ALEA, Lat. Am. J. Probab. Math. Stat. 12, 765–792 (2015)

# Discrete approximations for sums of m-dependent random variables

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**Abstract.** Sums of $m$-dependent integer-valued random variables are approximated by compound Poisson, negative binomial and binomial distributions and signed compound Poisson measures. Estimates are obtained for the total variation metric. The results are then applied to statistics of $m$-dependent $(k_1, k_2)$ events and 2-runs. Heinrich's method and smoothing properties of convolutions are used for the proofs.

## 1. The setup

In this paper, we consider sums $S_n = X_1 + X_2 + \cdots + X_n$ of non-identically distributed 1-dependent random variables concentrated on nonnegative integers. Our aim is to estimate the closeness of $S_n$ to compound Poisson, negative binomial and binomial distributions, under some conditions for factorial moments. For the proof of the main results, we use Heinrich's (see Heinrich, 1982, 1987) version of the characteristic function method. Though this method does not allow to obtain small absolute constants, it is flexible enough for obtaining asymptotically sharp constants, as demonstrated for 2-runs statistic. Moreover, our approach allows for construction of asymptotic expansions.

We recall that the sequence of random variables $\{X_k\}_{k\geq 1}$ is called $m$-dependent if, for $1 < s < t < \infty$, $t - s > m$, the sigma algebras generated by $X_1, \dots, X_s$ and $X_t, X_{t+1} \dots$ are independent. It is clear that, by grouping consecutive summands, we can reduce the sum of $m$-dependent variables to the sum of 1-dependent ones. Therefore, the results of this paper can be applied for some cases of $m$-dependent variables, as exemplified by binomial approximation to $(k_1, k_2)$ events.

*Received by the editors February 19, 2015; accepted October 19, 2015.*

2010 *Mathematics Subject Classification.* 60F05, 60G50.

*Key words and phrases.* Compound Poisson distribution, signed compound Poisson measure, negative binomial, binomial, m-dependent variables, total variation norm.

Let us introduce some necessary notations. Let $\{Y_k\}_{k\geq 1}$ be a sequence of arbitrary real or complex-valued random variables. We assume that $\widehat{\mathrm{E}}(Y_1)=\mathrm{E}Y_1$ and, for $k\geqslant 2$, define $\widehat{\mathrm{E}}(Y_1,Y_2,\cdots Y_k)$ by

$$\widehat{\mathrm{E}}(Y_1,Y_2,\cdots,Y_k)=\mathrm{E}Y_1Y_2\cdots Y_k-\sum_{j=1}^{k-1}\widehat{\mathrm{E}}(Y_1,\cdots,Y_j)\mathrm{E}Y_{j+1}\cdots Y_k.$$

We define $j$-th factorial moment of $X_k$ by $\nu_j(k)=\mathrm{E}X_k(X_k-1)\cdots(X_k-j+1)$, $(k=1,2,\ldots,n,\ j=1,2,\ldots)$. Let

$$\begin{aligned}
\Gamma_1 &= \mathrm{E}S_n=\sum_{k=1}^{n}\nu_1(k),\\
\Gamma_2 &= \frac{1}{2}(\mathrm{Var}S_n-\mathrm{E}S_n)=\frac{1}{2}\sum_{k=1}^{n}\big(\nu_2(k)-\nu_1^2(k)\big)+\sum_{k=2}^{n}\widehat{\mathrm{E}}(X_{k-1},X_k),\\
\Gamma_3 &= \frac{1}{6}\sum_{k=1}^{n}\big(\nu_3(k)-3\nu_1(k)\nu_2(k)+2\nu_1^3(k)\big)\\
&\quad-\sum_{k=2}^{n}\big(\nu_1(k-1)+\nu_1(k)\big)\widehat{\mathrm{E}}(X_{k-1},X_k)\\
&\quad+\frac{1}{2}\sum_{k=2}^{n}\big(\widehat{\mathrm{E}}(X_{k-1}(X_{k-1}-1),X_k)+\widehat{\mathrm{E}}(X_{k-1},X_k(X_k-1))\big)\\
&\quad+\sum_{k=3}^{n}\widehat{\mathrm{E}}(X_{k-2},X_{k-1},X_k).
\end{aligned}$$

For the sake of convenience, we assume that $X_k\equiv 0$ and $\nu_j(k)=0$ if $k\leqslant 0$ and $\sum_k^n=0$ if $k>n$. We denote the distribution and characteristic function of $S_n$ by $F_n$ and $\widehat{F}_n(t)$, respectively. Below we show that $\Gamma_1$, $2\Gamma_2$ and $6\Gamma_3$ are factorial cumulants of $F_n$, that is,

$$\widehat{F}_n(t)=\exp\{\Gamma_1(\mathrm{e}^{\mathrm{i}t}-1)+\Gamma_2(\mathrm{e}^{\mathrm{i}t}-1)^2+\Gamma_3(\mathrm{e}^{\mathrm{i}t}-1)^3+\ldots\}.$$

For approximation of $F_n$, it is natural to use measures or distributions which allow similar expressions.

Let $I_a$ denote the distribution concentrated at real $a$ and set $I=I_0$. Henceforth, the products and powers of measures are understood in the convolution sense. Further, for a measure $M$, we set $M^0=I$ and

$$\mathrm{e}^M:=\exp\{M\}=\sum_{k=0}^{\infty}\frac{1}{k!}\,M^k.$$

The total variation norm of measure $M$ is denoted by

$$\|M\|=\sum_{k=-\infty}^{\infty}|M\{k\}|.$$

We use symbol $C$ to denote all (in general, different) positive absolute constants. We use symbols $\theta$ and $\Theta$ to denote all real or complex quantities satisfying $|\theta|\leqslant 1$ and all measures of finite variation satisfying $\|\Theta\|=1$, respectively.

Next we define approximations of this paper. Let

$$\mathrm{Pois}(\Gamma_1) = \exp\{\Gamma_1(I_1 - I)\}, \quad \mathrm{G} = \exp\{\Gamma_1(I_1 - I) + \Gamma_2(I_1 - I)^2\}.$$

It is easy to see that $\mathrm{Pois}(\Gamma_1)$ is Poisson distribution with parameter $\Gamma_1$. In general, G is a signed measure, since $\Gamma_2$ can be negative. Signed compound Poisson measures similar to G are used in numerous papers, see Barbour and Čekanavičius (2002); Barbour and Xia (1999); Čekanavičius and Vellaisamy (2010); Roos (2003), and the references therein. In comparison to the Poisson distribution, the main benefit of G is matching of two moments, which then allows for the accuracy comparable to the one achieved by the normal approximation. This fact is illustrated in the next two sections. From a practical point of view, signed measures are not always convenient to use, since for calculation of their 'probabilities' one needs inverse Fourier transform or recursive algorithms. Therefore, we also prove estimates for such widely used distributions as binomial and negative binomial. We define the binomial distribution of this paper as

$$\mathrm{Bi}(N, \bar{p}) = (I + \bar{p}(I_1 - I))^N, \quad N = \lfloor \tilde{N} \rfloor, \quad \tilde{N} = \frac{\Gamma_1^2}{2|\Gamma_2|}, \quad \bar{p} = \frac{\Gamma_1}{N}.$$

Here, we use $\lfloor \tilde{N} \rfloor$ to denote the integer part of $\tilde{N}$, that is, $\tilde{N} = N + \epsilon$, for some $0 \le \epsilon < 1$. Also, we define negative binomial distribution and choose its parameters in the following way:

$$\begin{aligned} \mathrm{NB}(r, \bar{q})\{j\} &= \frac{\Gamma(r+j)}{j!\Gamma(r)}\, \bar{q}^r (1-\bar{q})^j, \quad (j \in \mathbb{Z}_+), \\ \frac{r(1-\bar{q})}{\bar{q}} &= \Gamma_1, \quad r\left(\frac{1-\bar{q}}{\bar{q}}\right)^2 = 2\Gamma_2. \end{aligned} \tag{1.1}$$

Note that symbols $\bar{q}$ and $\bar{p}$ are not related and, in general, $\bar{q} + \bar{p} \neq 1$.

## 2. Known results

There are many results dealing with approximations to the sum of dependent integer-valued random variables. Note, however, that with very few exceptions: a) all papers are devoted to the sums of indicator variables only; b) results are not related to $k$-dependent variables. For example, indicators connected in a Markov chain are investigated in Čekanavičius and Vellaisamy (2010); Xia and Zhang (2009). The most general results, containing $k$-dependent variables as partial cases, are obtained for birth-death processes with some stochastic ordering, see Brown and Xia (2001); Daly et al. (2012); Eichelsbacher and Roos (1999) and the references therein.

Arguably the best explored case of sums of $k$-dependent integer-valued random variables is $k$-runs. Let $\eta_i \sim Be(p_i)$ (i=1,2,…) be independent Bernoulli variables. Let us define $\xi_i = \prod_{s=i}^{i+k-1} \eta_i$, $S^* = \sum_{i=1}^{n} \xi_i$. The sum $S^*$ is called $k$-runs statistic. Note that frequently $\eta_{i+nm}$ is treated as $\eta_i$ for $1 \leqslant i \leqslant n$ and $m = \pm 1, \pm 2, \ldots$. Approximations of 2 or $k$-runs statistic by Poisson, negative binomial distribution or signed compound Poisson measure are considered in Barbour and Xia (1999); Brown and Xia (2001); Daly et al. (2012); Röllin (2005); Wang and Xia (2008). Particularly in Brown and Xia (2001) it was proved that, if $k = 2$ and $p_i \equiv p$, $n \geqslant 2$ and $p < 2/3$, then

$$\|\mathcal{L}(S^*) - \mathrm{NB}(\tilde{r}, \tilde{q})\| \leqslant \frac{64.4p}{\sqrt{(n-1)(1-p)^3}}. \tag{2.1}$$

Here $\tilde{q} = (2p - 3p^2)/(1 + 2p - 3p^2)$ and $(1-\tilde{q})/\tilde{q} = np^2$.

The $k$-runs statistic has very explicit dependency of summands. Meanwhile, our aim is to obtain a general result which includes sums of independent random variables as a particular case. Except for examples, no specific assumptions about the structure of summands are made. For bounded and identically distributed random variables a similar approach is taken in Petrauskienė and Čekanavičius (2011). We give one example from Petrauskienė and Čekanavičius (2011) in the notation of the previous Section. Let the $X_i$ be identically distributed, $|X_1| \leqslant C$, and, for $n \to \infty$,

$$\nu_1(1) = o(1), \quad \nu_2(1) = o(\nu_1(1)), \quad \mathrm{E}X_1X_2 = o(\nu_1(1)), \quad n\nu_1(1) \to \infty. \tag{2.2}$$

Then

$$\|F_n - G\| = O\bigg(\frac{\tilde{R}}{\nu_1(1)\sqrt{n\nu_1(1)}}\bigg),$$

where

$$\begin{aligned}\tilde{R} \;=\; & \nu_3(1) + \nu_1(1)\nu_2(1) + \nu_1^3(1) + \mathrm{E}(X_1(X_1-1)X_2 + X_1X_2(X_2-1)) \\ & + \nu_1(1)\mathrm{E}X_1X_2 + \mathrm{E}X_1X_2X_3.\end{aligned}$$

Condition (2.2) implies that $X_i$ form a triangular array and $P(X_i = k) = o(1)$, $k > 1$. Thus, the classical case of a sequence of random variables, so typical for CLT, is completely excluded. Moreover, assumption $|X_1| \leqslant C$ seems rather strong. For example, then one can not consider Poisson or geometric random variables as possible summands.

## 3. Results

All results are obtained under the following conditions:

$$\nu_1(k) \;\leqslant\; 1/100, \quad \nu_2(k) \leqslant \nu_1(k), \quad \nu_4(k) < \infty, \quad (k = 1, 2, \ldots, n), \tag{3.1}$$

$$\lambda \;:=\; \sum_{k=1}^n \nu_1(k) - 1.52\sum_{k=1}^n \nu_2(k) - 12\sum_{k=2}^n \mathrm{E}X_{k-1}X_k > 0. \tag{3.2}$$

The last condition is satisfied, if the following two assumptions hold

$$\sum_{k=1}^n \nu_2(k) \leqslant \frac{\Gamma_1}{20}, \qquad \sum_{k=2}^n |Cov(X_{k-1}, X_k)| \leqslant \frac{\Gamma_1}{20}. \tag{3.3}$$

Moreover, if (3.1) and (3.3) hold, then $\lambda > 0.2\Gamma_1$. Indeed, then

$$\mathrm{E}X_{k-1}X_k \leqslant |Cov(X_{k-1}, X_k) + \nu_1(k-1)\nu_1(k)| \leqslant |Cov(X_{k-1}, X_k)| + 0.01\nu_1(k).$$

Conditions above are weaker than (2.2). For example, $X_j$ are not necessarily bounded by some absolute constant.

