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Trimming a Tree and the Two-Sided Skorohod Reflection

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Abstract. The h-trimming of a tree is a natural regularization procedure which consists in pruning the small branches of a tree: given $h \geq 0$, it is obtained by only keeping the vertices having at least one leaf above them at a distance greater or equal to h.

The h-cut of a function f is the function of minimal total variation satisfying the constraint $0 \leq f - g \leq h$, and can be explicitly constructed via the two-sided Skorohod reflection of f on the interval $[0, h]$.

In this work, we show that the contour path of the h-trimming of a rooted real tree is given by the h-cut of its original contour path. We provide two applications of this result. First, we recover a famous result of Neveu and Pitman (1989), which states that the h-trimming of a tree coded by a Brownian excursion is distributed as a standard binary tree. In addition, we provide the joint distribution of this Brownian tree and its trimmed version in terms of the local time of the two-sided reflection of its contour path. As a second application, we relate the maximum of a sticky Brownian motion to the local time of its driving process.

1. Introduction and Main Results.

In a rooted tree, there is a natural partial ordering on the set of vertices – $x \preceq y$ iff the unique path from the root to vertex y passes through vertex x. Under this ordering, the children of a given node are not ordered. However, one can always specify some arbitrary ordering of the children of each vertex of the tree (from left to right) and by doing so, one defines an object called a rooted *plane* tree – see Le Gall (2005) for a formal definition.

Every rooted plane tree can be encoded by its contour path, where the contour path can be loosely understood by envisioning the tree as embedded in the plane,

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with each of its edges having unit length. We can then imagine a particle starting from the root, traveling along the edges of the tree at speed 1 and exploring the tree from left to right — see Fig 1.1. The contour path of the tree is simply defined as the current distance of the exploration particle to the root — see Fig 1.2.

In this paper, we show that the contour path of the h-trimming of a rooted plane tree (and more generally the h-trimming of rooted real trees) is given by the h-cut of the original contour path; where the h-cut is constructed from the two-sided Skorohod reflection of the original contour path – see (1.4).

Real rooted trees. As already discussed, every rooted plane tree can be encoded by its contour path which is a function in $C_0^+(\mathbb{R}^+)$ – the set of continuous non-negative functions on $\mathbb{R}^+$ with $f(0) = 0$ and compact support. Conversely, it is now well established that any $f \in C_0^+(\mathbb{R}^+)$ encodes a *real rooted tree* in the following natural way – see again Le Gall (2005) for more details. Define

$$\forall s,t \in \mathbb{R}^+, \quad d_f(s,t) = f(s) + f(t) - 2 \inf_{[s\wedge t, s\vee t]} f,$$

and the equivalence relation $\sim$ on $\mathbb{R}^+$ as follows

$$s \sim t \Longleftrightarrow d_f(s,t) = 0.$$

The equivalence relation $\sim$ defines a quotient space

$$\mathcal{T}_f \ = \ \mathbb{R}^+/\sim$$

referred to as the tree encoded by f. The function d_f induces a distance on $\mathcal{T}_f$, and we keep the notation d_f for this distance. It can be shown that the pair $(\mathcal{T}_f, d_f)$ defines a real tree in the sense that the two following properties are satisfied. For every $a, b \in \mathcal{T}_f$:

(i) **(Unique geodesics.)** There is a unique isometric map $\psi^{a,b}$ from $[0, d_f(a,b)]$ into $\mathcal{T}_f$ such that $\psi^{a,b}(0) = a$ and $\psi^{a,b}(d_f(a,b)) = b$.
(ii) **(Loop free.)** If q is a continuous injective map from $[0,1]$ into $\mathcal{T}_f$, such that $q(0) = a$ and $q(1) = b$, we have $q([0,1]) = \psi^{a,b}([0, d_f(a,b)])$.

In the following, for any $x, y \in \mathcal{T}_f$, $[x,y]$ will denote the geodesic from x to y, i.e., $[x,y]$ is the image of $[0, d_f(x,y)]$ by $\psi^{x,y}$. We will denote by p_f the canonical projection from $\mathbb{R}^+$ to $\mathcal{T}_f$ which can be thought of as the position of the exploration particle at time t. In the following, $\rho_f = p_f(0)$ will be referred to as the root of the tree $\mathcal{T}_f$. In what follows, real trees will always be rooted, even if this is not mentioned explicitly.

d_f induces a natural partial ordering on the rooted tree $\mathcal{T}_f$: $v' \preceq v$ (v' is an ancestor of v) iff

$$d_f(v,v') \ = \ d_f(\rho_f, v) - d_f(\rho_f, v').$$

We note that this partial ordering is directly related to the sub-excursions nested in the function f. Indeed, for any $s,t \geq 0$, $p_f(t) \preceq p_f(s)$ if and only if $\inf_{[t\wedge s, t\vee s]} f = f(t)$, which is equivalent to saying that t is the ending time or starting time of a sub-excursion of f starting from level $f(t)$ and straddling time s – see Fig 1.1 and 1.2.

Finally, for any $x, y \in \mathcal{T}_f$, the most recent common ancestor of x and y – denoted by $x \wedge y$ – is defined as $\sup\{z \in \mathcal{T}_f : z \preceq x, y\}$. From the definition of our genealogy, for any $t_1, t_2 \in \mathbb{R}^+$, we must have

$$p_f(t_1) \wedge p_f(t_2) = p_f(s), \quad \text{for any } s \in \operatorname{argmin}_{[t_1\wedge t_2, t_1\vee t_2]} f, \tag{1.1}$$

with the height of the most recent common ancestor being given by

$$f(s) = \min_{[t_1 \wedge t_2, t_1 \vee t_2]} f.$$

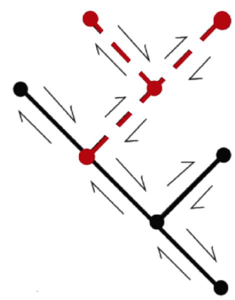

FIGURE 1.1. *Exploration of a plane tree. The exploration particle travels along each branch twice : first on the left and away from the root, and then on the right and towards the root. The root of the red sub-tree belongs to the 2-trimming of the tree.*

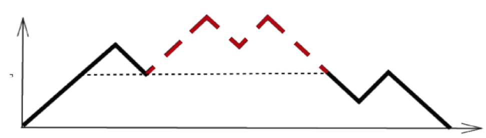

FIGURE 1.2. *Contour path. The red portion of the curve is a sub-excursion of height 2 corresponding to the exploration of the red sub-tree on the left panel.*

Trimming and the two-sided Skorohod reflection. As in Evans (2005), for every $h > 0$, we define the h-trimming of the real tree $(\mathcal{T}_f, d_f)$ as the (possibly empty) sub-tree

$$\mathrm{Tr}^h(\mathcal{T}_f) := \{x \in \mathcal{T}_f \ : \ \sup_{y \in \mathcal{T}_f \ : \ y \succeq x} d_f(x,y) \geq h\}, \tag{1.2}$$

which consists of all the points in $\mathcal{T}_f$ having at least one leaf above them at distance greater or equal to h. (Note that $\mathrm{Tr}^h(\mathcal{T}_f)$ is not empty if and only if $\sup_{[0,\infty)} f \geq h$.) As already mentioned, one of the main results of this paper is the relation between the h-trimming of a real rooted tree and the two-sided Skorohod reflection of its contour path. The one-sided Skorohod reflection is well known among probabilists. Given a continuous function f starting from $x \geq 0$, it is simply defined as the following transformation

$$\Gamma^0(f)(t) \ := \ f(t) \ - (\inf_{[0,t]} f \wedge 0). \tag{1.3}$$

The resulting path obviously remains non-negative and the function $c(t) := -(\inf_{[0,t]} f \wedge 0)$ is easily seen to be the unique solution of the so-called (one-sided) Skorohod equation, i.e., c is the unique continuous function c on $\mathbb{R}^+$ such that $c(0) = 0$ and

(1) $\Gamma^0(f)(t) := f(t) + c(t)$ is non-negative.
(2) c is non-decreasing.
(3) c does not vary off the set $\{t \ : \ \Gamma^0(f)(t) = 0\}$, i.e., the support of the measure dc is contained in $\Gamma^0(f)^{-1}(\{0\})$.

