ALEA, Lat. Am. J. Probab. Math. Stat. **12** (1), 115–127 (2015)

Windings of the stable Kolmogorov process

Christophe Profeta and Thomas Simon

Laboratoire de Mathématiques et Modélisation d'Evry (LaMME), Université d'Evry-Val-d'Essonne, UMR CNRS 8071, F-91037 Evry Cedex.

Laboratoire Paul Painlevé, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex and Laboratoire de physique théorique et modèles statistiques, Université Paris-Sud, F-91405 Orsay Cedex.

E-mail address: christophe.profeta@univ-evry.fr, simon@math.univ-lille1.fr

Abstract. We investigate the windings around the origin of the two-dimensional Markov process (X, L) having the stable Lévy process L and its primitive X as coordinates, in the non-trivial case when $|L|$ is not a subordinator. First, we show that these windings have an almost sure limit velocity, extending the result of McKean (1963) in the Brownian case. Second, we evaluate precisely the upper tails of the distribution of the half-winding times, connecting the results of our recent papers Profeta (2014); Profeta and Simon (2014).

1. Introduction and statement of the results

A celebrated theorem by Spitzer (1958) states that the angular part $\{\omega(t),\, t \geq 0\}$ of a two-dimensional Brownian motion starting away from the origin satisfies the following limit theorem

$$\frac{2\,\omega(t)}{\log t} \xrightarrow{d} \mathcal{C} \qquad \text{as } t \to +\infty,$$

where $\mathcal{C}$ denotes the standard Cauchy law. An analogue of this result for isotropic stable Lévy processes was given in Bertoin and Werner (1996), with a slower speed in $\sqrt{\log t}$ and a centered Gaussian limit law. Notice that both these results can be obtained as functional limit theorems with respect to the Skorohod topology. We refer to Doney and Vakeroudis (2013) for a recent paper revisiting these problems, with further results and an updated bibliography.

Received by the editors July 6, 2014; accepted February 17, 2015.

2010 *Mathematics Subject Classification.* 60F99, 60G52, 60J50.

Key words and phrases. Integrated process - Harmonic measure - Hitting time - Stable Lévy process - Winding.

Ce travail a bénéficié d'une aide de la Chaire *Marchés en Mutation*, Fédération Bancaire Française.

In a different direction, McKean (1963) had observed that the windings of the Kolmogorov diffusion, which is the two-dimensional process Z having a linear Brownian motion as second coordinate and its running integral as first coordinate, obey an almost sure limit theorem. More precisely, if $\{\omega(t),\, t \geq 0\}$ denotes the angular part of the process Z starting away from the origin, it is shown in Section 4.3 of McKean (1963) that

$$\frac{\omega(t)}{\log t} \xrightarrow{a.s.} -\frac{\sqrt{3}}{2} \qquad \text{as } t \to +\infty$$

(the constant which is given in McKean (1963) is actually $-\sqrt{3}/8$, but it will be observed below that the evaluation of the relevant improper integral in McKean (1963) was slightly erroneous). Of course, the degeneracy of the Kolmogorov diffusion makes it wind in a very particular way, since this process visits a.s. alternatively and clockwise the left and right half-planes. The regularity of this behaviour, which contrasts sharply with the complexity of planar Brownian motion, makes it possible to use the law of large numbers and to get an almost sure limit theorem.

The first aim of this paper is to obtain an analogue of McKean's result in replacing Brownian motion by a strictly $\alpha-$stable Lévy process $L = \{L_t,\, t \geq 0\}$. Without loss of generality, we choose the following normalization for the characteristic exponent

$$\Psi(\lambda) \;=\; \log(\mathbb{E}[e^{\mathrm{i}\lambda L_1}]) \;=\; -(\mathrm{i}\lambda)^{\alpha} e^{-\mathrm{i}\pi\alpha\rho\,\mathrm{sgn}(\lambda)}, \qquad \lambda \in \mathbb{R}, \tag{1.1}$$

where $\alpha \in (0, 2]$ is the self-similarity parameter and $\rho = \mathbb{P}[L_1 \geq 0]$ is the positivity parameter. We refer to Samorodnitsky and Taqqu (1994); Zolotarev (1986) for accounts on stable laws and processes, and to the introduction of our previous paper Profeta and Simon (2014) for a discussion on this specific parametrization. Recall that if $\alpha = 2$, then necessarily $\rho = 1/2$ and $L = \{\sqrt{2}B_t,\, t \geq 0\}$ is a rescaled Brownian motion. Introduce the primitive process

$$X_t \;=\; \int_0^t L_s\, ds, \qquad t \geq 0,$$

and denote by $\mathbb{P}_{(x,y)}$ the law of the strong Markov process $Z = (X, L)$ started from (x, y). By analogy with the classical Kolmogorov diffusion - see Kolmogoroff (1934), this process may and will be called the stable Kolmogorov process. When $(x, y) \neq (0, 0)$, it can be shown without much difficulty - see Lemma 2.3 below - that under $\mathbb{P}_{(x,y)}$, the process Z never hits $(0, 0)$. Filling in the gaps made by the jumps of L by vertical lines - see the figure below - and reasoning exactly as in Bertoin and Werner (1996) p.1270 it is possible to define the algebraic angle

$$\omega(t) \;=\; (\widehat{Z_0, Z_t})$$

measured in the trigonometric orientation.
If $\rho = 1$ resp. $\rho = 0$, then $|L|$ is a stable subordinator and it is easy to see that Z stays for large times within the positive resp. the negative quadrant with a.s. $X_t/L_t \to +\infty$, so that $\omega(t)$ converges a.s. to a finite limit which is

$$(\widehat{Z_0, \mathrm{O}x}) \qquad \text{resp.} \qquad (\widehat{-Z_0, \mathrm{O}x}).$$

When $\rho \in (0, 1)$ and $(x, y) \neq (0, 0)$, the Lévy process L oscillates and the Kolmogorov process Z winds clockwise and infinitely often around the origin as soon

