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Contact processes with random vertex weights on oriented lattices

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Abstract. In this paper we are concerned with contact processes with random vertex weights on oriented lattices. In our model, we assume that each vertex x of $\mathbb{Z}^d$ takes i. i. d. positive random value $\rho(x)$. Vertex y infects vertex x at rate proportional to $\rho(x)\rho(y)$ when and only when there is an oriented edge from y to x. We give the definition of the critical value λ_c of infection rate under the annealed measure and show that $\lambda_c = [1 + o(1)]/(dE\rho^2)$ as d grows to infinity. Classic contact processes on oriented lattices and contact processes on clusters of oriented site percolation are two special cases of our model.

1. Introduction

In this paper we are concerned with contact processes with random vertex weights on oriented lattices. For d-dimensional oriented lattice $\mathbb{Z}^d$, there is an oriented edge from x to $x + e_i$ for each $x \in \mathbb{Z}^d$ and $1 \leq i \leq d$, where

$$e_i = (0, \ldots, 0, \underset{i\text{th}}{1}, 0, \ldots, 0).$$

For $x, y \in \mathbb{Z}^d$, we write $x \to y$ when $y - x \in \{e_i\}_{1\leq i\leq d}$. We denote by O the origin of $\mathbb{Z}^d$.

Let ρ be a positive random variable such that $P(\rho > 0) > 0$ and $P(\rho \leq M) = 1$ for some $M \in (0, +\infty)$. Let $\{\rho(x)\}_{x\in\mathbb{Z}^d}$ be i. i. d. random variables such that $\rho(O)$ and ρ have the same distribution. When $\{\rho(x)\}_{x\in\mathbb{Z}^d}$ is given, the contact process with random vertex weights on oriented lattice $\mathbb{Z}^d$ is a spin system with state space

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$\{0,1\}^{\mathbb{Z}^d}$ and flip rates function given by

$$c(x,\eta)=\begin{cases}1 & \text{if } \eta(x)=1,\\ \lambda\sum\limits_{y:y\to x}\rho(x)\rho(y)\eta(y) & \text{if } \eta(x)=0\end{cases}\tag{1.1}$$

for each $(x,\eta)\in\mathbb{Z}^d\times\{0,1\}^{\mathbb{Z}^d}$, where $\lambda>0$ is a positive parameter called the infection rate. More details on the definition of spin systems can be found in Chapter 3 of Liggett (1985).

Intuitively, this contact process describes the spread of an infection disease. Vertices in state 0 are healthy and vertices in state 1 are infected. An infected vertex waits for an exponential time with rate one to become healthy. An healthy vertex x may be infected by an infected vertex y when and only when there is an oriented edge from y to x. The infection between y and x occurs at rate proportional to $\rho(x)\rho(y)$.

Please note that the assumption $P(\rho<M)=1$ for some $M<+\infty$ ensures the existence of our process according to the basic theory constructed in Harris (1972) and Liggett (1972).

The contact processes with random vertex weights is introduced by Peterson in Peterson (2011) on finite complete graphs. He proves that the infection rate λ has a critical value $\lambda_c=\frac{1}{E\rho^2}$ such that the disease survives for a long time with high probability when $\lambda>\lambda_c$ or dies out quickly with high probability when $\lambda<\lambda_c$.

Recently, contact processes in random environments or random graphs is a popular topic. In Chatterjee and Durrett (2009), Chatterjee and Durrett show that contact processes on random graphs with power law degree distributions have critical value 0. This result disproves the guess in Pastor-Satorras and Vespignani (2001a,b) that the critical value is strictly positive according to a non-rigorous mean-field analysis. In Peterson (2011), Peterson shows that contact processes with random vertex weights on complete graphs have critical value $\frac{1}{E\rho^2}$, which is consistent with the estimation given by the mean-field calculation. In Chen and Yao (2009) and Yao and Chen (2012), Yao and Chen show that complete convergence theorem holds for contact processes in a random environment on $\mathbb{Z}^d\times\mathbb{Z}^+$. The random environment they set includes the bond percolation model as a special case.

In our model, if ρ satisfies $P(\rho=1)=1-P(\rho=0)=p$, then our model can be regarded as contact processes on clusters of oriented site percolation on $\mathbb{Z}^d$, which is similar with the model in Bertacchi et al. (2011) with $N=1$. In Bertacchi et al. (2011), Bertacchi, Lanchier and Zucca study contact processes on $C_\infty\times K_N$, where C_∞ is the unique infinite open cluster of site percolation and K_N is the complete graph with N vertices. They give detailed criteria to judge whether the disease will survive. In Kesten (1990), Kesten shows that site percolation on $\mathbb{Z}^d$ has critical probability $[1+o(1)]/2d$. We are inspired a lot by this result.

2. Main result

Before giving our main results, we introduce some notations. We assume that the random variables $\{\rho(x)\}_{x\in\mathbb{Z}^d}$ are defined on a probability space $\big(\Omega,\mathcal{F},P\big)$. We denote by E the expectation operator with respect to P.

For any $\omega\in\Omega$, we denote by P_λ^ω the probability measure of our contact process on oriented lattice $\mathbb{Z}^d$ with infection rate λ and vertex weights $\{\rho(x,\omega)\}_{x\in\mathbb{Z}^d}$. The probability measure P_λ^ω is called the quenched measure. We denote by E_λ^ω the expectation operator with respect to P_λ^ω. We define

$$P_{\lambda,d}(\cdot)=E\big[P_\lambda^\omega(\cdot)\big],$$

which is called the annealed measure. We denote by $E_{\lambda,d}$ the expectation operator with respect to $P_{\lambda,d}$.

