad (4.80)$$

Combining (4.78) and (4.80),

$$\sum_{x\in B_{d_{N,1}(t)}, x'\in D_1(x)} \overline{\pi}_N^t(x)\overline{\pi}_N^t(x')\mathbb{E}(Q_{N,0,1}(x)Q_{N,0,1}(x')) \leq K(u,t)e^{-c'(\log \theta_{N,0})^2}, \qquad (4.81)$$

which is smaller than the right hand side of (4.61). Let $x' \in D_2(x)$. By Cauchy-Schwarz inequality, $\mathbb{E}(Q_{N,0,1}(x)Q_{N,0,1}(x')) \leq \rho_N(d)$ and so, by (4.79),

$$(k_{N,1}(t))^2 \sum_{x\in B_{d_{N,1}(t)}, x'\in D_2(x)} \overline{\pi}_N^t(x)\overline{\pi}_N^t(x')\rho_N(d) \leq c_3 k_{N,1}(t)\rho_N(d)(\log \theta_{N,1})^2, \quad (4.82)$$

as desired in (4.61). This finishes the proof of Lemma 4.2 for $d \geq 3$.

Let $d = 2$. Writing $\overline{Q}_{N,r_{i+1},r_i}(x) \equiv Q_{N,r_{i+1},r_i}(x) - \mathbb{E}Q_{N,r_{i+1},r_i}(x)$, the left hand side of (4.62) is given by

$$k_{N,1}^4(t) \sum_{(x_1,x_2,x_3,x_4):\min_{i,j}|x_i-x_j|\leq 2\theta_{N,1}} \mathbb{E}\big(\prod_{j=1}^4 \inf_{r\in\mathcal{I}_i} \overline{\pi}_{N,r}^t(x_j)\overline{Q}_{N,r_{i+1},r_i}(x_j)\big), \quad (4.83)$$

where we used that the $\overline{Q}_{N,r_{i+1},r_i}$'s are independent whenever $\min_{i,j}|x_i - x_j| > 2\theta_{N,1}$. Let us distinguish three cases, depending on the size of $\max_{i,j}|x_i - x_j|$. Suppose first that $\max_{i,j}|x_i - x_j| \leq \theta_{N,1}^{1/2}\log\log\theta_{N,1}$. Then, by Cauchy-Schwarz inequality and Lemma 4.3,

$$\mathbb{E}\big(\prod_{j=1}^4 \overline{Q}_{N,r_{i+1},r_i}(x_j)\big) \leq \rho_N(2,4) + 24(\rho_N(2,2))^2. \quad (4.84)$$

By (3.47) and since $\rho_N(2,4) \geq (\rho_N(2,2))^2$, it suffices to control the following quantity:

$$k_{N,1}^4(t) \sum_{(x_1,\ldots,x_4):\max_{i,j}|x_i-x_j|\leq\theta_{N,1}^{1/2}\log\log\theta_{N,1}} \prod_{j=1}^4 \overline{\pi}_N^t(x_j)\rho_N(2,4). \quad (4.85)$$

This is smaller than

$$k_{N,1}^4(t) \sum_{x_1\in B_{d_{N,1}(t)}} \overline{\pi}_N^t(x_1) \prod_{j=2}^4 \sum_{x_l\in B_{4\theta_{N,1}^{1/2}\log\log\theta_{N,1}}(x_1)} \overline{\pi}_N^t(x_j)\rho_N(2,4). \quad (4.86)$$

By (3.39) of Lemma 3.6 we have that

$$k_{N,1}(t) \sum_{x\in B_r(y)} \overline{\pi}_N^t(x) \leq \theta_{N,0}^{-1}(r^2 \wedge a_{N,1})\big(\log(a_{N,1}/|y|^2) \vee \log\log a_{N,1}\big). \quad (4.87)$$

This proves that the contribution to (4.83) coming from x_i's such that $\max_{i,j}|x_i - x_j| \leq \theta_{N,1}^{1/2}\log\log\theta_{N,1}$ is bounded above by $K(u,t)(\log\log a_{N,1})^6(\log\theta_{N,0})^{-4}$. Now suppose that $\max_{i,j}|x_i-x_j| > \theta_n^{1/2}\log\theta_{N,1}$. Assume that $|x_1-x_2| = \max_{i,j}|x_i-x_j|$. For $j = 3,4$ we have the bound $|\overline{Q}_{N,r_{i+1},r_i}(x_j)| \leq \theta_{N,1}^d$. Then, the only term that remains in the expectation in (4.83) is $\mathbb{E}\overline{Q}_{N,r_{i+1},r_i}(x_1)\overline{Q}_{N,r_{i+1},r_i}(x_2)$, which is smaller than

$$\mathbb{E}(Q_{N,0,1}(x_1)Q_{N,0,1}(x_2)\mathbb{1}_{A_{N,1}}) + e^{-cN} \leq 2\rho_N(2,2)e^{-c_4(\log\theta_{N,0})^2}, \quad (4.88)$$

where $c \in (0,\infty)$, and where the second inequality is proved as in (4.78). Hence, the contribution to (4.83) coming from x_i's such that $\max_{i,j}|x_i-x_j| > \theta_{N,1}^{1/2}\log\theta_{N,1}$ is bounded above by $e^{-c(\log\theta_{N,0})^2}$. Finally, let $\theta_{N,1}^{1/2}\log\log\theta_{N,1} \leq \max_{i,j}|x_i - x_j| \leq \theta_{N,1}^{1/2}\log\theta_{N,1}$. Suppose again that $|x_1 - x_2| = \max_{i,j}|x_i - x_j|$. By Cauchy-Schwarz and Lemma 4.3,

$$\mathbb{E}\left(\prod_{j=1}^4 \overline{Q}_{N,r_{i+1},r_i}(x_j)\right) \leq \{\mathbb{E}(Q_{N,0,1}(x_1)Q_{N,0,1}(x_2))^2\rho_N(2,4)\}^{1/2}$$
$$+ c(\rho_N(2,2))^2. \quad (4.89)$$

As in (4.88) and (4.78) we know that the first summand in (4.89) is bounded above by

$$\mathbb{E}(Q_{N,0,1}(x_1)Q_{N,0,1}(x_2))^2 \leq \rho_N(2,2)e^{-c_4(\log\log\theta_{N,0})^2}, \quad (4.90)$$

which is larger than the second summand. Hence, by (4.87), the contribution to (4.83) coming from x_i's such that $\theta_{N,1}^{1/2}\log\log\theta_{N,1} \leq \max_{i,j}|x_i - x_j| \leq \theta_{N,1}^{1/2}\log\theta_{N,1}$ is bounded above by $K'(u,t)e^{-c_4/4(\log\log\theta_{N,0})^2}$. The proof of Lemma 4.2 is finished.