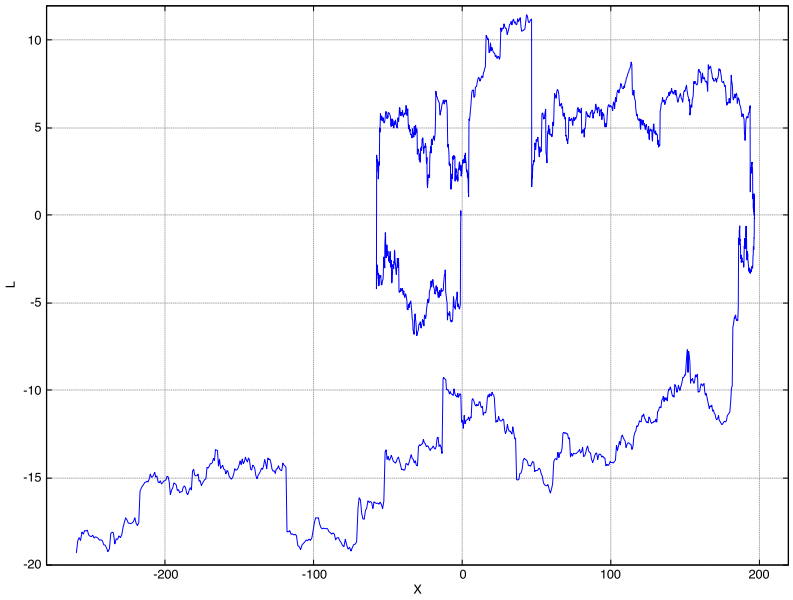

FIGURE 1.1. One path of $((X_t, L_t), t \leq 100)$ with $\alpha = 3/2$ and $\rho = 1/2$, starting from $X_0 = -1$ and $L_0 = 0$.

as $(x, y) \neq (0, 0)$. Indeed, considering the partition $\mathbb{R}^2 \setminus \{(0, 0)\} = \mathcal{P}_- \cup \mathcal{P}_+$ with

$$\mathcal{P}_- \;=\; \{x < 0\} \cup \{x = 0, y < 0\} \qquad \text{and} \qquad \mathcal{P}_+ \;=\; \{x > 0\} \cup \{x = 0, y > 0\},$$

we see that if $(x, y) \in \mathcal{P}_-$ the continuous process X visits alternatively the negative and positive half-lines, starting negative, and that its speed when it hits zero is alternatively positive and negative, starting positive. When $(x, y) \in \mathcal{P}_+$ the same alternating scheme occurs, with opposite signs. In particular, the function $\omega(t)$ is a.s. negative for all t large enough. In order to state our first result, which computes the a.s. limit velocity of $\omega(t)$, let us finally introduce the parameters

$$\gamma \;=\; \frac{\rho\alpha}{1+\alpha} \;\in\; (0, 1/2) \qquad \text{and} \qquad \overline{\gamma} \;=\; \frac{(1-\rho)\alpha}{1+\alpha} \;\in\; (0, 1/2).$$

Theorem A. Assume $\rho \in (0, 1)$ and $(x, y) \neq (0, 0)$. Then, under $\mathbb{P}_{(x,y)}$, one has

$$\frac{\omega(t)}{\log t} \;\xrightarrow{a.s.}\; -\frac{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})}{\alpha\sin(\pi(\gamma+\overline{\gamma}))} \qquad \text{as } t \to +\infty.$$

Note that in the Brownian motion case $\alpha = 2$, we have $\rho = 1/2$ and $\gamma = \overline{\gamma} = 1/3$, so that

$$\frac{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})}{\alpha\sin(\pi(\gamma+\overline{\gamma}))} \;=\; \frac{\sqrt{3}}{2}.$$

The constant $-\sqrt{3}/8$ which is given in Section 4.3 of McKean (1963) is not the right one because of the erroneous evaluation of the integral in (3.8.a) therein: this integral equals actually $\pi/\sqrt{3}$, as can be checked by an appropriate contour integration. The proof of Theorem A goes basically along the same lines as in McKean (1963). We consider the successive hitting times of 0 for the integrated

process X :

$$T_0^{(1)} \;=\; T_0 \;=\; \inf\{t>0,\; X_t=0\} \qquad \text{and} \qquad T_0^{(n)} \;=\; \inf\{t>T_0^{(n-1)},\; X_t=0\},$$

which can be viewed as the half-winding times of Z. We first check that $n \mapsto T_0^{(n)}$ increases a.s. to $+\infty$ as soon as $(x,y) \neq (0,0)$. The exact exponential rate of escape of $T_0^{(n)}$, which yields the exact winding velocity, is computed thanks to an elementary large deviation argument involving the law of L_{T_0} under $\mathbb{P}_{(x,0)}$, a certain transform of the half-Cauchy distribution as observed in Profeta and Simon (2014). Notice that contrary to McKean (1963) where the proof is only sketched, we provide here an argument with complete details.

In the Brownian case $\alpha = 2$, an expression of the law of the bivariate random variable $(T_0^{(n)}, |L_{T_0^{(n)}}|)$ under $\mathbb{P}_{(0,y)}$ has been given in Theorem 1 of Lachal (1997), in terms of the modified Bessel function of the first kind. This expression becomes very complicated under $\mathbb{P}_{(x,y)}$ when $x \neq 0$, even for $n = 1$ - see Formula (2) p.4 in Lachal (1997). In all cases, this expression is not informative enough to evaluate the upper tails of $T_0^{(n)}$. In Profeta (2014) it was shown that

$$\mathbb{P}_{(x,y)}[T_0^{(n)} \geq t] \;\asymp\; t^{-1/4}\,(\log t)^{n-1} \qquad \text{as } t \to +\infty$$

where, here and throughout, the notation $f(t) \asymp g(t)$ means that there exist two constants $0 < \kappa_1 \leq \kappa_2 < +\infty$ such that $\kappa_1 f(t) \leq g(t) \leq \kappa_2 f(t)$ as $t \to +\infty$. On the other hand, Theorem A in our previous paper Profeta and Simon (2014) shows the non-trivial asymptotics

$$\mathbb{P}_{(x,y)}[T_0 > t] \;\asymp\; t^{-\theta} \qquad \text{as } t \to +\infty$$

for $(x,y) \in \mathcal{P}_-$, with $\theta = \rho/(1+\alpha(1-\rho))$. By symmetry, the latter result also shows that $\mathbb{P}_{(x,y)}[T_0 > t] \asymp t^{-\overline{\theta}}$ as $t \to +\infty$ for $(x,y) \in \mathcal{P}_+$, with $\overline{\theta} = (1-\rho)/(1+\alpha\rho)$. The second main result of this paper connects the two above estimates.