See Lemma 6.17 in Karatzas and Shreve (1991) for a proof of this statement. Intuitively, the function c, which will be referred to as the *compensator* of the reflection in the rest of this paper, can be thought of as the minimal amount of upward push that one needs to exert on the path f to keep it away from negative values. The Skorohod equation states that the reflected path is completely driven by f when it is away from the origin, while it is repealed from negative values by the compensator upon reaching level 0. The following theorem is a generalization of the Skorohod equation to the two-sided case. See also Fig. 1.3.

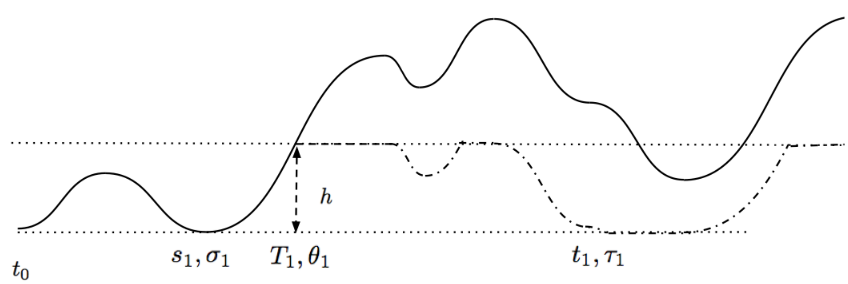

FIGURE 1.3. *A path (plain line) and its two-sided reflection (dashed line) on the interval* $[0, h]$.

Theorem 1.1 (Two-Sided Skorohod Reflection). *Let $h \geq 0$ and let f be a continuous function with $f(0) \in [0, h]$. There exists a unique pair of continuous functions $(c^0(f), c^h(f))$ with $c^0(f)(0) = c^h(f)(0) = 0$ satisfying the three following properties.*

(1) $\Lambda_{0,h}(f)(t) \ := \ f(t) + c^h(f)(t) + c^0(f)(t)$ *is valued in* $[0, h]$.
(2) $c^0(f)$ *(resp., $c^h(f)$) is a non-decreasing (resp., non-increasing) function.*
(3) $c^0(f)$ *(resp., $c^h(f)$) does not vary off the set $\Lambda_{0,h}(f)^{-1}(\{0\})$ (resp., $\Lambda_{0,h}(f)^{-1}(\{h\})$)*

As noted by Kruk et al. (2007), existence and uniqueness to the Skorohod problem follow directly from Lemma 2.1, 2.3 and 2.6 in Tanaka (1979). In the rest of this paper, $\Lambda_{0,h}(f)$ will be referred to as the two-sided Skorohod reflection of the path f on $[0, h]$, while the pair of functions $(c^0(f), c^h(f))$ will be referred to as the compensators associated with the function f. In the same spirit as the one-sided reflection, the compensator $c^h(f)$ (resp., $c^0(f)$) can be thought of as the minimal amount of downward (resp., upward) push at level h (resp., 0) that one has to exert on f to keep the path $\Lambda_{0,h}(f)$ inside the interval $[0,h]$. In other words, adding the compensators $c^0(f)$ and $c^h(f)$ to f is the "laziest way" of keeping f inside the interval $[0,h]$.

Let f be a continuous function on $\mathbb{R}^+$ with $f(0)=0$ (with no restriction on the support and on the sign of f). For such a function, define the h-cut of the function f as

$$f_h \ := \ f - \Lambda_{0,h}(f) = -c^0(f) - c^h(f). \tag{1.4}$$

f_h is also characterized by the following interesting variational property.

Lemma 1.2. *Let f be a non-negative continuous function on $\mathbb{R}^+$ with $f(0)=0$. For every $T \geq 0$, f_h is a solution of the following minimization problem:*

$$\mathit{Inf}\{\,TV(g,[0,T]) \ : \ g \quad s.t. \quad g(0)=0, \quad 0 \leq f-g \leq h\} \tag{1.5}$$

where $TV(g,[0,T])$ denotes the total variation of g on the interval $[0,T]$

Proof: Let $g := f - h/2$ and $(c^{-h/2}(g), c^{h/2}(g))$ the compensators at $-h/2$ and $h/2$ for the two-sided reflection reflection of g on $[-h/2,h/2]$. As an immediate corollary of Proposition 2 in Miłoś (2013), we get that when $f \geq 0$, the function

$$h/2 - c^{-h/2}(g) - c^{h/2}(g), \quad \text{where } g := f - h/2$$

is a minimizer of the following problem:

$$\mathrm{Inf}\{\mathrm{TV}^0(g,[0,T]) \ : \ g \ \text{ s.t. } 0 \leq ||f-g||_{\infty,[0,T]} \leq h/2\}. \tag{1.6}$$

(regardless of the value of T) where $||\cdot||_{\infty,[0,T]}$ is the sup norm on $[0,T]$.

The reflection of $g = f - h/2$ on the interval $[-h/2,h/2]$ is obtained from the reflection of f on the interval $[0,h]$ by a translation of $-h/2$ which implies that

$$(c^{-h/2}(g), c^{h/2}(g)) \ = \ (c^0(f), c^h(f)).$$

Combining this with the result in Miłoś (2013) stated above yields that $h/2 - c^0(f) - c^h(f)$ is a minimizer of (1.6). Since the total variation is invariant under translation by a constant, our result follows. ∎

Informally, the previous result states that up to a linear transformation, f_h is the function of minimal total variation satisfying the constraint $0 \leq f - g \leq h$. [1] Our main theorem states that the contour path of the h-trimming of a tree is simply given by the h-cut of its original contour path.

Theorem 1.3. *Let $f \in C_0^+(\mathbb{R}^+)$ and let us assume that the h-trimming of $\mathcal{T}_f$ is not empty.*

1. *The h-cut f_h belongs to $C_0^+(\mathbb{R}^+)$.*
2. *The h-trimming of the real tree $(\mathcal{T}_f, d_f)$ is identical to the real tree $(\mathcal{T}_{f_h}, d_{f_h})$ (up to a root preserving isometry).*

[1] For more information on the interesting minimization problem (1.5), we refer the reader to Łochowski (2011, 2013) – see also Łochowski and Miłoś (2013).

To state our next result, we need to introduce some extra notations. For a continuous function f with $f(0) = 0$, define $t_n(f) \equiv t_n$ (resp., $T_n(f) \equiv T_n$) to be the n^{th} returning time at level 0 (resp., h) of $\Lambda_{0,h}(f)$ and $s_n(f) \equiv s_n$ to be the n^{th} exit time at 0 of $\Lambda_{0,h}(f)$ as follows. $t_0 = 0$, and for $n \geq 1$

$$\begin{aligned} T_n &:= \inf\{u > t_{n-1} &&: \Lambda_{0,h}(f)(u) = h\} \\ t_n &:= \inf\{u > T_n &&: \Lambda_{0,h}(f)(u) = 0\} \\ s_n &:= \sup\{u \in [t_{n-1}, t_n) &&: \Lambda_{0,h}(f)(u) = 0\} \end{aligned} \tag{1.7}$$

with the convention that $\sup\{\emptyset\}, \inf\{\emptyset\} = \infty$. See also Fig. 1.3. Let $N_h(f)$ be the number of returns of $\Lambda_{0,h}(f)$ to 0, i.e.,

$$N_h(f) \ := \ \sup\{n : \ t_n < \infty\}.$$

Finally, define $\{X_n(f)\} \equiv \{X_n\}$ and $\{Y_n(f)\} \equiv \{Y_n\}$,

$$\begin{aligned} \forall n \geq 1, \quad X_n(f) &:= \quad f_h(t_n) - f_h(s_n), \\ Y_n(f) &:= \quad f_h(t_{n-1}) - f_h(s_n). \end{aligned} \tag{1.8}$$

As we shall see below (see Theorem 1.5(2)), when f is a Brownian excursion, the quantity $X_n(f)$ (resp., $Y_n(f)$) simply coincides with the amount of Brownian local time accumulated by the reflected path $\Lambda_{0,h}(f)$ at h (resp., 0) on the interval $[t_{n-1}, t_n]$.