$\square$

4.3. *Convergence of $\widetilde{\sigma}_n^t(u,\infty)$.* We establish that, $\mathbb{P}$-a.s., $\lim_{n\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)=0$.

As in the proof of Lemma 3.4 we consider subsequences $c_{N,r}=\exp(N+r)$ first (see the paragraph below (3.18)). With the notation of (4.54) and (3.47),

$$\mathbb{E}\sup_{r\in[0,1)}\widetilde{\sigma}_{N,r}^t(u,\infty)\leq k_{N,1}(t)\sum_{x\in B_{d_{N,1}(t)}}\overline{\pi}_N^t(x)\mathbb{E}(Q_{N,0,1}^u(x))^2\equiv\mathbb{E}\sigma_N^t(u,\infty).$$
(4.91)

As in Lemma 4.3 one can show that $\mathbb{E}(Q_{N,0,1}^u(0))^2\leq\rho_N^u(d)$, and so, by Lemma 3.6 there exists $K'\in(0,\infty)$ such that

$$\mathbb{E}\sigma_N^t(u,\infty)\leq K'tu^{-\alpha}((\log\theta_{N,0})^{-1}\mathbb{1}_{d=2}+\theta_{N,1}^{-1/2}\mathbb{1}_{d\geq3}).$$
(4.92)

For $d\geq3$, this is summable in N and we get that, $\mathbb{P}$-a.s., $\lim_{n\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)=0$. When $d=2$, the right hand side of (4.92) is not summable in N. Thus, to prove that $\widetilde{\sigma}_n^t(u,\infty)$ vanishes $\mathbb{P}$-a.s., we bound the variance of $\sigma_N^t(u,\infty)$. Using the same calculations as in the proof of (4.62) of Lemma 4.2 one can show that the variance of $\sigma_N^t(u,\infty)$ is bounded above by $N^{-4+\varepsilon}$. Since this is summable in N, we obtain that, $\mathbb{P}$-a.s., $\lim_{n\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)=0$ for $d=2$ as well.

4.4. *Verification of Condition (C-4).* We follow the same strategy as in the verification of (C-2). We first prove that $\lim_{n\to\infty}\mathbb{E}m_n^t(\varepsilon)\leq C(t)\varepsilon^{1-\alpha}$. Then, one can show as in Section 4.2 that, $\mathbb{P}$-a.s., $\lim_{n\to\infty}m_n^t(\varepsilon)\leq C(t)\varepsilon^{1-\alpha}$. Since this is very similar to Section 4.2, we leave the details to the interested reader. Let us bound $\mathbb{E}m_n^t(\varepsilon)$. Since the τ's are i.i.d. and since $a_n\theta_n^d\mathbb{P}(A_n^c)\leq n^{-3}$, it suffices to find $c\in(0,\infty)$ such that

$$\sum_{y\in B_{\theta_n}}\mathbb{E}(M_n^\varepsilon(0,y)\mathbb{1}_{A_n})\leq c\theta_na_n^{-1}\varepsilon^{1-\alpha}.$$
(4.93)

Fix $y\in B_{\theta_n}$ and set $h(\theta_n)=\theta_n-k(\theta_n)$ for $k(\theta_n)=\theta_n^{3/4}$. As in (4.4) and (4.9), by the Markov property,

$$\begin{aligned}M_n^\varepsilon(0,y)&\leq&\varepsilon P(\sigma(y)\in(h(\theta_n),\theta_n))\mathbb{1}_{y\in T_n}\\&&+P(\sigma(y)\leq h(\theta_n))\gamma_n(y)\mathcal{E}_y\big(\ell_{\theta_n}(y)\mathbb{1}_{\ell_{k(\theta_n)}(y)\gamma_n(y)\leq\varepsilon}\big)\mathbb{1}_{y\in T_n}\\&\equiv&M_{n,1}(y)+M_{n,2}(y)\end{aligned}$$
(4.94)

Let us first establish, that the sum over $M_{n,1}(y)$ is as in (4.93). Following the same argumentation as between (4.12) and (4.14), we can show that

$$M_{n,1}(y)\leq\varepsilon(P(\min_{y'\sim y}\sigma(y)\in(h(\theta_n),\theta_n))+c_n^{-\theta/2})\mathbb{1}_{y\in T_n}.$$
(4.95)

Since $\min_{y'\sim y}\sigma(y)$ is independent of $\gamma_n(y)$, we get by Lemma 3.3 that

$$\sum_y\mathbb{E}(M_{n,1}(y)\mathbb{1}_{A_n})\leq\sum_y\mathbb{E}\big[\mathbb{1}_{A_n}P(\sigma(y)\in(h(\theta_n),\theta_n))\mathbb{1}_{y\in T_n}\big]\leq c_5c_n^{-\alpha}\epsilon_n^{-\alpha}\theta_n^{3/4},$$
(4.96)

as desired. Let us now bound the expectation of $M_{n,2}(y)$. First we calculate the expected value with respect to $\mathcal{E}_y$ in $M_{n,2}(y)$. As in (4.10) and (4.11), we get that, up to an error of the order of $e^{-c_4(\log\theta_n)^2}$, on A_n, we can bound $\ell_{\eta(B_n^1)}(y)\leq\ell_{k(\theta_n)}(y)$ and $\ell_{\theta_n}(y)\leq\ell_{\eta(B_n^2)}(y)$, where $B_n^1=B_{k(\theta_n)^{1/2}(\log\theta_n)^{-2}}(y)$ and $B_n^2=B_{\theta_n^{1/2}\log\theta_n}(y)$, for all $y\in B_{\theta_n}\cap T_n$. Setting $\varepsilon(y)\equiv\{\ell_{\eta(B_n^1)}(y)\gamma_n(y)\leq\varepsilon\}$, we get

$$\begin{aligned}&\mathcal{E}_y\ell_{\theta_n}(y)\mathbb{1}_{\ell_{k(\theta_n)}(y)\gamma_n(y)\leq\varepsilon}\\&=\big(\mathcal{E}_y\mathbb{1}_{\varepsilon(y)}\ell_{\eta(B_n^1)}(y)+\mathcal{E}_y\mathbb{1}_{\varepsilon(y)}\big(\ell_{\eta(B_n^2)}(y)-\ell_{\eta(B_n^1)}(y)\big)\big).\end{aligned}$$
(4.97)