For any $t\geq 0$, we denote by η_t the configuration of our process at the moment t. In this paper, we mainly deal with the case where all the vertices are infected at $t=0$. In later sections, if we need deal with the case where

$$A=\{x:\eta_0(x)=1\}\neq\mathbb{Z}^d,$$

then we will point out the initial infected set A and write η_t as η_t^A. When η_t is with no upper script, we refer to the case where

$$\{x:\eta_0(x)=1\}=\mathbb{Z}^d.$$

According to basic coupling of spin systems, it is easy to see that

$$P_{\lambda_1,d}(\eta_t(O)=1)\leq P_{\lambda_2,d}(\eta_s(O)=1)$$

for all $t\geq s$ and $\lambda_1\leq\lambda_2$. As a result, it is reasonable to define the following critical value of the infection rate.

$$\lambda_c(d)=\sup\big\{\lambda:\lim_{t\rightarrow+\infty}P_{\lambda,d}(\eta_t(O)=1)=0\big\}. \tag{2.1}$$

Please note that our process is symmetric under the annealed measure $P_{\lambda,d}$. So, $P_{\lambda,d}(\eta_t(x)=1)$ does not depend on the choice of x. As a result, when $\lambda<\lambda_c(d)$,

$$\lim_{t\rightarrow+\infty}P_{\lambda,d}(\eta_t(x)=1\text{ for some }x\in A)=0$$

for any finite $A\subseteq\mathbb{Z}^d$ and hence η_t converges weakly to the configuration where all the vertices are healthy as t grows to infinity.

Our main result is the following limit theorem for $\lambda_c(d)$.

Theorem 2.1. *Assume that $P(\rho>0)>0$ and $P(\rho<M)=1$ for some $M\in(0,+\infty)$, then*

$$\lim_{d\rightarrow+\infty}d\lambda_c(d)=\frac{1}{E\rho^2}. \tag{2.2}$$

Theorem 2.1 shows that the critical value $\lambda_c(d)$ is approximately inversely proportional to the dimension d, the ratio of which is the reciprocal of the second moment of ρ.

When $\rho\equiv 1$, our process is the classic contact process on oriented lattice. In this case, we write $\lambda_c(d)$ as λ_d. When ρ satisfies

$$P(\rho=1)=1-P(\rho=0)=p$$

for some $p\in(0,1)$, our process is the contact process on clusters of oriented site percolation on $\mathbb{Z}^d$. In this case, we write $\lambda_c(d)$ as $\lambda_c(d,\text{site},p)$. There are two direct corollaries of Theorem 2.1.

Corollary 2.2.

$$\lim_{d\rightarrow+\infty}d\lambda_d=1. \tag{2.3}$$

We say y is x's neighbor when $y \to x$, then Corollary 2.2 shows that λ_d is approximately the reciprocal of the number of neighbors. In Griffeath (1983), Holley and Liggett (1981), Pemantle (1992), Holley, Liggett, Griffeath and Pemantle show that this conclusion holds for contact processes on non-oriented lattices and regular trees. In Xue (2014a), Xue shows that the same conclusion holds for threshold one contact processes on lattices and regular trees.

Corollary 2.3. *For $p \in (0,1)$,*

$$\lim_{d\to+\infty} dp\lambda_c(d, site, p) = 1. \tag{2.4}$$

Corollary 2.3 shows that $\lambda_c = [1 + o(1)]/(dp)$ as d grows to infinity for contact processes on clusters of oriented site percolation. In Xue (2014b), Xue claims that the same conclusion holds for contact process on clusters of oriented bond percolation on $\mathbb{Z}^d$.

Please note that the critical value $\lambda_c(d)$ we define is under the annealed measure $P_{\lambda,d}$. We can also define critical value $\lambda_c(\omega, d)$ under the quenched measure such that

$$\lambda_c(\omega, d) = \sup\left\{\lambda : \forall\ x \in \mathbb{Z}^d, \lim_{t\to+\infty} P_\lambda^\omega(\eta_t(x) = 1) = 0\right\}$$

for any $\omega \in \Omega$. $\lambda_c(\omega, d)$ is a random variable. However, according to the ergodic theorem for i. i. d. random variables, it is easy to see that

$$P(\omega : \lambda_c(\omega, d) = \lambda_c(d)) = 1.$$

So we only need to deal with the critical value under the annealed measure.

The proof of Theorem 2.1 is divide into two sections. In Section 3, we will prove that

$$\liminf_{d\to+\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}.$$

The fact that $\rho(O)$ and $\rho(y)\eta_t(y)$ are independent for $y \in \{z : z \to O\}$ is crucial for the proof. Hille-Yosida Theorem and Grönwall inequality are two main tools of the proof.

In Section 4, we will prove that

$$\limsup_{d\to+\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2}.$$

In the proof, we will introduce another process ζ_t to control η_t from below and define the set L of infected paths. The upper bound of λ_c will be given by the Hölder inequality $P(|L| > 0) \geq \frac{(E|L|)^2}{E|L|^2}$.

3. Lower bound

In this section we give a lower bound for $\lambda_c(d)$.

Lemma 3.1. *For each $d \geq 1$,*

$$\lambda_c(d) \geq \frac{1}{dE\rho^2}$$

and hence

$$\liminf_{d\to+\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}.$$

Proof: We use f_t to denote

$$f_t := E_{\lambda,d}\big[\rho(O)\eta_t(O)\big] = E\big[\rho(O,\omega)P_\lambda^\omega(\eta_t(O)=1)\big].$$

According to the flip rates function of η_t given by (1.1), $\rho(O,\omega)$ and $P_\lambda^\omega(\eta_t(O)=1)$ are positive correlated. Therefore,

$$f_t \geq E\rho(O,\omega)E[P_\lambda^\omega(\eta_t(O)=1)] = E\rho P_{\lambda,d}(\eta_t(O)=1).$$

Hence,

$$P_{\lambda,d}(\eta_t(O)=1) \leq \frac{f_t}{E\rho}. \tag{3.1}$$

Please note that the assumption $P(\rho>0)>0$ ensures that $E\rho>0$.