Theorem B. Assume that $\rho \in (0,1)$ and $(x,y) \in \mathcal{P}_-$. For every $n \geq 2$, the following asymptotics hold as $t \to +\infty$:

$$\begin{cases} \mathbb{P}_{(x,y)}[T_0^{(n)} > t] \;\asymp\; t^{-\theta}\,(\log t)^{\left[\frac{n-1}{2}\right]} & \text{if } \rho < 1/2, \\ \mathbb{P}_{(x,y)}[T_0^{(n)} > t] \;\asymp\; t^{-\overline{\theta}}\,(\log t)^{\left[\frac{n}{2}\right]-1} & \text{if } \rho > 1/2, \\ \mathbb{P}_{(x,y)}[T_0^{(n)} > t] \;\asymp\; t^{-\theta}\,(\log t)^{n-1} & \text{if } \rho = 1/2. \end{cases}$$

By symmetry, the same result holds for $(x,y) \in \mathcal{P}_+$ with θ and $\overline{\theta}$ switched. In the above statement, the separation of cases is intuitively clear, since for $\rho < 1/2$ resp. $\rho > 1/2$ the negative resp. positive excursions below 0 will tend to prevail. The main difficulty in the proof of Theorem B is to show that the required asymptotic behaviour does not depend on the starting point $(x,y) \in \mathcal{P}_-$. This is handled thanks to a uniform estimate on the Mellin transform of the harmonic measure $\mathbb{P}_{(x,y)}[L_{T_0} \in .]$, and a general estimate on the upper tails of the product of two positive independent random variables. These two estimates both have independent interest. Note also that, except in the Brownian case where the law of $T_0^{(n)}$ is explicitly known, getting exact asymptotics in Theorem B seems a difficult task.

This is essentially due to the fact that, in Profeta and Simon (2014), the estimate for T_0 is obtained from that of L_{T_0} via a scaling argument :

$$\mathbb{P}_{(x,y)}[T_0 > t] \asymp \mathbb{P}_{(x,y)}[|L_{T_0}|^\alpha > t] \qquad \text{as } t \to +\infty.$$

Hence, although the asymptotics of L_{T_0} is exactly known is some particular cases, as such, the change of scale does not allow us to get the analogous constant for T_0.

2. Proofs

2.1. *Preliminary results.* As mentioned before, we first establish some estimates on the harmonic measure of the left half-plane with respect to the stable Kolmogorov process. When starting from $(x, y) \in \mathcal{P}_-$, the process (X, L) ends up in exiting $\mathcal{P}_-$ on the positive vertical axis, and its exit distribution is given by the law of L_{T_0} under $\mathbb{P}_{(x,y)}$. This distribution is called the harmonic measure since by the generalized Poisson formula, it allows one to construct harmonic functions with respect to the degenerate operator

$$\mathcal{L}_y^{\alpha,\rho} + y\,\frac{\partial}{\partial x}$$

on the half-plane, where $\mathcal{L}_y^{\alpha,\rho}$ is the generator of the stable Lévy process L. However, we shall not pursue these lines of research here.

Lemma 2.1. *Assume that $(x, y) \in \mathcal{P}_-$. The Mellin transform $s \mapsto \mathbb{E}_{(x,y)}[L_{T_0}^{s-1}]$ is real-analytic on $(1/(\gamma-1), 1/(1-\gamma))$, with two simple poles at $1/(\gamma-1)$ and $1/(1-\gamma)$. In particular, the random variable L_{T_0} has a smooth density $f^0_{x,y}$ under $\mathbb{P}_{(x,y)}$, and there exist $c_1, c_2 > 0$ such that*

$$f^0_{x,y}(z) \underset{z\to 0}{\sim} c_1\, z^{\alpha\theta/\gamma} \qquad \text{and} \qquad f^0_{x,y}(z) \underset{z\to+\infty}{\sim} c_2\, z^{-\alpha\theta-1}.$$

Proof: Observe first that the smoothness and the asymptotic behaviour of the density function of L_{T_0} are a direct consequence of the statement on the Mellin transform, thanks to the converse mapping theorem stated e.g. as Theorem 4 in Flajolet et al. (1995). This latter statement is also a direct consequence of Theorem B in Profeta and Simon (2014) when either $x = 0$ or $y = 0$. From now on we shall therefore assume that $xy \neq 0$. By Proposition 2 (i) and Equation (3.2) in Profeta and Simon (2014) we have

$$\mathbb{E}_{(x,y)}\left[L_{T_0}^{s-1}\right] = \frac{\pi \displaystyle\int_0^{+\infty} \mathbb{E}_{(x,y)}\left[X_t^{-\nu}\mathbf{1}_{\{X_t>0\}}\right] dt}{(1+\alpha)^{1-\nu}\left(\Gamma\left(1-\nu\right)\right)^2 \Gamma(1-s)\sin(\pi s(1-\gamma))} \tag{2.1}$$

with $s = (1-\nu)(1+\alpha) \in (0,1)$. However, it does not seem easy to study the poles of the right-hand side directly since the integral is not expressed in closed form, and for this reason we shall perform a further Mellin transformation in space. First, we know from Proposition 1 in Profeta and Simon (2014) that

$$\mathbb{E}_{(x,y)}[X_t^{-\nu}\mathbf{1}_{\{X_t>0\}}] = \frac{\Gamma(1-\nu)}{\pi}\int_0^\infty \lambda^{\nu-1} e^{-c_{\alpha,\rho}\lambda^\alpha t^{\alpha+1}} \sin(\lambda(x+yt) + s_{\alpha,\rho}\lambda^\alpha t^{\alpha+1} + \pi\nu/2)\, d\lambda$$

for every $\nu \in (0,1)$, with

$$s_{\alpha,\rho} = \frac{\sin(\pi\alpha(\rho-1/2))}{\alpha+1} \in (-1,1) \quad \text{and} \quad c_{\alpha,\rho} = \frac{\cos(\pi\alpha(\rho-1/2))}{\alpha+1} \in (0,1).$$