By the strong Markov property, the second term in (4.97) is given by

$$\sum_{z \in \partial B_n^1} \mathcal{E}_y \big(\mathbb{1}_{\varepsilon(y)} \mathbb{1}_{J(\eta(B_n^1))=z} \big) E_z \int_0^{\eta(B_n^2)} \mathbb{1}_{J(s)=y} ds \le g_{B_n^2}(y) \mathcal{P}_y(\varepsilon(y)), \qquad (4.98)$$

where we used $E_z \int_0^{\eta(B_n^2)} \mathbb{1}_{J(s)=y} ds \le g_{B_n^2}(y)$. The first term in (4.97) equals

$$g_{B_n^1}(y) \big[1 - \exp \big(-\varepsilon/(\gamma_n(y) g_{B_n^1}(y)) \big) \big] = g_{B_n^1}(y) \mathcal{P}_y(\varepsilon(y)). \qquad (4.99)$$

Using (4.98) and (4.99) and the fact that $g_{B_n^1}(y) \le g_{B_n^2}(y)$, (4.97) is bounded by

$$2 g_{B_n^2}(y) \big[1 - \exp \big(-\varepsilon/(\gamma_n(y) g_{B_n^1}(y)) \big) \big]$$
$$\le \quad 2 c_8/c_7 \bar{g}_n^d(y) \big[1 - \exp \big(-\varepsilon/(\gamma_n(y) \bar{g}_n^d(y)) \big) \big] \mathbb{1}_{A_n} \qquad (4.100)$$
$$+ \quad c_1 (\log \theta_n \mathbb{1}_{d=2} + \mathbb{1}_{d \ge 3}) \mathbb{1}_{A_n^c} \big],$$

where by (4.20) (for $d \ge 3$) and Lemma 3.3 in Černý (2011) (for $d = 2$), $\bar{g}_n^d(y) \equiv c_7 (\log \theta_n \mathbb{1}_{d=2} + g_\infty(y) \mathbb{1}_{d \ge 3})$. Together with (4.12), $\mathbb{E} M_{n,2}(y) \mathbb{1}_{A_n}$ is bounded above by

$$\varepsilon \, \mathbb{E} \big[P(\min_{y' \sim y} \sigma(y') \le \theta_n) \bar{g}_n^d(y) \gamma_n(y) \big(1 - e^{-\varepsilon(\gamma_n(y) \bar{g}_n^d(y))^{-1}} \big) \mathbb{1}_{y \in T_n} \mathbb{1}_{A_n} \big]. \quad (4.101)$$

An asymptotic analysis and the fact that $\bar{g}_n^d(y) \le c_7 (\log \theta_n \mathbb{1}_{d=2} + c_6 \mathbb{1}_{d \ge 3})$ yield

$$(4.101) \le c' \varepsilon \, \mathbb{E} \Big[P(\min_{y' \sim y} \sigma(y') \le \theta_n) e^{-2\varepsilon(c' \gamma_n(y) \bar{g}_n^d(y))^{-1}} \mathbb{1}_{y \in T_n} \mathbb{1}_{A_n} \Big]$$
$$\le c'' (\log \theta_n \mathbb{1}_{d=2} + c_6 \mathbb{1}_{d \ge 3})^\alpha c_n^{-\alpha} \varepsilon^{1-\alpha} \mathbb{E}(P(\min_{y' \sim y} \sigma(y') \le \theta_n) \mathbb{1}_{A_n}). \quad (4.102)$$

for some $c', c'' \in (0, \infty)$. Thus, evoking Lemma 3.3, we get that

$$\sum_y \mathbb{E} M_{n,2}(y) \mathbb{1}_{A_n} \le c'' c_6 c_5 t \varepsilon^{1-\alpha}, \qquad (4.103)$$

i.e. (4.93) is satisfied. Thus, $\lim_{n \to \infty} \mathbb{E} m_n^t(\varepsilon) \le c \varepsilon^{1-\alpha}$. The verification of (C-4) now follows as in Section 4.2.

4.5. *Verification of Condition (C-5).* We proceed as in the verification of (C-2) and (C-4) to establish that (C-5) is satisfied. Namely, we first take the expected value in the left hand side of (3.36) and (3.37) and prove that both are bounded above by $C(u,t) \varepsilon$ for some $C(u,t) \in (0, \infty)$. Then, one can proceed as in Section 4.2 to obtain $\mathbb{P}$-a.s. upper bounds. Since the proofs are similar, we only prove the claim for (3.36). The expectation of the left hand side of (3.36) is given by

$$\sum_{(x,k) \in \mathcal{A}_n} (k\theta_n)^{-d/2} e^{-c_2 |x|^2/k\theta_n} \sum_y \mathbb{E} Q_n^u(x,y)$$

$$\le 2\nu(u,\infty) a_n^{-1} \theta_n \sum_{k=1}^{k_n(t)} (k\theta_n)^{-d/2} \sum_{|x|^2 \ll k\theta_n \, \vee \, |x|^2 > k\theta_n/\varepsilon} e^{-c_2 |x|^2/k\theta_n}, \quad (4.104)$$

where we used that by (4.2), for n large enough, the second sum in (4.104) is smaller than $2\nu(u,\infty)\theta_n/a_n$. Let us first control the contribution to the right hand side of (4.104) coming from $x \in B_{\sqrt{\varepsilon k\theta_n}}$. Bounding the exponential term by one and using the fact that $|\{x : |x|^2 < \varepsilon k\theta_n\}| \le C \varepsilon^{d/2} (k\theta_n)^{d/2}$ for some $C \in (0, \infty)$, this contribution is bounded above by $K(u,t) \varepsilon^{d/2}$ for $K(u,t) \in (0, \infty)$. Also, the sum over $x \in B_{d_n(t)} \setminus B_{(k\theta_n/\varepsilon)^{1/2}}$ of $(k\theta_n)^{-d/2} e^{-c_2 |x|^2/k\theta_n}$ is bounded above by $K'(t) e^{-c_2/\varepsilon}$. Thus, (4.104) is bounded above by $K(u,t) \varepsilon^{d/2}$. The verification of (C-5) can now be finished as in Section 4.2.