According to Hille-Yosida Theorem and (1.1),

$$\begin{aligned}\frac{d}{dt}P_\lambda^\omega(\eta_t(O)=1) = &-P_\lambda^\omega(\eta_t(O)=1)\\ &+\lambda\sum_{y:y\to O}\rho(O)\rho(y)P_\lambda^\omega(\eta_t(O)=0,\eta_t(y)=1)\\ \leq &-P_\lambda^\omega(\eta_t(O)=1)+\lambda\sum_{y:y\to O}\rho(O)\rho(y)P_\lambda^\omega(\eta_t(y)=1).\end{aligned} \tag{3.2}$$

Multiply (3.2) by $\rho(O,\omega)$, then

$$\frac{d}{dt}f_t \leq -f_t + \lambda\sum_{y:y\to O}E\big[\rho^2(O)\rho(y)P_\lambda^\omega(\eta_t(y)=1)\big]. \tag{3.3}$$

For each y such that $y\to O$, $\eta_t(y)$ is only influenced by the vertices from which there are oriented paths to y. Therefore, $\rho(O)$ is independent of $\rho(y)P_\lambda^\omega(\eta_t(y)=1)$ and hence

$$E\big[\rho^2(O)\rho(y)P_\lambda^\omega(\eta_t(y)=1)\big] = E\rho^2(O)E[\rho(y)P_\lambda^\omega(\eta_t(y)=1)] = E\rho^2 f_t. \tag{3.4}$$

Please note that in (3.4), we utilize the fact that

$$E[\rho(y)P_\lambda^\omega(\eta_t(y)=1)] = E[\rho(O)P_\lambda^\omega(\eta_t(O)=1)] = f_t$$

since the process η_t is symmetric for $\mathbb{Z}^d$ under the annealed measure.

By (3.3) and (3.4),

$$\frac{d}{dt}f_t \leq (d\lambda E\rho^2-1)f_t. \tag{3.5}$$

According to Grönwall inequality and (3.5),

$$f_t \leq f_0\exp\{(d\lambda E\rho^2-1)t\}$$

and hence

$$\lim_{t\to+\infty} f_t = 0 \tag{3.6}$$

when $\lambda < \frac{1}{dE\rho^2}$.

Lemma 3.1 follows from (3.1) and (3.6).

□

4. Upper bound

In this section we will prove that $\liminf_{d\to+\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$.

First we define the contact process $\widehat{\eta}_t$ where disease spreads through the opposite direction of the oriented edges. For any $\omega \in \Omega$, The flip rates of $\widehat{\eta}_t$ with random vertex weights $\{\rho(x,\omega)\}_{x\in\mathbb{Z}^d}$ is given by

$$\widehat{c}(x,\eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum\limits_{y:x\to y} \rho(x)\rho(y)\eta(y) & \text{if } \eta(x) = 0. \end{cases} \tag{4.1}$$

Hence, for $\widehat{\eta}_t$, y may infect x when and only when there is an edge from x to y.

According to the graphical representation of contact processes introduced by Harris in Harris (1978), we have the duality relationship

$$P_\lambda^\omega\big(\eta_t(O) = 1\big) = P_\lambda^\omega\big(\widehat{\eta}_t^O \neq \emptyset\big), \tag{4.2}$$

where $\widehat{\eta}_t^O$ is $\widehat{\eta}_t$ with that $\{x \in \mathbb{Z}^d : \widehat{\eta}_0(x) = 1\} = \{O\}$.

Please note that in (4.2) we utilize the identification of $\widehat{\eta}_t^O$ with

$$\big\{x \in \mathbb{Z}^d : \widehat{\eta}_t^O(x) = 1\big\}.$$

We put the rigorous proof of (4.2) in the appendix. We find (4.2) according to the approach of graphical method, but to avoid too much details, in the proof we resort to the tool of generator.

Since $\{\rho(x)\}_{x\in\mathbb{Z}^d}$ are i. i. d., the events $\{\eta_t^O \neq \emptyset\}$ and $\{\widehat{\eta_t}^O \neq \emptyset\}$ have the same distribution under the annealed measure $P_{\lambda,d}$. Therefore, according to (4.2),

$$P_{\lambda,d}\big(\eta_t(O) = 1\big) = P_{\lambda,d}\big(\eta_t^O \neq \emptyset\big). \tag{4.3}$$

To control the size of η_t^O from below, we introduce a Markov process ζ_t with state space $\{-1,0,1\}^{\mathbb{Z}^d}$. For given $\{\rho(x)\}_{x\in\mathbb{Z}^d}$, ζ_t evolves as follows. For each $x \in \mathbb{Z}^d$, if $\zeta(x) = -1$, then x is frozen in the state -1 forever. If $\zeta(x) = 1$, then the value of x waits for an exponential time with rate one to become -1. If $\zeta(x) = 0$, then the value of x flips to 1 at rate

$$\lambda \sum_{y:y\to x} \rho(x)\rho(y)1_{\{\zeta(y)=1\}}.$$

So for ζ_t, when an infected vertex becomes healthy, then it is removed and will never be infected again.

We use ζ_t^O to denote ζ_t with $\{x \in \mathbb{Z}^d : \zeta_0(x) = 1\} = \{O\}$ and $\{x \in \mathbb{Z}^d : \zeta_0(x) = -1\} = \emptyset$. According to the basic coupling of Markov processes, there is a coupling of η_t and ζ_t under quenched measure P_λ^ω such that

$$\eta_t^O \supseteq \{x \in \mathbb{Z}^d : \zeta_t^O(x) = 1\} \tag{4.4}$$

for any $t > 0$.

