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Functional limit theorems for the multivariate Hawkes Process with different exciting functions

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Abstract. In this work, we prove functional laws of large numbers and central limit theorem for an extension of the classical multivariate Hawkes process, which assumes the clusters of the process are generated by different exciting functions. Namely, by rescaling this process, we prove laws of large numbers for the processes associated to the extension of this Hawkes process. Also, a central limit theorem for this normalized process is proved and a type Donsker theorem is showed when unpredictable marks of the process are considered.

1. Introduction

The classical Hawkes process (HP) is a counting process having clustering effect and self-exciting temporal property. This process was firstly introduced by Hawkes (1971a), but the seminal ideas are also found in Hawkes (1971a,b) and Hawkes and Oakes (1974). Useful reviews on the topic are provided in Daley and Vere-Jones (2003) and Zhu (2013c). Applications to finance, genetics, neuroscience and seismology can be found in Carstensen et al. (2010), Embrechts et al. (2011), Gusto and Schbath (2005), Ogata (1988, 1998) and Pernice et al. (2012). On the other hand, the classical HP has been the subject of various studies such as large and moderate deviations Zhu (2013b, 2014b) and central limit theorems Zhu (2013a). Also, nonlinear versions of the HP has been considered in Zhu (2014b, 2013a).

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In Fierro et al. (2015) we introduced an extension of the standard Hawkes process, which considers different cluster could have different exciting functions. Indeed, this extension is defined as a superposition of a sequence $\{N^n\}_{n\in\mathbb{N}}$ of counting processes so that each N^n has an intensity depending on N^{n-1} and an exciting function γ_n eventually different for different $n \in \mathbb{N}$. In applications, this process has more flexibility for modeling. As a matter of fact, in seismology, main shocks produce aftershocks with possibly different intensities and thus a model as we are considering here could be more appropriate.

Asymptotical normality of this more general version of the HP has been proved by Fierro et al. (2015) and large and moderate deviations for this process are considered in Zhu (2014a). In this work, functional limit theorems for the multivariate HP with different exciting functions are stated and proved. By mean of a scale change in time and a suitable normalization of this process, we prove that it satisfies a functional law of large numbers and a central limit theorem. Moreover a type of Donsker theorem for this process with unpredictable marks is proved. Our results extend some limit theorems by Bacry et al. (2013), who showed a functional law of large numbers and a central limit theorem for the classical multivariate Hawkes process. These extensions are nontrivial, due to different exciting functions do not allow to state, as in Bacry et al. (2013), a renewal equation for the means of the process. This forces us to carry out proofs which are essentially different, but it provides alternatives which could eventually be applied to other results.

The paper is organized as follows. In Section 2, we define the multivariate HP with different exciting functions, which is considered in this work. Also, some crucial assumptions and results for proving the main theorems are stated in this section. In Section 3, some functional laws of large numbers are stated. Section 4 is devoted for introducing the central limit theorem for a normalization of the rescaled process. Two remarkable cases are summarized in Section 5, namely, those corresponding to the classical multivariate HP and the process defined by a finite number of exciting functions. Finally, in Section 6 we prove a type of Donsker theorem for the process with unpredictable marks.

2. Preliminaries

For $d \in \mathbb{N} \setminus \{0\}$, $\mathbb{R}^d$ stands for the Euclidean space endowed with its usual norm, which we denote by $\|\cdot\|$. Let $\mathbb{R}_+^{d\times d}$ (respectively $\mathbb{R}_+^d$) be the set of all matrices (respectively vectors in $\mathbb{R}^d$) with real and nonnegative entries. For each $\gamma \in \mathbb{R}_+^{d\times d}$, the norm of γ is defined by $\|\gamma\| = \sup\{\|\gamma x\|; x \in \mathbb{R}^d, \|x\| = 1\}$. In what follows, $\{\gamma_p\}_{p\in\mathbb{N}\setminus\{0\}}$ stands for a sequence of functions from $\mathbb{R}_+$, the set of all real nonnegative numbers, into $\mathbb{R}_+^{d\times d}$, such that $\left\|\int_0^\infty \gamma_p(u)\,\mathrm{d}u\right\| < \infty$, for each $p \geq 1$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be stochastic basis satisfying the usual Dellacherie conditions, where a sequence $\{N^p\}_{p\in\mathbb{N}}$ of $\mathbb{F}$-adapted multivariate counting processes, without common jumps, is recursively defined as follows: a) N^0 is a nonhomogeneous Poisson process (NHPP) with intensity $\gamma_0 : \mathbb{R}_+ \to \mathbb{R}_+^d$ such that for each $t \geq 0$, $\left\|\int_0^t \gamma_0(u)\,\mathrm{d}u\right\| < \infty$ and b) for each $p \geq 1$, N^p is a counting process with predictable intensity λ^p given by