Next we define remainder terms. Let

$$
\begin{aligned}
R_0 &= \sum_{k=1}^{n}\Big\{\nu_2(k)+\nu_1^2(k)+\mathrm{E}X_{k-1}X_k\Big\},\\
R_1 &= \sum_{k=1}^{n}\Big\{\nu_1^3(k)+\nu_1(k)\nu_2(k)+\nu_3(k)+\widehat{\mathrm{E}}_2^+(X_{k-1},X_k)\\
&\quad+[\nu_1(k-2)+\nu_1(k-1)+\nu_1(k)]\mathrm{E}X_{k-1}X_k+\widehat{\mathrm{E}}^+(X_{k-2},X_{k-1},X_k)\Big\},\\
R_2 &= \sum_{k=1}^{n}\Big\{\nu_1^4(k)+\nu_2^2(k)+\nu_4(k)+\Big(\mathrm{E}X_{k-1}X_k\Big)^2\\
&\quad+[\nu_1(k-2)+\nu_1(k-1)+\nu_1(k)][\nu_3(k)+\widehat{\mathrm{E}}_2^+(X_{k-1},X_k)]\\
&\quad+\sum_{l=0}^{3}\nu_1(k-l)\widehat{\mathrm{E}}^+(X_{k-2},X_{k-1},X_k)+\widehat{\mathrm{E}}_2^+(X_{k-2},X_{k-1},X_k)\\
&\quad+\widehat{\mathrm{E}}_3^+(X_{k-1},X_k)+\widehat{\mathrm{E}}^+(X_{k-3},X_{k-2},X_{k-1},X_k)\Big\}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\widehat{\mathrm{E}}^+(X_1) &= \mathrm{E}X_1, \qquad \widehat{\mathrm{E}}^+(X_1,X_2)=\mathrm{E}X_1X_2+\mathrm{E}X_1\mathrm{E}X_2,\\
\widehat{\mathrm{E}}^+(X_1,\dots,X_k) &= \mathrm{E}X_1\dots X_k\\
&\quad+\sum_{j=1}^{k-1}\widehat{\mathrm{E}}^+(X_1,X_2,\dots,X_j)\mathrm{E}X_{j+1}X_{j+2}\cdots X_k,\\
\widehat{\mathrm{E}}_2^+(X_{k-1},X_k) &= \widehat{\mathrm{E}}^+(X_{k-1}(X_{k-1}-1),X_k)+\widehat{\mathrm{E}}^+(X_{k-1},X_k(X_k-1)),\\
\widehat{\mathrm{E}}_2^+(X_{k-2},X_{k-1},X_k) &= \widehat{\mathrm{E}}^+(X_{k-2}(X_{k-2}-1),X_{k-1},X_k)\\
&\quad+\widehat{\mathrm{E}}^+(X_{k-2},X_{k-1}(X_{k-1}-1),X_k)\\
&\quad+\widehat{\mathrm{E}}^+(X_{k-2},X_{k-1},X_k(X_k-1)),\\
\widehat{\mathrm{E}}_3^+(X_{k-1},X_k) &= \widehat{\mathrm{E}}^+(X_{k-1}(X_{k-1}-1)(X_{k-1}-2),X_k)\\
&\quad+\widehat{\mathrm{E}}^+(X_{k-1}(X_{k-1}-1),X_k(X_k-1))\\
&\quad+\widehat{\mathrm{E}}^+(X_{k-1},X_k(X_k-1)(X_k-2)).
\end{aligned}
$$

For better understanding of the order of remainder terms, let us consider the case of Bernoulli variables $P(X_i=1)=1-P(X_i=0)=p_i$. If all $X_i$ are independent, then $R_0=C\sum_1^n p_i^2$ and $R_1=C\sum_1^n p_i^3$. If $X_i$ are 1-dependent, then at least $R_0\leqslant C\sum_1^n p_i$ and $R_1\leqslant C\sum_1^n p_k^{3/2}$. If some additional information about $X_i$ is available (for example, that they form 2-runs), then the estimates are somewhat in between.

Our aim is investigation of approximations with at least two parameters. However, for the sake of completeness, we start with the Poisson approximation. Note that Poisson approximation (for indicator variables) is considered in Arratia et al. (1990); Barbour et al. (1992) under much more general conditions than assumed in this paper.

**Theorem 3.1.** *Let conditions (3.1) and (3.2) be satisfied. Then, for all $n$,*

$$\|F_n - \mathrm{Pois}(\Gamma_1)\| \leqslant C R_0 \big\{1 + \Gamma_1 \min(1, \lambda^{-1})\big\} \min(1, \lambda^{-1}), \tag{3.4}$$

$$\begin{aligned} \|F_n - \mathrm{Pois}(\Gamma_1)(I + \Gamma_2(I_1 - I)^2)\| \quad &\leqslant \quad C\big\{1 + \Gamma_1 \min(1, \lambda^{-1})\big\}(R_0^2 \min(1, \lambda^{-2}) \\ &\quad + R_1 \min(1, \lambda^{-3/2})). \end{aligned} \tag{3.5}$$

If all $X_i \sim Be(1, p_i)$ are independent, then the order of accuracy in (3.4) is correct (see, for example, Barbour et al., 1992) and is equal to $C \sum_1^n p_i^2 (1 \vee \sum_1^n p_i)^{-1}$. Similarly, in (3.5) the order of accuracy is $C(\max p_i)^2$. As one can expect, the accuracy of approximation is trivial, if all $p_i$ are uniformly bounded from zero, i.e., $p_i > C$. The accuracy of approximation is much better for G.

**Theorem 3.2.** *Let conditions (3.1) and (3.2) be satisfied. Then, for all $n$,*

$$\|F_n - \mathrm{G}\| \leqslant C R_1 \big\{1 + \Gamma_1 \min(1, \lambda^{-1})\big\} \min(1, \lambda^{-3/2}), \tag{3.6}$$

$$\begin{aligned} \|F_n - \mathrm{G}(I + \Gamma_3(I_1 - I)^3)\| \quad &\leqslant \quad C\big\{1 + \Gamma_1 \min(1, \lambda^{-1})\big\}\big(R_1^2 \min(1, \lambda^{-3}) \\ &\quad + R_2 \min(1, \lambda^{-2})\big). \end{aligned} \tag{3.7}$$

If, instead of (3.2), we assume (3.3), then $\lambda \geqslant C\Gamma_1$ and $1 + \Gamma_1 \min(1, \lambda^{-1}) \leqslant C$. If, in addition, all $X_i$ do not depend on $n$ and are bounded, then estimates in (3.6) and (3.7) are of orders $O(n^{-1/2})$ and $O(n^{-1})$, respectively. Thus, the order of accuracy is comparable to CLT and Edgeworth's expansion. If all $X_i \sim Be(1, p_i)$ are independent, then the order of accuracy in (3.6) is the right one (see Kruopis, 1986) and is equal to $C \sum_1^n p_i^3 (1 \vee \sum_1^n p_i)^{-3/2}$.

Approximation G has two parameters, but: a) is not always a distribution, b) its "probabilities" are not easily calculable. Some authors argue (see, for example, Brown and Xia, 2001) that, therefore, probabilistic approximations are more preferable. We start from the negative binomial approximation. Observe, that the negative binomial approximation is meaningful only if $\mathrm{Var}S_n > \mathrm{E}S_n$.

**Theorem 3.3.** *Let conditions (3.1) and (3.3) be satisfied and let $\Gamma_2 > 0$. Then, for all $n$,*

$$\|F_n - \mathrm{NB}(r, \bar{q})\| \quad \leqslant \quad C \min(1, \Gamma_1^{-3/2})(R_1 + \Gamma_2^2 \Gamma_1^{-1}), \tag{3.8}$$

$$\begin{aligned} \|F_n - \mathrm{NB}(r, \bar{q})\big(I + [\Gamma_3 - 4\Gamma_2^2(3\Gamma_1)^{-1}](I_1 - I)^3\big)\| &\leqslant C\big\{R_1^2 \min(1, \Gamma_1^{-3}) \\ + R_2 \min(1, \Gamma_1^{-2}) + \Gamma_2^2 \Gamma_1^{-1} &|\Gamma_3 - 4\Gamma_2^2 (3\Gamma_1)^{-1}| \min(1, \Gamma_1^{-3}) \\ + \Gamma_2^3 \Gamma_1^{-2} \min(1, \Gamma_1^{-2})\big\}. \end{aligned} \tag{3.9}$$

It seems that asymptotic expansion for the negative binomial approximation was so far never considered in the context of 1-dependent summands. If all $X_i$ do not depend on $n$ and are bounded, the accuracies of approximation in (3.8) and (3.9) are $O(n^{-1/2})$ and $O(n^{-1})$, respectively.

If $\mathrm{Var}S_n < \mathrm{E}S_n$, it is more natural to use the binomial approximation.

**Theorem 3.4.** *Let conditions (3.1) and (3.3) be satisfied, $\Gamma_1 \geqslant 1$ and $\Gamma_2 < 0$. Then, for all $n$,*

$$\|F_n - \mathrm{Bi}(N, \bar{p})\| \quad \leqslant \quad C(\Gamma_2^2 \Gamma_1^{-5/2} + R_1 \Gamma_1^{-3/2}), \tag{3.10}$$

$$\begin{aligned} \|F_n - \mathrm{Bi}(N, \bar{p})\big(I + [\Gamma_3 - N\bar{p}^3/3](I_1 - I)^3\big)\| &\leqslant C\big\{R_1^2 \Gamma_1^{-3} \\ + R_2 \Gamma_1^{-2} + |\Gamma_2|^3 \Gamma_1^{-4} + \Gamma_2^2 \Gamma_1^{-3} &+ \Gamma_2^2 \Gamma_3 \Gamma_1^{-4}\big\}. \end{aligned} \tag{3.11}$$

If all the $X_i$ do not depend on $n$ and are bounded, the accuracies of approximation in (3.10) and (3.11) are $O(n^{-1/2})$ and $O(n^{-1})$, respectively.

In this paper we consider the total variation norm only. It must be noted that formula of inversion for probabilities allows to prove local estimates too. If $\lambda > 1$, then local estimates are equal to (3.4) – (3.11) multiplied by factor $\lambda^{-1/2}$.

## 4. Applications

**1. Asymptotically sharp constant for the negative binomial approximation to 2-runs.** As already mentioned above, the 2- runs statistic is one of the best investigated cases of sums of 1-dependent discrete random variables. It is easy to check that the rate of accuracy in (2.1) is $O(pn^{-1/2})$. However, the constant 64.4 is not particularly small. Here, we shall show, that, on the other hand, *asymptotically* sharp constant is small. Asymptotically sharp constant can be used heuristically to get the idea about the magnitude of constant in (3.8). We shall consider 2-runs with edge effects, which we think to be more realistic case than $S^*$. Let $S_\xi = \xi_1+\xi_2+\cdots+\xi_n$, where $\xi_i = \eta_i\eta_{i+1}$ and $\eta_i \sim Be(p)$, $(i = 1, 2, \ldots, n+1)$ are independent Bernoulli variables. The sum $S^*$ differs from $S_\xi$ by the last summand only, which is equal to $\eta_n\eta_1$. As shown in Petrauskienė and Čekanavičius (2010), for $S_\xi$ we have

$$\Gamma_1 = np^2, \quad \Gamma_2 = \frac{np^3(2-3p) - 2p^3(1-p)}{2},$$
$$\Gamma_3 = \frac{np^4(3-12p+10p^2) - 6p^4(1-p)(1-2p)}{3}.$$

Let $\mathrm{NB}(r, \bar{q})$ be defined as in (1.1) and

$$\tilde{C}_{TV} = \frac{1}{3}\sqrt{\frac{2}{\pi}}(1 + 4\mathrm{e}^{-3/2}) = 0.5033...$$

**Theorem 4.1.** *Let $p \leqslant 1/20$, $np^2 \geqslant 1$. Then*

$$\Big| \|\mathcal{L}(S_\xi) - \mathrm{NB}(r,\bar{q})\| - \tilde{C}_{TV}\frac{p}{\sqrt{n}} \Big| \leqslant C\Big(\frac{p^2}{\sqrt{n}} + \frac{1}{n}\Big).$$

We now get the following corollary.

**Corollary 4.2.** *Let $p \to 0$ and $np^2 \to \infty$, as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{\|\mathcal{L}(S_\xi) - \mathrm{NB}(r,\bar{q})\|\sqrt{n}}{p} = \tilde{C}_{TV}.$$

**2. Binomial approximation to $N(k_1, k_2)$ events.** Let $\eta_i \sim Be(p)$,$(0 < p < 1)$ be independent Bernoulli variables and let $Y_j = (1-\eta_{j-m+1})\cdots(1-\eta_{j-k_2})\eta_{j-k_2+1}\cdots\eta_{j-1}\eta_j$, $j = m, m+1, \ldots, n$, $k_1 + k_2 = m$. Further, we assume that $k_1 > 0$ and $k_2 > 0$. Let $N(n; k_1, k_2) = Y_m + Y_{m+1} + \cdots + Y_n$. We denote the distribution of $N(n; k_1, k_2)$ by H. Let $a(p) = (1-p)^{k_1}p^{k_2}$. It is well known that $N(n; k_1, k_2)$ has limiting Poisson distribution and the accuracy of Poisson approximation is $O(a(p))$, see Huang and Tsai (1991) and Vellaisamy (2004), respectively. However, Poisson approximation has just one parameter. Consequently, the closeness of $p$ to zero is crucial. We can expect any two-parametric approximation to be more universal. It is proved in Upadhye (2009) that

$$\begin{aligned}\mathrm{E}N(n;k_1,k_2) &= (n-m+1)a(p),\\ \mathrm{Var}N(n;k_1,k_2) &= (n-m+1)a(p)+(1-4m+3m^2-n(2m-1))a^2(p).\end{aligned}$$

Under quite mild assumptions $\mathrm{Var}N(n;k_1,k_2) < \mathrm{E}N(n;k_1,k_2)$. Consequently, the natural probabilistic approximation is binomial one. The binomial approximation to $N(n;k_1,k_1)$ was already considered in Upadhye (2009). Regrettably, the estimate in Upadhye (2009) contains expression which is of the constant order when $a(p)\to 0$.