4.6. *Conclusion of the proof.* We are now ready to conclude the proof of Theorem 1.5. By Lemma 4.1, for all $u > 0$, $t > 0$, $\mathbb{E}\tilde{\nu}_n^t(u, \infty) \to t\nu(u, \infty)$ as $n \to \infty$. Together with the results of Section 4.2 this shows that (C-2) is satisfied. By the results of Section 4.3, $\tilde{\sigma}_n^t(u, \infty)$ tends $\mathbb{P}$-a.s. to zero, proving (C-3). By the same arguments as those used to establish (C-2), (C-4) follows from the results of Section 4.4. Finally, it follows from the results of Section 4.5 that (C-5) is satisfied. Hence, we may apply Proposition 3.5 and get that $\overline{S}_n^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$. By Lemma 3.4 this proves that $S_n^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$, as claimed in Theorem 1.5.

5. Aging in BATM

In this section we present the proofs of Theorem 1.6 and Theorem 1.7. Section 5.1, respectively Section 5.2 and Section 5.3, contains the proof of Theorem 1.6 for $i = 1$, respectively $i = 2$ and $i = 3$. We then prove Theorem 1.6 in Section 5.4. The proofs in Sections 5.1-5.3 follow a common scheme. We show that for each $i \in \{1, 2, 3\}$, as $s \to \infty$, $\mathcal{C}_s^i(1, \rho)$, coincides with the probability of $\mathcal{M}_{s,\rho} \equiv \{\mathcal{R}_s \cap (1, 1 + \rho) = \emptyset\}$, where $\mathcal{R}_s = \{S_s^{J,b}(t), t \geq 0\}$ is the range of $S_s^{J,b}$. We then use that $\mathbb{P}$-a.s.,

$$\lim_{s \to \infty} \mathcal{P}(\mathcal{M}_{s,\rho}) = \mathrm{Asl}_\alpha(1/(1 + \rho)). \tag{5.1}$$

The proof of (5.1) closely follows that of Theorem 1.6 in Gayrard (2012). We thus only sketch it here. Namely, it relies on the continuity of the overshoot function that maps $Y \in D[0, \infty)$ to

$$\chi_u(Y) = Y(\mathcal{L}_u(Y)) - u, \quad u > 0, \tag{5.2}$$

where $\mathcal{L}_u \equiv \inf\{t \geq 0 : Y(t) > u\}$ is the time of the first passage of Y beyond the level $u > 0$. For Lévy motions having $\mathcal{P}$-a.s. diverging paths, this mapping is $\mathcal{P}$-a.s. continuous on $D[0, \infty)$ equipped with Skorohod's J_1 topology. Now, $\mathcal{M}_{s,\rho} = \{\chi_1(S_s^{J,b}) \geq 1 + \rho\}$ and by Theorem 1.5, $\mathbb{P}$-a.s., $S_s^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$. Since V_α has $\mathcal{P}$-a.s. diverging paths we deduce that, $\mathbb{P}$-a.s.,

$$\lim_{s \to \infty} \mathcal{P}(\mathcal{M}_{s,\rho}) = \mathcal{P}(\xi_1(V_\alpha) \geq 1 + \rho) = \mathrm{Asl}_\alpha(1/(1 + \rho)), \quad \rho > 0, \tag{5.3}$$

where the last equality follows from the arcsine law for stable subordinators (see Section III in Bertoin (1996)). Given (5.3), it remains to establish that, $\mathbb{P}$-a.s.,

$$\lim_{s \to \infty} |\mathcal{P}(\mathcal{M}_{s,\rho}) - \mathcal{P}(\mathcal{M}_{s,\rho}^i)| = 0, \quad \forall \rho > 0, \tag{5.4}$$

where $\mathcal{M}_{s,\rho}^i$ stands for the events appearing in the right hand sides of (1.41)-(1.43), namely $\mathcal{C}_s^i(1, \rho) = \mathcal{P}(\mathcal{M}_{s,\rho}^i)$. We will verify (5.4) in Sections 5.1-5.3.

For this, let $A' \subseteq A$ (where A as in (3.8)) such that $\mathbb{P}(A') = 1$ and such that for all $\omega \in A'$, $S_s^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$. Fix $\omega \in A'$. We write $a_s \equiv a_{\lfloor s \rfloor}$ and similarly for $\theta_s, k_s, \epsilon_s$, and δ_s (see Theorem 1.6 and Section 3.2 for their definitions).

5.1. *Convergence of* $\mathcal{C}_s^1(1, \rho)$. In this section we prove that (5.4) holds for $i = 1$.
Step 1. Let $\delta > 0$ and $s > 0$. We prove that for all s large enough, $\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^1)^c) \leq \delta$. For $k \in \mathbb{N}$ we define

$$B_k \equiv \{\textstyle\sum_{i=1}^k Z_{s,i}^J < 1, \quad \text{and} \quad \sum_{i=1}^{k+1} Z_{s,i}^J > (1 + \rho)\}. \tag{5.5}$$

Then, $\mathcal{M}_{s,\rho} = \bigcup_{k \geq 1} B_k$. For s large enough, there exists $T > 0$ large enough such that

$$\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^1)^c) \leq \mathcal{P}(\bigcup_{k \leq k_s(T)} B_k, (\mathcal{M}_{s,\rho}^1)^c) + \delta, \quad \forall s > s', \qquad (5.6)$$

where $k_s(T) = \lfloor a_s T \rfloor / \theta_s$. To see the claim of (5.6) note that, since $S_s^{J,b} \overset{J_1}{\Longrightarrow} V_\alpha$,

$$\mathcal{P}(\bigcup_{k \geq k_s(T)} B_k, (\mathcal{M}_{s,\rho}^1)^c) \leq \mathcal{P}(S_s^{J,b}(T) < 1) \leq \mathcal{P}(V_\alpha(T) < 1 + \delta) + \delta, \qquad (5.7)$$

which vanishes as $T \to \infty$ and proves (5.6). Let us now study B_k for fixed k. By Lemma 3.4, we know that for s large enough, the contribution to $Z_{s,k+1}$ coming from $y \notin T_s$ is bounded above by δ_s and therefore, on B_k, there exists $x \in T_s$ such that $\ell_{\theta_s(k+1)}(x) - \ell_{\theta_s k}(x) > 0$. Moreover, as in the proof (C-2) $\Rightarrow$ (B-2) (see proof of Proposition 3.5), one can show that this x is unique. Therefore, we have that $\ell_{a_s T}(x)\gamma_s(x) > \rho - \delta_s$. We prove now that, for all s large enough, with $\mathcal{P}$-probability larger than $1 - \delta$, we know on B_k that $\gamma_s(x) > \delta_s^{1/2}$. For this, note that by definition of A' (last paragraph of Section 5) and Lemma 3.2, for all but finitely many values of $\lfloor s \rfloor$, $\mathcal{P}(\eta(B_{d_s(T)}) > a_s T) \geq 1 - \delta$. Moreover, by the same lemma and (3.2), we have for all s large enough, $g_{B_{d_s(T)}(x)}(x) \leq c_9 \log a_s \mathbb{1}_{d=2} + c_6 \mathbb{1}_{d \geq 3}$ for all $x \in B_{d_s(T)}$. Therefore, for s large enough,