We use C_t to denote $\{x \in \mathbb{Z}^d : \zeta_t^O(x) = 1\}$. Then, by (4.3) and (4.4),

$$\lim_{t\to+\infty} P_{\lambda,d}\big(\eta_t(O) = 1\big) \geq P_{\lambda,d}\big(\forall\, t, C_t \neq \emptyset\big). \tag{4.5}$$

We give another description of $\{\forall\, t, C_t \neq \emptyset\}$. When the random environment ω is given, we let $\{T_x\}_{x\in\mathbb{Z}^d}$ be i. i. d. exponential times with rate 1 and let U_{xy} be exponential time with rate $\lambda\rho(x,\omega)\rho(y,\omega)$ for any $x \to y$. We assume that all these exponential times are independent under the quenched measure P^{ω}. For

$$O = x_0 \to x_1 \to x_2 \to \ldots \to x_n = x,$$

if $U_{x_j x_{j+1}} \leq T_{x_j}$ for each $0 \leq j \leq n-1$, then we say that there is an infected path with length n from O to x, which is denoted by $O \Rightarrow_n x$.

If x becomes infected at some moment t, then it waits for T_x to be removed and waits for U_{xy} to infect neighbor y. If $U_{xy} > T_x$, then the infection between x and y will not occur. Else if y has been infected by other vertices before $t + U_{xy}$, then the infection between x and y has no effect. As a result, for each $x \in \mathbb{Z}^d$, x has ever been infected when and only when there is an oriented path $O = x_0 \to x_1 \to x_2 \to \ldots \to x_n = x$ such that $U_{x_j x_{j+1}} \leq T_{x_j}$ for each $0 \leq j \leq n-1$. Therefore, in the sense of coupling,

$$\{O \Rightarrow_n x\} = \{\exists\ t, x \in C_t\}.$$

Let $I_n = \{x : O \Rightarrow_n x\}$ and L_n be the set of infected paths with length n from O. $\{\forall\ t, C_t \neq \emptyset\}$ is the event that that there are infinite many vertices which have ever been infected. Therefore,

$$\{\forall\ t, C_t \neq \emptyset\} = \{\forall\ n, I_n \neq \emptyset\}.$$

Since $\{I_n \neq \emptyset\} \supseteq \{I_m \neq \emptyset\}$ for any $n \leq m$ and $\{\forall\ n, I_n \neq \emptyset\} = \bigcap_{n=1}^{+\infty} \{I_n \neq \emptyset\}$, by Monotone Convergence Theorem and Hölder inequality,

$$\begin{aligned} P_{\lambda,d}\big(\forall\ t, C_t \neq \emptyset\big) &= \lim_{n\to+\infty} P_{\lambda,d}\big(I_n \neq \emptyset\big) \\ &= \lim_{n\to+\infty} P_{\lambda,d}\big(|L_n| > 0\big) \\ &\geq \limsup_{n\to+\infty} \frac{\big(E_{\lambda,d}|L_n|\big)^2}{E_{\lambda,d}|L_n|^2}. \end{aligned} \tag{4.6}$$

To calculate $E_{\lambda,d}|L_n|$ and $E_{\lambda,d}|L_n|^2$, we utilize the simple random walk S_n on oriented lattice $\mathbb{Z}^d$ with $S_0 = O$ and

$$P(S_{n+1} - S_n = e_i) = \frac{1}{d}$$

for $1 \leq i \leq d$. Let $\{\widehat{S}_n\}_{n=0}^{+\infty}$ be an independent copy of $\{S_n\}_{n=0}^{+\infty}$. We assume that $\{S_n\}_{n=0}^{+\infty}$ and $\{\widehat{S}_n\}_{n=0}^{+\infty}$ are defined on probability space $(\widetilde{\Omega}, \mathcal{G}, \widetilde{P})$ and are independent of $\{\rho(x)\}_{x\in\mathbb{Z}^d}$, $\{T_x\}_{x\in\mathbb{Z}^d}$ and $\{U_{xy}\}_{x\to y}$. We denote by $\widetilde{E}$ the expectation operator with respect to $\widetilde{P}$.

For a given path $O \to x_1 \to x_2 \to \ldots \to x_n$,

$$P_{\lambda}^{\omega}(U_{x_j x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n-1) = \prod_{j=0}^{n-1} \Big[\frac{\lambda\rho(x_j,\omega)\rho(x_{j+1},\omega)}{1 + \lambda\rho(x_j,\omega)\rho(x_{j+1},\omega)}\Big] \tag{4.7}$$

and hence

$$P_{\lambda,d}(U_{x_j x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n-1) = E \prod_{j=0}^{n-1} \Big[\frac{\lambda\rho(x_j)\rho(x_{j+1})}{1 + \lambda\rho(x_j)\rho(x_{j+1})}\Big]. \tag{4.8}$$

Please note that $U_{x_j, x_{j+1}}$ and $U_{x_{j+1}, x_{j+2}}$ are not independent under the annealed measure $P_{\lambda,d}$ but are independent under the quenched measure P^{ω}, since $\{\rho(x)\}_{x\in\mathbb{Z}^d}$

are not random when ω is given. As a result, we can get a product under P^ω in (4.7) and then utilize the fact that

$$P_{\lambda,d}(\cdot) = EP^\omega(\cdot)$$

to obtain (4.8). We will use this technic several times.

It is obviously that the right-hand side of (4.8) does not depend on the choice of the oriented path $x_1, x_2, \ldots, x_n$.