$$\lambda_t^p = \int_0^t \gamma_p(t-s)\,\mathrm{d}N_s^{p-1}, \quad t \geq 0.$$

For each $p \in \mathbb{N}$, $\Lambda^p = \{\Lambda^p_t\}_{t\geq 0}$ is defined as $\Lambda^0_t = \int_0^t \gamma_0(s)\,\mathrm{d}s$ and for $p \geq 1$, $\Lambda^p_t = \int_0^t \lambda^p_s\,\mathrm{d}s$, $(t \geq 0)$. Let $M^p = N^p - \Lambda^p$, for $p \in \mathbb{N}$. It follows by induction that for each $p \in \mathbb{N}$ and each $t \geq 0$, $\|\Lambda^p_t\| < \infty$, $\mathbb{P}$-a.s. Consequently, for each $p \in \mathbb{N}$, M^p is a d-dimensional local $(\mathbb{F}, \mathbb{P})$-martingale and N^p is a $\mathbb{P}$-nonexplosive counting process. Let $\{H_t\}_{t\geq 0}$ be the process defined as $H = \sum_{p=0}^{\infty} N^p$. Hence, H is a d-dimensional counting process with $(\mathbb{F}, \mathbb{P})$-predictable intensity λ given by

$$\lambda_t = \gamma_0(t) + \sum_{p=1}^{\infty} \lambda^p_t, \quad t \geq 0.$$

Let $\widetilde{H} = \sum_{p=0}^{\infty} \Lambda^p$ and note that if for each $t \geq 0$, $\|\widetilde{H}_t\| < \infty$, $\mathbb{P}$-a.s., then H is $\mathbb{P}$-nonexplosive and $H - \widetilde{H}$ is a d-dimensional local $(\mathbb{F}, \mathbb{P})$-martingale.

Let f be a function from $\mathbb{R}_+$ to $\mathbb{R}_+^{d\times d}$, and g be a function from $\mathbb{R}_+$ to E, where E denotes $\mathbb{R}_+^d$ or $\mathbb{R}_+^{d\times d}$, two componentwise locally integrable functions. In the sequel, $f * g$ denotes the convolution between f and g, i.e., $(f*g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$, for $t \geq 0$.

Lemma 2.1. *Let f and g be two measurable functions from $\mathbb{R}_+$ into $\mathbb{R}_+^{d\times d}$ and $\mathbb{R}_+^d$, respectively. Then, for each $t \geq 0$, the following two conditions hold:*

(1) $\displaystyle\int_0^t \left(\int_0^u f(u-s)\,\mathrm{d}g(s)\right)\mathrm{d}u = (f*g)(t)$, *whenever g is componentwise increasing and $g(0) = 0$.*

(2) $\displaystyle\left\|\int_0^t (f*g)(u)\,\mathrm{d}u\right\| \leq \left\|\int_0^{\infty} f(u)\,\mathrm{d}u\right\| \left\|\int_0^t g(s)\,\mathrm{d}s\right\|.$

Proof: It is skiped due to both conditions are easy to prove. □

Assumptions

(A1) There exists $\overline{\gamma}_0 = (\overline{\gamma}_0^1, \ldots, \overline{\gamma}_0^d)^\top \in \mathbb{R}_+^d$ such that, $\lim_{n\to\infty} \frac{1}{n}\int_0^{nt} \gamma_0(u)\,\mathrm{d}u = t\overline{\gamma_0}$, for each $t \in [0,1]$.

(A2) $\limsup_{k\to\infty} \left\|\int_0^{\infty} \gamma_k(u)\,\mathrm{d}u\right\| < 1$.

Notations Through this work, we maintain the following notations:

$$\begin{array}{rcl} h_p &=& \gamma_p * \cdots * \gamma_1, \\ h &=& \sum_{p=1}^{\infty} h_p, \\ m_0 &=& \overline{\gamma}_0, \\ m_p &=& \left(\int_0^{\infty} h_p(u)\,\mathrm{d}u\right)\overline{\gamma}_0, \text{ for } p \geq 1, \text{ and} \\ m &=& \sum_{p=0}^{\infty} m_p. \end{array}$$

In the sequel, for any $(\mathbb{F}, \mathbb{P})$-square integrable d-dimensional martingale M, $\langle M\rangle$ stands for the predictable quadratic variation matrix associated to M. When D is a $d\times d$-diagonal matrix with corresponding entries $a \in \mathbb{R}^d$ in its diagonal, we denote $D = \mathrm{diag}(a)$. As usual, $\mathrm{D}([0,1], \mathbb{R}^d)$ stands for the Skorohod space of all right continuous and left hand limited function from $[0,1]$ to $\mathbb{R}^d$. For any $x \in \mathrm{D}([0,1], \mathbb{R}^d)$, we denote $\Delta x(t) = x(t) - x(t-)$, where $x(t-) = \lim_{s\uparrow t} x(s)$. The following continuity module at $(x, \delta) \in \mathrm{D}([0,1], \mathbb{R}^d) \times (0, \infty)$ is defined as

$$\omega(x, \delta) = \sup\{\|x(v) - x(u)\| : 0 \leq u < v \leq 1, v - u < \delta\}.$$

For each $n \in \mathbb{N} \setminus \{0\}$, H^n and $\widetilde{H}^n$ stand for the processes defined, for each $t \geq 0$, as $H_t^n = H_{nt}/n$ and $\widetilde{H}_t^n = \widetilde{H}_{nt}/n$. We are interested in studying the asymptotic properties of H^n, as $n \to \infty$.