Proposition 1.4. *Let $f \in C_0^+(\mathbb{R}^+)$ and let us assume that the h-trimming of $\mathcal{T}_f$ is not empty. The h-trimming of $\mathcal{T}_f$ is equal (up to a root preserving isometry) to the tree generated inductively according to the following algorithm – see Fig 1.4.*

(Step 1.) *Start with a single branch of length $X_1(f)$.*

(Step n, $n \geq 2$) *If $n = N_h(f)$ stop. Otherwise, let z_{n-1} be the tip of the $(n-1)^{th}$ branch. On the ancestral line $[\rho, z_{n-1}]$, graft a branch of length $X_n(f)$ at a distance $Y_n(f)$ from the leaf z_{n-1}.*

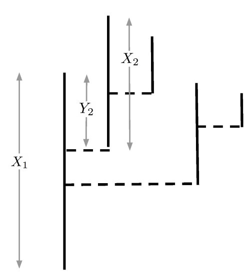

FIGURE 1.4. *Schematic representation of the algorithm generating the h-trimming of a tree from the two-sided reflection of its contour path.*

Relation with standard binary trees. Recall that standard binary trees have branches (1) that have i.i.d. exponential life time with mean α, and (2) when they die, they either give birth to two new branches, or have no offspring with equal probability 1/2. The algorithm described in Proposition 1.4 is reminiscent of a classical construction of standard binary trees (see e.g., Le Gall (1989)), for which $(X_1(f), \{(X_n(f), Y_n(f))\}_{2\le n\le N_h(f)})$ is replaced with $(\tilde{X}_1, \{(\tilde{X}_n, \tilde{Y}_n)\}_{2\le n\le \tilde{N}})$ where $\tilde{X}_1, \tilde{Y}_2, \tilde{X}_2, \tilde{Y}_3, \cdots$ is an infinite sequence of independent exponential r.v.'s with parameter α and the algorithm stops at step $\tilde{N}$, with

$$\tilde{N} := \inf\{n \ : \ \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) < 0\}. \tag{1.9}$$

(Note that this stopping condition is quite natural: the quantity $\sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1})$ is the height of the n^{th} intercalated branching point. We stop the algorithm once the branching point has negative height.) Using Proposition 1.4, we easily recover a result due to Neveu and Pitman (1989), relating the h-trimming of the tree encoded by a Brownian excursion e with standard binary trees (see item 1 in the following theorem). Further, the next theorem provides the joint distribution of the tree $\mathcal{T}_e$ and its trimmed version $\mathrm{Tr}^h(\mathcal{T}_e)$ (see item 2). In the following, for a path w, we define

$$\begin{aligned} l^h(w)(t) \ &:= \ \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}|\{s\in[0,t] \ : \ \Lambda_{0,h}(w)(s)\in[h-\varepsilon,h]\}| \\ l^0(w)(t) \ &:= \ \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}|\{s\in[0,t] \ : \ \Lambda_{0,h}(w)(s)\in[0,\varepsilon]\}|, \end{aligned} \tag{1.10}$$

provided that those limits exist. $l^h(w)$ (resp., $l^0(w)$) will be referred to as the local time of $\Lambda_{0,h}(w)$ at h (resp., at 0).

Theorem 1.5. *Let e be a Brownian excursion conditioned on having a height larger than h.*

(1) *The h-trimming of the tree $(\mathcal{T}_e, d_e)$ is a standard binary tree with parameter $\alpha = h/2$.*
(2) *For $1 \le i \le N_h(e)$,*
- *$X_i(e)$ a.s. coincides with the local time of $\Lambda_{0,h}(e)$ at h accumulated between $[t_{i-1}(e), t_i(e)]$, i.e., $X_i(e) = l^h(e)(t_i) - l^h(e)(t_{i-1})$.*
- *$Y_i(e)$ a.s. coincides with the local time of $\Lambda_{0,h}(e)$ at 0 accumulated between $[t_{i-1}(e), t_i(e)]$.*

The maximum of a sticky Brownian motion. Our final application of Theorem 1.3 relates to the sticky Brownian motion. Given a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\ge 0}, \mathbb{P})$, a sticky Brownian motion, with parameter $\theta > 0$, is defined as the adapted process taking value on $[0,\infty)$ solving the following stochastic differential equation (SDE):

$$dz^\theta(t) \ = \ 1_{z^\theta(t)>0}dw(t) \ + \ \theta \ 1_{z^\theta(t)=0}dt, \tag{1.11}$$

where $(w(t); \ t\ge 0)$ is a standard $\mathcal{G}_t$-Brownian motion. Intuitively, z^θ is driven by w away from level 0, and gets an upward push upon reaching this level, keeping the process away from negative values. Sticky Brownian motion were first investigated by Feller (1957) on strong Markov processes taking values in $[0,\infty)$ that behave like Brownian motion away from 0. We refer the reader to Varadhan (2001) for a good introduction on this object.

Ikeda and Watanabe showed that (1.11) admits a unique weak solution. The result was later straightened by Chitashvili (1989) and Warren (1999) who showed that z^θ is not measurable with respect to w and that, in order to construct the process z^θ, one needs to add some extra randomness to the driving Brownian motion w. In Warren (2002), Warren did exhibit this extra randomness and showed that it can be expressed in terms of a certain marking procedure of the random tree induced by the one-sided Skorohod reflection of the driving Brownian motion w (more on that in Section 4).

Among the first applications related to sticky Brownian motions, we cite Yamada (1994) and Harrison and Lemoine (1981) who studied sticky random walks as the limit of storage processes. More recently, Sun and Swart (2008) introduced a new object called the Brownian net which can be thought of as an infinite family of one-dimensional coalescing-branching Brownian motions and in which sticky Brownian motions play an essential role (see also Newman et al. (2010)).

Building on the approach of Warren (2002), and using Theorem 1.3, we will show that the law of the maximum of a sticky Brownian motion can be expressed in terms of the local time of the two-sided reflection of its driving Brownian motion w on the interval $[0,h]$.

In the following, $\lambda_{0,h}(\cdot)$ will refer to the linear function reflected at 0 and h, i.e., the function obtained by a linear interpolation of the points $\{(2n\cdot h, 0)\}_{n\in\mathbb{Z}}$ and $\{((2n+1)\cdot h), h\}_{n\in\mathbb{Z}}$. For any continuous function f, the *standard* reflection of f on $[0,h]$ (as opposed to the two-sided Skorohod reflection) will refer to the transformation $\lambda_{0,h}(f)$. In Warren (1999), the one-dimensional distribution of a sticky Brownian motion conditionally on its driving process was given. The following theorem provides the one-dimensional distribution of the maximum of a sticky Brownian motion conditionally on its driving process.

Theorem 1.6. *Let $h>0$ and let $(z^\theta(t), w(t); t\geq 0)$ be a weak solution of equation (1.11) starting at $(0,0)$. $\Lambda_{0,h}(w)$ is distributed as a Brownian motion reflected (in the standard way) on $[0,h]$ and we denote by $l^h(w)$ its local time at h (see (1.10)). Then,*

$$\mathbb{P}\left(\max_{[0,t]} z^\theta \ \leq\ h \mid \sigma(w)\right) \ =\ \exp\left(-2\theta\ l^h(w)(t)\right),$$

where $l^h(w)$ is the local time at h for the path $\Lambda_{0,h}(w)$ (see (1.10)).

2. Proof of Theorem 1.3 and Proposition 1.4

In the following, a nested sub-excursion of the function $f\in C_0^+(\mathbb{R}^+)$ will refer to any section of the path f on an interval $[t_-,t_+]$ such that $\forall t\in[t_-,t_+], \inf_{[t_-,t]} f = f(t-) = f(t_+)$ – see Fig 1.2. The height of such a sub-excursion is defined as $\max_{[t_-,t_+]} f - f(t_+)$.

Let $p_f(t)$ be the canonical projection from $\mathbb{R}^+$ to $\mathcal{T}_f$, which can be thought of as the position of the exploration particle at time t. By definition, $p_f(t)$ belongs to $\mathrm{Tr}^h(\mathcal{T}_f)$ if and only if there exists s such that $\inf_{[t\wedge s, t\vee s]} f = f(t)$ and $f(s)-f(t)\geq h$, which is equivalent to saying that t is the ending time or starting time of a sub-excursion (nested in f) of height at least h starting from level $f(t)$. We claim that the extreme points of $\mathrm{Tr}^h(\mathcal{T}_f)$ – or *leaves* – are contained in the set of points of the form $z=p_f(t)$, where t is the time extremity of a sub-excursion of height *exactly* h.