Note that $Y_1, Y_2, \dots$ are $m$-dependent. Consequently, results of the previous Section can not be applied directly. However, one can group summands in the following natural way:

$$\begin{aligned}N(n;k_1,k_2) &= (Y_m+Y_{m+1}+\dots+Y_{2m-1})+(Y_{2m}+Y_{2m+1}+\dots+Y_{3m-1})+\dots\\ &= X_1+X_2+\dots\end{aligned}$$

Each $X_j$, with probable exception of the last one, contains $m$ summands. It is not difficult to check that $X_1, X_2, \dots$ are 1-dependent Bernoulli variables. All parameters can be written explicitly. Set $N=\lfloor \tilde N\rfloor$ be the integer part of $\tilde N$,

$$\begin{aligned}\tilde N &= \frac{(n-m+1)^2}{(n-m+1)(2m-1)-m(m-1)}, \quad \tilde N = N+\epsilon, \quad 0\leqslant \epsilon<1,\\ \bar p &= \frac{(n-m+1)a(p)}{N}.\end{aligned}$$

For the asymptotic expansion we need the following notation

$$A=\frac{a^3(p)}{6}(n-m+1)m(m-1).$$

The two-parametric binomial approximation is more natural, when $\mathrm{E}N(n;k_1,k_2)\geqslant 1$, which means that we deal with large values of $n$ only.

**Theorem 4.3.** *Let $(n-m+1)a(p)\geqslant 1$ and $ma(p)\leqslant 0.01$. Then*

$$\|\mathrm{H}-\mathrm{Bi}(N,\bar p)\| \leqslant C\frac{a^{3/2}(p)m^2}{\sqrt{n-m+1}}, \tag{4.1}$$

$$\|\mathrm{H}-\mathrm{Bi}(N,\bar p)\big(I+A(I_1-I)^3\big)\| \leqslant C\frac{a(p)m^2(a(p)m+1)}{n-m+1}. \tag{4.2}$$

Note that the assumption $ma(p)\leqslant 0.01$ in Theorem 4.3 is not very restrictive on $p$ when $k_1,k_2>1$. For example, it is satisfied for $p\leqslant 1/4$ and $N(n;4,4)$.

**Theorem 4.4.** *Let $(n-m+1)a(p)\geqslant 1$ and $ma(p)\leqslant 0.01$. Then*

$$\begin{aligned}&\Big|\|\mathrm{H}-\mathrm{Bi}(N,\bar p)\| - \tilde C_{TV}\frac{a^{3/2}(p)m(m-1)}{2\sqrt{n-m+1}}\Big|\\ &\qquad\leqslant C(m)\frac{a^{3/2}(p)m(m-1)}{\sqrt{n-m+1}}\bigg(\frac{1}{\sqrt{(n-m+1)a(p)}}+\frac{1}{\tilde N-1}+a(p)\bigg).\end{aligned}$$

Constant $C(m)$ depends on $m$.

**Corollary 4.5.** *Let $m$ be fixed, $a(p)\to 0$, $(n-m+1)a(p)\to\infty$, as $n\to\infty$. Then*

$$\lim_{n\to\infty}\frac{\|\mathrm{H}-\mathrm{Bi}(N,\bar p)\|\sqrt{n-m+1}}{a^{3/2}(p)m(m-1)} = \frac{\tilde C_{TV}}{2}.$$

## 5. Auxiliary results

In this section, some auxiliary results from other papers are collected. For the sake of brevity, further we will use the notation $U = I_1 - I$. First, we need representation of the characteristic function $\widehat{F}(t)$ as product of functions.

**Lemma 5.1.** *Let conditions (3.1) and (3.2) be satisfied. Then*

$$\widehat{F}(t) = \varphi_1(t)\varphi_2(t)\dots\varphi_n(t),$$

*where* $\varphi_1(t) = \mathrm{E}\mathrm{e}^{\mathrm{i}tX_1}$ *and, for* $k = 2, \dots, n$,

$$\varphi_k(t) = 1 + \mathrm{E}(e^{\mathrm{i}tX_k} - 1) + \sum_{j=1}^{k-1} \frac{\widehat{\mathrm{E}}((e^{\mathrm{i}tX_j} - 1), (e^{\mathrm{i}tX_{j+1}} - 1), \dots (e^{\mathrm{i}tX_k} - 1))}{\varphi_j(t)\varphi_{j+1}(t)\dots\varphi_{k-1}(t)}.$$

Lemma 5.1 follows from more general Lemma 3.1 in Heinrich (1982). Representation holds for all $t$, since the assumption of Lemma 3.1

$$\sqrt{\mathrm{E}|\mathrm{e}^{\mathrm{i}tX_k} - 1|^2} \leqslant \sqrt{2\mathrm{E}|\mathrm{e}^{\mathrm{i}tX_k} - 1|} \leqslant \sqrt{2\nu_1(k)} \leqslant \sqrt{0.02} < 1/6$$

is satisfied for all $t$.

**Lemma 5.2.** *Let* $t \in (0, \infty)$, $0 < p < 1$ *and* $n, j = 1, 2, \dots$. *We then have*

$$\|U^2\mathrm{e}^{tU}\| \leqslant \frac{3}{t\mathrm{e}}, \qquad \|U^j\mathrm{e}^{tU}\| \leqslant \Big(\frac{2j}{t\mathrm{e}}\Big)^{j/2},$$
$$\|U^j(I + pU)^n\| \leqslant \binom{n+j}{j}^{-1/2} (p(1-p))^{-j/2}.$$

The first inequality was proved in Roos (2001), formula (29). The second bound follows from formula (3.8) in Deheuvels and Pfeifer (1988) and the properties of the total variation norm. For the proof of the third estimate, see Lemma 4 from Roos (2000).

**Lemma 5.3.** *Let* $t > 0$ *and* $p \in (0, 1)$. *Then*

$$\Big|\|U^3\mathrm{e}^{tU}\| - \frac{3\tilde{C}_{TV}}{t^{3/2}}\Big| \leqslant \frac{C}{t^2}, \qquad \Big|\|U^3(I + pU)^n\| - \frac{3\tilde{C}_{TV}}{(np(1-p))^{3/2}}\Big| \leqslant \frac{C}{(np(1-p))^2}.$$

The statements in Lemma 5.3 follow from a more general Proposition 4 in Roos (1999) and from Čekanavičius and Roos (2006).

**Lemma 5.4.** *Let* $\lambda > 0$ *and* $k = 0, 1, 2, \dots$. *Then*

$$|\sin(t/2)|^k\mathrm{e}^{-\lambda\sin^2(t/2)} \leqslant \frac{C(k)}{\lambda^{k/2}}, \qquad \int_{-\pi}^{\pi} |\sin(t/2)|^k\mathrm{e}^{-\lambda\sin^2(t/2)}\mathrm{d}t \leqslant \frac{C(k)}{\max(1, \lambda^{(k+1)/2})}.$$

Both estimates are trivial. Note that, for $|t| \leqslant \pi$, we have $|\sin(t/2)| \geqslant |t|/\pi$.

**Lemma 5.5.** *Let* $M$ *be finite variation measure concentrated on integers,* $\sum_k |k||M\{k\}| < \infty$. *Then for any* $v \in \mathbb{R}$ *and* $u > 0$ *the following inequality is valid*

$$\|M\| \leqslant \big(1 + u\pi\big)^{1/2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{M}(t)|^2 + \frac{1}{u^2}|\big(\mathrm{e}^{-\mathrm{i}tv}\widehat{M}(t)\big)'|^2\mathrm{d}t\Big)^{1/2}. \tag{5.1}$$

The estimate (5.1) is well-known; see, for example, Presman (1986).

**Lemma 5.6.** *(Bergström, 1951) For all numbers $A, B > 0$, $s = 0, 1, 2, \dots, n$, the following identity holds:*

$$A^n = \sum_{m=0}^{s} \binom{n}{m} B^{n-m}(A-B)^m + \sum_{m=s+1}^{n} \binom{m-1}{s} A^{n-m}(A-B)^{s+1}B^{m-s-1}. \quad (5.2)$$

**Lemma 5.7.** *Let $s = 1, 2, 3$. For all $t \in \mathbb{R}$,*

$$\begin{aligned} \mathrm{E}\exp\{\mathrm{i}tX_k\} &= 1 + \sum_{l=1}^{s} \nu_l(k)\frac{(\mathrm{e}^{\mathrm{i}t}-1)^l}{l!} + \theta\nu_{s+1}(k)\frac{|\mathrm{e}^{\mathrm{i}t}-1|^{s+1}}{s!}, \\ \mathrm{E}(\exp\{\mathrm{i}tX_k\})' &= \sum_{l=1}^{s} \nu_l(k)\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}(\mathrm{e}^{\mathrm{i}t}-1)^{l-1}}{l!} + \theta\nu_{s+1}(k)\frac{|\mathrm{e}^{\mathrm{i}t}-1|^{s}}{(s-1)!}. \end{aligned}$$

Lemma 5.7 is a particular case of Lemma 3 from Šiaulys and Čekanavičius (1988).

**Lemma 5.8.** *(Heinrich, 1982) Let $Z_1, Z_2, \dots, Z_k$ be 1-dependent complex-valued random variables with $\mathrm{E}|Z_m|^2 < \infty$, $1 \leqslant m \leqslant k$. Then*

$$|\widehat{\mathrm{E}}(Z_1, Z_2, \cdots, Z_k)| \leqslant 2^{k-1} \prod_{m=1}^{k} (\mathrm{E}|Z_m|^2)^{1/2}.$$

## 6. Preliminary results

Let $z = \mathrm{e}^{\mathrm{i}t} - 1$ and $Z_j = \exp\{\mathrm{i}tX_j\} - 1$. As before we assume that $\nu_j(k) = 0$ and $X_k = 0$ for $k \leqslant 0$. Also, we omit the argument $t$, wherever possible and, for example, write $\varphi_k$ instead of $\varphi_k(t)$.

The next lemma can easily be proved by induction.

**Lemma 6.1.** *For all $t \in \mathbb{R}$ and $k \geqslant 2$, the following estimate holds:*

$$\widehat{\mathrm{E}}^+(|Z_1|, \dots, |Z_k|) \leqslant 4\widehat{\mathrm{E}}^+(|Z_1|, \dots, |Z_{k-1}|). \quad (6.1)$$

**Lemma 6.2.** *Let $\max_k \nu_1(k) \leqslant 0.01$. Then, for $k = 1, 2, \dots, n$,*

$$\begin{aligned} |\varphi_k - 1| &\leqslant \frac{1}{10}, \quad \frac{1}{|\varphi_k|} \leqslant \frac{10}{9}, & (6.2) \\ |\varphi_k - 1| &\leqslant |z|[(0.66)\nu_1(k-1) + (4.13)\nu_1(k)], & (6.3) \\ |\varphi_k - 1 - \mathrm{E}Z_k| &\leqslant \sin^2(t/2)[(0.374)\nu_1(k) + (0.288)\nu_1(k-1) \\ &\quad +(15.58)\mathrm{E}X_{k-1}X_k + (0.1)\mathrm{E}X_{k-2}X_{k-1}]. & (6.4) \end{aligned}$$

*Proof:* We repeatedly apply below the following trivial inequalities:

$$|z| \leqslant 2, \qquad |Z_k| \leqslant 2, \qquad |Z_k| \leqslant X_k|z|. \quad (6.5)$$

The second estimate in (6.2) follows from the first estimate:

$$|\varphi_k| \geqslant |1 - |\varphi_k - 1|| \geqslant 1 - (1/10) = 9/10.$$

The first estimate in (6.2) follows from (6.3) and (6.5) and by the assumption of the lemma. It remains to prove the (6.3) and (6.4). Both proofs are very similar.

From Lemma 5.1 and equation (6.2), we get

$$\begin{aligned}
|\varphi_k - 1 - \mathrm{E}Z_k| \quad &\leqslant \quad \frac{|\widehat{\mathrm{E}}(Z_{k-1}, Z_k)|}{|\varphi_{k-1}|} + \frac{|\widehat{\mathrm{E}}(Z_{k-2}, Z_{k-1}, Z_k)|}{|\varphi_{k-3}\varphi_{k-1}|} \\
&\quad + \frac{|\widehat{\mathrm{E}}(Z_{k-3}, \dots, Z_k)|}{|\varphi_{k-3}\cdots\varphi_{k-1}|} + \frac{|\widehat{\mathrm{E}}(Z_{k-4}, \dots, Z_k)|}{|\varphi_{k-4}\cdots\varphi_{k-1}|} \\
&\quad + \frac{|\widehat{\mathrm{E}}(Z_{k-5}, \dots, Z_k)|}{|\varphi_{k-5}\cdots\varphi_{k-1}|} + \sum_{j=1}^{k-6} \frac{|\widehat{\mathrm{E}}(Z_j, \dots, Z_k)|}{|\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}|} \\
&\leqslant \quad \left(\frac{10}{9}\right)|\widehat{\mathrm{E}}(Z_{k-1}, Z_k)| + \left(\frac{10}{9}\right)^2 |\widehat{\mathrm{E}}(Z_{k-2}, Z_{k-1}, Z_k)| \\
&\quad + \left(\frac{10}{9}\right)^3 |\widehat{\mathrm{E}}(Z_{k-3}, \dots, Z_k)| + \left(\frac{10}{9}\right)^4 |\widehat{\mathrm{E}}(Z_{k-4}, \dots, Z_k)| \\
&\quad + \left(\frac{10}{9}\right)^5 |\widehat{\mathrm{E}}(Z_{k-5}, \dots, Z_k)| \\
&\quad + \sum_{j=1}^{k-6} \left(\frac{10}{9}\right)^{k-j} |\widehat{\mathrm{E}}(Z_j, \dots, Z_k)|. \qquad (6.6)
\end{aligned}$$