$$\mathcal{P}(\exists x : \gamma_s(x)\ell_{a_s T}(x) > \rho - \delta_s, \gamma_s(x) \leq \delta_s^{1/2})$$
$$\leq \delta + \sum_{x \in B_{d_s(T)}} P_x(\ell_{\eta(B_{d_s(T)}(x))}(x) > \rho \delta_s^{-1/2}) \leq \delta + |B_{d_s(T)}| \exp(-c'(\log a_s)^2), (5.8)$$

where we used that $\delta_s^{-1/2} \geq c_0(\log a_s)^3$. Therefore, for s large enough, we know with high $\mathcal{P}$-probability that, on B_k, there exists a unique $x \in T_s' \equiv \{x \in T_s : \gamma_s(x) > \delta_s^{1/2}\}$ that contributes to $Z_{s,k+1}$. Since this holds for all $1 \leq k \leq M$, we get for s large enough that

$$\mathcal{P}(\mathcal{M}_{s,\rho} \cap (\mathcal{M}_{s,\rho}^1)^c) \leq \mathcal{P}(\bigcup_{s' \in \{s,s(1+\rho)\}} M_{s'}) + 3\delta, \qquad (5.9)$$

where

$$M_{s'} \equiv \{X(s') \notin T_s'\} \cap \{\exists s'(1 - \delta_s) < v < s' : v : X(v) \in T_s'\}. \qquad (5.10)$$

Let us now bound $\mathcal{P}(M_{s'})$ for $s' = s$ and fixed $x \in T_s'$; the proof for $s' = s(1 + \rho)$ is the same. We distinguish two cases with respect to θ. We begin with $\theta > 0$. Fix v such that $s(1 - \delta_s) < v < s$ and $X(v) = x$. Let us first bound the probability that there exists v' such that $v \leq v' \leq s$ and $X(v') \notin B_1(x) \equiv \{x\} \cup \{y \sim x\}$. Writing N_x for the number of returns to x before J escapes $B_1(x)$, we have

$$\mathcal{P}(X(v) = x, \exists v' : v \leq v' \leq s : X(v') \notin B_1(x))$$
$$\leq P_x(N_x \leq \delta(s\delta_s^{1/2}\epsilon_s^{-2/\alpha})^\theta) + \mathcal{P}_x(\sum_{i=1}^{\delta(s\delta_s^{1/2}\epsilon_s^{-2/\alpha})^\theta} (\lambda(x))^{-1} e_i \leq \delta_s s). \quad (5.11)$$

Since $\max_{y \sim x}(1 - p(y, x)) \leq (s\delta_s^{1/2}\epsilon_s^{-2/\alpha})^{-\theta}$, the first probability in (5.11) is, smaller than δ. Since $\lambda(x) \leq 2ds^{-1+\theta}\epsilon_s^{-2\theta/\alpha}\delta_s^{-(1-\theta)/2}$ the law of large numbers implies that also the second probability in (5.11) is bounded above by δ, for s large enough. It remains to bound

$$\mathcal{P}(X(v) = x, X(s) \neq x, \forall v \leq v' \leq s : X(v') \in B_1(x)). \qquad (5.12)$$

By definition of T_s', $\max_{y \sim x}(\lambda(y))^{-1} \leq \epsilon_s^{-2\theta/\alpha}\delta_s^{\theta/2}s^{-\theta}$, and so, with probability larger than $1 - \exp(-\delta_s^{-\theta/2})$, there exists v' such that $s - v' \leq s^{-\theta}\epsilon_s^{-2\theta/\alpha}$ and

$X(v') = x$. By the Markov property we have for all such v',

$$\mathcal{P}_x(X(s - v') \neq x) \leq \mathcal{P}_x(e_1 \lambda^{-1}(x) < s - v') \leq 1 - e^{-s^{-1}\delta_s^{(1-\theta)/2}}, \qquad (5.13)$$

which tends to zero. Thus, (5.12) tends to zero as $s \to \infty$. This finishes the proof for $\theta > 0$. When $\theta = 0$, one can bound (5.9) directly as in (5.13). This shows that for all s large enough, $\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^1)^c) \leq \delta$.

Step 2. Let us now show that $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^1) \to 0$. Let $m_{s,\rho} \equiv (S^J)^{\leftarrow}(s(1 + \rho)) - (S^J)^{\leftarrow}(s)$, where $(S^J)^{\leftarrow}(t) = \inf\{v \geq 0 : S^J(v) > t\}$. Notice that $(\mathcal{M}_{s,\rho})^c \subseteq \{m_{s,\rho} \geq \theta_s\} \cup \{Z_{s,1} > 1\}$. By (A-0), $\mathcal{P}(Z_{s,1} > \rho)$ tends to zero and so, for all $\delta > 0$ there exists s large enough such that $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^1) \leq \mathcal{P}(m_{s,\rho} \geq \theta_s, \mathcal{M}_{s,\rho}^1) + \delta$. Let us distinguish whether $d \geq 3$ or $d = 2$. In the first case we use the identity $X(t) = J((S^J)^{\leftarrow}(t))$ and get by (3.2) of Theorem 3.1, uniformly in $x \in \mathbb{Z}^d$,

$$\mathcal{P}_x(X(s(1 + \rho)) = x, m_{s,\rho} \geq \theta_s) = \mathcal{P}_x(J(m_{s,\rho}) = x, m_{s,\rho} \geq \theta_s) \leq \int_{\theta_s}^{\infty} v^{-d/2} dv, \qquad (5.14)$$

which is smaller than $\theta_s^{-d/2+1}$ and shows that $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^1) \to 0$ for $d \geq 3$.