3. Laws of large numbers

Lemma 3.1. *Let $\{A^n\}_{n\in\mathbb{N}}$ be a sequence of componentwise increasing d-dimensional processes starting at zero and $A \in \mathrm{C}([0,1], \mathbb{R}^d)$ such that for each $t \in [0,1]$, $\lim_{n\to\infty} \mathbb{E}(\|A_t^n - A(t)\|) = 0$. Then,*

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \|A_t^n - A(t)\|\right) = 0.$$

Proof: It follows by a slight modification of Dini's theorem. □

Lemma 3.2. *Suppose assumptions (A1) and (A2) hold. Then, for each $q \in (0,2]$,*

$$\sum_{p=0}^{\infty} \sup_{n\geq 1} \mathbb{E}\left(\sup_{0\leq u\leq 1} \|M_{nu}^p/\sqrt{n}\|^q\right) < \infty.$$

Proof: From (A2), there exists $k_0 \in \mathbb{N}$ such that $\rho \triangleq \sup_{k>k_0} \left\|\int_0^\infty \gamma_k(u)\,\mathrm{d}u\right\| < 1$.

From Lemma 2.1, for $n \geq 1$ and $p > k_0$, we have

$$\|\mathbb{E}(\Lambda_n^p)\| \leq \rho^{p-k_0} \left\|\int_0^\infty h_{k_0}(u)\,\mathrm{d}u \int_0^n \gamma_0(u)\,\mathrm{d}u\right\|.$$

Hence the Jensen and Doob inequalities imply

$$\sup_{n\geq 1} \mathbb{E}\left(\sup_{0\leq u\leq 1} \|M_{nu}^p/\sqrt{n}\|^q\right) \leq \sup_{n\geq 1} \mathbb{E}\left(\sup_{0\leq u\leq 1} \|M_{nu}^p/\sqrt{n}\|^2\right)^{q/2} \leq 2^q C \rho^{(p-k_0)q/2},$$

where $C = \left\|\int_0^\infty h_{k_0}(u)\,\mathrm{d}u\right\|^{q/2} \sup_{n\geq 1} \left\|\frac{1}{n}\int_0^n \gamma_0(u)\,\mathrm{d}u\right\|^{q/2}$. Consequently

$$\sum_{p=k_0}^{\infty} \sup_{n\geq 1} \mathbb{E}\left(\sup_{0\leq u\leq 1} \|M_{nu}^p/\sqrt{n}\|^q\right) \leq \frac{2^q C}{1-\rho^{q/2}} < \infty,$$

which completes the proof. □

Theorem 3.3. *Under condition (A1), for each $p \geq 1$,*

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \left\|\frac{\Lambda_{nt}^p}{n} - tm_p\right\|\right) = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \left\|\frac{N_{nt}^p}{n} - tm_p\right\|\right) = 0. \quad (3.1)$$

If, additionally, condition (A2) holds, then

$$\lim_{n\to\infty} \sum_{p=0}^{\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \left\|\frac{\Lambda_{nt}^p}{n} - tm_p\right\|\right) = 0 \text{ and } \lim_{n\to\infty} \sum_{p=0}^{\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \left\|\frac{N_{nt}^p}{n} - tm_p\right\|\right) = 0. \quad (3.2)$$

In particular,

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \|\widetilde{H}_t^n - tm\|\right) = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} \|H_t^n - tm\|\right) = 0. \quad (3.3)$$

Proof: Since for each $p \geq 1$, $\Lambda^p = \gamma_p * N^{p-1}$, it follows by induction that

$$\Lambda^p = \sum_{j=0}^{p-1} (\gamma_p * \cdots * \gamma_{j+1}) * M^j + \mathrm{I} * h_p * \gamma_0,$$

where I es the $d \times d$-identity matrix. Hence for each $t \in [0,1]$, we have

$$\frac{1}{n}\Lambda^p_{nt} - tm_p = \alpha^{n,p}_t + \left(\frac{1}{n}(\mathrm{I} * h_p * \gamma_0)(nt) - tm_p\right), \tag{3.4}$$

where

$$\alpha^{n,p}_t = \sum_{j=0}^{p-1} \int_0^t (\gamma_p * \cdots * \gamma_{j+1})(nu) M^j_{n(t-u)} \,\mathrm{d}u.$$

Consequently

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq 1} \|\alpha^{n,p}_t\|\right) \leq & \sqrt{d/n} \sum_{j=0}^{p-1} \mathbb{E}\left(\sup_{0\leq u\leq 1} \left\|M^j_{nu}/\sqrt{n}\right\|\right) \\ & \times \left\|\int_0^\infty (\gamma_p * \cdots * \gamma_{j+1})(u)\,\mathrm{d}u\right\|.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq 1} |\alpha^{n,p}_t|\right) = 0. \tag{3.5}$$

Next we prove that for each $t \in [0,1]$,

$$\lim_{n\to\infty} \frac{1}{n}(\mathrm{I} * h_p * \gamma_0)(nt) = tm_p. \tag{3.6}$$

It is clear that (3.3) holds for $t = 0$, thus we assume $t \in (0,1]$. We have

$$\begin{aligned}\frac{1}{n}(\mathrm{I} * h_p * \gamma_0)(nt) - tm_p \quad = \quad & -\int_0^{nt} h_p(s)\left(\frac{1}{n}\int_{nt-s}^{nt} \gamma_0(u)\,\mathrm{d}u\right)\mathrm{d}s \\ & +\left(\int_0^{nt} h_p(s)\,\mathrm{d}s\right)\left(\frac{1}{n}\int_0^{nt} (\gamma_0(u) - \overline{\gamma}_0)\,\mathrm{d}u\right) \\ & -t\int_{nt}^\infty h_p(u)\overline{\gamma}_0\,\mathrm{d}u.\end{aligned} \tag{3.7}$$