By (6.5) and Lemma 5.8, we obtain

$$\begin{aligned}
\mathrm{E}|Z_j| \quad &\leqslant \quad \nu_1(j)|z| \leqslant 0.02|\sin(t/2)|, \\
\mathrm{E}|Z_j|^2 \quad &\leqslant \quad 2\mathrm{E}|Z_j| \leqslant 2\nu_1(j)|z| = 4\nu_1(j)|\sin(t/2)|, \qquad (6.7) \\
|\widehat{\mathrm{E}}(Z_j, \dots Z_k)| \quad &\leqslant \quad 2^{k-j} 2^{(k-j+1)/2} |z|^{(k-j+1)/2} \prod_{l=j}^{k} \sqrt{\nu_1(l)} \\
&\leqslant \quad 2^{2(k-j)-1} |z|^2 \sqrt{\nu_1(k)\nu_1(k-1)} 0.1^{k-j-1} \\
&\leqslant \quad 4^{k-j} \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)] 0.1^{k-j-1} \\
&= \quad 10 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)] (0.4)^{k-j}. \qquad (6.8)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{j=1}^{k-6} \left(\frac{10}{9}\right)^{k-j} |\widehat{\mathrm{E}}(Z_j, \dots, Z_k)| \quad &\leqslant \quad 10 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)] \sum_{j=1}^{k-6} \left(\frac{4}{9}\right)^{k-j} \\
&\leqslant \quad 2 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)] (0.0694). \qquad (6.9)
\end{aligned}$$

By 1-dependence, (6.5) and Hölder's inequality (see also Heinrich, 1982), we have for $j \geqslant 3$,

$$\begin{aligned}
|\mathrm{E}Z_{k-j} \cdots Z_k| \quad &\leqslant \quad \prod_{i=k-j}^{k} \sqrt{\mathrm{E}|Z_i|^2} \leqslant \prod_{i=k-j}^{k} \sqrt{2\nu_1(i)|z|} \\
&\leqslant \quad 2^{(j+1)/2} |z|^{(j+1)/2} \sqrt{\nu_1(k-1)\nu_1(k)} (0.1)^{j-1} \\
&\leqslant \quad 2^{j-1} |z|^2 \frac{\nu_1(k-1) + \nu_1(k)}{2} (0.1)^{j-1} \\
&= \quad 2^j \sin^2(t/2) [\nu_1(k-1) + \nu_1(k)] (0.1)^{j-1}. \qquad (6.10)
\end{aligned}$$

Moreover, for any $j$,

$$\begin{aligned} |\mathrm{E}Z_{j-1}Z_j| &\leqslant 2\mathrm{E}|Z_j| \leqslant 4|\sin(t/2)|\nu_1(j),\\ |\mathrm{E}Z_{j-1}Z_j| &\leqslant |z|^2\mathrm{E}X_{j-1}X_j = 4\sin^2(t/2)\mathrm{E}X_{j-1}X_j \end{aligned} \tag{6.11}$$

and

$$|\mathrm{E}Z_{j-2}Z_{j-1}Z_j| \leqslant 2\mathrm{E}|Z_{j-1}Z_j| \leqslant 8\sin^2(t/2)\mathrm{E}X_{j-1}X_j. \tag{6.12}$$

Therefore, from (6.7), we have

$$\begin{aligned} |\widehat{\mathrm{E}}(Z_{j-1},Z_j)| &\leqslant \mathrm{E}|Z_{j-1}Z_j| + \nu_1(j-1)\nu_1(j)|z|^2\\ &\leqslant 2.02|z|\nu_1(j) \leqslant 0.0404|\sin(t/2)|. \end{aligned} \tag{6.13}$$

Similarly, applying (6.10), (6.11), (6.12) and (6.15), we obtain the following rough estimates:

$$\begin{aligned} |\widehat{\mathrm{E}}(Z_{j-2},Z_{j-1},Z_j)| &\leqslant |z||\sin(t/2)|\{0.2\nu_1(j-1) + 0.2804\nu_1(j)\}\\ &\leqslant 0.01\sin^2(t/2),\\ |\widehat{\mathrm{E}}(Z_{j-3},\dots,Z_j)| &\leqslant |z||\sin(t/2)|\{0.044\nu_1(j-1) + 0.1348\nu_1(j)\}\\ &\leqslant 0.0036\sin^2(t/2),\\ |\widehat{\mathrm{E}}(Z_{j-4},\dots,Z_j)| &\leqslant |z||\sin(t/2)|\{0.0169\nu_1(j-1) + 0.0405\nu_1(j)\}\\ &\leqslant 0.00115\sin^2(t/2). \end{aligned} \tag{6.14}$$

Taking into account that $\nu_1(k-1) \leqslant 0.01$, we get

$$\begin{aligned} |\widehat{\mathrm{E}}(Z_{k-1},Z_k)| &\leqslant \mathrm{E}|Z_{k-1}Z_k| + \nu_1(k-1)\nu_1(k)|z|^2\\ &\leqslant \sin^2(t/2)\{4\mathrm{E}X_{k-1}X_k + 0.04\nu_1(k)\}. \end{aligned} \tag{6.15}$$

Similarly, taking into account (6.10)–(6.14), we get

$$\begin{aligned} |\widehat{\mathrm{E}}(Z_{k-2},Z_{k-1},Z_k)| &\leqslant \sin^2(t/2)\{8.08\mathrm{E}X_{k-1}X_k + 0.08\mathrm{E}X_{k-2}X_{k-1}\\ &\quad +0.0008\nu_1(k-1)\},\\ |\widehat{\mathrm{E}}(Z_{k-3},\dots,Z_k)| &\leqslant \sin^2(t/2)\{0.3216\mathrm{E}X_{k-1}X_k + 0.08\nu_1(k-1)\\ &\quad +0.1\nu_1(k)\},\\ |\widehat{\mathrm{E}}(Z_{k-4},\dots,Z_k)| &\leqslant \sin^2(t/2)\{0.3632\mathrm{E}X_{k-1}X_k + 0.0176\nu_1(k-1)\\ &\quad +0.0248\nu_1(k)\},\\ |\widehat{\mathrm{E}}(Z_{k-5},\dots,Z_k)| &\leqslant \sin^2(t/2)\{0.0944\mathrm{E}X_{k-1}X_k + 0.0068\nu_1(k-1)\\ &\quad +0.0091\nu_1(k)\}. \end{aligned} \tag{6.16}$$

Combining (6.9), (6.15)–(6.16) with (6.6) we prove (6.4).

For the proof of (6.3), we apply mathematical induction. Let us assume that (6.2) holds for first $k-1$ functions and let $k \geqslant 6$. Then the proof is almost identical to the proof of (6.4). We expand $\varphi_k$ just like in (6.6):

$$|\varphi_k - 1| \leqslant \mathrm{E}|Z_k| + \left(\frac{10}{9}\right)|\widehat{\mathrm{E}}(Z_{k-1},Z_k)| + \dots + \sum_{j=1}^{k-4}\left(\frac{10}{9}\right)^{k-j}|\widehat{\mathrm{E}}(Z_j,\dots,Z_k)|.$$

Applying (6.10), (6.7) and (6.13)–(6.14), we easily complete the proof of (6.3). The proof for $k < 6$ is analogous. □

**Lemma 6.3.** *Let $\nu_1(k) \leqslant 0.01$, $\nu_2(k) < \infty$, for $1 \le k \le n$. Then, for all $t \in \mathbb{R}$,*

$$|\varphi_k| \leqslant 1 - \lambda_k \sin^2(t/2) \leqslant \exp\{-\lambda_k \sin^2(t/2)\}$$

*and*

$$|\widehat{F}(t)| \leqslant \prod_{k=1}^{n} |\varphi_k| \leqslant \exp\{-1.3\lambda \sin^2(t/2)\}.$$

*Here $\lambda_k = 1.606\nu_1(k) - 0.288\nu_1(k-1) - 2\nu_2(k) - 0.1\mathrm{E}X_{k-2}X_{k-1} - 15.58\mathrm{E}X_{k-1}X_k$ and $\lambda$ is defined by (3.2).*

*Proof:* From Lemma 5.7 it follows that

$$1 + \mathrm{E}Z_k = \mathrm{E}\exp\{\mathrm{i}tX_k\} = 1 + \nu_1(k)z + \theta\frac{\nu_2(k)|z|^2}{2}.$$

Therefore

$$|\varphi_k| \leqslant |1 + \mathrm{E}Z_k| + |\varphi_k - 1 - \mathrm{E}Z_k| \leqslant |1 + \nu_1(k)z| + \frac{\nu_2(k)}{2}|z|^2 + |\varphi_k - 1 - \mathrm{E}Z_k|.$$

Applying the definition of the square of the absolute value for complex number we get

$$|1 + \nu_1(k)z|^2 = (1 - \nu_1(k)\cos t)^2 + (\nu_1(k)\sin t)^2 = 1 - 4\nu_1(k)(1 - \nu_1(k))\sin^2(t/2).$$

Consequently,

$$|1 + \nu_1(k)z| \leqslant \sqrt{1 - 4\nu_1(k)(1 - \nu_1(k))\sin^2(t/2)} \leqslant 1 - 2\nu_1(k)(1 - \nu_1(k))\sin^2(t/2).$$

Combining the last estimate with (6.4), we get the first estimate of the lemma. The second estimate follows immediately. □

For expansions of $\varphi_k$ in powers of $z$, we use the following notation:

$$\begin{aligned}
\gamma_2(k) &= \frac{\nu_2(k)}{2} + \widehat{\mathrm{E}}(X_{k-1}, X_k),\\
\gamma_3(k) &= \frac{\nu_3(k)}{6} + \frac{\widehat{\mathrm{E}}_2(X_{k-1}, X_k)}{2} + \widehat{\mathrm{E}}(X_{k-2}, X_{k-1}, X_k) - \nu_1(k-1)\widehat{\mathrm{E}}(X_{k-1}, X_k),\\
r_0(k) &= \nu_2(k) + \sum_{l=0}^{3} \nu_1^2(k-l) + \mathrm{E}X_{k-1}X_k,\\
r_1(k) &= \nu_3(k) + \sum_{l=0}^{5} \nu_1^3(k-l) + \nu_1(k-1)\mathrm{E}X_{k-1}X_k + \widehat{\mathrm{E}}_2^+(X_{k-1}, X_k)\\
&\quad + \widehat{\mathrm{E}}^+(X_{k-2}, X_{k-1}, X_k),\\
r_2(k) &= \nu_4(k) + \sum_{l=0}^{7} \nu_1^4(k-l) + \nu_2^2(k) + \nu_2^2(k-1) + (\mathrm{E}X_{k-1}X_k)^2\\
&\quad + (\mathrm{E}X_{k-2}X_{k-1})^2 + (\nu_1(k-2) + \nu_1(k-1))\widehat{\mathrm{E}}_2^+(X_{k-1}, X_k)\\
&\quad + \sum_{l=0}^{3} \nu_1(k-l)\widehat{\mathrm{E}}^+(X_{k-2}, X_{k-1}, X_k) + \widehat{\mathrm{E}}_3^+(X_{k-1}, X_k)\\
&\quad + \widehat{\mathrm{E}}_2^+(X_{k-2}, X_{k-1}, X_k) + \widehat{\mathrm{E}}^+(X_{k-3}, X_{k-2}, X_{k-1}, X_k).
\end{aligned}$$

**Lemma 6.4.** *Let condition (3.1) be satisfied, $k = 1, \dots, n$. Then, for all $t \in \mathbb{R}$,*

$$\begin{aligned}
\varphi_k &= 1 + \nu_1(k)z + \theta C|z|^2 r_0(k), && (6.17)\\
\varphi_k &= 1 + \nu_1(k)z + \gamma_2(k)z^2 + \theta C|z|^3 r_1(k), && (6.18)\\
\varphi_k &= 1 + \nu_1(k)z + \gamma_2(k)z^2 + \gamma_3(k)z^3 + \theta C|z|^4 r_2(k), && (6.19)\\
\frac{1}{\varphi_{k-1}} &= 1 + C\theta|z|\{\nu_1(k-2) + \nu_1(k-1)\}, && (6.20)\\
\frac{1}{\varphi_{k-1}} &= 1 - \nu_1(k-1)z + C\theta|z|^2\Big\{\nu_2(k-1)\\
&\quad + \sum_{l=1}^{4}\nu_1^2(k-l) + \mathrm{E}X_{k-2}X_{k-1}\Big\}, && (6.21)\\
\frac{1}{\varphi_{k-1}} &= 1 - \nu_1(k-1)z - \left(\frac{\nu_2(k-1)}{2} - \nu_1^2(k-1) + \widehat{\mathrm{E}}(X_{k-2}, X_{k-1})\right)z^2\\
&\quad + C\theta|z|^3\Big\{\nu_3(k-1) + \sum_{l=1}^{6}\nu_1^3(k-l) + \widehat{\mathrm{E}}_2^+(X_{k-2}, X_{k-1})\\
&\quad + \widehat{\mathrm{E}}^+(X_{k-3}, X_{k-2}, X_{k-1})\\
&\quad + (\nu_1(k-2) + \nu_1(k-1))[\nu_2(k-1) + \mathrm{E}X_{k-2}X_{k-1}]\Big\}, && (6.22)\\
\frac{1}{\varphi_{k-1}\varphi_{k-2}} &= 1 + C\theta|z|\{\nu_1(k-3) + \nu_1(k-2) + \nu_1(k-1)\}, && (6.23)\\
(\varphi_k - 1)^2 &= \nu_1^2(k)z^2 + C\theta|z|^3[\nu_1(k-1)\\
&\quad + \nu_1(k)]\Big\{\nu_2(k) + \sum_{l=0}^{3}\nu_1^2(k-l) + \mathrm{E}X_{k-1}X_k\Big\}, && (6.24)\\
(\varphi_k - 1)^3 &= \nu_1^3(k)z^3\\
&\quad + C\theta|z|^4\Big\{\nu_2^2(k) + \sum_{l=0}^{3}\nu_1^4(k-l) + (\mathrm{E}X_{k-1}X_k)^2\Big\}. && (6.25)
\end{aligned}$$

*Proof:* Further on we assume that $k \geqslant 7$. For smaller values of $k$, all proofs just become shorter. The lemma is proved in four steps. First, we prove (6.17), (6.18), (6.20) and (6.23). Second, we obtain (6.21) and (6.24). Then we prove (6.22) and (6.25). The final step is the proof of (6.19). At each step, we employ results from the previous step. Since all proofs are very similar, we give just some of them.
Due to (6.2), we have

$$\frac{1}{\varphi_{k-1}} = \frac{1}{1 - (1 - \varphi_{k-1})} = 1 + \sum_{j=1}^{\infty}(1 - \varphi_{k-1})^j = 1 + C\theta|1 - \varphi_{k-1}|.$$

Therefore, (6.20) and (6.23) follow from (6.3) and (6.2).
From Lemmas 5.1, 5.7, 6.1, equation (6.2) and second estimate in (6.8), we get

$$\begin{aligned}
|\varphi_k| &= 1 + \mathrm{E}Z_k + \frac{|\widehat{\mathrm{E}}(Z_{k-1}, Z_k)|}{|\varphi_{k-1}|} + \frac{|\widehat{\mathrm{E}}(Z_{k-2}, Z_{k-1}, Z_k)|}{|\varphi_{k-2}\varphi_{k-1}|} + \sum_{j=1}^{k-3}\frac{|\widehat{\mathrm{E}}(Z_j, \dots, Z_k)|}{|\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}|}\\
&\leqslant 1 + \nu_1(k)z + C\theta|z|^2\nu_2(k) + C\theta\widehat{\mathrm{E}}^+(|Z_{k-1}|, |Z_k|)
\end{aligned}$$

$$
\begin{aligned}
&+C\theta|z|^2\sqrt{\nu_1(k-3)\nu_1(k-2)\nu_1(k-1)\nu_1(k)}\\
=\;& 1+\nu_1(k)z+C\theta|z|^2\Big\{\nu_2(k)+\mathrm{E}X_{k-1}X_k+\nu_1(k-1)\nu_1(k)+\sum_{l=0}^{3}\nu_1^2(k-l)\Big\}\\
=\;& 1+\nu_1(k)z+C\theta|z|^2 r_0(k),
\end{aligned}
$$

which proves (6.17).