Let $d = 2$. We construct a more precise bound for $\mathcal{P}((\mathcal{M}_{s,\rho})^c \cap \mathcal{M}_{s,\rho}^1)$ than $\mathcal{P}(\{m_{s,\rho} \geq \theta_s\} \cap \mathcal{M}_{s,\rho}^1) + \delta$. Assume first that $\text{dist}(\mathcal{R}_s, 1 + \rho) > \delta$ and that there are $t, t' > 0$ such that $S_s^{J,b}(t), S_s^{J,b}(t + t') \in (1, 1 + \rho - \delta/2)$. Then, $s < S^J(k_s(t)\theta_s) < S^J(k_s(t + t')\theta_s) < s(1 + \rho)$ and so $m_{s,\rho} \geq \theta_s(k_s(t + t') - k_s(t'))$. Moreover, by (5.7) there exists $T > 0$ such that $m_{s,\rho} \leq \theta_s k_s(T)$. Since $\text{dist}(\mathcal{R}_s, 1 + \rho) > \delta$ one can show as in Step 1 that $X(s(1 + \rho)) = x \in T_s$. But then, on $(\mathcal{M}_{s,\rho})^c \cap \mathcal{M}_{s,\rho}^1$, we have with probability larger than $1 - (\log \theta_s)^{-2}$ that $\ell_{m_{s,\rho}}(x) - \ell_{m_{s,\rho} - \theta_s}(x) > c \log \theta_s / \log \log \theta_s$ for some $c \in (0, \infty)$. $\mathcal{R}_s \cap (1, 1 + \rho) = \emptyset$ or $X(s) \neq x$.) By (3.2) of Theorem 3.1 we get, for all $x \in \mathbb{Z}^d$,

$$\mathcal{P}_x\big(J(m_{s,\rho}) = x, m_{s,\rho} \in (\theta_s k_s(t), \theta_s k_s(T)), \ell_{m_{s,\rho}}(x) - \ell_{m_{s,\rho} - \theta_s}(x) > \tfrac{c \log \theta_s}{\log \log \theta_s}\big)$$

$$\leq P_x(\ell_{\theta_s k_s(T)}(x) - \ell_{\theta_s k_s(t) - \theta_s}(x) > \tfrac{c \log \theta_s}{\log \log \theta_s}) \leq c \tfrac{c \log \log \theta_s}{\log \theta_s} \log(T/t), \qquad (5.15)$$

which tends, as $s \to \infty$, to zero. It remains to establish that for all $\delta' > 0$ there exist $\delta > 0$, $t > 0$ such that $\mathcal{P}(\text{dist}(\mathcal{R}_s, 1 + \rho) > \delta) \leq 1 - \delta'$ and $\mathcal{P}(S_s^{J,b}(t + t') \in (1, 1 + \rho - \delta/2)|S_s^{J,b}(t) \in (1, 1 + \rho - \delta)) \geq 1 - \delta'$. This can be derived from the convergence of $S_s^{J,b}$ to V_α and properties of V_α (see Section III in Bertoin (1996)). Thus, $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^1) \leq \delta + 2\delta'$ for $d \geq 2$. Together with Step 1 this finishes the proof of (5.4) for $i = 1$.

5.2. *Convergence of* $\mathcal{C}_s^2(1, \rho)$. In this section we prove the claim of (5.4) for $i = 2$.
Step 1. We show that $\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^2)^c) \to 0$. Note that $\mathcal{M}_{s,\rho} \subseteq \{m_{s,\rho} \leq \theta_s\}$. Let $x \in B_{a_s}$. Using $X(s) = J((S^J)^{\leftarrow}(s))$ and the Markov property

$$\mathcal{P}(X(s) = x, (\mathcal{M}_{s,\rho}^2)^c, m_{s,\rho} \leq \theta_s)$$

$$\leq \mathcal{P}(X(s) = x) P_x(\eta(B_{(\theta_s \log \theta_s)^{1/2}}(x)) \leq \theta_s), \qquad (5.16)$$

where $\eta(B)$ is the exit time of B for J as defined in Section 3.1. By definition of A' (last paragraph of Section 5) and Lemma 3.2 we have for all but finitely many values of $\lfloor s \rfloor$, for all $x \in B_{a_s}$,

$$P_x(\eta(B_{(\theta_s \log \theta_s)^{1/2}}(x)) \leq \theta_s) \leq \exp(-c_4 \log \theta_s). \qquad (5.17)$$

If we can show that $\mathcal{P}(X(s) \notin B_{a_s}) \leq \delta$, then (5.16)-(5.17) imply that $\mathcal{P}(\mathcal{M}_{s,\rho}(\mathcal{M}_{s,\rho}^2)^c) \leq \delta$. To bound $\mathcal{P}(X(s) \notin B_{a_s})$ we recall that by (5.7), with

probability larger than $1 - \delta$, $(S^J)^{\leftarrow}(s) \leq k_s(T)\theta_s$ for $T > 0$ and so, for s large enough,

$$\mathcal{P}(X(s) \notin B_{a_s}, (S^J)^{\leftarrow}(s) \leq a_s T) \leq P(\eta(B_{a_s}) \leq a_s T) \leq e^{-c_4 a_s^{1/2} T^{-2}}, \qquad (5.18)$$

by Lemma 3.2. This tends to zero and we conclude that $\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^2)^c) \leq \delta$.
Step 2. Now we prove that $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^2)$ vanishes. On $(\mathcal{M}_{s,\rho})^c$ one can show as in Section 5.1 (Step 2) that there exist t, t' such that $m_{s,\rho} \geq \theta_s(k_s(t + t') - k_s(t)) \gg \theta_s^2$. Moreover, by (5.18) we know that, with probability larger than $1 - \delta$, $X(s) \in B_{a_s}$. By definition of A' and Lemma 3.2, for all but finitely many values of $\lfloor s \rfloor$, for all $x \in B_{a_s}$

$$P_x(\eta(B_{(\theta_s \log \theta_s)^{1/2}}(x)) > \theta_s^2) \leq \exp(-c_4 \log \theta_s). \qquad (5.19)$$

As in (5.16) we thus get $\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^2) \to 0$. Together with Step 1, the proof of (5.4) is finished for $i = 2$.

5.3. Convergence of $\mathcal{C}_s^3(1, \rho)$.
We now show that (5.4) holds for $\mathcal{C}_s^3(1, \rho)$. This follows readily from Sections 5.1 and 5.2. Indeed on the one hand,

$$\mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^3) \leq \mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^1) + \mathcal{P}((\mathcal{M}_{s,\rho})^c, \mathcal{M}_{s,\rho}^2), \qquad (5.20)$$

and on the other hand, $\mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^3)^c) \leq \mathcal{P}(\mathcal{M}_{s,\rho}, (\mathcal{M}_{s,\rho}^1)^c)$. Both upper bounds tend by Sections 5.1 and 5.2 $\mathbb{P}$-a.s. to zero, which proves (5.4) for $\mathcal{C}_s^3(1, \rho)$.