For every $\beta \in (0,\nu)$ this yields

$$\begin{aligned}
&\frac{\pi}{\Gamma(1-\nu)} \int_{-\infty}^{0} |x|^{\beta-1} \mathbb{E}_{(x,y)}[X_t^{-\nu} \mathbf{1}_{\{X_t>0\}}] dx \\
&= \int_0^{+\infty} \lambda^{\nu-1} e^{-c_{\alpha,\rho}\lambda^\alpha t^{\alpha+1}} \int_{-\infty}^{0} |x|^{\beta-1} \sin(\lambda(x+yt) + s_{\alpha,\rho}\lambda^\alpha t^{\alpha+1} + \pi\nu/2)\, dx\, d\lambda \\
&= \Gamma(\beta) \int_0^{+\infty} \lambda^{\nu-\beta-1} e^{-c_{\alpha,\rho}\lambda^\alpha t^{\alpha+1}} \sin(\lambda y t + s_{\alpha,\rho}\lambda^\alpha t^{\alpha+1} + \pi(\nu-\beta)/2)\, d\lambda \\
&= \frac{\pi\Gamma(\beta)}{\Gamma(1-\nu+\beta)} \mathbb{E}_{(0,y)}\left[X_t^{-\nu+\beta} \mathbf{1}_{\{X_t>0\}}\right]
\end{aligned}$$

where the switching of the first equality is justified exactly as in Lemma 1 of Profeta and Simon (2014), and the second equality follows from trigonometry and generalized Fresnel integrals - see (2.1) and (2.2) in Profeta and Simon (2014). Assume first that $y<0$. From Proposition 2 (ii) in Profeta and Simon (2014), we obtain

$$\begin{aligned}
&\frac{\pi}{\Gamma(1-\nu)} \int_0^{+\infty} \int_{-\infty}^{0} |x|^{\beta-1} \mathbb{E}_{(x,y)}[X_t^{-\nu} \mathbf{1}_{\{X_t>0\}}]\, dx\, dt \\
&= \Gamma(\beta)\Gamma(1-\nu+\beta)\,(\alpha+1)^{1-\nu+\beta}\,\Gamma(1-s-\beta(1+\alpha))\,\sin(\pi\rho\alpha\beta+\pi\gamma s)\,|y|^{s+\beta(1+\alpha)-1}
\end{aligned}$$

for every $\beta \in (0,(1-s)/(\alpha+1))$. Putting this together with (2.1), we finally deduce

$$\begin{aligned}
&\int_{-\infty}^{0} |x|^{\beta-1} \mathbb{E}_{(x,y)}\left[L_{T_0}^{s-1}\right] dx \\
&= (\alpha+1)^\beta \frac{\Gamma(\beta)\sin(\pi\rho\alpha\beta+\pi\gamma s)\Gamma(1-\nu+\beta)\Gamma(1-s-\beta(1+\alpha))}{\Gamma(1-\nu)\Gamma(1-s)\sin(\pi s(1-\gamma))} |y|^{s+\beta(1+\alpha)-1}.
\end{aligned} \tag{2.2}$$

We shall now invert this Mellin transform in the variable β in order to get a suitable integral expression for $\mathbb{E}_{(x,y)}[L_{T_0}^{s-1}]$. Fix $\beta \in (0,(1-s)/(\alpha+1))$. On the one hand, since $\rho\alpha<1$, we have

$$\Gamma(\beta)\cos((\pi\rho\alpha\beta+\pi\gamma s)/2) = \int_0^\infty x^{\beta-1}\, e^{-x\cos(\pi\rho\alpha/2)} \cos(x\sin(\pi\rho\alpha/2)+\pi\gamma s/2)\, dx$$

and

$$\begin{aligned}
\Gamma(1-\nu+\beta)\sin((\pi\rho\alpha\beta+\pi\gamma s)/2) &= \Gamma(1-\nu+\beta)\sin\left(\frac{\pi\rho\alpha}{2}(\beta+1-\nu)\right) \\
&= \int_0^\infty x^{\beta-\nu} e^{-x\cos(\pi\rho\alpha/2)} \sin(x\sin(\pi\rho\alpha/2))\, dx.
\end{aligned}$$

On the other hand, a change of variable in the definition of the Gamma function shows that

$$\Gamma(1-s-\beta(1+\alpha)) = \frac{1}{1+\alpha}\int_0^{+\infty} x^{\beta-1} x^{\frac{s-1}{1+\alpha}} e^{-x^{-1/(1+\alpha)}} dx.$$

Setting

$$\begin{aligned}
K_s(\xi) = \int_0^{+\infty} &z^{\frac{s}{1+\alpha}-1} e^{-\cos(\pi\rho\alpha/2)(\xi/z+z)} \\
&\times \cos\left(\frac{\xi}{z}\sin(\pi\rho\alpha/2)+\pi\gamma s/2\right)\sin(z\sin(\pi\rho\alpha/2))\, dz,
\end{aligned}$$

we can now invert (2.2) and obtain, applying Fubini's theorem and using the notation $\widetilde{x} = x(1+\alpha)|y|^{1+\alpha}$, a new expression for the Mellin transform of L_{T_0} :

$$\mathbb{E}_{(\widetilde{x},y)}\left[L_{T_0}^{s-1}\right] = \frac{2|y|^{s-1}|x|^{\frac{s-1}{1+\alpha}}(1-s)}{\Gamma\left(\frac{s}{1+\alpha}\right)\Gamma(2-s)\sin(\pi s(1-\gamma))}\int_0^{+\infty}\xi^{\frac{1-s}{1+\alpha}-1}e^{-\left(\frac{\xi}{|x|}\right)^{1/(1+\alpha)}}K_s(\xi)\,d\xi.$$

Since $\frac{1}{1-\gamma} = 1 + \frac{\alpha\rho}{\alpha+(1-\alpha\rho)} < 1+\rho < 2$, it remains to prove that the function

$$H_s(x) = (1-s)\int_0^{\infty}\xi^{\frac{1-s}{1+\alpha}-1}e^{-\left(\frac{\xi}{|x|}\right)^{1/(1+\alpha)}}K_s(\xi)\;d\xi$$

admits an analytic continuation on $[1/(\gamma-1), 1/(1-\gamma)]$. Observe first that for any $s > -1-\alpha$, the function K_s is uniformly bounded on $[0,+\infty)$ by

$$\begin{aligned}|K_s(\xi)| &\le \sin(\pi\rho\alpha/2)\int_0^{+\infty} z^{\frac{s}{1+\alpha}}e^{-\cos(\pi\rho\alpha/2)z}\,dz\\ &= \frac{\sin(\pi\rho\alpha/2)}{(\cos(\pi\rho\alpha/2))^{\frac{s}{1+\alpha}+1}}\Gamma\left(\frac{s}{1+\alpha}+1\right).\end{aligned}$$