The proof of (6.18) is almost identical. We take longer expansion in Lemma 5.1 and note that due to (5.2)

$$Z_k=X_kz+\theta X_k(X_k-1)\frac{|z|^2}{2}.$$

Therefore,

$$
\begin{aligned}
\widehat{\mathrm{E}}(Z_{k-1},Z_k) &= \widehat{\mathrm{E}}(X_{k-1}z+\theta|z|^2X_{k-1}(X_{k-1}-1),Z_k)=z\widehat{\mathrm{E}}X_{k-1}Z_k\\
&\quad +C\theta|z|^3\widehat{\mathrm{E}}^+(X_{k-1}(X_{k-1}-1),X_k)\\
&= z^2\widehat{\mathrm{E}}(X_{k-1},X_k)+C\theta|z|^3\widehat{\mathrm{E}}_2^+(X_{k-1},X_k).
\end{aligned}
$$

The other proofs are simple repetition of the given ones with the only exception that results from previous steps are used. For example, for the proof of (6.19), we apply Lemma 5.1 and get

$$|\varphi_k|=1+\mathrm{E}Z_k+\sum_{j=1}^{k-1}\frac{|\widehat{\mathrm{E}}(Z_j,\dots,Z_k)|}{|\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}|}=1+\mathrm{E}Z_k+\sum_{j=k-2}^{k-1}+\sum_{j=k-6}^{k-3}+\sum_{j=1}^{k-7}.$$

By (6.8),

$$\sum_{j=1}^{k-7}\frac{|\widehat{\mathrm{E}}(Z_j,\dots,Z_k)|}{|\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}|}\leqslant C|z|^4\sqrt{\nu_1(k-1)\cdots\nu_1(k)}\leqslant C|z|^4\sum_{l=0}^{7}\nu_1^4(k-l)$$

and by (6.1)

$$
\begin{aligned}
\Big|\sum_{j=k-6}^{k-3}\Big| &\leqslant C\sum_{j=k-6}^{k-3}\widehat{\mathrm{E}}^+(|Z_j|,\dots,|Z_k|)\leqslant C\widehat{\mathrm{E}}^+(|Z_{k-3}|,\dots,|Z_k|)\\
&\leqslant C|z|^4\widehat{\mathrm{E}}^+(X_{k-3},\dots,X_k).
\end{aligned}
$$

For other summands, we apply Lemma 5.7 and use the previous estimates. □

Hereafter, the prime denotes the derivative with respect to $t$.

**Lemma 6.5.** *Let condition (3.1) hold. Then, for all $t\in\mathbb{R}$,*

$$
\begin{aligned}
(\widehat{\mathrm{E}}(Z_j,\dots,Z_k))' &= \sum_{i=j}^{k}\widehat{\mathrm{E}}(Z_j,\dots,Z_i',\dots,Z_k),\\
|\widehat{\mathrm{E}}(Z_j,\dots,Z_i',\dots,Z_k)| &\leqslant 2^{[3(k-j)+1]/2}|z|^{(k-j)/2}\prod_{l=j}^{k}\sqrt{\nu_1(l)}.
\end{aligned}
$$

*Proof:* The first identity was proved in Heinrich (1982). Applying (6.10) we obtain

$$|\widehat{\mathrm{E}}(Z_j,\dots,Z_i',\dots,Z_k)|\leqslant 2^{k-j}\sqrt{\mathrm{E}|Z_i'|^2}\prod_{l\neq i}^{k}\sqrt{\mathrm{E}|Z_l|^2}.$$

Due to assumption (3.1), $\nu_2(l) \leqslant \nu_1(l)$. Therefore,

$$\mathrm{E}|Z_l'|^2 = \mathrm{E}|\mathrm{i}\mathrm{e}^{\mathrm{i}tX_l}X_l|^2 = \mathrm{E}X_l^2 = \mathrm{E}X_l(X_l-1+1) = \nu_2(l)+\nu_1(l) \leqslant 2\nu_1(l).$$

Combining the last estimate with $\mathrm{E}|Z_l|^2 \leqslant 2\mathrm{E}|Z_l| \leqslant 2|z|\nu_1(l)$, the proof follows. □

**Lemma 6.6.** *Let condition (3.1) be satisfied, $k=1,\dots,n$ and $\varphi_k$ be defined as in Lemma 5.1. Then, for all $t\in\mathbb{R}$,*

$$\varphi_k' = 33\theta[\nu_1(k)+\nu_1(k-1)], \tag{6.26}$$

$$\varphi_k' = \nu_1(k)z' + \theta C|z|(r_0(k)+\widehat{\mathrm{E}}^+(X_{k-2},X_{k-1})), \tag{6.27}$$

$$\begin{aligned}\varphi_k' &= \nu_1(k)z' + \gamma_2(k)(z^2)' + \theta C|z|^2\Big([\nu_1(k-2)+\nu_1(k)]\mathrm{E}X_{k-1}X_k\\ &\quad +\widehat{\mathrm{E}}^+(X_{k-4},X_{k-3},X_{k-2})+\widehat{\mathrm{E}}^+(X_{k-3},X_{k-2},X_{k-1})+r_1(k)\Big),\end{aligned} \tag{6.28}$$

$$\begin{aligned}\varphi_k' &= \nu_1(k)z' + \gamma_2(k)(z^2)' + \gamma_3(k)(z^3) + \theta C|z|^3\Big(r_2(k)\\ &\quad +\widehat{\mathrm{E}}^+(X_{k-4},\dots,X_{k-1})+\widehat{\mathrm{E}}^+(X_{k-5},\dots,X_{k-2})\Big).\end{aligned} \tag{6.29}$$

*Proof:* Note that

$$\left(\frac{\widehat{\mathrm{E}}(Z_j,\dots,Z_k)}{\varphi_j\cdots\varphi_{k-1}}\right)' = \frac{(\widehat{\mathrm{E}}(Z_j,\dots,Z_k))'}{\varphi_j\cdots\varphi_k} - \frac{\widehat{\mathrm{E}}(Z_j,\dots,Z_k)}{\varphi_j\cdots\varphi_k}\sum_{m=j}^{k-1}\frac{\varphi_m'}{\varphi_m}.$$

Now the proof is just a repetition of the proof of Lemma 6.4. For example, (6.26) is easily verifiable for $k=0,1$. Let us assume that it holds for $1,2,\dots,k-1$. From Lemmas 5.1 and 5.7 and equation (6.2), we get

$$\begin{aligned}|\varphi_k'| &\leqslant \nu_1(k) + \sum_{j=1}^{k-1}\frac{|(\widehat{\mathrm{E}}(Z_j,\dots,Z_k))'|}{|\varphi_j\cdots\varphi_{k-1}|} + \sum_{j=1}^{k-1}\frac{|\widehat{\mathrm{E}}(Z_j,\dots,Z_k)|}{|\varphi_j\cdots\varphi_{k-1}|}\sum_{m=j}^{k-1}\frac{|\varphi_m'|}{|\varphi_m|}\\ &\leqslant \nu_1(k) + \sum_{j=1}^{k-1}\left(\frac{10}{9}\right)^{k-j}\sum_{i=j}^{k}|\widehat{\mathrm{E}}(Z_j,\dots,Z_i',\dots,Z_k)|\\ &\quad + \sum_{j=1}^{k-1}\left(\frac{10}{9}\right)^{k-j}|\widehat{\mathrm{E}}(Z_j,\dots,Z_k)|(k-j)33\cdot 0.02\cdot\left(\frac{10}{9}\right).\end{aligned}$$

By Lemma 6.5,

$$|\widehat{\mathrm{E}}(Z_j,\dots,Z_i',\dots,Z_k)| \leqslant [\nu_1(k-1)+\nu_1(k)](0.04)^{k-j}\frac{10}{\sqrt{2}}.$$

Combining the last two estimates and (6.8), the proof of (6.26) is completed. We omit the proofs of remaining expansions and note only that

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}tX}-1)' &= \mathrm{i}X\mathrm{e}^{\mathrm{i}tX} = \mathrm{i}\mathrm{e}^{\mathrm{i}t}X\mathrm{e}^{\mathrm{i}t(X-1)}\\ &= z'X\Big(1+(X-1)z+\theta\frac{(X-1)(X-2)}{2}|z|^2\Big),\end{aligned}$$

due to Bergström's identity. □

Let, for $j=1,\dots,n$ and $l=2,\dots,n$,

$$g_j(t) = \exp\Big\{\nu_1(j)(\mathrm{e}^{\mathrm{i}t}-1) + \Big(\frac{\nu_2(j)-\nu_1^2(j)}{2} + \widehat{\mathrm{E}}(X_{j-1},X_j)\Big)(\mathrm{e}^{\mathrm{i}t}-1)^2\Big\}.$$

**Lemma 6.7.** *Let conditions in (3.1) be satisfied, $k=1,2,\dots,n$. Then, for all $t\in\mathbb{R}$,*

$$
\begin{aligned}
g_k &= 1+C\theta|z|[\nu_1(k-1)+\nu_1(k)], && (6.30)\\
g_k' &= C\theta[\nu_1(k-1)+\nu_1(k)], && (6.31)\\
g_k &= 1+\nu_1(k)z+\gamma_2(k)z^2+C\theta|z|^3\big\{\nu_1^3(k-1)+\nu_1^3(k)\\
&\quad +\nu_1(k)\nu_2(k)+[\nu_1(k-1)+\nu_1(k)]\mathrm{E}X_{k-1}X_k\big\}, && (6.32)\\
g_k' &= \nu_1(k)z'+\gamma_2(k)(z^2)'+C\theta|z|^2\big\{\nu_1^3(k-1)+\nu_1^3(k)\\
&\quad +\nu_1(k)\nu_2(k)+[\nu_1(k-1)+\nu_1(k)]\mathrm{E}X_{k-1}X_k\big\}, && (6.33)\\
g_k &= 1+\nu_1(k)z+\gamma_2(k)z^2+\tilde\gamma_3(k)z^3\\
&\quad +C\theta|z|^4\big\{\nu_1^4(k-1)+\nu_1^4(k)+\nu_2^2(k)+(\mathrm{E}X_{k-1}X_k)^2\big\}, && (6.34)\\
g_k' &= \nu_1(k)z'+\gamma_2(k)(z^2)'+\tilde\gamma_3(k)(z^3)'\\
&\quad +C\theta|z|^3\big\{\nu_1^4(k-1)+\nu_1^4(k)+\nu_2^2(k)+(\mathrm{E}X_{k-1}X_k)^2\big\}, && (6.35)\\
|g_k| &\leqslant \exp\{-\lambda_k\sin^2(t/2)\}. && (6.36)
\end{aligned}
$$

*Here $\lambda_k$ is as in Lemma 6.3 and*

$$\tilde\gamma_3(k)=\frac{\nu_1(k)\nu_2(k)-\nu_1^3(k)}{2}+\nu_1(k)\widehat{\mathrm{E}}(X_{k-1},X_k)+\frac{\nu_1^3(k)}{6}.$$

*Proof:* For any complex number $b$, we have

$$\mathrm{e}^b=1+b+\frac{b^2}{2}+\cdots+\frac{b^s}{s!}+\theta\frac{|b|^{s+1}}{(s+1)!}\mathrm{e}^{|b|}.$$

Due to (3.1), $\nu_2(j)\leqslant\nu_1(j)$. Therefore,

$$
\begin{aligned}
\mathrm{E}X_{j-1}X_j &\leqslant \sqrt{\mathrm{E}X_{j-1}^2\mathrm{E}X_j^2}\leqslant\sqrt{[\nu_2(j-1)+\nu_1(j-1)][\nu_2(j)+\nu_1(j)]}\\
&\leqslant 2\sqrt{\nu_1(j-1)\nu(j)}\leqslant 2[\nu_1(j-1)+\nu_1(j)]. && (6.37)
\end{aligned}
$$