5.4. Convergence of $\mathcal{C}_s^\varepsilon(1, \rho)$ and $\mathcal{C}^\varepsilon(1, \rho)$.
In this section we prove Theorem 1.7. The convergence of $\mathcal{C}_s^\varepsilon(1, \rho)$ can be proved as that of $\mathcal{C}_s^2(1, \rho)$ and we only establish the claim of Theorem 1.7 for $\mathcal{C}^\varepsilon(1, \rho)$. Let us write in short $\mathcal{M}^\varepsilon(\rho)$ for the event in the right hand side of (1.49), i.e. $\mathcal{C}^\varepsilon(1, \rho) = \mathcal{P}(\mathcal{M}^\varepsilon(\rho))$. One can show that

$$\mathcal{C}_s^{\varepsilon/2}\left(1 - \varepsilon^2, \tfrac{\rho + 2\varepsilon^2}{1 - \varepsilon^2}\right) - \left[1 - \mathcal{C}_s^{\varepsilon/2}\left(1 - \varepsilon^2, \tfrac{2\varepsilon^2}{1 - \varepsilon^2}\right)\right] - \delta_s \leq \mathcal{C}^\varepsilon(1, \rho) \leq \mathcal{C}_s^\varepsilon(1, \rho) + \delta_s, (5.21)$$

where $\delta_s \equiv \delta_s(\rho, \varepsilon)$ is given by

$$\delta_s = \mathcal{P}(\max_{v \in (1 - \varepsilon^2, 1 + \rho + \varepsilon^2)} \max_{v' \in (1 - \varepsilon^2, 1 + \varepsilon^2)} |X_s(v') - X_s(v)| \leq \varepsilon/2, (\mathcal{M}^\varepsilon(\rho))^c). \qquad (5.22)$$

Now, by Theorem 1.3 in Barlow and Černý (2011a) (see the erratum Barlow and Černý (2011b) to this theorem) and Theorem 1.1 in Černý (2011), $\mathbb{P}$-a.s., $X_s \overset{J_1}{\Longrightarrow} Z_{d,\alpha}$. By definition of Skorohod's metric, there exists $\delta > 0$ such that, for s large enough and $\lambda : [0, 1 + \rho] \to [0, 1 + \rho]$ strictly increasing and continuous,

$$\mathcal{P}\left(\max\{\max_{v \in [0, 1 + \rho]} |X_s(\lambda(v)) - Z_{d,\alpha}(v)|, \max_{v \in [0, 1 + \rho]} |\lambda(v) - v|\} > \varepsilon^2\right) \leq \delta, \qquad (5.23)$$

and so δ_s vanishes as first $s \to \infty$ and then $\varepsilon \to 0$. By the statement of Theorem 1.6 for $\mathcal{C}_s^\varepsilon(1, \rho)$, $\mathcal{C}_s^\varepsilon(1, \rho)$ and $\mathcal{C}_s^{\varepsilon/2}\left(1 - \varepsilon^2, \tfrac{\rho + 2\varepsilon^2}{1 - \varepsilon^2}\right)$ tend to $\mathrm{Asl}_\alpha(1/(1 + \rho))$ as first $s \to \infty$ and then $\varepsilon \to 0$. It remains to show that $1 - \mathcal{C}_s^{\varepsilon/2}\left(1 - \varepsilon^2, \tfrac{2\varepsilon^2}{1 - \varepsilon^2}\right)$ vanishes. But this can be done as in Section 5.2 (Step 2). The proof of Theorem 1.7 is finished.

6. Appendix

Proof of Lemma 3.2: Fix $x \in B_{a_n}$ and take $\omega \in A_n$. We use Proposition 2.18 in Barlow and Deuschel (2010) to prove (3.9). This proposition states that there exists $c_4 \in (0, \infty)$ such that for all $m_n \gg r_n$ for which $U_z \leq m_n/r_n$ for all $z \in B_{r_n}(x)$, $P_x(\eta(B_{r_n}(x)) \leq m_n) \leq e^{-c_4 r_n^2 m_n^{-1}}$, as desired in (3.9). Since we assume $m_n \gg r_n$, it remains to verify whether $U_z \leq m_n/r_n$ for all $z \in B_{r_n}(x)$. But $B_{r_n}(x) \subseteq B_{2a_n}$ and so (3.7) implies that, $U_z \leq c_0(\log a_n)^3 \leq m_n/r_n$ for all $z \in B_{r_n}(x)$. This finishes the proof of (3.9). The proof of (3.10) is as the proof of Lemma 3.2 in Černý (2011), where the claim is proved for $d = 2$. $\qquad\square$

Proof of Lemma 3.3: Let us first establish (3.12). We begin with the contribution to $\mathbb{E}ER_{m_n}^k$ coming from $y \notin B_{\sqrt{m_n}\log m_n}$. By (3.9) of Lemma 3.2 and (3.3) of Theorem 3.1,

$$\sum_{|y| \geq \sqrt{m_n}\log m_n} \mathbb{E}(P(\sigma(y) \leq m_n))$$
$$\leq \quad \sum_{|y| \geq \sqrt{m_n}\log m_n} (|y|^{d-1} e^{-c_4(\log|y|)^2} + |y|^{2d} \exp(-c_2|y|^{1/3})), \qquad (6.1)$$

where we used that $P(\sigma(y) \leq m_n) \leq P(\sigma(y) \leq |y|^2)$ for $|y| \geq \sqrt{m_n}\log m_n$. Hence, the contribution to $\mathbb{E}ER_{m_n}^k$ coming from such y's tends to zero. Now, let $y \in B_{\sqrt{m_n}\log m_n}$. Since $|B_{\sqrt{m_n}\log m_n}|\mathbb{P}(A_n^c) \ll n^{-2}$, it suffices to bound $\mathbb{E}[P(\sigma(y) \leq m_n)\mathbb{1}_{A_n}]$. We have that,

$$P(\sigma(y) \leq m_n) \leq P(\eta(B_{\sqrt{m_n}\log m_n}) \leq m_n) + P(\sigma(y) \leq \eta(B_{\sqrt{m_n}\log m_n})). \ (6.2)$$

By (3.9) of Lemma 3.2, on A_n, the first probability in (6.2) is smaller than $e^{-c_4(\log m_n)^2}$. By the strong Markov property,

$$P(\sigma(y) \leq \eta(B_{\sqrt{m_n}\log m_n})) = g_{B_{\sqrt{m_n}\log m_n}}(0, y)(g_{B_{\sqrt{m_n}\log m_n}}(y, y))^{-1}, \qquad (6.3)$$

where $g_B(x, z) = E_x\left(\int_0^{\eta(B)} \mathbb{1}_{J(t)=z} dt\right)$. Write

$$D_1 = B_{\sqrt{m_n}/2}, \text{ and } D_2 = B_{\sqrt{m_n}\log m_n} \setminus D_1.$$