In order to see that, let t be the time extremity of a sub-excursion of height strictly larger than h. By continuity, this sub-excursion must contain another sub-excursion of height greater or equal to h. Thus, $p_f(t)$ must have at least one descendant and can not be a leaf. As claimed earlier, this shows that the leaves must be visited at the time extremities of some sub-excursion of height *exactly* h. [2]

Let us now define inductively $\{\tau_n(f)\}_{n\geq 0} \equiv \{\tau_n\}_{n\geq 0}$, $\{\theta_n(f)\}_{n\geq 1} \equiv \{\theta_n\}_{n\geq 1}$ and $\{\sigma_n(f)\}_{n\geq 0} \equiv \{\sigma_n\}_{n\geq 0}$ as follows : $\tau_0 = 0$, $\sigma_0 = 0$ and

$$\begin{aligned} \theta_{n+1} &= \inf\{s > \tau_n : \ f(s) - h = \inf_{[\tau_n, s]} f\}. \\ \tau_{n+1} &= \inf\{s > \theta_{n+1} : \sup_{[\theta_{n+1}, s]} f = f(t) + h\}, \\ \sigma_{n+1} &= \sup\{s \in [\tau_n, \tau_{n+1}] : f(s) = \inf_{[\tau_n, s]} f\}. \end{aligned} \tag{2.1}$$

with the convention that $\inf\{\emptyset\}, \sup\{\emptyset\} = \infty$. See also Fig. 1.3.

As already noted in Neveu and Pitman (1989) (although under a slightly different form), the sequences $\{\tau_n\}_{\{n\,:\,\tau_n<\infty\}}$ and $\{\sigma_n\}_{\{n\,:\,\sigma_n<\infty\}}$ play a key role in the tree $\mathrm{Tr}^h(\mathcal{T}_f)$, being respectively related to the exploration times for the leaves and the branching points respectively.

First, the reader can easily convince herself that the set of finite τ_n's coincide with the completion times of all the sub-excursions nested in the function f which are exactly of height h (see Neveu and Pitman (1989) for more details). As already discussed, this implies that $\{p_f(\tau_n)\}_{\{n:\tau_n<\infty\}}$ contains the set of leaves of the tree $\mathrm{Tr}^h(\mathcal{T}_f)$.

Secondly, the very definition of the σ_i's implies that for every $m < n$, $\inf_{[\tau_m,\tau_n]} f = f(\sigma_k)$ for some $k \in \{m+1, \cdots, n\}$. From (1.1), this implies that $p_f(\tau_n) \wedge p_f(\tau_m)$ – the most recent common ancestor of $p_f(\tau_n)$ and $p_f(\tau_m)$ – is given by some $p_f(\sigma_k)$.

We now show that the times $\sigma_n(f)$ and $\tau_n(f)$ also appear quite naturally in the two-sided Skorohod reflection. Recall from the introduction, that t_n (resp., s_n) refers to the n^{th} returning time (exit time) of $\Lambda_{0,h}(f)$ at level 0 (see (1.7)).

Proposition 2.1. *For every continuous function f with $f(0) = 0$ and for every $n \geq 1$,*

(1) *$\tau_n(f)$ is the n^{th} returning time to level 0 of $\Lambda_{0,h}(f)$, i.e., $\tau_n(f) = t_n(f)$.*
(2) *$\sigma_n(f)$ is the n^{th} exit time at level 0 of $\Lambda_{0,h}(f)$, i.e., $\sigma_n(f) = s_n(f)$.*
(3) *The function $f_h = f - \Lambda_{0,h}(f)$ is non-decreasing (resp., non-increasing) on $[\sigma_{n+1}(f), \tau_{n+1}(f)]$ (resp., on $[\tau_n(f), \sigma_{n+1}(f)]$). In particular, $\{f_h(\sigma_i)\}$ (resp., $\{f_h(\tau_i)\}$) coincide with the local minima (resp., local maxima) of f_h.*

See also Fig. 1.3 for a pictorial representation of the next proposition. As we shall see, this proposition is a consequence of elementary results on the two-sided Skorohod reflection that we now expose. We start by introducing some notations:

$$\forall f,\ \forall t > 0,\ R^T(f)(t) \ := \ 1_{t\geq T}\left(f(t) - f(T)\right).$$

In other words, $R^T(f)$ is constant on $[0, T]$ and follows the variation of f afterwards. The next elementary lemma states that the reflection of a path can be obtained

[2] Note that we only have an inclusion. For example, let t be the starting time of a sub-excursion of height exactly h, and let t_e be the ending time of this excursion. If t_e is also the starting time of another sub-excursion of size $> h$, then $p_f(t)$ is not a leaf of the h-trimming of the tree.

by successively reflecting the path up to some T and then reflecting the remaining portion of the path from T to ∞.

Lemma 2.2. *For any continuous function f with $f(0) \in [0,h]$ and $T \geq 0$,*

$$\begin{array}{lcl} \forall t < T, \quad \Lambda_{0,h}(f)(t) & = & \Lambda_{0,h}\left(f(\cdot \wedge T)\right)(t), \\ \forall t \geq T, \quad \Lambda_{0,h}(f)(t) & = & \Lambda_{0,h}\left(R^T(f) + \Lambda_{0,h}(f)(T)\right)(t). \end{array}$$

Proof: In the following, we write

$$L^T(f)(t) \quad := \quad R^T(f)(t) + \Lambda_{0,h}(f)(T),$$

and for any continuous function F with $F(0) \in [0,h]$, we denote by $(c^0(F), c^h(F))$ the pair of compensators solving the Skorohod equation for the two-sided reflection of F on the interval $[0,h]$. For $y = 0, h$, define

$$\forall t \geq 0, \quad \tilde{c}^y(t) \ := \ c^y(f(\cdot \wedge T))(t) \ + \ c^y(L^T(f))(t).$$

We will show that $(\tilde{c}^0, \tilde{c}^h)$ solves the two-sided Skorohod equation for f. We first need to prove that the function $G(t) := f(t) \ + \ \tilde{c}^h(t) + \tilde{c}^0(t)$ is valued in $[0,h]$. First

$$\begin{array}{lcl} G(t) & = & \left(f(t\wedge T) + \displaystyle\sum_{y=0,h} c^y(f(\cdot \wedge T))(t)\right) + \left(L^T(f)(t) + \displaystyle\sum_{y=0,h} c^y(L^T(f))(t)\right) \\ & & -\Lambda_{0,h}(f)(T) \\ & = & \Lambda_{0,h}(f(\cdot \wedge T))(t) \ + \ \Lambda_{0,h} \circ L^T(f)(t) \ - \Lambda_{0,h}(f)(T) \\ & = & \Lambda_{0,h}(f)(t \wedge T) \ + \ \Lambda_{0,h} \circ L^T(f)(t) - \Lambda_{0,h}(f)(T) \end{array}$$

where the first equality follows from the fact $f(t) = f(t \wedge T) + L^T(f)(t) - \Lambda_{0,h}(f)(T)$ and the last equality only states that the reflection of the function $f(\cdot \wedge T)$ (the function f "stopped" at T) is the reflection of f stopped at T (this can directly be checked from the definition of the two-sided Skorohod reflection).

The function $L^T(f)$ is constant and equal to $\Lambda_{0,h}(f)(T)$ on the interval $[0,T]$. This easily implies that its reflection is also identically $\Lambda_{0,h}(f)(T)$ on the same interval. Thus, the latter equality implies that

$$G(t) := \begin{cases} \Lambda_{0,h}(f)(t) & \text{if } t < T, \\ \Lambda_{0,h} \circ L^T(f)(t) & \text{otherwise.} \end{cases} \tag{2.2}$$