As a consequence, the function H_s has an analytic continuation on $(-1-\alpha,1) \supset [1/(\gamma-1),1)$ since $\frac{1}{\gamma-1} = -\frac{1+\alpha}{1+\alpha(1-\rho)} > -1-\alpha$. Next for every $s \in (0,1)$ an integration by parts shows that

$$\begin{aligned}\frac{H_s(x)}{1+\alpha} &= \int_0^{\infty}\xi^{\frac{1-s}{1+\alpha}}\frac{d}{d\xi}\left(e^{-\left(\frac{\xi}{|x|}\right)^{1/(1+\alpha)}}K_s(\xi)\right)d\xi\\ &= \int_0^{\infty}\xi^{\frac{1-s}{1+\alpha}}e^{-\left(\frac{\xi}{|x|}\right)^{\frac{1}{1+\alpha}}}K_s'(\xi)\;d\xi - \frac{1}{|x|^{\frac{1}{1+\alpha}}}\int_0^{+\infty}\xi^{\frac{1-s-\alpha}{1+\alpha}}e^{-\left(\frac{\xi}{|x|}\right)^{\frac{1}{1+\alpha}}}K_s(\xi)\,d\xi,\end{aligned}$$

where K_s' is well-defined on $[0,+\infty)$ for any $s>0$, and bounded by

$$\begin{aligned}|K_s'(\xi)| &\le 2\sin(\pi\rho\alpha/2)\int_0^{\infty} z^{\frac{s}{1+\alpha}-1}e^{-\cos(\pi\rho\alpha/2)z}dz\\ &= \frac{2\sin(\pi\rho\alpha/2)}{(\cos(\pi\rho\alpha/2))^{s/(1+\alpha)}}\Gamma\left(\frac{s}{1+\alpha}\right).\end{aligned}$$

Consequently, the function H_s also admits an analytic continuation on $(0,2) \supset (0,1/(1-\gamma)]$. This completes the proof in the case $y<0$. The case $y>0$ may be dealt with in an entirely similar way, and we leave the details to the reader.

□

Our second preliminary result is elementary, but we could not find any exact reference in the literature - see however the corollary after Theorem 3 in Embrechts and Goldie (1980) and Theorem 4 (v) in Cline (1986) for tightly related estimates - and we hence provide a proof.

Lemma 2.2. *Let $\mu \ge \nu > 0$ and $n,p \in \mathbb{N}$. Assume that X and Y are two independent positive random variables such that :*

$$\mathbb{P}[X \ge z] \underset{z\to+\infty}{\asymp} z^{-\nu}(\log z)^n \qquad \text{and} \qquad \mathbb{P}[Y \ge z] \underset{z\to+\infty}{\asymp} z^{-\mu}(\log z)^p.$$

Then

$$\begin{cases}\mathbb{P}[XY \ge z] \underset{z\to+\infty}{\asymp} z^{-\nu}(\log z)^{n+p+1} & \text{if } \mu = \nu,\\ \mathbb{P}[XY \ge z] \underset{z\to+\infty}{\asymp} z^{-\nu}(\log z)^{n} & \text{if } \mu > \nu.\end{cases}$$

Proof: We first decompose the product as

$$\mathbb{P}[XY \geq z] \;=\; \int_0^\infty \mathbb{P}[X \geq zy^{-1}]\,\mathbb{P}[Y \in dy].$$

Therefore, for $z > A$ large enough,

$$\begin{aligned}\mathbb{P}[XY \geq z] &\geq \int_A^{\sqrt{z}} \mathbb{P}[X \geq zy^{-1}]\,\mathbb{P}[Y \in dy] \\ &\geq \frac{\kappa_1}{z^\nu}\int_A^{\sqrt{z}} y^\nu(\log(zy^{-1}))^n\,\mathbb{P}[Y \in dy] \;\geq\; \frac{\kappa_1(\log(z))^n}{2^n z^\nu}\int_A^{\sqrt{z}} y^\nu\,\mathbb{P}[Y \in dy].\end{aligned}$$

Then, integrating by parts,

$$\int_A^{\sqrt{z}} y^\nu\,\mathbb{P}[Y \in dy] \;=\; A^\nu\mathbb{P}[Y \geq A] - z^{\nu/2}\mathbb{P}[Y \geq \sqrt{z}] \;+\; \nu\int_A^{\sqrt{z}} y^{\nu-1}\,\mathbb{P}[Y \geq y]dy.$$

Now, if $\nu < \mu$, this expression remains bounded as $z \to +\infty$. Assume therefore that $\mu = \nu$. In this case, we have :

$$\int_A^{\sqrt{z}} y^\nu\,\mathbb{P}[Y \in dy] \geq A^\nu\mathbb{P}[Y \geq A] \;-\; \frac{\kappa_2}{2^p}(\log z)^p \;+\; \nu\kappa_1\int_A^{\sqrt{z}} \frac{(\log y)^p}{y}dy$$

where, as $z \to +\infty$, the right-hand side is seen to be equivalent to $\dfrac{\nu\kappa_1\,(\log z)^{p+1}}{2^{p+1}(p+1)}$. This gives the lower bound. To obtain the upper bound, we separate the integral in three parts and proceed similarly, with ε small enough:

$$\begin{aligned}&\int_0^{1/\varepsilon} \mathbb{P}\left[X \geq \frac{z}{y}\right]\mathbb{P}[Y \in dy] + \int_{1/\varepsilon}^{\varepsilon z} \mathbb{P}\left[X \geq \frac{z}{y}\right]\mathbb{P}[Y \in dy] + \int_{\varepsilon z}^\infty \mathbb{P}\left[X \geq \frac{z}{y}\right]\mathbb{P}[Y \in dy] \\ &\qquad\leq\; \mathbb{P}[X \geq \varepsilon z] + \frac{\kappa_2(\log(\varepsilon z))^n}{z^\nu}\int_{1/\varepsilon}^{\varepsilon z} y^\nu\mathbb{P}[Y \in dy] + \mathbb{P}[Y \geq \varepsilon z]\end{aligned}$$

and the proof is concluded as before, using an integration by parts and looking separately at both cases $\nu < \mu$ and $\nu = \mu$.