We distinguish whether $d \geq 3$ or $d = 2$. Let $d \geq 3$ first. Take $y \in D_1$. By (3.2) and (3.4) of Theorem 3.1, $g_{B_{\sqrt{m_n}\log m_n}}(0, y) \leq c_3(m_n^{-d/2+1} \wedge |y|^{2-d})$, and so

$$\sum_{y \in D_1} \mathbb{E}(P(\sigma(y) \leq m_n))^k \leq \sum_{y \in D_1} c_3(m_n^{-d/2+1} \wedge |y|^{2-d})^k \mathbb{E}\left(g_{B_{\sqrt{m_n}\log m_n}}(y, y)\right)^{-k}.$$
$$(6.4)$$

Since $B_{\sqrt{m_n}}(y) \subseteq B_{\sqrt{m_n}\log m_n}$ and since the τ's are identically distributed, (6.4) is bounded by

$$\mathbb{E}g_{B_{\sqrt{m_n}}}^{-k}(0, 0) \sum_{y \in D_1} c_3(m_n^{-d/2+1} \wedge |y|^{2-d})^k \leq c_6 m_n^{1/k} \mathbb{E}g_{B_{\sqrt{m_n}}}^{-k}(0, 0) \leq c_5 m_n^{1/k},$$
$$(6.5)$$

where we used (3.3) and (3.5) of Theorem 3.1 to bound $\mathbb{E}g_{B_{\sqrt{m_n}}}^{-k}(0, 0) \leq c$ for $c \in (0, \infty)$. Thus, the contribution to $\mathbb{E}ER_{m_n}^k$ coming from $y \in D_1$ satisfies (3.12). Now let $y \in D_2$. We bound $P(\sigma(y) \leq m_n)$ by

$$\sum_{z: |z|=|y|/2} P(J(\eta(B_{|z|/2})) = z, \eta(B_{|y|/2}) \leq m_n, \sigma(y) \leq m_n)$$
$$\leq \sum_{z: |z|=|y|/2} P_z(\sigma(y) \leq m_n) P(J(\eta(B_{|y|/2})) = z, \eta(B_{|y|/2}) \leq m_n). \quad (6.6)$$

As in (6.2) and (6.3), $P_z(\sigma(y) \leq m_n) \leq 2c_3|z-y|^{2-d}(g_{B_{\sqrt{m_n}}(y)}(y,y))^{-1}$. For $c \in (0,\infty)$ large enough, $2c_3|z-y|^{2-d} \leq c|y|^{2-d}$, and so we get that

$$
\begin{aligned}
&P(\sigma(y) \leq m_n) \\
&\leq \quad \frac{c|y|^{2-d}}{g_{B_{\sqrt{m_n}}(y)}(y,y)}P(\eta(B_{|y|/2}) \leq m_n)\frac{c|y|^{2-d}}{g_{B_{\sqrt{m_n}}(y)}(y,y)}\exp(-\tfrac{c_4}{4}|y|^2 m_n^{-1}), \quad (6.7)
\end{aligned}
$$

where we used (3.9) of Lemma 3.2 in the last step. Using (6.7) and proceeding as in (6.5), one sees that the contribution to $\mathbb{E}ER_{m_n}^k$ coming from $y \in D_2$ is as claimed in (3.12). The proof of (3.12) is finished for $d \geq 3$.

Let $d = 2$. Recall that for $y \in D_1 \cup D_2$ it is sufficient to bound the expected value on A_n. In fact, we will bound $P(\sigma(y) \leq \theta_n)$ for every $\omega \in A_n$ by a function $f_{m_n}(|y|)$ that is as in (3.14) and this way also prove that the contribution coming from $y \in D_1 \cup D_2$ satisfies (3.12). Let $y \in D_1$. By (3.4) of Theorem 3.1, one can show that $g_{\sqrt{m_n}\log m_n}(0,y) \leq c_3(\log \sqrt{m_n}/|y|)$, and so

$$
\sum_{y \in D_1} P(\sigma(y) \leq \theta_n) \leq \sum_{y \in D_1} c_3 g_{B_{\sqrt{m_n}\log m_n}}^{-1}(y,y)(\log \sqrt{m_n}/|y|). \quad (6.8)
$$

As in Lemma 3.3 in Černý (2011), one sees that $g_{\sqrt{m_n}\log m_n}(y,y) \geq g_{B_{\sqrt{m_n}}(y)}(y,y) \geq c_7 \log m_n$. We set $f_{m_n}(|y|) \equiv 2c_3/c_7(1-\log(|y|/\sqrt{m_n}))$. A simple calculations shows that hence the contribution coming from $y \in D_1$ is as claimed in (3.12) and (3.14). Let $y \in D_2$. As in (6.6) - (6.7) we bound

$$
P_z(\sigma(y) \leq m_n) \leq 2g_{B_{\sqrt{m_n}\log m_n}}(z,y)/g_{B_{\sqrt{m_n}}(y)}(y,y). \quad (6.9)
$$

Since $|z-y| \geq \sqrt{m_n}/2$, one can check that $g_{B_{\sqrt{m_n}\log m_n}}(z,y) \leq c$ to get that

$$
P(\sigma(y) \leq m_n) \leq c\exp(-\tfrac{c_4}{4}|y|^2 m_n^{-1})(c_7 \log m_n)^{-1} \equiv f_{m_n}(|y|), \quad (6.10)
$$

where we used that, $g_{B_{\sqrt{m_n}}(y)}(y,y) \geq c_7 \log m_n$. The sum over $y \in D_2$ of $(f_{m_n}(|y|))^k$ satisfies (3.12) and (3.14) for $k = 1,2,4$. The proof of (3.12) is complete.

To finish the proof of Lemma 3.3, it remains to prove (3.14) for $y \in B_{m_n} \setminus (D_1 \cup D_2)$. By (3.10) of Lemma 3.2, on A_n, we know that we may set $f_{m_n}(|y|) = e^{-c_4(\log|y|)^2}$ for $y \in B_{m_n} \setminus (D_1 \cup D_2)$. Eq. (6.1) shows that the contribution to the sum in (3.14) coming from these y's vanishes. This finishes the proof of Lemma 3.3. $\qquad\square$

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