(2.2) implies that $G(t)$ belongs to $[0,h]$, hence proving that the first requirement of the Skorohod equation (see Theorem 1.1) is satisfied. The second requirement – the function $\tilde{c}^h$ (resp., $\tilde{c}^0$) non-increasing (resp., non-decreasing) – is obviously satisfied since the function $\tilde{c}^h$ (resp., $\tilde{c}^0$) is constructed out of a compensator at h (resp., at 0). Finally, for $y = 0, h$, we need to show that the support of the measure $d\tilde{c}^y$ is included in the set $G^{-1}(\{y\})$. In order to see that, we use the fact that the support of $dc^y(L^T(f))$ and $dc^y(f(\cdot \wedge T))$ are respectively included in $[T, \infty]$ and $[0,T]$ — using the fact that if a function g is constant on some interval, its compensator does not vary on this interval. As a consequence, for $y = 0, h$

$$\mathrm{Supp}(\tilde{c}^y) \cap [0,T] \ = \ \mathrm{Supp}(c^y(f(\cdot \wedge T))) \cap [0,T]$$

where Supp denotes the support of the measure under consideration. Further, $\mathrm{Supp}(c^y(f(\cdot \wedge T)) \subset \{t \ : \ \Lambda_{0,h}(f(\cdot \wedge T))(t) = y\}$. Since $\Lambda_{0,h}(f(\cdot \wedge T))$ and G coincides on $[0,T]$ (by (2.2)), we get that on $[0,T]$ the compensator $\tilde{c}^y_t$ only varies on $G^{-1}(\{y\})$. By an analogous argument, one can show that the same holds on the interval $[T, \infty]$. Hence, the third and final requirement of the Skorohod equation holds for $\tilde{c}^y$, $y = 0, h$. This shows that $(\tilde{c}^0, \tilde{c}^h)$ solves the two-sided Skorohod reflection. Combining this with (2.2) ends the proof of our lemma. ∎

Let $h \geq 0$. For any continuous function f with $f(0) \leq h$, let us define the one-sided reflection (with downward push) at h – denoted by $\Gamma^h(f)$ – as

$$\Gamma^h(f) := f - (\sup_{[0,t]} f - h) \vee 0.$$

Along the same lines as the one-sided reflection at 0 (as introduced in (1.3)), the function $c(t) = -(\sup_{[0,t]} f - h) \vee 0$ can be interpreted as the minimal amount of downward push necessary to keep the path f below level h. More precisely, this function is easily seen to be the only continuous function c with $c(0) = 0$ satisfying the following requirements: (1) $f + c \leq h$, (2) c is non-increasing and, (3) c does not vary off the set $\{t \geq 0 \ : \ f(t) + c(t) = h\}$.

Lemma 2.3. *For every $T \geq 0$ and every continuous function F with $F(0) \in [0, h]$ such that $\Gamma^h(F) \geq 0$ (resp., $\Gamma^0(F) \leq h$) on $[0, T]$, we must have*

$$\forall t \in [0, T], \quad \Lambda_{0,h}(F)(t) \ = \ \Gamma^h(F)(t) \quad (\textit{resp.}, \ \Lambda_{0,h}(F)(t) \ = \ \Gamma^0(F)(t)).$$

Proof: Let us consider a continuous F with $F(0) \in [0, h]$ and such that $\Gamma^h(F) \geq 0$ on $[0, T]$. Let us prove that $\Lambda_{0,h}(F) = \Gamma^h(F)$ on $[0, T]$. We aim at showing that $(0, -(\sup_{[0,t]} F - h)^+)$ coincides with the pair of compensators of F on the time interval $[0, T]$. First,

$$\Gamma^h(F) \ = \ F \ + \ 0 \ + (-(\sup_{[0,t]} F - h) \vee 0)$$

belongs to $[0, h]$ since $\Gamma^h(F) \leq h$ and under the conditions of our lemma $\Gamma^h(F) \geq 0$. Secondly, using the fact that $-(\sup_{[0,t]} F - h) \vee 0$ is the compensator for the one-sided case (at h), this function is non-increasing and only decreases when $\Gamma^h(F)$ is at level h. This shows that $\Gamma^h(F)$ coincides with the two sided reflection of f on the interval $[0, h]$. The case $\Gamma^0(F) \leq h$ can be handled similarly. ∎

Proof of Proposition 2.1: In order to prove Proposition 2.1, we will now proceed by induction on n.

Step 1. We first claim that $\sigma_1 \leq \theta_1$. When $\theta_1 = \infty$, this is obvious. Let us assume that $\theta_1 < \infty$ and let us assume that $\sigma_1 > \theta_1$. The definition of θ_1 implies that $\Gamma^0(f)(\theta_1) = h$ and thus θ_1 belongs to an excursion of $\Gamma^0(f)$ away from 0 (of height at least h), whose time interval we denote by $[t_-, t_+]$. Since σ_1 was defined as the last visit at 0 of $\Gamma^0(f)$ before time τ_1 (see (2.1)) and σ_1 is assumed to be greater than θ_1, $\sigma_1 \geq t_+$ and the excursion of $\Gamma^0(f)$ on $[t_-, t_+]$ must be completed before τ_1. On the other hand,

$$\begin{aligned} h &= \left(f(\theta_1) - \inf_{[0,\theta_1]} f \right) - \left(f(t_+) - \inf_{[0,t_+]} f \right) \\ &= f(\theta_1) - f(t_+) \\ &\leq \sup_{[\theta_1, t+]} f - f(t_+), \end{aligned}$$

where we used the fact that $\inf_{[0,t]} f$ must remain constant during an excursion of $\Gamma^0(f)$ away from 0 in the second equality. By continuity of f, there must exist $s \in [\theta_1, t_+]$ such that $\sup_{[\theta_1,s]} f - f(s) = h$, which implies that $\tau_1 \le t_+$, thus yielding a contradiction and proving that $\sigma_1 \le \theta_1$.

Next, the strategy for proving the first step of our proposition consists in breaking the intervals $[0,\tau_1]$ into three pieces: $[0,\sigma_1]$, $[\sigma_1,\theta_1]$ and $[\theta_1,\tau_1]$. In the following, we assume that $\tau_1 < \infty$. The complementary case is obvious.

First, on $[0,\sigma_1]$, we must have $\Gamma^0(f) < h$ since $\sigma_1 < \theta_1$, and θ_1 was defined as the first time $\Gamma^0(f)(t) = h$. By Lemma 2.3, this implies that

$$\forall t \in [0,\sigma_1],\ \Lambda_{0,h}(f)(t) = \Gamma^0(f)(t) = f - \inf_{[0,t]} f, \quad \text{and } \Lambda_{0,h}(f)(\sigma_1) = 0, \tag{2.3}$$

where the latter equality follows directly from the definition of σ_1. Next by Lemma 2.2, we must have

$$\forall t \in [\sigma_1,\infty],\ \Lambda_{0,h}(f)(t) \quad = \quad \Lambda_{0,h}(1_{\cdot \ge \sigma_1}(f - f(\sigma_1))(t).$$

Using the fact that $\inf_{[0,t]} f$ remains constant during an excursion of $\Gamma^0(f)$ away from 0 and the fact that $f - \inf_{[0,\cdot]} f < h$ on $[0,\theta_1]$, it is easy to see that θ_1 coincides with the first visit of $1_{\cdot \ge \sigma_1}(f(\cdot) - f(\sigma_1))$ at h. Further, since σ_1 is the *last* visit at 0 of $\Gamma^0(f)$ before τ_1, we must have

$$\forall t \in (\sigma_1,\tau_1), \quad f(t) - \inf_{[0,t]} f = f(t) - f(\sigma_1) > 0.$$

In particular, $f - f(\sigma_1) \in (0,h]$ on the interval $(\sigma_1,\theta_1]$ and thus, $(0,0)$ solves the Skorohod equation for $1_{\cdot \ge \sigma_1}(f - f(\sigma_1))$ on this interval. This yields

$$\forall t \in (\sigma_1,\theta_1),\ \Lambda_{0,h}(f)(t) \ = f(t) - f(\sigma_1) > 0. \tag{2.4}$$

Finally, we look at $\Lambda_{0,h}(f)$ on $[\theta_1,\tau_1]$. Using $\Lambda_{0,h}(f)(\theta_1) = h$, Lemma 2.2 implies that

$$\forall t \in [\theta_1,\tau_1], \quad \Lambda_{0,h}(f)(t) \quad = \quad \Lambda_{0,h}(h + 1_{\cdot \ge \theta_1}(f(\cdot) - f(\theta_1)).$$

A straightforward computation yields

$$\forall t \le \tau_1,\ \Gamma^h\left(h + 1_{\cdot \ge \theta_1}(f(\cdot) - f(\theta_1)\right)(t) \quad = \quad h + 1_{t \ge \theta_1}(f(t) - \sup_{[\theta_1,t]} f)$$