Condition (A1) and the Dominated Convergence Theorem (DCT), which is applied componentwise, imply

$$\lim_{n\to\infty} \int_0^{nt} h_p(s)\left(\frac{1}{n}\int_{nt-s}^{nt} \gamma_0(u)\,\mathrm{d}u\right)\mathrm{d}s = 0$$

and since $t > 0$, (3.6) holds. This fact along with (3.4) and (3.5) imply that for each $t \in [0,1]$,

$$\lim_{n\to\infty} \mathbb{E}\left(\left\|\frac{\Lambda^p_{nt}}{n} - tm_p\right\|\right) = 0$$

and from Lemma 3.1, $\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{\Lambda^p_{nt}}{n} - tm_p\right\|\right) = 0$. Hence by the Jensen and Doob inequalities, we have

$$\mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{N^p_{nt}}{n} - tm_p\right\|\right) \le \frac{2d}{\sqrt{n}}\,\mathbb{E}\left(\left\|\frac{\Lambda^p_n}{n}\right\|\right)^{1/2} + \mathbb{E}\left(\sup_{0\le t\le 1}\left|\frac{\Lambda^p_{nt}}{n} - tm_p\right|\right)$$

and consequently (3.1) holds.

Next, we assume condition (A2). From (3.4) and Lemma 2.1, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{\Lambda^p_{nt}}{n} - tm_p\right\|\right) &\le \mathbb{E}\left(\sup_{0\le t\le 1}\|\alpha^{n,p}_t\|\right)\\ &\quad + \left\|\int_0^\infty h_p(u)\,\mathrm{d}u\right\|\left(\left\|\frac{1}{n}\int_0^n \gamma_0(u)\,\mathrm{d}u\right\| + \|\overline{\gamma}_0\|\right).\end{aligned}$$

On the one hand,

$$\begin{aligned}\sum_{p=1}^\infty \sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le t\le 1}\|\alpha^{n,p}_t\|\right) &\le \sum_{j=0}^\infty \sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le u\le 1}\left\|M^j_{nu}/\sqrt{n}\right\|\right)\\ &\quad \times \sup_{j\ge 1}\sum_{p=j}^\infty \|\int_0^\infty (\gamma_p * \cdots * \gamma_j)(u)\,\mathrm{d}u\|.\end{aligned}$$

Lemma 3.2 and (A2) imply that $\sum_{p=1}^\infty \sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le t\le 1}\|\alpha^{n,p}_t\|\right) < \infty$. From (A1) and (A2), we have

$$\sum_{p=1}^\infty \left\|\int_0^\infty h_p(u)\,\mathrm{d}u\right\|\left(\sup_{n\ge 1}\left\|\frac{1}{n}\int_0^n \gamma_0(u)\,\mathrm{d}u\right\| + \|\overline{\gamma}_0\|\right) < \infty.$$

Hence, (3.1) and the DCT imply

$$\lim_{n\to\infty}\sum_{p=0}^\infty \mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{\Lambda^p_{nt}}{n} - tm_p\right\|\right) = 0 \tag{3.8}$$

and since

$$\begin{aligned}\sum_{p=0}^\infty \mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{N^p_{nt}}{n} - tm_p\right\|\right) &\le \frac{1}{\sqrt{n}}\sum_{p=1}^\infty \sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le u\le 1}\|M^p_{nu}/\sqrt{n}\|\right)\\ &\quad + \sum_{p=0}^\infty \mathbb{E}\left(\sup_{0\le t\le 1}\left\|\frac{\Lambda^p_{nt}}{n} - tm_p\right\|\right),\end{aligned}$$

(3.2) follows from Lemma 3.2 and (3.8). Finally, (3.3) is directly obtained from (3.2) and therefore the proof is complete. □

4. A central limit theorem

Theorem 4.1. *Let $\{X^n\}_{n\in\mathbb{N}}$ be the sequence of processes defined, for $t\in[0,1]$, as $X^n_t = \sqrt{n}(H^n_t - tm)$. Additionally to (A2), suppose the following two conditions holds:*

(B1) $\lim_{n\to\infty}\sup_{0\le t\le 1}\left\|\frac{1}{\sqrt{n}}\int_0^{nt}(\gamma_0(u) - \overline{\gamma}_0)\,\mathrm{d}u\right\| = 0.$

(B2) $\lim_{n\to\infty}\sqrt{n}\int_n^\infty h(u)\overline{\gamma}_0\,\mathrm{d}u = 0.$

Then, $\{X^n\}_{n\in\mathbb{N}}$ converges in law to a continuous d-dimensional martingale X, starting at zero, with predictable quadratic variation matrix $\langle X\rangle$ given by $\langle X\rangle_t = t\sum_{j=0}^{\infty} C_j \mathrm{diag}(m_j) C_j^\top$, where for each $j\in\mathbb{N}$,

$$C_j = \left(\mathrm{I} + \sum_{p=1}^{\infty}\int_0^{\infty}\gamma_{j+p}(u)\,\mathrm{d}u\cdots\int_0^{\infty}\gamma_{j+1}(u)\,\mathrm{d}u\right)$$