Therefore, the exponent of $g_k$ is bounded by some absolute constant $C$ and (6.30) and (6.31) easily follow. We have

$$
\begin{aligned}
g_k &= 1+\nu_1(k)z+\gamma_2(k)z^2+C\theta\big\{\nu_1^3(k)+\nu_2^2(k)+\nu_1(k)\nu_2(k)\\
&\quad +\nu_1(k)\widehat{\mathrm{E}}^+(X_{k-1},X_k)+\nu_1(k-1)\nu_1^2(k)+(\widehat{\mathrm{E}}^+(X_{k-1},X_k))^2\big\}.
\end{aligned}
$$

Moreover,

$$\nu_2^2(k)\leqslant\nu_1(k)\nu_2(k),\qquad \nu_1(k-1)\nu_1^2(k)\leqslant\nu_1^3(k-1)+\nu_1^3(k)$$

and

$$
\begin{aligned}
(\widehat{\mathrm{E}}^+(X_{k-1},X_k) &\leqslant 2(\mathrm{E}X_{k-1}X_k)^2+2\nu_1^2(k-1)\nu_1^2(k)\\
&\leqslant 2[\nu_1(k-1)+\nu_1(k)]\mathrm{E}X_{k-1}X_k+2\nu_1^3(k-1)+2\nu_1^3(k).
\end{aligned}
$$

Thus, (6.32) easily follows. The estimates (6.33) – (6.35) are proved similarly. For the proof of (6.36), note that

$$\widehat{\mathrm{E}}^+(X_{k-1},X_k)\leqslant\mathrm{E}X_{k-1}X_k+0.01\nu_1(k),\quad \nu_1^2(k)\leqslant 0.01\nu_1(k)$$

and

$$
\begin{aligned}
|g_k| &\leqslant \exp\Big\{-2\nu_1(k)\sin^2(t/2)+2[\nu_2(k)+\nu_1^2(k)+2\widehat{\mathrm{E}}^+(X_{k-1},X_k)]\sin^2(t/2)\Big\}\\
&\leqslant \exp\Big\{-1.92\nu_1(k)\sin^2(t/2)+2\nu_2(k)\sin^2(t/2)+4\mathrm{E}X_{k-1}X_k\sin^2(t/2)\Big\},
\end{aligned}
$$

which completes the proof. □

For asymptotic expansions, we need a few smoothing estimates.

**Lemma 6.8.** *Let conditions (3.1) and (3.2) be satisfied,* $0 \leqslant \alpha \leqslant 1$*, and* $M$ *be any finite (signed) measure. Then*

$$\|M\exp\{\Gamma_1 U + \alpha\Gamma_2 U^2\}\| \leqslant C\|M\exp\{0.9\lambda U\}\|.$$

*Proof:* Due to (3.1) and (3.2), we have

$$\Gamma_1 - 3.1|\Gamma_2| \geqslant \Gamma_1 - 1.55\sum_{k=1}^{n}\nu_2(k) - 0.0155\Gamma_1 - 3.1\sum_{k=1}^{n}\mathrm{E}X_{k-1}X_k - 0.031\Gamma_1 \geqslant 0.9\lambda.$$

Thus,

$$\begin{aligned}
&\|M\exp\{\Gamma_1 U + \alpha\Gamma_2 U^2\}\| \\
&\qquad \leqslant \|M\exp\{(\Gamma_1 - 3.1|\Gamma_2|)U\}\|\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\| \\
&\qquad \leqslant \|M\exp\{0.9\lambda U\}\|\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\|.
\end{aligned}$$

It remains to prove that the second exponent measure is bounded by some absolute constant. Note that the total variation of *any distribution* equals unity. Therefore, by Lemma 5.2

$$\begin{aligned}
\Big\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\Big\| &= \Big\|\exp\{3.1|\Gamma_2|U\}\Big(I + \sum_{m=1}^{\infty}\frac{(\alpha\Gamma_2 U^2)^m}{m!}\Big)\Big\| \\
&\leqslant 1 + \sum_{m=1}^{\infty}\frac{|\Gamma_2|^m}{m!}\|U^2\exp\{3.1|\Gamma_2|U/m\}\|^m \\
&\leqslant 1 + \sum_{m=1}^{\infty}\frac{|\Gamma_2|^m}{m^m \mathrm{e}^{-m}\sqrt{2\pi m}}\Big(\frac{3m}{3.1|\Gamma_2|\mathrm{e}}\Big)^m \leqslant C.
\end{aligned}$$

Combining both inequalities given above, we complete the proof of the lemma. □

**Lemma 6.9.** *Let conditions (3.1) and (3.3) be satisfied. Then*

$$\begin{aligned}
\mathrm{NB}(r,\overline{q}) &= \exp\Big\{\Gamma_1 U + \Gamma_2 U^2 + \frac{4\Gamma_2^2}{3\Gamma_1}U^3 + \frac{2\Gamma_2^3}{\Gamma_1^2}U^4\Theta\frac{1}{0.7}\Big\} \\
&= \exp\Big\{\Gamma_1 U + \Gamma_2 U^2 + \frac{4\Gamma_2^2}{3\Gamma_1}U^3\Theta\frac{1}{0.7}\Big\} = \exp\Big\{\Gamma_1 U + \Gamma_2 U^2\Theta\frac{1}{0.7}\Big\} \\
&= \exp\Big\{\Gamma_1 U + \Gamma_1 U^2\Theta\frac{3}{28}\Big\} = \exp\Big\{0.5\Gamma_1 U\Big\}\Theta C. \qquad (6.38)
\end{aligned}$$

*Proof:* Due to (3.3),

$$\begin{aligned}
\Gamma_2 &= \frac{1}{2}\sum_{k=1}^{n}(\nu_2^2(k) - \nu_1^2(k)) + \sum_{k=1}^{n}Cov(X_{k-1}, X_k) \\
&\leqslant \frac{1}{2}\sum_{k=1}^{n}\nu_2(k) + \sum_{k=1}^{n}|Cov(X_{k-1}, X_k)| \leqslant \frac{3}{40}\Gamma_1.
\end{aligned}$$

Therefore,

$$\frac{1-\overline{q}}{\overline{q}} = \frac{2\Gamma_2}{\Gamma_1} \leqslant 0.15, \quad \Big(\frac{1-\overline{q}}{\overline{q}}\Big)\|U\| \leqslant 0.15(\|I_1\| + \|I\|) \leqslant 0.3.$$

Consequently, from (1.1),

$$\begin{aligned}
\mathrm{NB}(r,\overline{q}) &= \exp\Big\{\sum_{j=1}^{\infty}\frac{r}{j}\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{j}U^{j}\Big\}\\
&= \exp\Big\{\Gamma_1 U + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{2}\frac{U^2}{2} + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{3}\frac{U^3}{3} + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{4}\frac{U^4}{4}\Theta\frac{1}{0.7}\Big\}\\
&= \exp\Big\{\Gamma_1 U + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{2}\frac{U^2}{2} + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{3}\frac{U^3}{3}\Theta\frac{1}{0.7}\Big\}\\
&= \exp\Big\{\Gamma_1 U + r\Big(\frac{1-\overline{q}}{\overline{q}}\Big)^{2}\frac{U^2}{2}\Theta\frac{1}{0.7}\Big\}.
\end{aligned}$$

Recalling that $r(1-\overline{q})/\overline{q} = \Gamma_1$, we obtain all equalities except the last one. The last equality is equivalent to

$$\Big\|\exp\Big\{0.5\Gamma_1 U + \Gamma_1 U^2\Theta\frac{3}{28}\Big\}\Big\| \leqslant C$$

which is proved similarly to Lemma 6.8. □

**Lemma 6.10.** *Let conditions (3.1) and (3.3) be satisfied. Then*

$$\begin{aligned}
\mathrm{Bi}(N,\overline{p}) &= \exp\Big\{-N\sum_{j=1}^{\infty}\frac{(-\overline{p}U)^j}{j}\Big\}\\
&= \exp\Big\{\Gamma_1 U + \Gamma_2 U^2 + U^2\theta\frac{50\Gamma_2^2\epsilon}{21\Gamma_1^2} + \frac{N\overline{p}^3U^3}{3} + \frac{N\overline{p}^4U^4}{4}\Theta\frac{5}{3}\Big\}\\
&= \exp\Big\{\Gamma_1 U + \Gamma_2 U^2 + U^2\theta\frac{50\Gamma_2^2\epsilon}{21\Gamma_1^2} + \frac{N\overline{p}^3U^3}{3}\Theta\frac{5}{3}\Big\}\\
&= \exp\Big\{\Gamma_1 U + \frac{N\overline{p}^2U^2}{2}\Theta\frac{5}{3}\Big\} = \exp\Big\{\Gamma_1 U + \Gamma_1 U^2\Theta\frac{1}{6}\Big\}\\
&= \exp\Big\{0.5\Gamma_1 U\Big\}\Theta C.
\end{aligned}$$

*Proof:* Due to (3.3),

$$\begin{aligned}
|\Gamma_2| &\leqslant \frac{1}{2}\sum_{k=1}^{n}(\nu_2(k)+0.01\nu_1(k)) + \sum_{k=1}^{n}|Cov(X_{k-1},X_k)|\\
&\leqslant \Gamma_1(0.025+0.005+0.05) = 0.08\Gamma_1.
\end{aligned}$$

Therefore,

$$\overline{p} = \frac{\Gamma_1}{\tilde{N}-\epsilon} \leqslant \frac{\Gamma_1}{\tilde{N}-1} \leqslant \frac{2|\Gamma_2|}{\Gamma_1 - 2|\Gamma_2|} \leqslant \frac{50|\Gamma_2|}{21\Gamma_1} < \frac{1}{5}. \tag{6.39}$$

and

$$\frac{\epsilon}{\tilde{N}} \leqslant \frac{1}{\tilde{N}} = \frac{2|\Gamma_2|}{\Gamma_1^2} \leqslant \frac{2|\Gamma_2|}{\Gamma_1} \leqslant 0.16.$$

Consequently,

$$N\overline{p}^2 = 2|\Gamma_2|\frac{\tilde{N}}{N} = 2|\Gamma_2|\frac{1}{1-\epsilon/\tilde{N}} = 2|\Gamma_2|\Big(1+\frac{\epsilon}{\tilde{N}}\theta\frac{100}{84}\Big)$$

and

$$-\frac{N\overline{p}^2}{2} = \Gamma_2 + \theta\frac{50\Gamma_2^2}{21\Gamma_1^2}\epsilon. \tag{6.40}$$

Taking into account (6.39), we prove

$$\begin{aligned}
\mathrm{Bi}(N,\overline{p}) &= \exp\Big\{-N\sum_{j=1}^{\infty}\frac{(-\overline{p}U)^j}{j}\Big\} \\
&= \exp\Big\{\Gamma_1 U - \frac{N(\overline{p}U)^2}{2} + \frac{N(\overline{p}U)^3}{3} + \frac{N\overline{p}^4U^4}{4}\Theta\frac{5}{3}\Big\} \\
&= \exp\Big\{\Gamma_1 U - \frac{N(\overline{p}U)^2}{2} + \frac{N(\overline{p}U)^3}{3}\Theta\frac{5}{3}\Big\} \\
&= \exp\Big\{\Gamma_1 U + \frac{N(\overline{p}U)^2}{2}\Theta\frac{5}{3}\Big\} = \exp\Big\{\Gamma_1 U + \Gamma_1 U^2\Theta\frac{1}{6}\Big\}.
\end{aligned}$$

Combining (6.40) with the last expansions, we obtain all equalities except the last one whose proof is similar to that of Lemma 6.8. □

## 7. Proofs

*Proof of Theorem 3.2:* Let $\widehat{M}(t) = J_1 + J_2$, where

$$\begin{aligned}
J_1 &= \prod_{j=1}^{n}\varphi_j - \prod_{j=1}^{n} g_j - \sum_{m=1}^{n}(\varphi_m - g_m)\prod_{j\neq m}^{n} g_m, \\
J_2 &= \prod_{j=1}^{n} g_j + \sum_{m=1}^{n}(\varphi_m - g_m)\prod_{j\neq m} g_j - \prod_{j=1}^{n} g_j(1+\Gamma_3 z^3).
\end{aligned}$$

We estimate $J_1$ and $J_2$ separately. Further we frequently apply the following estimate

$$\begin{aligned}
\prod_{j=1,j\neq m,l}^{n}\exp\{-\lambda_j\sin^2(t/2)\} &\leqslant \exp\{-1.3\lambda\sin^2(t/2)\}\exp\{(\lambda_m+\lambda_l)\sin^2(t/2)\} \\
&\leqslant C\exp\{-1.3\lambda\sin^2(t/2)\}, \qquad (7.1)
\end{aligned}$$

which is valid for any $m, l \in \{1, 2, \ldots, n\}$, since all $\lambda_j \leqslant C$.