By definition of τ_1, the RHS of the equality must remain positive on $[\theta_1,\tau_1)$. Using Lemma 2.3, we get that

$$\begin{aligned}\forall t \in [\theta_1,\tau_1), \quad \Lambda_{0,h}(f)(t) \ = \ \Gamma^h(f)(t) \ &= f(t) + h - \sup_{[\theta_1,t]} f > 0, \\ \text{and } \ \Lambda_{0,h}(f)(\tau_1) \ &= 0\end{aligned} \tag{2.5}$$

where the second equality follows from the very definition of τ_1. Finally, combining (2.3)–(2.5) yields that $\Lambda_{0,h}(f)$ is continuous on $[0,\tau_1]$, and further that

$$\begin{aligned}\forall t \in [0,\tau_1],\ \Lambda_{0,h}(f)(t) \quad = \quad f \ - \ \Bigg(& 1_{t\in[0,\sigma_1)} \cdot \inf_{[0,t]} f \ + \ 1_{t\in[\sigma_1,\theta_1)} \cdot f(\sigma_1) \\ & + \ 1_{t\in[\theta_1,\tau_1]} \cdot (\sup_{[\theta_1,t]} f - h)\Bigg).\end{aligned}$$

As a consequence, $f_h = f - \Lambda_{0,h}(f)$ is non-increasing (resp., non-decreasing) on $[0,\sigma_1]$ (resp., $[\sigma_1,\tau_1]$). Furthermore, the argument above also shows that τ_1 (resp.,

σ_1) is the first returning time (resp., exit time) at level 0. Indeed, piecing together the previous results, we proved

$$\Lambda_{0,h}(f)(t) < h, \text{ on } [0, \sigma_1) \text{ and } \Lambda_{0,h}(f)(\sigma_1) = 0$$
$$\Lambda_{0,h}(f) > 0 \text{ on } (\sigma_1, \tau_1), \text{ and } \Lambda_{0,h}(f)(\theta_1) = h, \ \Lambda_{0,h}(f)(\tau_1) = 0.$$

Step n+1. Let us assume Proposition 2.1 is valid up to rank n. Recall that $R^{\tau_n}(f) = 1_{t \geq \tau_n}(f(t) - f(\tau_n))$. By Lemma 2.2,

$$\forall t \in [\tau_n, \infty), \ \Lambda_{0,h}(f) = \Lambda_{0,h}(R^{\tau_n}(f)),$$

where we used the induction hypothesis to write $\Lambda_{0,h}(f)(\tau_n) = 0$. On the other hand, it is straightforward to check from the definitions of τ_{n+1}, θ_{n+1} and σ_{n+1} in (2.1) that

$$\sigma_{n+1}(f) = \sigma_1(R^{\tau_n}(f)), \quad \tau_{n+1}(f) = \tau_1(R^{\tau_n}(f)), \quad \theta_{n+1}(f) = \theta_1(R^{\tau_n}(f))$$

Applying the case $n = 1$ to the function $R^{\tau_n}(f)$ immediately implies that our proposition is valid at step $n + 1$.

∎

In order to prove Theorem 1.3, we will combine Proposition 2.1 with the following lemma.

Lemma 2.4. *Let $(\mathcal{T}_1, d_1)$ and $(\mathcal{T}_2, d_2)$ be two rooted real trees with only finitely many leaves. For $k = 1, 2$, let $S_k = (z_1^k, \cdots, z_N^k) \in \mathcal{T}_k$ such that the two following conditions hold.*

(1) *For $k = 1, 2$, S_k contains the leaves of $\mathcal{T}_k$.*
(2)
$$\forall i \leq N, \ d_1(\rho^1, z_i^1) = d_2(\rho^2, z_i^2) \text{ and } \forall i, j \leq N, \ d_1(\rho^1, z_i^1 \wedge z_j^1) = d_2(\rho^2, z_i^2 \wedge z_j^2),$$
where ρ^k is the root of $\mathcal{T}_k$.

Under those conditions, there exists a root preserving isometry from $\mathcal{T}_1$ onto $\mathcal{T}_2$.

Proof: For $k = 1, 2$ and $m \leq N$, let $I_m^k := [\rho^k, z_m^k]$. Using the second assumption of our lemma, the two ancestral lines I_m^1 and I_m^2 must have the same length. From there, it easy to construct a root preserving isometry from I_m^1 onto I_m^2 as follows. Since $\mathcal{T}_k$ is a tree, there exists a unique isometric map ψ_m^k from $[0, d_k(\rho^k, z_m^k)]$ onto I_m^k, such that $\psi_m^k(0) = \rho^k$ and $\psi_m^k(d_k(\rho^k, z_m^k)) = z_m^k$. Define

$$\forall a \in I_m^1, \quad \phi_m(a) := \psi_m^2 \circ (\psi_m^1)^{-1}(a).$$

Since $d_1(\rho^1, z_m^1) = d_2(\rho^2, z_m^2)$, it is straightforward to show that ϕ_m defines an isometry from I_m^1 onto I_m^2. Furthermore, ϕ_m preserves the root, $\phi_m(z_m^1) = z_m^2$ and it is order preserving, i.e., $\forall a \preceq b \in I_m^1, \ \phi_m(a) \preceq \phi_m(b)$.

Next, we claim that if $a \in I_m^1 \cap I_l^1$, then $\phi_m(a) = \phi_l(a)$. We first show the property for $a = z_m^1 \wedge z_l^1$. Using the isometry of ϕ_m and the root preserving property, we have

$$\begin{aligned} d_2(\rho^2, \phi_m(z_l^1 \wedge z_m^1)) &= d_1(\rho^1, z_l^1 \wedge z_m^1) \\ &= d_2(\rho^2, z_l^2 \wedge z_m^2), \end{aligned}$$

where the second equality follows from the second assumption of our lemma. Since $\phi_m(a) \in I_m^2$, it follows that $\phi_m(z_l^1 \wedge z_m^1) = z_l^2 \wedge z_m^2$ – on the segment I_m^2, a point is uniquely determined by its distance from the root. By the same reasoning, we get that $\phi_l(z_l^1 \wedge z_m^1) = z_l^2 \wedge z_m^2$. Let us now take any point $a \in I_m^1 \cap I_l^1$. Under this assumption, we must have $a \preceq z_m^1 \wedge z_l^1$. Using the isometry property, this implies

$$d_2(\rho^2, \phi_m(a)) \ = \ d_2(\rho^2, \phi_l(a))$$

and using the order preserving property

$$\phi_l(a), \phi_m(a) \preceq z_m^2 \wedge z_l^2,$$

since we showed that $\phi_m(z_m^1 \wedge z_l^1), \phi_l(z_m^1 \wedge z_l^1) = z_m^2 \wedge z_l^2$. It follows that $\phi_m(a) = \phi_l(a)$, as claimed earlier.

We are now ready to construct the isometry from $\mathcal{T}_1$ onto $\mathcal{T}_2$. First, for $k = 1, 2$, any point $a_k \in \mathcal{T}_k$ must belong to some ancestral line of the form $[\rho^k, l]$, for some leaf l in the tree $\mathcal{T}_k$. By what we just proved, and since S_1 contains all the leaves of $\mathcal{T}_1$, we can define a map ϕ from $\mathcal{T}_1$ into $\mathcal{T}_2$ as follows

$$\forall a \in \mathcal{T}_1, \quad \phi(a) \ := \ \phi_m(a) \ \text{ if } a \in I_m^1.$$

Since S_2 contains all the leaves of $\mathcal{T}_2$, and any ancestral line of the form $[\rho^1, z_m^1]$ is mapped onto $[\rho^2, z_m^2]$, the map ϕ is onto.

It remains to show that ϕ is isometric. Let $a, b \in \mathcal{T}_1$ and let us distinguish between two cases. First, let us assume that a and b belong to the same ancestral line I_m^1 for some $m \leq N$. Under this assumption, the property simply follows from the isometry of ϕ_m. Let us now consider the case where a and b belong to two distinct ancestral lines: $a \in I_m^1$ but $a \notin I_l^1$, and $b \in I_l^1$ but $b \notin I_m^1$; in such a way that $a \wedge b = z_l^1 \wedge z_m^1$. Using the fact that both ϕ_m and ϕ_l are isometric and order preserving, and $\phi(z_l^1 \wedge z_m^1) = z_l^2 \wedge z_m^2$, $\phi_l(a) \in I_l^2, \phi_m(a) \in I_m^2$ we get that $\phi(a) \wedge \phi(b) = z_l^2 \wedge z_m^2$. We can then write $[\phi(a), \phi(b)]$ as the union $[z_l^2 \wedge z_m^2, \phi(a)] \cup [z_l^2 \wedge z_m^2, \phi(b)]$ and write

$$\begin{aligned}
d_2(\phi(a), \phi(b)) \ &= \ d_2(z_l^2 \wedge z_m^2, \phi(a)) + d_2(z_l^2 \wedge z_m^2, \phi(b)) \\
&= \ d_1(z_l^1 \wedge z_m^1, a) + d_1(z_l^1 \wedge z_m^1, b) \\
&= \ d_1(a, b)
\end{aligned}$$

where the second equality follows by applying the previous case to the pairs of points $(z_l^1 \wedge z_m^1, a)$ and $(z_l^1 \wedge z_m^1, b)$.