Proof: Let

$$\begin{aligned}
Y_t^n &= \frac{1}{\sqrt{n}}\sum_{j=0}^{\infty} M_{nt}^j + \sum_{p=1}^{\infty}\sum_{j=0}^{p-1}\sqrt{n}\left(\int_0^1(\gamma_p*\cdots*\gamma_{j+1})(nu)\,\mathrm{d}u\right)M_{nt}^j,\\
D_{0,t}^n &= \sum_{p=1}^{\infty}\sqrt{n}\left(\frac{1}{n}\int_0^{nt}(h_p*\gamma_0)(u)\,\mathrm{d}u - tm_p\right),\\
D_{1,t}^n &= \sum_{p=1}^{\infty}\sum_{j=0}^{p-1}\sqrt{n}\int_0^t(\gamma_p*\cdots*\gamma_{j+1})(nu)(M_{n(t-u)}^j - M_{nt}^j)\,\mathrm{d}u \quad \text{and}\\
D_{2,t}^n &= \sum_{j=0}^{p-1}\sum_{p=1}^{\infty}\sqrt{n}\int_t^1(\gamma_p*\cdots*\gamma_{j+1})(nu)M_{nt}^j\,\mathrm{d}u.
\end{aligned}$$

We have

$$X_t^n = Y_t^n + \frac{1}{\sqrt{n}}\int_0^{nt}(\gamma_0(u)-\overline{\gamma}_0)\,\mathrm{d}u + D_{0,t}^n + D_{1,t}^n - D_{2,t}^n.$$

It is clear that $D_{0,0}^n = 0$ and from (3.7), for each $t\in(0,1]$, we have

$$\begin{aligned}
D_{0,t}^n &= -\int_0^{nt}h(s)\left(\frac{1}{\sqrt{n}}\int_{nt-s}^{nt}\gamma_0(u)\,\mathrm{d}u\right)\mathrm{d}s\\
&+ \left(\int_0^{nt}h(s)\,\mathrm{d}s\right)\left(\frac{1}{\sqrt{n}}\int_0^{nt}(\gamma_0(u)-\overline{\gamma}_0)\,\mathrm{d}u\right) + t\sqrt{n}\int_{nt}^{\infty}h(u)\overline{\gamma}_0\,\mathrm{d}u
\end{aligned}$$

and condition (B1) implies

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left\|\sup_{0\le t\le 1}\int_{nt-s}^{nt}\gamma_0(u)\,\mathrm{d}u\right\| = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\sup_{0\le t\le 1}\left\|\int_0^{nt}(\gamma_0(u)-\overline{\gamma}_0)\,\mathrm{d}u\right\| = 0.$$

Since for each $x\in\mathbb{R}_+^d$, each component of $\int_0^{\infty}h(s)x\,\mathrm{d}s$ is finite, from the DCT we have

$$\lim_{n\to\infty}\int_0^{nt}h(s)\left(\frac{1}{\sqrt{n}}\int_{nt-s}^{nt}\gamma_0(u)\,\mathrm{d}u\right)\mathrm{d}s = 0$$

and due to by (B2), $\lim_{n\to\infty}\sup_{0\le t\le 1}\left\|t\sqrt{n}\int_{nt}^{\infty}h(u)\overline{\gamma}_0\,\mathrm{d}u\right\| = 0$, the sequence $\{\sup_{0\le t\le 1}\|D_{0,t}^n\|\}_{n\in\mathbb{N}\setminus\{0\}}$ converges to zero.

For each $\delta \in (0,1)$ we have

$$\begin{aligned}\mathbb{E}(\sup_{0\le t\le 1}\|D^n_{1,t}\|) \quad &\le \quad \sum_{j=0}^{\infty}\sup_{n\ge 1}\mathbb{E}\left(\sup_{|u'-u''|\le\delta}\|(M^j_{nu'}-M^j_{nu''})/\sqrt{n}\|\right)\\ &\quad\times \sum_{p=j+1}^{\infty}\left\|\int_0^{\infty}(\gamma_p*\cdots*\gamma_{j+1})(u)\,\mathrm{d}u\right\|\\ &\quad+\quad 2\sum_{j=0}^{\infty}\sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le u\le 1}\|M^j_{nu}/\sqrt{n}\|\right)\\ &\quad\times \sum_{p=j+1}^{\infty}\left\|\int_{n\delta}^{\infty}(\gamma_p*\cdots*\gamma_{j+1})(u)\,\mathrm{d}u\right\|.\end{aligned}$$

Since

$$\sum_{j=0}^{\infty}\sup_{n\ge 1}\mathbb{E}\left(\sup_{0\le u\le 1}\|M^j_{nu}/\sqrt{n}\|\right)<\infty \text{ and } \sup_{j\ge 1}\sum_{p=j}^{\infty}\left\|\int_0^{\infty}(\gamma_p*\cdots*\gamma_j)(u)\,\mathrm{d}u\right\|<\infty,$$

by the DCT, in order to prove that $\{\mathbb{E}(\sup_{0\le t\le 1}\|D^n_{1,t}\|)\}_{n\in\mathbb{N}\setminus\{0\}}$ converges to zero, it suffices to prove that for each $j\ge 1$,

$$\lim_{\delta\to 0}\lim_{n\to\infty}\mathbb{E}\left(\omega(M^j_{n\cdot}/\sqrt{n},\delta)\right)=0 \tag{4.1}$$

and for each $j\ge 1$ and $\delta>0$,

$$\lim_{n\to\infty}\sum_{p=j+1}^{\infty}\int_{n\delta}^{\infty}(\gamma_p*\cdots*\gamma_{j+1})(u)\,\mathrm{d}u=0. \tag{4.2}$$