□

2.2. *Proof of Theorem A.* By symmetry, it is enough to show Theorem A for $(x, y) \in \mathcal{P}_-$. Consider the sequence

$$\left(T_0^{(n)}, |L_{T_0^{(n)}}|\right)_{n\geq 1}$$

and set $\{\mathcal{F}_n, n \geq 1\}$ for its natural completed filtration. It is easy to see from the strong Markov and scaling properties of Z that this sequence is Markovian. To be more precise, starting from $\mathcal{P}_-$ and taking into account the possible asymmetry of the process L, we have the following identities for all $p \geq 1$.

$$\left(T_0^{(2p)}, |L_{T_0^{(2p)}}|\right) \;\overset{d}{=}\; \left(T_0^{(2p-1)} + |L_{T_0^{(2p-1)}}|^\alpha\tau^+, |L_{T_0^{(2p-1)}}|\ell^+\right)$$

with $(\tau^+, \ell^+) \perp \mathcal{F}_{2p-1}$ distributed as $(T_0, |L_{T_0}|)$ under $\mathbb{P}_{(0,1)}$, and

$$\left(T_0^{(2p+1)}, |L_{T_0^{(2p+1)}}|\right) \;\overset{d}{=}\; \left(T_0^{(2p)} + |L_{T_0^{(2p)}}|^\alpha\tau^-, |L_{T_0^{(2p)}}|\ell^-\right)$$

with $(\tau^-, \ell^-) \perp \mathcal{F}_{2p}$ distributed as $(T_0, |L_{T_0}|)$ under $\mathbb{P}_{(0,-1)}$. The starting term $(T_0, |L_{T_0}|)$ has the same law as (τ^-, ℓ^-) if $x = 0$ and $y = -1$. By induction we deduce the identities

$$|L_{T_0^{(2p)}}| \stackrel{d}{=} |L_{T_0}| \times \prod_{k=1}^{p-1} \ell_k^- \times \prod_{k=1}^{p} \ell_k^+ \quad \text{and} \quad |L_{T_0^{(2p+1)}}| \stackrel{d}{=} |L_{T_0}| \times \prod_{k=1}^{p} \ell_k^- \times \prod_{k=1}^{p} \ell_k^+$$

where, here and throughout, $(\tau_k^\pm, \ell_k^\pm)_{k\geq 1}$ are two i.i.d. sequences distributed as $(\tau^\pm, \ell^\pm)$, and all products are assumed independent. From Theorem B (i) in Profeta and Simon (2014) and its symmetric version, the Mellin tranforms of $\ell^\pm$ are given by

$$\mathbb{E}\left[(\ell^-)^{s-1}\right] = \frac{\sin(\pi\gamma s)}{\sin(\pi(1-\gamma)s)} \qquad \text{and} \qquad \mathbb{E}\left[(\ell^+)^{s-1}\right] = \frac{\sin(\pi\overline{\gamma} s)}{\sin(\pi(1-\overline{\gamma})s)} \tag{2.3}$$

for each real s in the respective domain of definition, which is in both cases an open interval containing 1. This entails that $\mathbb{E}[|\log(\ell^\pm)|] < +\infty$, with

$$\mathbb{E}\left[\log(\ell^-)\right] = \pi\cot(\pi\gamma) > 0 \qquad \text{and} \qquad \mathbb{E}\left[\log(\ell^+)\right] = \pi\cot(\pi\overline{\gamma}) > 0. \tag{2.4}$$

The following lemma is intuitively obvious.

Lemma 2.3. *Assume $(x, y) \in \mathcal{P}_-$. Then one has $T_0^{(n)} \to +\infty$ and the process Z never hits the origin, a.s. under $\mathbb{P}_{(x,y)}$.*

Proof: To prove the first statement, it is enough to show that $S_n = T_0^{(n)} - T_0^{(n-1)} \to +\infty$ a.s. as $n \to \infty$. Set

$$\kappa_{\alpha,\rho} = \frac{\pi\alpha}{2}(\cot(\pi\gamma) + \cot(\pi\overline{\gamma})) = \frac{\pi\alpha\sin(\pi(\gamma+\overline{\gamma}))}{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})} > 0.$$

From the above discussion, we have

$$S_{2p} \stackrel{d}{=} |L_{T_0}|^\alpha \times \tau^+ \times \left(\prod_{k=1}^{p-1} \ell_k^- \times \ell_k^+\right)^\alpha \tag{2.5}$$

for every $p \geq 2$, with independent products on the right-hand side. For every $\varepsilon \in (0, \kappa_{\alpha,\rho})$, this entails

$$\begin{aligned}\mathbb{P}_{(x,y)}\left[S_{2p} \leq e^{2(p-1)(\kappa_{\alpha,\rho}-\varepsilon)}\right] \leq\ & \mathbb{P}_{(x,y)}\left[|L_{T_0}| \leq e^{-\varepsilon(p-1)/2\alpha}\right] + \mathbb{P}\left[\tau^+ \leq e^{-\varepsilon(p-1)/2}\right] \\ & + \mathbb{P}\left[\frac{1}{p-1}\sum_{k=1}^{p-1}\log(\ell_k^-) \leq \pi\cot(\pi\gamma) - \varepsilon/2\right] \\ & + \mathbb{P}\left[\frac{1}{p-1}\sum_{k=1}^{p-1}\log(\ell_k^+) \leq \pi\cot(\pi\overline{\gamma}) - \varepsilon/2\right].\end{aligned}$$

From Lemma 2.1, there exists $\theta_1(\varepsilon) > 0$ such that $\mathbb{P}_{(x,y)}\left[|L_{T_0}| \leq e^{-\varepsilon(p-1)/2\alpha}\right] < e^{-p\theta_1(\varepsilon)}$ for p large enough. On the other hand, we have

$$\begin{aligned}\mathbb{P}\left[\tau^+ \leq e^{-\varepsilon(p-1)/2}\right] &\leq \mathbb{P}_{(0,1)}\left[\inf\{L_t,\ t \leq e^{-\varepsilon(p-1)/2}\} < 0\right] \\ &= \mathbb{P}_{(0,0)}\left[\sup\{\hat{L}_t,\ t \leq 1\} > e^{\varepsilon(p-1)/2\alpha}\right] \leq e^{-p\theta_2(\varepsilon)}\end{aligned}$$

for some $\theta_2(\varepsilon) > 0$ and all p large enough, where we have set $\hat{L} = -L$, the equality following from translation invariance and self-similarity, and the second inequality

from the general estimate of Theorem 12.6.1 in Samorodnitsky and Taqqu (1994). Last, the existence of some $\theta_3(\varepsilon) > 0$ such that both remaining terms can be bounded from above by $e^{-p\theta_3(\varepsilon)}$ for p large enough is a standard consequence of (2.3), (2.4) and Cramér's theorem - see e.g. Theorem 1.4 in den Hollander (2000), recalling that the assumption (I.5) can be replaced by (I.17) therein. We can finally appeal to the Borel-Cantelli lemma to deduce, having let $\varepsilon \to 0$,

$$\liminf_{p\to\infty} \frac{1}{2p} \log(S_{2p}) \;\geq\; \kappa_{\alpha,\rho} \;>\; 0 \qquad \text{a.s.} \tag{2.6}$$

This shows that $S_{2p} \to +\infty$ a.s. and an entirely similar argument yields $S_{2p+1} \to +\infty$ a.s. This concludes the proof of the first part of the lemma.