∎

In the following, we make the assumption that the h-trimming of the tree $\mathcal{T}_f$ is non-empty, i.e., that $\sup_{[0,\infty)} f \geq h$.

Proof of Theorem 1.3: Recall that $C_0^+(\mathbb{R}^+)$ denotes the set of continuous non-negative functions with $f(0) = 0$ and compact support. We start by showing the first item of our theorem, i.e., that $f \in C_0^+(\mathbb{R}^+)$ implies that $f_h \in C_0^+(\mathbb{R}^+)$. First, as an easy corollary of Proposition 2.1, we get that for every continuous $f \geq 0$, the function $f_h = f - \Lambda_{0,h}(f)$ is non-negative. This simply follows from the fact that the local minima of f_h are attained on the set $\{\sigma_i\}$, on which $f(\sigma_i) = f_h(\sigma_i)$ since $\Lambda_{0,h}(f)(\sigma_i) = 0$. Since $f(\sigma_i) \geq 0$, the function f_h is non-negative. Secondly, we show that $f \geq 0$ implies that $\mathrm{Supp}(f_h) \subset \mathrm{Supp}(f)$. In order to see that, let t be such that $f(t) = 0$. We have $f_h(t) = -\Lambda_{0,h}(f)(t)$ and since $f_h \geq 0$ and $-\Lambda_{0,h}(f) \leq 0$, it follows that $f_h(t) = 0$ (and $\Lambda_{0,h}(f)(t) = 0$[3]). As a consequence, if $f \in C_0^+(\mathbb{R}^+)$ then $f_h \in C_0^+(\mathbb{R}^+)$.

[3]This also shows that $\mathrm{Supp}(\Lambda_{0,h}(f)) \subset \mathrm{Supp}(f)$.

Next, let us show that f_h is such that the real tree $(\mathcal{T}_{f_h}, d_{f_h})$ is isometric to the h-trimming of the tree $(\mathcal{T}_f, d_f)$. is the contour function of the h-trimming of the tree $\mathcal{T}_f$ (up to a root preserving isometry). For $k \leq N_h(f)$, Proposition 2.1 immediately implies that the maximum of f_h on $[\sigma_k, \sigma_{k+1})$ is attained at time τ_k and that the set

$$I_k := \{t \in [\sigma_k, \sigma_{k+1}) \ : \ f_h(t) = f_h(\tau_k) \ \}$$

is a closed interval. On the one hand, any time $t \in [\sigma_k, \sigma_{k+1})$ outside of this interval is the starting or ending time of a sub-excursion with (strictly) positive height, and for such t, $p_{f_h}(t)$ can not be a leaf. On the other hand, we have $p_{f_h}(t) = p_{f_h}(t')$ for $t, t' \in I_k$. This implies that the only possible leaf of $\mathcal{T}_{f_h}$ visited during the time interval $[\sigma_k, \sigma_{k+1})$ is given by $p_{f_h}(\tau_k)$ and thus, that the set of leaves of $\mathcal{T}_{f_h}$ is included in the finite set of points $\{p_{f_h}(\tau_n)\}_{\{n \ : \ \tau_n < \infty\}}$.

As explained at the beginning of this section, any leaf of the tree $\mathrm{Tr}^h(\mathcal{T}_f)$ must be explored at some finite τ_n, i.e., the set of leaves of $\mathrm{Tr}^h(\mathcal{T}_f)$ is a subset of $\{p_f(\tau_n)\}_{\{n \ : \ \tau_n < \infty\}}$.

In order to prove our result, we use Lemma 2.4 with $z_i^1 = p_f(\tau_i)$ and $z_i^2 = p_{f_h}(\tau_i)$ and

$$N = N_h(f) = |\{n \ : \ \tau_n(f) < \infty\}|.$$

First, item 1 of Proposition 2.1 implies that the height of the vertices $p_f(\tau_i)$ and $p_{f_h}(\tau_i)$ are identical, i.e., that $f(\tau_i) = f_h(\tau_i)$ since $\Lambda_{0,h}(f)(\tau_i) = 0$. In order to show that $\mathrm{Tr}^h(\mathcal{T}_f)$ and $\mathcal{T}_{f_h}$ are identical (up to a root preserving isomorphism), it is sufficient to check that

$$\forall i < j, \quad \inf_{[\tau_i, \tau_j]} f = \inf_{[\tau_i, \tau_j]} f_h,$$

i.e., that the height of the most recent common ancestor of the vertices visited at τ_i and τ_j is the same in both trees. To justify the latter relation, we first note that the definition of the σ_m's (see (2.1)) implies that $\inf_{[\tau_i, \tau_j]} f$ must be attained at some σ_k (for some $k \in \{i+1, \cdots, j\}$). On the other hand, the third item of the Proposition 2.1 implies that the same must hold for f_h since the set of local minima of f_h coincide with $\{f_h(\sigma_i)\}$. Since $f(\sigma_i) = f_h(\sigma_i)$, ($s_i = \sigma_i$ by the second item of Proposition 2.1 and $\Lambda_{0,h}(f)(s_i) = 0$), Theorem 1.3 follows.

∎

Proof of Proposition 1.4: Let us define

$$\mathcal{T}_{f_h}^n \ := \ \{z \in \mathcal{T}_{f_h} : \ \exists t \leq t_n, \ p_{f_h}(t) = z\},$$

the set of vertices in $\mathcal{T}_{f_h}$ visited up to time t_n. $\mathcal{T}_{f_h}$ can be constructed recursively by adding to $\mathcal{T}_{f_h}^n$ all the vertices in $\mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^n$ for every $n < N$, where $N \equiv N_h(f)$. (Indeed, by definition of a real tree from its contour path, if a point is visited at a given time, its ancestral line must have been explored before that time. Thus, if all the leaves have been explored at a given time – e.g., at t_N–vertex has been visited at least once before that.) For every $n < N$, let us show that the set $\mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^n$ is a branch (more precisely, a segment $[a, b] \setminus \{a\}$ with $a, b \in \mathcal{T}_{f_h}$ and $a \preceq b$)

(i) with tip $b = p_{f_h}(t_{n+1})$ – i.e., $\forall z \in \mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^n, \ z \preceq p_{f_h}(t_{n+1})$ –

(ii) attached to $a = p_{f_h}(s_{n+1})$ – i.e., $\forall z \in \mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^n, \ p_{f_h}(s_{n+1}) \preceq z$.

(iii) $p_f(s_{n+1})$ belongs to the ancestral line $[\rho_{f_h}, p_{f_h}(t_n)]$.

Let $n < N$ and let $t \in (t_n, t_{n+1}]$. The definition of the real tree $\mathcal{T}_{f_h}$ implies that the point $p_{f_h}(t)$ has been visited before t_n if and only if there exists $s \leq t_n$ such that

$$\inf_{[s,t]} f_h = f_h(t) = f_h(s), \tag{2.6}$$

i.e., t must be the ending time of a sub-excursion straddling t_n. On the one hand, since f_h is non-increasing on $[t_n, s_{n+1}]$ (by Proposition 2.1), we have

$$\forall t \in [t_n, s_{n+1}], \ \inf_{[t_n,t]} f_h = f_h(t). \tag{2.7}$$

Since $f_h(0) = 0$ and $f_h(t) \geq 0$, one can find $s \leq t_n$ such that (2.6) is satisfied (using the continuity of f_h). Thus, every point visited on the time interval $[t_n, s_{n+1}]$ has already been visited before t_n and does not belong to $\mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^{n}$.