We have

$$\mathbb{E}\left(\sup_{|u'-u''|\le\delta}\left\|\langle M^j_{n\cdot}/\sqrt{n}\rangle_{u'}-\langle M^j_{n\cdot}/\sqrt{n}\rangle_{u''}\right\|\right)\le 2\,\mathbb{E}\left(\sup_{0\le t\le 1}\left\|\widetilde{H}^n_t-tm\right\|\right)+\delta\|m\|.$$

Hence Theorem 3.3 implies that $\{\langle M^j_{n\cdot}/\sqrt{n}\rangle\}_{n\in\mathbb{N}\setminus\{0\}}$ satisfies

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{j\in\mathbb{N}}\mathbb{E}\left(\sup_{|u'-u''|\le\delta}\|\langle M^j_{n\cdot}/\sqrt{n}\rangle_{u'}-\langle M^j_{n\cdot}/\sqrt{n}\rangle_{u''}\|\right)=0.$$

From Theorem 2, Section II.3 in Rebolledo (1979), it is obtained that the sequence $\{M^j_{n\cdot}/\sqrt{n}\}_{j\in\mathbb{N},n\in\mathbb{N}\setminus\{0\}}$ is tight and consequently (4.1) holds.

Let k_0 and ρ as in the proof of Lemma 3.2. We have

$$\begin{aligned}&\sum_{p=j+1}^{\infty}\sup_{n\in\mathbb{N}}\left\|\int_{n\delta}^{\infty}(\gamma_p*\cdots*\gamma_{j+1})(u)\,\mathrm{d}u\right\|\\ &\le\begin{cases}\displaystyle\sum_{p=j+1}^{k_0}\prod_{i=j+1}^{p}\rho_i+\frac{\rho}{1-\rho}\prod_{i=j+1}^{k_0}\rho_i & \text{if}\quad j<k_0;\\ \displaystyle\frac{1}{1-\rho} & \text{if}\quad j\ge k_0,\end{cases}\end{aligned}$$

where $\rho_i=\left\|\int_0^{\infty}\gamma_i(u)\,\mathrm{d}u\right\|$. Hence, (4.2) follows from the DCT.

Notice that $\sup_{0\le t\le 1}\|D^n_{2,t}\| \le \sum_{j=0}^\infty A_{n,j}$, where

$$A_{n,j} = \sqrt{d}\sup_{0\le t\le 1}\|M^j_{nt}/\sqrt{n}\| \sum_{p=j+1}^\infty \left\| \int_{nt}^n (\gamma_p * \cdots * \gamma_{j+1})(u)\,\mathrm{d}u \right\|.$$

For each $\delta \in (0,1]$, we have

$$\begin{aligned} \mathbb{E}(A_{n,j}) \quad \le \quad & \sqrt{d}\,\mathbb{E}\left(\sup_{0\le t\le \delta}\|M^j_{nt}/\sqrt{n}\|\right)\sum_{p=j+1}^\infty \left\|\int_0^\infty (\gamma_{j+1} * \cdots * \gamma_p)(u)\,\mathrm{d}u\right\| \\ & +\sqrt{d}\,\mathbb{E}\left(\sup_{0\le t\le 1}\|M^j_{nt}/\sqrt{n}\|\right)\sum_{p=j+1}^\infty \left\|\int_{n\delta}^\infty (\gamma_{j+1} * \cdots * \gamma_p)(u)\,\mathrm{d}u\right\|. \end{aligned}$$

Hence Lemma 3.2, (4.1) and (A2) imply that for each $j \in \mathbb{N}$, $\lim_{n\to\infty}\mathbb{E}(A_{n,j}) = 0$. Consequently, $\lim_{n\to\infty}\mathbb{E}(\sup_{0\le t\le 1}\|D^n_{2,t}\|) = 0$.

We have

$$\sup_{0\le t\le 1}\|\Delta Y^n_t\| \le \frac{1}{\sqrt{n}}\left(1 + \sup_{j\ge 1}\sum_{p=j}^\infty \left\|\int_0^\infty (\gamma_p * \cdots * \gamma_j)(u)\,\mathrm{d}u\right\|\right)$$

and

$$\langle Y^n\rangle_t = \sum_{j=0}^\infty C^n_j \mathrm{diag}\left(\frac{\Lambda^j_{nt}}{n}\right) C^{n\top}_j,$$

where

$$C^n_j = \mathrm{I} + \sum_{p=1}^\infty \int_0^n (\gamma_{j+p} * \cdots * \gamma_{j+1})(u)\,\mathrm{d}u.$$

Hence, there exists a constant $C > 0$ tal que $\sup_{0\le t\le 1}\|\Delta Y^n_t\| \le C/\sqrt{n}$ and from Theorem 3.3, (A2) and the DCT, we have

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\le t\le 1}\left\|\langle Y^n\rangle_t - t\sum_{j=0}^\infty C_j \mathrm{diag}(m_j) C_j^\top\right\|\right) = 0.$$

Accordingly, Corollary 12, in Section II.5 by Rebolledo (1979), implies that $\{Y^n\}_{n\in\mathbb{N}}$ converges in law to a continuous d-dimensional martingale with predictable increasing process $\langle X\rangle$ given by $\langle X\rangle_t = t\sum_{j=0}^\infty C_j\mathrm{diag}(m_j)C_j^\top$, for $t \in [0,1]$. Therefore, the proof is complete. □

5. Two remarkable cases

In this section we consider two remarkable cases where Theorem 4.1 applies. The first one corresponds to the classical multivariate Hawkes process, i.e. when the matrix functions γ_k ($k \in \mathbb{N}\setminus\{0\}$) are assumed to be equal.