As a result,

$$E_{\lambda,d}|L_n| = d^n E \prod_{j=0}^{n-1} \Big[\frac{\lambda\rho(S_j)\rho(S_{j+1})}{1+\lambda\rho(S_j)\rho(S_{j+1})}\Big] \tag{4.9}$$

for any given first n steps $(S_0, S_1, \ldots, S_n)$ of the simple random walk $\{S_n\}_{n=0}^{+\infty}$.

Please note that we write E not $\widetilde{E}\times E$ in the right hand side of (4.9). We mean that the right hand side of (4.9) is a random variable with respect to $\mathcal{G}$ and is a constant with probability one.

To calculate $E_{\lambda,d}|L_n|^2$, we introduce the following notations.

$$\begin{aligned}
\tau_1 &= \inf\{n \geq 0 : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\},\\
\sigma_1 &= \inf\{n > \tau_1 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
D_1 &= \sigma_1 - \tau_1 + 1,\\
\tau_2 &= \inf\{n > \sigma_1 : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\},\\
\sigma_2 &= \inf\{n > \tau_2 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
D_2 &= \sigma_2 - \tau_2 + 1,\\
&\cdots\cdots\\
\tau_k &= \inf\{n > \sigma_{k-1} : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\},\\
\sigma_k &= \inf\{n > \tau_k : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
D_k &= \sigma_k - \tau_k + 1,\\
&\cdots\cdots\\
T &= \sup\{k : \tau_k < +\infty\}.
\end{aligned}$$

Please note that $P(T < +\infty) = 1$ for $d \geq 4$ according to the conclusion proven in Cox and Durrett (1983) that $P(\exists\ n > 0, S_n = \widehat{S}_n) < 1$ for $d \geq 4$. Therefore, τ_k, σ_k, D_k are finite for $k \leq T$.

Furthermore, we define

$$\begin{aligned}
A_0 &= \{0 \leq n < \tau_1 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
A_1 &= \{\sigma_1 < n < \tau_2 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
&\cdots\cdots\\
A_{T-1} &= \{\sigma_{T-1} < n < \tau_T : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\},\\
A_T &= \{n > \sigma_T : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}.
\end{aligned}$$

For $0 \leq i \leq T$, we use K_i to denote $|A_i|$.

Please note that there are no $n \in (\sigma_k, \tau_{k+1})$ which satisfy $S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}$ for $k \leq T$ according to the definition of σ_k and τ_k.

After all this prepare work, we give a lemma which is crucial for us to give an upper bound for $\lambda_c(d)$. We let

$$V_\lambda := \frac{2^{T+\sum\limits_{j=0}^{T} K_j} M^{6T+4\sum\limits_{j=0}^{T} K_j} \left(1+\lambda M^2\right)^{2\sum\limits_{j=1}^{T} D_j + 2\sum\limits_{j=0}^{T} K_j}}{\lambda^{\sum\limits_{j=1}^{T} D_j - T} \left(E\rho^2\right)^{\sum\limits_{j=1}^{T} D_j + 2T + 2\sum\limits_{j=0}^{T} K_j}}. \tag{4.10}$$

Lemma 4.1. *Assume that $P(\rho > 0) > 0$ and $P(\rho < M) = 1$. If λ satisfies*

$$\widetilde{E} V_\lambda < +\infty, \tag{4.11}$$

then

$$\lambda_c(d) \le \lambda,$$

where V_λ is given by (4.10).

Proof: For each $x \to z_1$ and $y \to z_2$, we define $F(x, y; z_1, z_2)$ as

$$P_\lambda^\omega\big(U_{xz_1} \le T_x, U_{yz_2} \le T_y\big).$$

By direct calculation,

$$F(x,y;z_1,z_2) \begin{cases} = \frac{\lambda^2\rho(x)\rho(y)\rho(z_1)\rho(z_2)}{[1+\lambda\rho(x)\rho(z_1)][1+\lambda\rho(y)\rho(z_2)]} & \text{if } x \ne y \text{ and } z_1 \ne z_2, \\ = \frac{\lambda^2\rho(x)\rho(y)\rho^2(z_1)}{[1+\lambda\rho(x)\rho(z_1)][1+\lambda\rho(y)\rho(z_1)]} & \text{if } x \ne y \text{ and } z_1 = z_2, \\ = \frac{\lambda\rho(x)\rho(z_1)}{1+\lambda\rho(x)\rho(z_1)} & \text{if } x = y \text{ and } z_1 = z_2, \\ \le \frac{2\lambda^2\rho^2(x)\rho(z_1)\rho(z_2)}{[1+\lambda\rho(x)\rho(z_1)][1+\lambda\rho(x)\rho(z_2)]} & \text{if } x = y \text{ and } z_1 \ne z_2. \end{cases} \tag{4.12}$$

Please note that in (4.12) we utilize the fact that U_{xz_1} and U_{yz_2} are independent under the quenched measure P^ω when $(x, z_1) \ne (y, z_2)$.