Applying the generalized Bergström identity from Čekanavičius (1998), (7.1), Lemmas 6.3, 6.4 , 6.7 and 5.4, we obtain

$$\begin{aligned}
|J_1| = \Big|\sum_{l=2}^{n}(\varphi_l - g_l)\prod_{j=l+1}^{n}\varphi_j\sum_{m=1}^{l-1}(\varphi_m - g_m)\prod_{j=1,j\neq m}^{l-1} g_j\Big| \\
\leqslant\ & C\sum_{l=2}^{n}|\varphi_l - g_l|\sum_{m=1}^{l-1}|\varphi_m - g_m|\prod_{j=1,j\neq m,l}^{n}\exp\{-1.3\lambda_j\sin^2(t/2)\} \\
\leqslant\ & C\exp\{-1.3\lambda\sin^2(t/2)\}\Big(\sum_{k=1}^{n}|\varphi_k - g_k|\Big)^2 \leqslant C\exp\{-1.3\lambda\sin^2(t/2)\}R_1^2|z|^6 \\
\leqslant\ & C\exp\{-\lambda\sin^2(t/2)\}R_1^2\min(1,\lambda^{-3}).
\end{aligned}$$

Similarly, taking into account (6.19),(6.30), (6.34) and (6.36), we get

$$\begin{aligned}
|J_2| &\leqslant \Big|\prod_{j=1}^n g_j(1+\Gamma_3 z^3) - \prod_{j=1}^n g_j - \prod_{j=1}^n g_j \sum_{m=1}^n(\varphi_m - g_m)\Big| \\
&\quad + \Big|\sum_{m=1}^n(\varphi_m - g_m)\Big(\prod_{j=1}^n g_j - \prod_{j\neq m}^n g_j\Big)\Big| \\
&= \Big|\prod_{j=1}^n g_j\Big(\sum_{m=1}^n(\varphi_m - g_m) - \Gamma_3 z^3\Big)\Big| + \Big|\sum_{m=1}^n(\varphi_m - g_m)\prod_{j\neq m} g_j(g_m - 1)\Big| \\
&\leqslant CR_2|z|^4\exp\{-1.3\lambda\sin^2(t/2)\} \leqslant C\exp\{-\lambda\sin^2(t/2)\}R_2\min(1,\lambda^{-2}).
\end{aligned}$$

Therefore,

$$|\widehat{M}(t)| \leqslant C\exp\{-\lambda\sin^2(t/2)\}(R_1^2\min(1,\lambda^{-3}) + R_2\min(1,\lambda^{-2})). \tag{7.2}$$

Let $\tilde{\varphi}_k = \varphi_k\exp\{-\mathrm{i}\nu_1(k)t\}$, $\tilde{g}_k = g_k\exp\{-\mathrm{i}\nu_1(k)t\}$ . Observe that $|\tilde{\varphi}_l' - \tilde{g}_l'| \leqslant C(|\varphi_l' - g_l'| + \nu_1(k)|\varphi_l - g_l|)$. Moreover, taking into account (6.27), (6.28) and (6.37), we get

$$\begin{aligned}
|\tilde{\varphi}_l'| &\leqslant |\varphi_l' - \nu_1(l)z'| + \nu_1(l)|\mathrm{e}^{\mathrm{i}t} - \varphi_l| \\
&\leqslant |\varphi_l' - \nu_1(l)z'| + \nu_1(l)|z| + \nu_1(l)|1-\varphi_l| \leqslant C|z|\sum_{j=0}^3 \nu_1(l-j)
\end{aligned}$$

and similar estimate holds for $|\tilde{g}_l'|$.

Taking into account (7.1), Lemmas 6.3, 6.4, 6.6 and 6.7 we prove that

$$\begin{aligned}
\Big|(\mathrm{e}^{-\mathrm{i}\Gamma_1 t}J_1)'\Big| &\leqslant \sum_{l=2}^n|\tilde{\varphi}_l' - \tilde{g}_l'|\prod_{j=l+1}^n|\tilde{\varphi}_l|\sum_{m=1}^{l-1}|\tilde{\varphi}_m - \tilde{g}_m|\prod_{j=1,j\neq m}^{l-1}|\tilde{g}_j| \\
&\quad + \sum_{l=2}^n|\tilde{\varphi}_l - \tilde{g}_l|\sum_{j=l+1}^n|\tilde{\varphi}_j'|\prod_{i=l+1,i\neq j}^n|\tilde{\varphi}_j|\sum_{m=1}^{l-1}|\tilde{\varphi}_m - \tilde{g}_m|\prod_{j=1,j\neq m}^{l-1}|\tilde{g}_j| \\
&\quad + \sum_{l=2}^n|\tilde{\varphi}_l - \tilde{g}_l|\prod_{j=l+1}^n|\tilde{\varphi}_j|\sum_{m=1}^{l-1}|\tilde{\varphi}_m' - \tilde{g}_m'|\prod_{j=1,j\neq m}^{l-1}|\tilde{g}_j| \\
&\quad + \sum_{l=2}^n|\tilde{\varphi}_l - \tilde{g}_l|\prod_{j=l+1}^n|\tilde{\varphi}_j|\sum_{m=1}^{l-1}|\tilde{\varphi}_m - \tilde{g}_m|\sum_{j=1,j\neq m}^{l-1}|\tilde{g}_j'|\prod_{k=1,k\neq m,j}^{l-1}|\tilde{g}_k| \\
&\leqslant C\exp\{-1.3\lambda\sin^2(t/2)\}\Big(\sum_{l=2}^n|\tilde{\varphi}_l' - \tilde{g}_l'|\sum_{m=1}^{l-1}|\tilde{\varphi}_m - \tilde{g}_m| \\
&\quad + \Big(\sum_{l=1}^n|\tilde{\varphi}_l - \tilde{g}_l|\Big)^2\sum_{j=1}^n(|\tilde{\varphi}_j'| + |\tilde{g}_j'|)\Big) \\
&\leqslant C\exp\{-1.3\lambda\sin^2(t/2)\}(R_1^2|z|^5(1+\Gamma_1|z|^2) + R_1^2|z|^7\Gamma_1) \\
&\leqslant C\exp\{-\lambda\sin^2(t/2)\}(1+\Gamma_1\min(1,\lambda^{-1}))R_1^2\min(1,\lambda^{-5/2}).
\end{aligned} \tag{7.3}$$

Similarly

$$
\begin{aligned}
|(\mathrm{e}^{-\mathrm{i}\Gamma_1 t}J_2)'| \leqslant\ & \Big|\Big(\mathrm{e}^{-\mathrm{i}t\Gamma_1}\prod_{j=1}^n g_j\sum_{m=1}^n(\varphi_m-g_m-\gamma_3(m)z^3)\Big)'\Big|\\
&+\Big|\Big(\mathrm{e}^{-\mathrm{i}t\Gamma_1}\sum_{m=1}^n(\varphi_m-g_m)\prod_{j\neq m}g_j(g_m-1)\Big)'\Big|\\
\leqslant\ & \Big|\Big(\prod_{j=1}^n\tilde g_j\Big)'\sum_{m=1}^n(\varphi_m-g_m-\gamma_3(m)z^3)\Big|\\
&+\Big|\prod_{j=1}^n\tilde g_j\sum_{m=1}^n(\varphi_m'-g_m'-\gamma_3(m)(z^3)')\Big|\\
&+\Big|\sum_{m=1}^n(\varphi_m'-g_m')\prod_{j\neq m}\tilde g_j(\tilde g_m-\mathrm{e}^{-\mathrm{i}t\nu_1(m)})\Big|\\
&+\sum_{m=1}^n|\varphi_m-g_m||g_m-1|\Big|\Big(\prod_{j\neq m}\tilde g_j\Big)'\Big|\\
&+\sum_{m=1}^n|\varphi_m-g_m|\Big|\prod_{k\neq m}\tilde g_j\Big|[|\tilde g_m'|+\nu_1(m)].
\end{aligned}
$$

Applying Lemmas 6.6, 6.7 and 5.4, it is not difficult to prove that the derivative given above is less than $C|z|^5\Gamma_1R_2\exp\{-1.3\lambda\sin^2(t/2)\}$. Combining this estimate with (7.3) we obtain

$$
\begin{aligned}
|(\mathrm{e}^{-\mathrm{i}\Gamma_1 t}\widehat{M}(t))'| \leqslant\ & C\exp\{-\lambda\sin^2(t/2)\}(1+\Gamma_1\min(1,\lambda^{-1})(R_1^2\min(1,\lambda^{-5/2})\\
&+R_2\min(1,\lambda^{-3/2})).
\end{aligned}
$$

For the proof of (3.7), we use (7.2), (5.1) with $v=\Gamma_1$ and $u=\max(1,\Gamma_1)$. For the proof of (3.6) we use identity

$$
\prod_{j=1}^n\varphi_j-\prod_{j=1}^n g_j=\sum_{j=1}^n(\varphi_j-g_j)\prod_{l=j+1}^n\varphi_l\prod_{l=1}^{j-1}g_l. \tag{7.4}
$$

The rest of the proof is very similar to the proof of (3.7) and, therefore, omitted. □

*Proof of Theorem 3.1:* For the proof of (3.4) we use (7.4) with $g_j$ replaced by $\exp\{\nu_1(j)z\}$. Now the proof is very similar to the proofs of (3.7) and (3.6) and, therefore, omitted. Applying Lemma 6.8 and using the following identity

$$
\mathrm{e}^b-1-b=b^2\int_0^1(1-\tau)\mathrm{e}^{\tau b}\mathrm{d}\tau, \tag{7.5}
$$

we get

$$
\begin{aligned}
\Big\|\mathrm{G}-\mathrm{Pois}(\Gamma_1)(I+\Gamma_2U^2)\Big\| =\ & \Big\|\exp\{\Gamma_1U\}\int_0^1(1-\tau)(\gamma_2U^2)^2\exp\{\tau\Gamma_2U^2\}\mathrm{d}\tau\Big\|\\
\leqslant\ & \int_0^1\|\Gamma_2^2U^4\exp\{\Gamma_1U+\tau\Gamma_2U^2\}\|\mathrm{d}\tau\\
\leqslant\ & C|\Gamma_2|^2\|U^4\exp\{0.9\lambda U\}\|\leqslant CR_0^2\min(1,\lambda^{-2}).
\end{aligned}
$$

Combining this estimate with Bergström expansion ($s = 1$) for G, we prove (3.5). □

*Proof of Theorem 3.3:* Applying (6.38) and Lemma 5.2, we obtain

$$\begin{aligned}
\Big\|\mathrm{G}-\mathrm{NB}(r,\overline{q})\Big\| &= \Big\|\mathrm{G}-\mathrm{G}\exp\Big\{\frac{4\Gamma_2^2}{\Gamma_1}U^3\Theta\frac{1}{0.7}\Big\}\Big\| \\
&= \Big\|\mathrm{G}\int_0^1\Big(\exp\Big\{\tau\frac{4\Gamma_2^2}{\Gamma_1}U^3\Theta\frac{1}{0.7}\Big\}\Big)'\mathrm{d}\tau\Big\| \\
&\leqslant C\int_0^1\frac{\Gamma_2^2}{\Gamma_1}\Big\|U^3\exp\Big\{\Gamma_1U+\Gamma_2U^2+\tau\frac{4\Gamma_2^2}{\Gamma_1}U^3\Theta\frac{1}{0.7}\Big\}\Big\|\mathrm{d}\tau \\
&\leqslant C\frac{\Gamma_2^2}{\Gamma_1}\Big\|U^3\exp\{0.5\Gamma_1U\}\Big\|\leqslant C\frac{\Gamma_2^2}{\Gamma_1}\min(1,\Gamma_1^{-3/2}).
\end{aligned}$$

Combining the last estimate with (3.6), we prove (3.8).

Let

$$M_1:=\frac{4\Gamma_2^2}{3\Gamma_1}U^3,\quad M_2:=\frac{2\Gamma_2^3}{\Gamma_1^2}U^4\Theta\frac{1}{0.7},\quad M_3:=\Gamma_3U^3-M_1.$$

Then by Lemmas 6.9 and 5.2 and using equation (7.5),

$$\begin{aligned}
&\mathrm{NB}(r,\overline{q})=\mathrm{G}\exp\{M_1+M_2\} \\
&\quad= \mathrm{G}\Big(I+M_1+M_1^2\int_0^1(1-\tau)\exp\{\tau M_1\}\mathrm{d}\tau\Big)\Big(I+M_2\int_0^1\exp\{xM_2\}\mathrm{d}x\Big) \\
&\quad= \mathrm{G}(I+M_1)+M_1^2\int_0^1(1-\tau)\mathrm{G}\exp\{\tau M_1\}\mathrm{d}\tau \\
&\qquad+\int_0^1\int_0^1M_2(I+M_1+M_1^2(1-\tau))\mathrm{G}\exp\{\tau M_1+xM_2\}\mathrm{d}\tau\mathrm{d}x \\
&\quad= \mathrm{G}(I+M_1)+\exp\{0.5\Gamma_1U\}(M_1^2\Theta C+[M_2+M_1M_2]\Theta C+M_1^2M_2\Theta C) \\
&\quad= \mathrm{G}(I+M_1)+\exp\{0.25\Gamma_1U\}\Gamma_2^3\Gamma_1^{-2}U^4\Theta C.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
&\|F_n-\mathrm{NB}(r,\overline{q})(I+M_3)\| \\
&\quad\leqslant \|F_n-\mathrm{G}(I+\Gamma_3U^3)\|+\|\mathrm{G}(I+\Gamma_3U^3)-\mathrm{G}(I+M_1)(I+M_3)\| \\
&\qquad+C\|\exp\{0.25\Gamma_1U\}\Gamma_2^3\Gamma_1^{-1}U^4(I+M_3)\|=:J_{31}+J_{32}+J_{33}.
\end{aligned}$$

By Lemmas 6.8 and 5.2,

$$\begin{aligned}
J_{32} &\leqslant C\|\exp\{0.9\lambda U\}\Gamma_2^2\Gamma_1^{-1}(\Gamma_3-4\Gamma_2^2(3\Gamma_1)^{-1})U^6\| \\
&\leqslant \Gamma_2^2\Gamma_1^{-1}|\Gamma_3-4\Gamma_2^2(3\Gamma_1)^{-1}|\min(1,\Gamma_1^{-3}).
\end{aligned}$$

Similarly

$$\begin{aligned}
J_{33} &\leqslant C\|\exp\{0.25\Gamma_1U\}\Gamma_2^3\Gamma_1^{-2}U^4\| \\
&\quad+C\|\exp\{0.25\Gamma_1U\}\Gamma_2^3\Gamma_1^{-2}(\Gamma_3-4\Gamma_2^2(3\Gamma_1)^{-1})U^7\| \\
&\leqslant C\Gamma_2^3\Gamma_1^{-2}\min(1,\Gamma_1^{-2})+C\Gamma_2^3\Gamma_1^{-2}|\Gamma_3-4\Gamma_2^2(3\Gamma_1)^{-1}|\min(1,\Gamma_1^{-7/2}).
\end{aligned}$$