Corollary 5.1. *Suppose for each $k \ge 1$, $\gamma_k = \gamma$ does not depend on k, condition (B1) holds, $\left\|\int_0^\infty \gamma(u)\,\mathrm{d}u\right\| < 1$ and $\left\|\int_0^\infty \sqrt{u}\gamma(u)\,\mathrm{d}u\right\| < \infty$. Then, $\{X^n\}_{n\in\mathbb{N}}$ converges in law to a d-dimensional continuous martingale X starting at zero and such that, for each $t \in [0,1]$, $\langle X\rangle_t = ts^2$, where*

$$s^2 = \left(\mathrm{I} - \int_0^\infty \gamma(u)\,\mathrm{d}u\right)^{-1}\mathrm{diag}\left[\left(\mathrm{I} - \int_0^\infty \gamma(u)\,\mathrm{d}u\right)^{-1}\overline{\gamma}_0\right]\left(\mathrm{I} - \int_0^\infty \gamma(u)\,\mathrm{d}u\right)^{-1}.$$

Note that in this case, which was studied by Bacry et al. (2013), the martingale X has the same distribution that

$$\left(\mathrm{I} - \int_0^\infty \gamma(u)\,\mathrm{d}u\right)^{-1} \mathrm{diag}\left[\left(\mathrm{I} - \int_0^\infty \gamma(u)\,\mathrm{d}u\right)^{-1} \overline{\gamma}_0\right]^{1/2} W,$$

where W is a standard d-dimensional Brownian motion.

Other particular case of Theorem 4.1 is when there exists $n^* \in \mathbb{N}$ such that $\gamma_{n^*+1} = 0$. The particular case $n^* = 1$ corresponds to a d-dimensional version of the Neyman-Scott cluster point process where the 'mother point process' is included (see e.g. Møller and Waagepetersen, 2004).

Corollary 5.2. *Suppose condition (B1) holds and that there exists $n^* \in \mathbb{N}$ such that $\gamma_{n^*+1} = 0$. In addition, we assume for each $k \in \{1, \ldots, n^*\}$, $\left\|\int_0^\infty u^{1/2}\gamma_k(u)\,\mathrm{d}u\right\| < \infty$. Then, $\{X^n\}_{n\in\mathbb{N}}$ converges in law to a d-dimensional continuous martingale X, starting at zero, with predictable quadratic variation matrix $\langle X\rangle$ given by*

$$\langle X\rangle_t = t\sum_{j=0}^{n^*} C_j \mathrm{diag}(m_j) C_j^\top,$$

where for each $j \in \mathbb{N}$, C_j is defined as in Theorem 4.1.

6. Unpredictable marks

The classical HP with unpredictable marks is defined in Daley and Vere-Jones (2003), Brémaud et al. (2002) and Møller and Rasmussen (2005). In this section, this situation is extended to the case of the HP with different exciting functions, as follows: let $\{\xi_k^i; k \in \mathbb{N}, 1 \le i \le d\}$ a set of i.i.d. random variables and independent of H with mean ν and variance σ^2. Let $H = (H^{(1)}, \ldots, H^{(d)})^\top$ and $\{S^n\}_{n\in\mathbb{N}}$ defined by $S^n = (S^{n,1}, \ldots, S^{n,d})^\top$, where for each $i \in \{1, \ldots, d\}$,

$$S_t^{n,i} = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{H_{nt}^{(i)}} \xi_k^i - \nu m^i t\right), \quad t \in [0,1]$$

and $m = (m^1, \ldots, m^d)^\top$.

The proof of the following lemma follows from a slight extension of Proposition 8.15 in Breiman (1968) and the Stone-Weierstrass theorem.

Lemma 6.1. *Let E and F be two metric spaces and $\{P_n\}_{n\in\mathbb{N}}$ a sequence of probability measures defined on the Borel σ-algebra of $E \times F$. Suppose there exists a probability measure P defined on the Borel σ-algebra of $E \times F$ such that for any bounded functions $u : E \to \mathbb{R}$ and $v : F \to \mathbb{R}$, $\lim_{n\to\infty} \int u(x)v(y)P_n(dxdy) = \int u(x)v(y)P(dxdy)$. Then, $\{P_n\}_{n\in\mathbb{N}}$ converges weakly to P.*

In Theorem 6.2 below, we maintain notations stated in Theorem 4.1.