We denote by P_n the set of all the oriented paths from O with length n, then

$$\begin{aligned} E_{\lambda,d}|L_n|^2 &= \sum_{\mathbf{x}\in P_n}\sum_{\mathbf{y}\in P_n} P_{\lambda,d}\big(\forall\ 0 \le i \le n-1, U_{x_ix_{i+1}} \le T_{x_i}, U_{y_iy_{i+1}} \le T_{y_i}\big) \\ &= \sum_{\mathbf{x}\in P_n}\sum_{\mathbf{y}\in P_n} EP_\lambda^\omega\big(\forall\ 0 \le i \le n-1, U_{x_ix_{i+1}} \le T_{x_i}, U_{y_iy_{i+1}} \le T_{y_i}\big) \\ &= \sum_{\mathbf{x}\in P_n}\sum_{\mathbf{y}\in P_n} E\big[\prod_{i=0}^{n-1} F(x_i, y_i; x_{i+1}, y_{i+1})\big] \\ &= d^{2n}\sum_{\mathbf{x}\in P_n}\sum_{\mathbf{y}\in P_n} \frac{1}{d^{2n}} E\big[\prod_{i=0}^{n-1} F(x_i, y_i; x_{i+1}, y_{i+1})\big] \\ &= d^{2n}(\widetilde{E}\times E)\big[\prod_{i=0}^{n-1} F(S_i, \widehat{S}_i; S_{i+1}, \widehat{S}_{i+1})\big], \end{aligned} \tag{4.13}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the oriented paths

$$O = x_0 \to x_1 \to x_2 \to \ldots \to x_n \quad \text{and} \quad O = y_0 \to y_1 \to y_2 \to \ldots \to y_n.$$

Please note that in the oriented paths $\mathbf{x}$ and $\mathbf{y}$, $\|x_i\| = \|y_i\| = i$ for $0 \le i \le n$, where $\|\cdot\|$ is the l_1 norm on $\mathbb{Z}^d$. Therefore, $x_i \ne x_j, y_j$ for $i \ne j$. As a result, $(U_{x_ix_{i+1}}, U_{y_iy_{i+1}}, T_{x_i}, T_{y_i})_{i=0}^{n-1}$ are independent random vectors under the quenched measure P^ω. That is why we can get a product in the third line of (4.13).

Therefore, by (4.9),

$$\frac{(E_{\lambda,d}|L_n|)^2}{E_{\lambda,d}|L_n|^2} = \left\{\widetilde{E}\left[\frac{E\prod_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})}{\left(E\prod_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\right)^2}\right]\right\}^{-1}. \tag{4.14}$$

Then by (4.5) and (4.6), $\lambda \geq \lambda_c(d)$ when

$$\limsup_{n\to+\infty}\widetilde{E}\left[\frac{E\prod_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})}{\left(E\prod_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\right)^2}\right] < +\infty.$$

Now we control

$$\widetilde{E}\left[\frac{E\prod_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})}{\left(E\prod_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\right)^2}\right]$$

from above.

For the denominator $\left(E\prod_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\right)^2$, if $S_i=\widehat{S}_i$ or $S_{i+1}=\widehat{S}_{i+1}$, then

$$1+\lambda\rho(S_i)\rho(S_{i+1}) \leq 1+\lambda M^2,$$

where $P(\rho < M)=1$ as we assumed.

For the numerator $E\prod_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})$, if $S_i=\widehat{S}_i$ or $S_{i+1}=\widehat{S}_{i+1}$, then

$$1+\lambda\rho(S_i)\rho(S_{i+1}) \geq 1.$$

If $i \in A_k$ for some k, then by (4.12),

$$2\lambda^2\rho^2(S_i)\rho(S_{i+1})\rho(\widehat{S}_{i+1}) \leq 2\lambda^2M^2\rho(S_{i+1})\rho(\widehat{S}_{i+1})$$

and

$$\lambda^2\rho(S_{i-1})\rho(\widehat{S}_{i-1})\rho^2(S_i) \leq \lambda^2\rho(S_{i-1})\rho(\widehat{S}_{i-1})M^2.$$

If $i=\tau_k$ for some k, then

$$\lambda^2\rho(S_{i-1})\rho(\widehat{S}_{i-1})\rho^2(S_i) \leq \lambda^2\rho(S_{i-1})\rho(\widehat{S}_{i-1})M^2$$

and

$$\lambda\rho(S_i)\rho(S_{i+1}) \leq \lambda M\rho(S_{i+1}).$$

If $i=\sigma_k$ for some k, then

$$\lambda\rho(S_{i-1})\rho(S_i) \leq \lambda\rho(S_{i-1})M$$

and

$$2\lambda^2\rho^2(S_i)\rho(S_{i+1})\rho(\widehat{S}_{i+1}) \leq 2\lambda^2M^2\rho(S_{i+1})\rho(\widehat{S}_{i+1}).$$

After all these operations, we can cancel many common factors in the numerator and denominator. For example, if $i,j \in A_k$ and $l \notin A_k$ for each $i<l<j$, then we

can abstract

$$\Big[E\frac{\prod\limits_{l=1}^{j-i-1}\rho_l^2}{\prod\limits_{l=1}^{j-i-2}\big(1+\lambda\rho_l\rho_{l+1}\big)}\Big]^2$$

from both numerator and denominator and cancel this common factor, where $\{\rho_l\}_{l=1}^{i-j-1}$ are i. i. d. and have the same distribution as that of ρ.

Therefore, after all the above operations, it is easy to see that

$$\limsup_{n\to+\infty}\frac{E\prod\limits_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})}{\Big(E\prod\limits_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\Big)^2}\leq V_\lambda$$

and

$$\limsup_{n\to+\infty}\widetilde{E}\left[\frac{E\prod\limits_{i=0}^{n-1}F(S_i,\widehat{S}_i;S_{i+1},\widehat{S}_{i+1})}{\Big(E\prod\limits_{i=0}^{n-1}\frac{\lambda\rho(S_i)\rho(S_{i+1})}{1+\lambda\rho(S_i)\rho(S_{i+1})}\Big)^2}\right]\leq \widetilde{E}V_\lambda. \tag{4.15}$$

Lemma 4.1 follows (4.5), (4.6), (4.14) and (4.15).

□

Finally, we give the proof of $\limsup_{n\to+\infty} d\lambda_c(d)\leq\frac{1}{E\rho^2}$.