The second part is easier. If $\alpha \leq 1$, it is well-known that L never hits zero, so that Z never hits the origin. If $\alpha > 1$, we see from Lemma 2.1 that L_{T_0} has no atom at zero under $\mathbb{P}_{(x,y)}$ and because $\ell^{\pm}$ are absolutely continuous, all $L_{T_0^{(n)}}$'s have no atom at zero. We finally get

$$\mathbb{P}_{(x,y)}\left[Z \text{ visits the origin}\right] \;=\; \mathbb{P}_{(x,y)}\left[\bigcup_{n\geq 1}\left\{L_{T_0^{(n)}} = 0\right\}\right] \;=\; 0$$

where the first identification comes from the fact that $T_0^{(n)} \to +\infty$ a.s. □

We can now finish the proof of Theorem A. Set $\theta_0 = \widehat{Z_0 Z_{T_0}} \in (-\pi, 0)$ a.s. Observing as in McKean (1963) the a.s. identifications

$$\{\omega(t) \geq -(n-1)\pi + \theta_0\} \;=\; \{T_0^{(n)} \geq t\}$$
$$\text{and} \qquad \{\omega(t) \leq -(n-2)\pi + \theta_0\} \;=\; \{T_0^{(n-1)} \leq t\},$$

we see that Theorem A amounts to show that

$$\frac{1}{n}\log(T_0^{(n)}) \xrightarrow{a.s.} \frac{\pi\alpha \sin(\pi(\gamma+\overline{\gamma}))}{2\sin(\pi\gamma)\sin(\pi\overline{\gamma})} \;=\; \kappa_{\alpha,\rho} \qquad \text{as } n \to +\infty.$$

Firstly, with the above notation, we have a.s. under $\mathbb{P}_{(x,y)}$

$$\liminf_{n\to\infty} \frac{1}{n}\log(T_0^{(n)}) \;\geq\; \liminf_{n\to\infty} \frac{1}{n}\log(S_n) \;\geq\; \kappa_{\alpha,\rho},$$

where the second inequality comes from (2.6) and its analogue for n odd. To obtain the upper bound, we will proceed as in the above Lemma 2.3. Fixing $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}_{(x,y)}\left[T_0^{(n)} \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon)}\right] &\leq \sum_{k=1}^{n} \mathbb{P}_{(x,y)}\left[S_k \geq n^{-1} e^{n(\kappa_{\alpha,\rho}+\varepsilon)}\right] \\ &\leq \sum_{k=1}^{n} \mathbb{P}_{(x,y)}\left[S_k \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon/2)}\right]\end{aligned}$$

for n large enough, with the above notation for S_k and having set $S_1 = T_0$. Recalling (2.5) we have for every $k = 2p \leq n$

$$\begin{aligned}\mathbb{P}_{(x,y)}\left[S_{2p} \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon/2)}\right] \;\leq\; & \mathbb{P}_{(x,y)}\left[|L_{T_0}| \geq e^{n\varepsilon/8\alpha}\right] \;+\; \mathbb{P}\left[\tau^+ \geq e^{n\varepsilon/8}\right] \\ & +\; \mathbb{P}\left[\frac{2}{n}\sum_{k=1}^{p-1}\log(\ell_k^-) \geq \pi\cot(\pi\gamma) + \varepsilon/4\right] \\ & +\; \mathbb{P}\left[\frac{2}{n}\sum_{k=1}^{p-1}\log(\ell_k^+) \geq \pi\cot(\pi\overline{\gamma}) + \varepsilon/4\right].\end{aligned}$$

Again from Lemma 2.1, there exists $\theta_4(\varepsilon) > 0$ such that $\mathbb{P}_{(x,y)}\left[|L_{T_0}| \geq e^{n\varepsilon/8\alpha}\right] < e^{-n\theta_4(\varepsilon)}$ for n large enough, whereas

$$\mathbb{P}\left[\tau^+ \geq e^{n\varepsilon/8}\right] \leq \mathbb{P}_{(0,0)}\left[\sup\{\hat{L}_t,\ t \leq 1\} < e^{-n\varepsilon/8\alpha}\right] \;\leq\; e^{-n\theta_5(\varepsilon)}$$

for some $\theta_5(\varepsilon) > 0$ and n large enough, the second inequality following e.g. from Proposition VIII.2 in Bertoin (1996). To handle the third term, we separate according as $p \leq \sqrt{n}$ or $p > \sqrt{n}$. In the first case, we have the upper bound

$$\begin{aligned}\mathbb{P}\left[\frac{2}{n}\sum_{k=1}^{p-1}\log(\ell_k^-) \geq \pi\cot(\pi\gamma) + \varepsilon/4\right] \;&\leq\; \sum_{k=1}^{\sqrt{n}}\mathbb{P}\left[\frac{2}{\sqrt{n}}\log(\ell_k^-) \geq \pi\cot(\pi\gamma) + \varepsilon/4\right] \\ &\leq\; e^{-\theta_6(\varepsilon)\sqrt{n}}\end{aligned}$$

for some $\theta_6(\varepsilon) > 0$ and n large enough, using Lemma 2.1 for the second inequality. In the second case, applying Cramér's theorem exactly as in Lemma 2.3 gives the upper bound

$$\mathbb{P}\left[\frac{2}{n}\sum_{k=1}^{p-1}\log(\ell_k^-) \geq \pi\cot(\pi\gamma) + \varepsilon/4\right] \;\leq\; e^{-\theta_7(\varepsilon)\sqrt{n}}$$

for some $\theta_7(\varepsilon) > 0$ and n large enough. The fourth term is estimated in the same way and we finally get the existence of some $\theta(\varepsilon) > 0$ such that, for n large enough :

$$\mathbb{P}_{(x,y)}\left[S_{2p} \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon/2)}\right] \;\leq\; e^{-\theta(\varepsilon)\sqrt{n}}.$$

An analogous estimate is obtained for $\mathbb{P}_{(x,y)}\left[S_{2p+1} \geq e^{n(\kappa_{\alpha,\rho}+\varepsilon/2)}\right]$ and we can apply as usual the Borel-Cantelli lemma to show the required upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log(T_0^{(n)}) \;\leq\; \kappa_{\alpha,\rho} + \varepsilon,$$

for all $\varepsilon > 0$, a.s. under $\mathbb{P}_{(x,y)}$.