Theorem 6.2. *Suppose that conditions (A1), (A2), (B1) and (B2) hold. Then, $\{S^n\}_{n\in\mathbb{N}}$ converges in law to the continuous d-dimensional semi-martingale $S = \nu X + V$, where V is independent of X and for each $t \in [0,1]$, $V_t = \mathrm{diag}(tm)^{1/2}W$, being W a normal random vector with mean vector zero and variance and covariance matrix $\sigma^2\mathrm{I}$.*

Proof: Let $D = \mathbb{N}^d$ be with the natural order $\preceq$ defined componentwise. For each $n = (n_1, \ldots, n_d) \in D \setminus \{0\}$ and $t \geq 0$, let $W_n = (W^1_{n_1}, \ldots, W^d_{n_d})^\top$, where for $i \in \{1, \ldots, d\}$, $W^i_{n_i} = \sum_{k=0}^{n_i} (\xi^i_k - \nu)/\sqrt{n_i}$. By Theorem 4.1, $\{X^n\}_{n\in\mathbb{N}}$ converges in distribution to X and by the standard Central Limit Theorem, $\{W_n\}_{n\in D}$ converges in distribution to a normal random vector W with mean zero and variance and covariance matrix $\sigma^2\mathrm{I}$. We assume X and W are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and hence they are independent. By defining $W^i_0 = 0$, we have $S^n = \nu X^n + \mathrm{diag}(H^n)^{1/2} W_{nH^n}$. Moreover, $\{X^n\}_{n\in\mathbb{N}}$ and $\{W_n\}_{n\in D}$ are independent. Let $T^i = \inf\{t > 0 : H^{(i)}_t > 0\}$ and for each $n \in \mathbb{N} \setminus \{0\}$, $\tau^i_n = T^i/\sqrt{n}$, $J^{n,i}_t = H^{(i)}_{n(\tau^i_n \vee t)}$ and

$$\widetilde{S}^{n,i}_t = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{J^{n,i}_t} \xi^i_k - \nu m^i t\right), \quad t \in [0,1].$$

Let $\widetilde{S}^n = (\widetilde{S}^{n,1}, \ldots, \widetilde{S}^{n,d})^\top$ and note that, for each $i \in \{1, \ldots, d\}$,

$$\sup_{0\leq t\leq 1} |\widetilde{S}^{n,i}_t - S^{n,i}_t| \leq 2\nu m^i \tau^i_n + |\xi^i_1|/\sqrt{n}.$$

From (A1), $\gamma_0 = (\gamma^1_0, \ldots, \gamma^d_0)^\top$ satisfies $\int_0^\infty \gamma^i_0(u)\,\mathrm{d}u = \infty$, for all $i \in \{1, \ldots, d\}$. Accordingly $T^i < \infty$, $\mathbb{P}$-a.s. and hence $\lim_{n\to\infty} \sup_{0\leq t\leq 1} |\widetilde{S}^{n,i}_t - S^{n,i}_t| = 0$. Moreover $\widetilde{S}^n = \nu X^n + \mathrm{diag}(H^n)^{1/2} W_{J_n}$, where $J^n = (J^{n,1}, \ldots, J^{n,d})^\top$ and since, from Theorem 3.3, $\{\mathrm{diag}(H^n)^{1/2}\}_{n\in\mathbb{N}}$ converges in probability to d, where $d(t) = \mathrm{diag}(tm)^{1/2}$, it only remains to prove that $\{(X^n, W_{J^n})\}_{n\in\mathbb{N}}$ converges in distribution to (X, W).

Let u and v be two uniformly continuous and bounded functions from $\mathrm{D}([0,1], \mathbb{R}^d)$ to $\mathbb{R}$ and $\epsilon > 0$. There exists $n^* \in D$ such that $|\,\mathbb{E}(v(W_n)) - \mathbb{E}(v(W))| < \epsilon$, whenever $n^* \preceq n$. Let c_u and c_v be upper bounds of $|u|$ and $|v|$, respectively. We have

$$\begin{aligned} |\mathbb{E}(u(X^n)v(W_{J^n}) - u(X)v(W))| \quad &\leq \quad c_v\,|\mathbb{E}\,(u(X^n) - u(X))| \\ &\quad + |\mathbb{E}(u(X^n)[v(W_{J^n}) - v(W)]|\,. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} |\mathbb{E}(u(X^n)v(W_{J^n}) - u(X)v(W))| \leq \limsup_{n\to\infty} |\mathbb{E}(u(X^n)[v(W_{J^n}) - v(W)]|\,. \tag{6.1}$$

Let $A_n = \{\omega \in \Omega :\ \text{for all } t \in [0,1], n^* \preceq J^n_t(\omega)\}$. Due to the independence of W^n and H, we have

$$|\mathbb{E}(u(X^n)[v(W_{J^n}) - v(W)]| \quad \leq \quad \epsilon c_u + 2c_u c_v(1 - \mathbb{P}(B_n)),$$

where $B_n = \{\omega \in \Omega : n^* \preceq (H^{(1)}_{\sqrt{n}T^1}(\omega), \ldots, H^{(d)}_{\sqrt{n}T^d}(\omega))^\top\}$. Since for each $i \in \{1, \ldots, d\}$, $T^i > 0$, $\mathbb{P}$-a.s., Theorem 3.3 implies that $\lim_{n\to\infty} \mathbb{P}(B_n) = 1$. This fact along with (6.1) imply that $\limsup_{n\to\infty} \left|\mathbb{E}(u(X^n)v(\widetilde{V}^n) - u(X)v(V))\right| \leq 2\epsilon c_u$. But $\epsilon > 0$ is arbitrary and consequently,

$$\lim_{n\to\infty} \left|\mathbb{E}\left(u(U^n)v(\widetilde{V}^n) - u(U)v(V)\right)\right| = 0.$$

Therefore, by Lemma 6.1 the proof is complete. $\square$

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