Proof of $\limsup_{n\to+\infty} d\lambda_c(d)\leq\frac{1}{E\rho^2}$: Let

$$\tau=\inf\{n>0:S_n=\widehat{S}_n\}.$$

Then according to (2.9) of Cox and Durrett (1983),

$$P(2\leq\tau<+\infty)\leq\frac{C_1}{d^2},$$

where C_1 does not depend on d. Therefore, according to strong Markov property,

$$\begin{aligned}&P(T=m,K_i=k_i\text{ for }0\leq i\leq m,D_i=l_i\text{ for }1\leq i\leq m)\\&\leq\big(\frac{C_1}{d^2}\big)^{\sum\limits_{i=0}^{m}k_i+m-1}\big(\frac{1}{d}\big)^{\sum\limits_{i=1}^{m}l_i-m}\end{aligned}\tag{4.16}$$

for all possible m,k_i,l_i. Please note that k_0 may take 0 but $l_i\geq 1$ and $k_i\geq 1$ for $1\leq i\leq m$.

Let

$$\lambda=\frac{\gamma}{dE\rho^2}$$

for some fixed $\gamma > 1$. Then by (4.16),

$$\begin{aligned}\widetilde{E}V_\lambda &\leq \sum_{m=0}^{+\infty}\sum_{k_0=0}^{+\infty}\sum_{k_1=1}^{+\infty}\cdots\sum_{k_m=1}^{+\infty}\sum_{l_1=1}^{+\infty}\cdots\sum_{l_m=1}^{+\infty}\big(\frac{C_1}{d^2}\big)^{\sum\limits_{i=0}^{m}k_i+m-1}\big(\frac{1}{d}\big)^{\sum\limits_{i=1}^{m}l_i-m}\\ &\times\frac{2^{m+\sum\limits_{j=0}^{m}k_j}M^{6m+4\sum\limits_{j=0}^{m}k_j}\big(1+\lambda M^2\big)^{2\sum\limits_{j=1}^{m}l_j+2\sum\limits_{j=0}^{m}k_j}}{\lambda^{\sum\limits_{j=1}^{m}l_j-m}\big(E\rho^2\big)^{\sum\limits_{j=1}^{m}l_j+2m+2\sum\limits_{j=0}^{m}k_j}}.\\ &=\sum_{m=0}^{+\infty}\sum_{k_0=0}^{+\infty}\Big(\frac{2C_1M^6\lambda}{d(E\rho^2)^2}\Big)^m\Big[\frac{2C_1M^4(1+\lambda M^2)^2}{d^2(E\rho^2)^2}\Big]^{k_0} \qquad (4.17)\\ &\times\Big[\sum_{l=1}^{+\infty}\big(\frac{2C_1M^4(1+\lambda M^2)^2}{d^2(E\rho^2)^2}\big)^l\Big]^m\Big[\sum_{l=1}^{+\infty}\big(\frac{(1+\lambda M^2)^2}{d\lambda E\rho^2}\big)^l\Big]^m\frac{d^2}{C_1}.\end{aligned}$$

Since $\lambda = \frac{\gamma}{dE\rho^2}$ for some $\gamma > 1$,

$$\frac{2C_1M^6\lambda}{d(E\rho^2)^2} \leq \frac{C_2}{d^2}$$

and

$$\frac{2C_1M^4(1+\lambda M^2)^2}{d^2(E\rho^2)^2} \leq \frac{C_3}{d^2}$$

for sufficiently large d, where C_2 and C_3 do not depend on d (but may depend on γ and ρ).

We choose $\widehat{\gamma}$ such that $1 < \widehat{\gamma} < \gamma$, then for sufficiently large d,

$$\frac{(1+\lambda M^2)^2}{d\lambda E\rho^2} = \frac{\big(1+\frac{\gamma M^2}{dE\rho^2}\big)^2}{\gamma} < \frac{1}{\widehat{\gamma}}.$$

Then, by (4.17),

$$\widetilde{E}V_\lambda \leq \frac{d^2}{C_1}\sum_{m=0}^{+\infty}\sum_{k_0=0}^{+\infty}\big(\frac{C_2}{d^2}\big)^m\big(\frac{C_3}{d^2}\big)^{k_0}\big[\sum_{l=1}^{+\infty}\big(\frac{C_3}{d^2}\big)^l\big]^m\big[\sum_{l=1}^{+\infty}\big(\frac{1}{\widehat{\gamma}}\big)^l\big]^m$$

for sufficiently large d.

For sufficiently large d,

$$\sum_{k_0=0}^{+\infty}\big(\frac{C_3}{d^2}\big)^{k_0} = \frac{d^2}{d^2-C_3} \leq 2$$

and

$$\sum_{l=1}^{+\infty}\big(\frac{C_3}{d^2}\big)^l = \frac{C_3}{d^2-C_3} \leq \frac{C_4}{d^2},$$

where C_4 does not depend on d.

Therefore,

$$\widetilde{E}V_\lambda \leq \frac{2d^2}{C_1}\sum_{m=0}^{+\infty}\big(\frac{C_2}{d^2}\big)^m\big(\frac{C_4}{d^2}\big)^m\big[\frac{1}{\widehat{\gamma}-1}\big]^m = \frac{2d^2}{C_1}\sum_{m=0}^{+\infty}\big[\frac{C_2C_4}{d^4(\widehat{\gamma}-1)}\big]^m. \qquad (4.18)$$

For sufficiently large d, $\frac{C_2C_4}{d^4(\widehat{\gamma}-1)} < 1$ and therefore

$$\widetilde{E}V_\lambda < +\infty \qquad (4.19)$$

when $\lambda = \frac{\gamma}{dE\rho^2}$ for some fixed $\gamma > 1$.

Then according to Lemma 4.1,

$$\lambda_c(d) \leq \frac{\gamma}{dE\rho^2}$$

for sufficiently large d and hence

$$\limsup_{d\to+\infty} d\lambda_c(d) \leq \frac{\gamma}{E\rho^2}$$

for any $\gamma > 1$.