□

2.3. *Proof of Theorem B.* Recall the decomposition $T_0^{(n)} = S_1 + \cdots + S_n$ with

$$S_1 \stackrel{d}{=} T_0,$$
$$S_{2p} \stackrel{d}{=} |L_{T_0}|^\alpha \times \tau^+ \times \left(\prod_{k=1}^{p-1} \ell_k^- \times \ell_k^+\right)^\alpha,$$
$$S_{2p+1} \stackrel{d}{=} |L_{T_0}|^\alpha \times \tau^- \times \left(\prod_{k=1}^{p-1} \ell_k^- \times \prod_{k=1}^{p} \ell_k^+\right)^\alpha,$$

and the above notation for $(\tau^\pm, \ell^\pm)$. Let us first investigate the upper tails of the distribution of each S_k under $\mathbb{P}_{(x,y)}$. We know that

$$\mathbb{P}_{(x,y)}[T_0 > t] \asymp \mathbb{P}_{(x,y)}[|L_{T_0}|^\alpha > t] \asymp \mathbb{P}[\tau^- > t] \asymp \mathbb{P}[(\ell^-)^\alpha > t] \asymp t^{-\theta}$$

and

$$\mathbb{P}[\tau^+ > t] \asymp \mathbb{P}[(\ell^+)^\alpha > t] \asymp t^{-\overline{\theta}}$$

as $t \to +\infty$. Supposing first $\rho = 1/2$ viz. $\theta = \overline{\theta}$, a successive application of Lemma 2.2 shows that

$$\mathbb{P}_{(x,y)}[S_k > t] \asymp t^{-\theta}(\log t)^{k-1} \qquad \text{as } t \to +\infty$$

for every $k \geq 1$. Suppose then $\rho < 1/2$ viz. $\theta < \overline{\theta}$, we obtain in a similar way

$$\mathbb{P}_{(x,y)}[S_{2p} > t] \asymp t^{-\theta}(\log t)^{p-1} \quad \text{and} \quad \mathbb{P}_{(x,y)}[S_{2p+1} > t] \asymp t^{-\theta}(\log t)^{p} \quad \text{as } t \to +\infty,$$

for every $p \geq 1$. Last, if $\rho > 1/2$ we find

$$\mathbb{P}_{(x,y)}[S_{2p} > t] \asymp t^{-\overline{\theta}}(\log t)^{p-1} \text{ and } \mathbb{P}_{(x,y)}[S_{2p+1} > t] \asymp t^{-\overline{\theta}}(\log t)^{p-1} \text{ as } t \to +\infty,$$

for every $p \geq 1$. All in all, for all $k \geq 2$, this shows that

$$\begin{cases} \mathbb{P}_{(x,y)}[S_k > t] \asymp t^{-\theta}\,(\log t)^{\left[\frac{k-1}{2}\right]} & \text{if } \rho < 1/2 \\ \mathbb{P}_{(x,y)}[S_k > t] \asymp t^{-\overline{\theta}}\,(\log t)^{\left[\frac{k}{2}\right]-1} & \text{if } \rho > 1/2 \\ \mathbb{P}_{(x,y)}[S_k > t] \asymp t^{-\theta}\,(\log t)^{k-1} & \text{if } \rho = 1/2, \end{cases}$$

and we also know that $\mathbb{P}_{(x,y)}[S_1 > t] \asymp t^{-\theta}$. The immediate estimate $\mathbb{P}_{(x,y)}[T_0^{(n)} > t] \geq \mathbb{P}_{(x,y)}[S_n > t]$ yields the required lower bound. To get the upper bounds, it suffices to write

$$\mathbb{P}_{(x,y)}[T_0^{(n)} > t] \leq \sum_{k=1}^{n} \mathbb{P}_{(x,y)}[S_k > t/n]$$

and to control the sum separately according as $\rho < 1/2, \rho > 1/2$ and $\rho = 1/2$. We leave the details to the reader.

□

References

J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge (1996). ISBN 0-521-56243-0. MR1406564.

J. Bertoin and W. Werner. Stable windings. *Ann. Probab.* **24** (3), 1269–1279 (1996). MR1411494.

D.B.H. Cline. Convolution tails, product tails and domains of attraction. *Probab. Theory Relat. Fields* **72** (4), 529–557 (1986). MR847385.

R.A. Doney and S. Vakeroudis. Windings of planar stable processes. In *Séminaire de Probabilités XLV*, volume 2078 of *Lecture Notes in Math.*, pages 277–300. Springer, Cham (2013). MR3185918.

P. Embrechts and C.M. Goldie. On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc. Ser. A* **29** (2), 243–256 (1980). MR566289.

P. Flajolet, X. Gourdon and P. Dumas. Mellin transforms and asymptotics: harmonic sums. *Theoret. Comput. Sci.* **144** (1-2), 3–58 (1995). Special volume on mathematical analysis of algorithms. MR1337752.

F. den Hollander. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI (2000). ISBN 0-8218-1989-5. MR1739680.

A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)* **35** (1), 116–117 (1934). MR1503147.

A. Lachal. Les temps de passage successifs de l'intégrale du mouvement brownien. *Ann. Inst. H. Poincaré Probab. Statist.* **33** (1), 1–36 (1997). MR1440254.

H.P. McKean, Jr. A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.* **2**, 227–235 (1963). MR0156389.

C. Profeta. Some limiting laws associated with the integrated Brownian motion. *To appear in ESAIM: P&S* (2014).

C. Profeta and T. Simon. Persistence of integrated stable processes. *To appear in Probab. Theory and Related Fields* (2014).

G. Samorodnitsky and M.S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York (1994). ISBN 0-412-05171-0. Stochastic models with infinite variance. MR1280932.

F. Spitzer. Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, 187–197 (1958). MR0104296.

V.M. Zolotarev. *One-dimensional stable distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI (1986). ISBN 0-8218-4519-5. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver. MR854867.