Combining the last two estimates and applying (3.7) for $J_{31}$, we prove (3.9). □

*Proof of Theorem 3.4:* Let

$$\tilde{M}_1:=\frac{N\overline{p}^3U^3}{3},\quad \tilde{M}_2:=\frac{N\overline{p}^4U^4}{4}\Theta\frac{5}{3}+U^2\theta\frac{50\Gamma_2^2\epsilon}{21\Gamma_1^2},\quad \tilde{M}_3:=\Gamma_3U^3-\tilde{M}_1. \tag{7.6}$$

Since the proof is almost identical to that of Theorem 3.3, it is omitted. □

*Proof of Theorem 4.1:* Let $\tilde{M}_3$ be defined as (7.6). Observe that

$$\nu_1(k)=p^2,\quad \nu_2(k)=\nu_3(k)=0,\quad \mathrm{E}X_{k-1}X_k\leqslant Cp^3,\quad \mathrm{E}X_{k-2}X_{k-1}X_k\leqslant Cp^4,$$
$$\mathrm{E}X_{k-3}\cdots X_k\leqslant Cp^5,\quad \Gamma_2\leqslant Cnp^3,\quad \Gamma_3\leqslant Cnp^4,\quad R_1\leqslant Cnp^4,\quad R_2\leqslant Cnp^5.$$

and

$$\tilde{M}_3=-\frac{np^4}{3}U^3+U^3\theta Cnp^5.$$

From Lemmas 6.9 and 5.2, we have

$$\begin{aligned}
&\Big\|(\mathrm{NB}(r,\overline{q})-\exp\{np^2U\})U^3\Big\|\\
&\qquad\leqslant\quad \Big\|\exp\{np^2U\}\int_0^1(\Gamma_2U^2\Theta/0.7)\exp\{\tau(\Gamma_2U^2\Theta/0.7)\}\mathrm{d}\tau U^3\Big\|\\
&\qquad\leqslant\quad Cnp^3\|\exp\{0.5np^2U\}U^5\|\leqslant\frac{C}{p^2n\sqrt{n}}. \qquad (7.7)
\end{aligned}$$

Applying (3.9), (7.7) and Lemmas 6.9 and 6.8, we obtain

$$\begin{aligned}
&\left|\|F-\mathrm{NB}(r,\overline{q})\|-\frac{\tilde{C}_{TV}p}{\sqrt{n}}\right|\leqslant\|F-\mathrm{NB}(r,\overline{q})(I+M_3)\|+\left|\|\mathrm{NB}(r,\overline{q})M_3\|-\frac{\tilde{C}_{TV}p}{\sqrt{n}}\right|\\
&\quad\leqslant\ \frac{Cp}{n}+\|\mathrm{NB}(r,\overline{q})(M_3+np^4U^3/3)\|+\left|\frac{np^4}{3}\|\mathrm{NB}(r,\overline{q})U^3\|-\frac{\tilde{C}_{TV}p}{\sqrt{n}}\right|\\
&\quad\leqslant\ \frac{Cp^2}{\sqrt{n}}+\frac{np^4}{3}\|(\mathrm{NB}(r,\overline{q})-\exp\{np^2U\})U^3\|\\
&\qquad+\left|\frac{np^4}{3}\|\exp\{np^2U\}U^3\|-\frac{\tilde{C}_{TV}p}{\sqrt{n}}\right|\\
&\quad\leqslant\ \frac{Cp^2}{\sqrt{n}}+\frac{np^4}{3}\left|\|\exp\{np^2U\}U^3\|-\frac{3\tilde{C}_{TV}}{(np^2)^{3/2}}\right|\leqslant\frac{Cp^2}{\sqrt{n}}+\frac{C}{n}.
\end{aligned}$$

□

*Proof of Theorem 4.3:* The direct consequence of conditions $(n-m+1)a(p)\geqslant 1$ and $ma(p)\leqslant 0.01$ are the following estimates

$$(n-m+1)\geqslant 100m,\quad \tilde{N}=\frac{(n-m+1)}{2m-1-m(m-1)/(n-m+1)}\geqslant\frac{100m}{2m}=50.$$

We have

$$\begin{aligned}
\overline{p}\ &=\ \frac{(n-m+1)a(p)}{\tilde{N}}+\frac{(n-m+1)a(p)}{\tilde{N}}\left(\frac{\tilde{N}}{N}-1\right)\\
&=\ \frac{(n-m+1)a(p)}{\tilde{N}}\left(1+\frac{\epsilon}{\tilde{N}-\epsilon}\right)\\
&=\ a(p)\left(2m-1-\frac{m(m-1)}{n-m+1}\right)\left(1+\frac{\epsilon}{\tilde{N}-\epsilon}\right)\leqslant a(p)\left(2m+\frac{m}{100}\right)\left(1+\frac{1}{49}\right)\\
&\leqslant\ 2.05a(p)m\leqslant 0.03. \qquad (7.8)
\end{aligned}$$

The sum $\tilde{N}$ has $n-m+1$ summands. After grouping, we get $K$ 1-dependent random variables containing $m$ initial summands each, and (possibly) one additional variable, equal to the sum of $\delta m$ initial summands. Here

$$K = \left\lfloor \frac{n-m+1}{m} \right\rfloor, \quad \frac{n-m+1}{m} = K + \delta, \quad 0 \leqslant \delta < 1. \tag{7.9}$$

The analysis of the structure of new variables $X_j$ shows that, for $j = 1, \dots, K$

$$\begin{aligned} X_j &= \begin{cases} 1, & \text{with probability } ma(p), \\ 0, & \text{with probability } 1 - ma(p), \end{cases} \\ X_{K+1} &= \begin{cases} 1, & \text{with probability } \delta ma(p), \\ 0, & \text{with probability } 1 - \delta ma(p). \end{cases} \end{aligned}$$

Consequently,

$$\nu_2(j) = \nu_3(j) = \nu_4(j) = \widehat{\mathrm{E}}_2^+(X_1, X_2) = \widehat{\mathrm{E}}_2^+(X_1, X_2, X_3) = \widehat{\mathrm{E}}_3(X_1, X_2) = 0.$$

For calculation of $\mathrm{E}X_1X_2$, note that there are the following non-zero product events: a) the first summand of $X_1$ equals 1 and any of the summands of $X_2$ equals 1 ($m$ variants); b) the second summand of $X_1$ equals 1 and any of the summands of $X_2$, beginning from the second one, equals 1 ($m-1$ variant) and etc. Each event has the probability of occurrence $a^2(p)$. Therefore,

$$\mathrm{E}X_1X_2 = a^2(p)(m + (m-1) + (m-2) + \cdots + 1) = \frac{a(p)^2 m(m+1)}{2}.$$

Similarly arguing we obtain the following relations for $j = 1, \dots, K$, ($j = 2, \dots, K$ and $j = 3, \dots, K$ if more variables are involved) and $X_{K+1}$ (if $\delta > 0$):

$$\mathrm{E}X_j = ma(p), \quad \mathrm{E}X_{j-1}X_j = \frac{m(m+1)a^2(p)}{2}, \quad \mathrm{E}X_{K+1} = \delta ma(p),$$

$$\begin{aligned} \widehat{\mathrm{E}}(X_{j-1}, X_j) &= -\frac{m(m-1)a^2(p)}{2}, \\ \mathrm{E}X_{j-2}X_{j-1}X_j &= \frac{m(m+1)(m+2)a^3(p)}{6}, \\ \widehat{\mathrm{E}}(X_{j-2}, X_{j-1}, X_j) &= \frac{a^3(p)m(m-1)(m-2)}{6}, \\ \mathrm{E}X_K X_{K+1} &= \frac{\delta m(\delta m+1)a^2(p)}{2}, \\ \widehat{\mathrm{E}}(X_K, X_{K+1}) &= \frac{a^2(p)\delta m(\delta m + 1 - 2m)}{2}, \\ \mathrm{E}X_{K-1}X_K X_{K+1} &= \frac{\delta m(\delta m+1)(\delta m+2)a^3(p)}{6} \\ \widehat{\mathrm{E}}X_{K-1}X_K X_{K+1} &= \frac{a^3(p)\delta m(9m^2 - 9m + 2)}{6}. \end{aligned} \tag{7.10}$$

It is obvious, that $\Gamma_1 = (n-m+1)a(p)$. Taking into account (7.9) and (7.10) we can calculate $\Gamma_2$:

$$\begin{aligned} \Gamma_2 &= -\frac{1}{2}[Km^2a^2(p) + \delta^2 m^2 a^2(p)] - \frac{(K-1)m(m-1)a^2(p)}{2} \\ &\quad + \frac{\delta m a^2(p)(\delta m + 1 - 2m)}{2} = -\frac{a^2(p)m}{2}[2m(K+\delta) - (K+\delta) - (m-1)] \\ &= -\frac{a^2(p)}{2}[(n-m+1)(2m-1) - m(m-1)]. \end{aligned} \tag{7.11}$$

Similarly,

$$\Gamma_3 = \frac{a^3(p)}{6}[(n-m+1)(3m-1)(3m-2) - 4m(2m-1)(m-1)].$$

Making use of all the formulas given above and noting that $m \geqslant 2$, we to get the estimate

$$\begin{aligned} R_1 &\leqslant K(ma(p))^3 + 3ma(p)[(K-2)m(m+1)a^2(p)/2 + \delta m(\delta m+1)a^2(p)/2] \\ &\quad +(\delta ma(p))^3 + C(K+\delta)m^3a^3(p) \leqslant Cm^3a^3(p)(K+\delta) \\ &\leqslant C(n-m+1)m^2a^3(p). \end{aligned}$$

Similarly,

$$\widehat{\mathrm{E}}^+(X_1,X_2,X_3,X_4) \leqslant Cm^4a^4(p), \quad R_2 \leqslant C(n-m+1)m^3a^4(p).$$

Using (7.11) and (7.8), we get

$$\begin{aligned} \frac{N\overline{p}^3}{3} &= \frac{\Gamma_1\overline{p}^2}{3} = \frac{\Gamma_1}{3}\frac{4\Gamma_2^2}{\Gamma_1^2}\left(1+\frac{\epsilon}{\tilde{N}-\epsilon}\right)^2 = \frac{4\Gamma_2^2}{3\Gamma_1} + \frac{4\Gamma_2^2}{3\Gamma_1}\frac{\epsilon}{\tilde{N}-\epsilon}\left(2+\frac{\epsilon}{\tilde{N}-\epsilon}\right) \\ &= \frac{a^3(p)}{3}(n-m+1)(2m-1)^2 + \theta Cm^3a^3(p). \end{aligned}$$

Similarly,

$$\Gamma_3 = \frac{a^3(p)}{6}(n-m+1)(3m-1)(3m-2) + \theta Cm^3a^3(p).$$

Therefore,

$$\Gamma_3 - \frac{N\overline{p}^3}{3} = A + C\theta m^3a^3(p).$$

By Lemma 5.2

$$\begin{aligned} m^3a^3(p)\|U^3\mathrm{Bi}(N,\overline{p})\| &\leqslant C\frac{m^3a^3(p)}{(n-m+1)a(p)\sqrt{(n-m+1)a(p)}} \\ &\leqslant C\frac{m^3a^2(p)}{n-m+1}. \end{aligned} \tag{7.12}$$

Next, we check the conditions in (3.3). Indeed, we already noted that $\nu_2(j)=0$. Now

$$\begin{aligned} (K-1)|\widehat{\mathrm{E}}(X_1,X_2)| + |\widehat{\mathrm{E}}(X_{K-1},X_K)| &\leqslant \frac{Km(m-1)a^2(p)}{2} + \frac{\delta m2ma^2}{2} \\ &\leqslant (K+\delta)2m^2a^2(p) \leqslant \frac{2ma^2}{n-m+1} = 2ma\Gamma_1 \leqslant 0.02\Gamma_1. \end{aligned}$$

It remains to apply Theorem 3.4 and (7.12). □

*Proof of Theorem 4.4:* We have

$$\begin{aligned} &\left|\|\mathrm{H}-\mathrm{Bi}(N,\overline{p})\| - \tilde{C}_{TV}\frac{a^{3/2}(p)m(m-1)}{2\sqrt{n-m+1}}\right| \leqslant \|\mathrm{H}-\mathrm{Bi}(N,\overline{p})\big(I+AU^3\big)\| \\ &\quad + \left\|\mathrm{Bi}(N,\overline{p})U^3\Big(A - \frac{a^3(p)}{6}(n-m+1)m(m-1)\Big)\right\| \\ &\quad + \frac{a^3(p)}{6}(n-m+1)m(m-1)\left|\|\mathrm{Bi}(N,\overline{p})U^3\| - \frac{3\tilde{C}_{TV}}{(N\overline{p}(1-\overline{p}))^{3/2}}\right| \\ &\quad + \frac{a^3(p)}{6}(n-m+1)m(m-1)\frac{3\tilde{C}_{TV}}{(N\overline{p})^{3/2}}\left|\frac{1}{(1-\overline{p})^{3/2}} - 1\right|. \end{aligned}$$

We easily check that

$$\begin{aligned}\frac{1}{(1-\overline{p})^{3/2}}-1 &= \frac{1-(1-\overline{p})^3}{(1-\overline{p})^{3/2}[1+(1-\overline{p})^{3/2}]} = \frac{\overline{p}[1+(1-\overline{p})+(1-\overline{p})^2]}{(1-\overline{p})^{3/2}[1+(1-\overline{p})^{3/2}]}\\ &= a(p)C(m)\theta.\end{aligned}$$

All that now remains is to apply (4.2) and use Lemmas 5.2 and 5.3. □


### Acknowledgements

We are grateful to the referees for useful remarks, which helped to improve the paper.


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