Let γ decrease to 1, then we accomplish the proof.

□

Since we have shown that $\liminf_{d\to+\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$ in Section 3, the whole proof of Theorem 2.1 is completed.

Appendix A. Appendix

Proof of (4.2)*:* We identify $\widehat{\eta}_t$ as the set

$$\{x : \widehat{\eta}_t(x) = 1\}.$$

Then, $\widehat{\eta}_t$ is a Markov process with state space $2^{\mathbb{Z}^d} = \{A : A \subseteq \mathbb{Z}^d\}$. According to (4.1), the generator $\widehat{\Omega}$ of $\widehat{\eta}_t$ is given by

$$\widehat{\Omega} f(A) = \sum_{x\in A}[f(A\setminus x) - f(A)] + \lambda\sum_{x\in A}\sum_{y:y\to x}\rho(x)\rho(y)[f(A\cup y) - f(A)] \tag{A.1}$$

for any $f \in C(2^{\mathbb{Z}^d})$ and $A \subseteq \mathbb{Z}^d$.

For any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $A \subseteq \mathbb{Z}^d$, we define

$$H(\eta, A) = \begin{cases} 1 & \text{if } \{x : \eta(x) = 1\} \cap A = \emptyset, \\ 0 & \text{else if.} \end{cases}$$

We write $H(\eta, A)$ as $H(\cdot, A)(\eta)$ when we consider H as a function of η with fixed A and write $H(\eta, A)$ as $H(\eta, \cdot)(A)$ when we consider H as a function of A with fixed η. Then, $H(\cdot, A) \in C(\{0,1\}^{\mathbb{Z}^d})$ for each $A \subseteq \mathbb{Z}^d$ and $H(\eta, \cdot) \in C(2^{\mathbb{Z}^d})$ for each $\eta \in \{0,1\}^{\mathbb{Z}^d}$.

We denote by Ω the generator of η_t, then

$$\Omega f(\eta) = \sum_{x\in\mathbb{Z}^d} c(x,\eta)[f(\eta^x) - f(\eta)] \tag{A.2}$$

for any $f \in C(\{0,1\}^{\mathbb{Z}^d})$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$, where $c(x,\eta)$ is given by (1.1) and

$$\eta^x(y) = \begin{cases} 1 - \eta(x) & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x. \end{cases}$$

For any $A, B \subseteq \mathbb{Z}^d$ and $x \in A$, it is easy to see that

$$\begin{aligned} H(\eta, A\cup B) &= H(\eta, A)H(\eta, B) \quad \text{and} \\ H(\eta^x, A) &= H(\eta, A\setminus x) - H(\eta, A). \end{aligned} \tag{A.3}$$

Then, by (A.2), (A.3) and direct calculation,

$$\begin{aligned}
&\Omega H(\cdot,A)(\eta)\\
&=\sum_{x\in A}c(x,\eta)[H(\eta^x,A)-H(\eta,A)]\\
&=\sum_{x\in A}c(x,\eta)H(\eta^x,A)-\sum_{x\in A}c(x,\eta)H(\eta,A)\\
&=\sum_{x\in A}c(x,\eta)H(\eta,A\setminus x)1_{\{\eta(x)=1\}}-\sum_{x\in A}c(x,\eta)H(\eta,A\setminus x)1_{\{\eta(x)=0\}}\\
&=\sum_{x\in A}H(\eta,A\setminus x)1_{\{\eta(x)=1\}}-\sum_{x\in A}\sum_{y:y\to x}\lambda\rho(x)\rho(y)1_{\{\eta(y)=1\}}H(\eta,A\setminus x)1_{\{\eta(x)=0\}}\\
&=\sum_{x\in A}H(\eta^x,A)-\sum_{x\in A}\sum_{y:y\to x}\lambda\rho(x)\rho(y)[1-H(\eta,y)]H(\eta,A)\\
&=\sum_{x\in A}[H(\eta,A\setminus x)-H(\eta,A)]-\sum_{x\in A}\sum_{y:y\to x}\lambda\rho(x)\rho(y)[H(\eta,A)-H(\eta,A\cup y)]\\
&=\sum_{x\in A}[H(\eta,A\setminus x)-H(\eta,A)]+\lambda\sum_{x\in A}\sum_{y:y\to x}\rho(x)\rho(y)[H(\eta,A\cup y)-H(\eta,A)]\\
&=\widehat{\Omega}H(\eta,\cdot)(A)
\end{aligned}$$

for any $\eta\in\{0,1\}^{\mathbb{Z}^d}$ and $A\subseteq\mathbb{Z}^d$.

Then, according to Theorem 3.39 of Liggett (2010),

$$E_\eta H(\eta_t,A)=E_A H(\eta,\widehat{\eta}_t) \tag{A.4}$$

for any $\eta\in\{0,1\}^{\mathbb{Z}^d}$ and $A\subseteq\mathbb{Z}^d$.

We denote by δ_1 the configuration where all the vertices are in state 1. In (A.4), let $\eta=\delta_1$ and $A=\{O\}$, then we have

$$P\big(\eta_t(O)=0\big)=P\big(\{x:\delta_1(x)=1\}\cap\widehat{\eta}_t^O=\emptyset\big). \tag{A.5}$$

Since $\delta_1(x)=1$ for each $x\in\mathbb{Z}^d$,

$$\Big\{\{x:\delta_1(x)=1\}\cap\widehat{\eta}_t^O=\emptyset\Big\}=\{\widehat{\eta}_t^O=\emptyset\}. \tag{A.6}$$

(4.2) follows from (A.5) and (A.6).

□

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