On the other hand, the function f_h is non-decreasing on $[s_{n+1}, t_{n+1}]$ (again by Proposition 2.1). Let us define

$$\bar{\theta}_{n+1} = \sup\{t \in [s_{n+1}, t_{n+1}] \ : \ f_h(t) = f_h(s_{n+1})\}$$

(with the convention $\sup\{\emptyset\} = t_{n+1}$). First, the definition of our real tree $\mathcal{T}_{f_h}$ implies that any $p_{f_h}(t)$ with $t \in [s_{n+1}, \bar{\theta}_{n+1}]$ coincides with $p_{f_h}(s_{n+1})$. Secondly, for any $t \in [\bar{\theta}_{n+1}, t_{n+1}]$, and any $s \leq t_n$

$$\inf_{[s,t]} f_h \leq f_h(s_{n+1}) < f_h(t)$$

which implies that any point visited during the interval $(\bar{\theta}_{n+1}, t_{n+1}]$ belongs to $\mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^{n}$. Furthermore, the previous inequality implies that

$$\forall t \in (\bar{\theta}_{n+1}, t_{n+1}], \quad p_{f_h}(t) \succ p_{f_h}(s_{n+1}). \tag{2.8}$$

Finally,

$$\forall t \in [\bar{\theta}_{n+1}, t_{n+1}], \ \inf_{[t,t_{n+1}]} f_h = f_h(t),$$

which implies that

$$\forall t \in [\bar{\theta}_{n+1}, t_{n+1}], \quad p_{f_h}(t) \preceq p_{f_h}(t_{n+1}). \tag{2.9}$$

Combining the results above, we showed the claims (i)–(iii) made earlier: $\mathcal{T}_{f_h}^{n+1} \setminus \mathcal{T}_{f_h}^{n}$ is a branch with tip $p_f(t_{n+1})$ (see (2.9)) attached at the point $p_{f_h}(s_{n+1})$ (see (2.8)), which belongs to $[\rho_{f_h}, p_{f_h}(t_n)]$ (see (2.7) applied to $t = s_{n+1}$). The length of the branch is given by

$$f_h(t_{n+1}) - f_h(s_{n+1}) = X_{n+1}(f), \tag{2.10}$$

(height of the $(n+1)^{th}$ leaf $-$ height of the attachment point) and the distance of the attachment point from the leaf $p_{f_h}(t_n)$ is given by

$$f_h(t_n) - f_h(s_{n+1}) = Y_{n+1}(f). \tag{2.11}$$

(Height of the n^{th} leaf $-$ height of the attachment point.) This completes the proof of our proposition.

∎

3. Proof of Theorem 1.5

Next, let e be a Brownian excursion conditioned on having a height greater than h and let $\{(X_n(e), Y_n(e))\}_{i\leq N_h(e)}$ be defined as in (1.8), i.e.,

$$\forall n \geq 1, \quad X_n(e) = \quad e_h(t_n) - e_h(s_n)$$
$$Y_n(e) = \quad e_h(t_{n-1}) - e_h(s_n),$$

with $t_n \equiv t_n(e)$, $s_n \equiv s_n(e)$ and let $N_h(e)$ be the number of returns of $\Lambda_{0,h}(e)$ at level 0. As discussed in the introduction (see the discussion preceding Theorem 1.5), in order to prove that the trimmed tree $\mathrm{Tr}^h(\mathcal{T}_e)$ is a binary tree, we need to show that $(X_1(e), \{(X_i(e), Y_i(e))\}_{2\leq i\leq N_h(e)})$ is identical in law with a sequence $(\tilde{X}_1, \{(\tilde{X}_i, \tilde{Y}_i)\}_{2\leq i\leq \tilde{N}})$, where $\tilde{X}_1, \tilde{Y}_2, \tilde{X}_2, \cdots$ is an infinite sequence of independent exponential random variables with mean $h/2$ and

$$\tilde{N} \ := \ \inf\{n \ : \ \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) < 0\}.$$

The idea of the proof consists in constructing a coupling between $(X_1(e), \{(X_i(e), Y_i(e))\}_{2\leq i\leq N_h(e)})$ and $(\tilde{X}_1, \tilde{Y}_2, \tilde{X}_2, \cdots)$ as follows. Let w be a Brownian motion with $w(0) = 0$, independent of the excursion e, and define

$$\tilde{w}(t) \ := \ e(t) + w\left((t - K(e)) \vee 0\right) \quad \text{where } K(e) := \sup\{t > 0 : \ e(t) > 0\}, \tag{3.1}$$

obtained by pasting the process w at the end of the excursion e. Finally, for $n \geq 1$, define $\tilde{X}_n := X_n(\tilde{w})$ and $\tilde{Y}_n := Y_n(\tilde{w})$.

First, the support of $\Lambda_{0,h}(e)$ is included in the support of e. (This was established in the course of the proof of Theorem 1.3.) As a consequence, for every $n \leq N_h(e)$, we must have $t_n(e), s_n(e) \leq K(e)$ (recall that for $n \leq N_h(e)$, $t_n(e)$ and $s_n(e)$ coincide with the n^{th} *finite* returning and exit times at 0). Since e and $\tilde{w}$ (and their reflections) coincide up to $K(e)$, this implies that $s_n(e) = s_n(\tilde{w}), t_n(e) = t_n(\tilde{w})$ and that $X_n(e) = \tilde{X}_n, Y_n(e) = \tilde{Y}_n$ for $n \leq N_h(e)$. Theorem 1.5 is then a direct consequence of our coupling and the two following lemmas.

Lemma 3.1. (1) *$\tilde{X}_1, \tilde{Y}_2, \tilde{X}_2, \tilde{Y}_3, \cdots$ is an i.i.d. sequence of independent exponential variables with parameter $h/2$. Further, $\tilde{Y}_1 = 0$.*
(2) *For $i \geq 1$,*
- $\tilde{X}_i \ = \ l^h(\tilde{w})(\tilde{t}_i) - l^h(\tilde{w})(\tilde{t}_{i-1})$,
- $\tilde{Y}_i \ = \ l^0(\tilde{w})(\tilde{t}_i) - l^0(\tilde{w})(\tilde{t}_{i-1})$,

where $\tilde{t}_i := t_i(\tilde{w})$.

Lemma 3.2. *Under our coupling,*

$$N_h(e) = \ \inf\{n : \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) < 0\} \quad a.s..$$

Proof of Lemma 3.1: Let us first prove that $\tilde{Y}_1 = 0$ and that $\tilde{Y}_1 = l^0(\tilde{w})(\tilde{t}_1) - l^0(\tilde{w})(\tilde{t}_0)$. Let

$$\bar{T}_1 := \inf\{t : \ \tilde{w}(t) = e(t) = h\}.$$

Since e is a Brownian excursion with height larger than h, $\bar{T}_1 < \infty$ and $\tilde{w} \in (0, h]$ on $(0, \bar{T}_1]$. From there, it immediately follows that $\Lambda_{0,h}(\tilde{w}) = \tilde{w}$ on $[0, \bar{T}_1]$ and that $\bar{T}_1$ is the first returning time at level h for the reflected process, i.e., $\bar{T}_1 = T_1(\tilde{w})$

(see (1.7) for a definition of $T_1(\tilde w)$). Further, $\tilde s_1$ – the first exit time of $\Lambda_{0,h}(\tilde w)$ at level 0 – is equal to 0. Since

$$\tilde Y_1 \quad = \quad -(c^0+c^h)(\tilde w)(0) + (c^0+c^h)(\tilde w)(\tilde s_1),$$

this implies $\tilde Y_1 = 0$. Finally, we also get that $l^0(\tilde w)(\tilde t_1) - l^0(\tilde w)(\tilde t_0) = 0$, since $\tilde t_1$ coincides with the first returning time of the reflected process at 0, and this process never hits 0 on the interval $(0,\tilde t_1)$.

Before proceeding with the rest of the proof, we start with a preliminary discussion. Let w' be a one-dimensional Brownian motion starting at $x\in[0,h]$. Recall that the one-sided Skorohod reflection $\Gamma^0(w')$ is distributed as the absolute value of a standard Brownian motion, and the compensator $c(w')(t) = -(\inf_{[0,t]} w \wedge 0)$ is the local time at 0 of $\Gamma^0(w')$. A proof of this statement can be found in Karatzas and Shreve (1991). By following the exact same steps, one can prove an analogous statement for the two-sided case, i.e., that for any Brownian motion w' starting at some $x\in[0,h]$, $\Lambda_{0,h}(w')$ is identical in law with $\lambda_{0,h}(w')$ (the standard reflection of the Brownian motion w' – see Section 1 in the discussion preceding Theorem 1.6 for a description of $\lambda_{0,h}(w')$) and that $-c^h(w')$ and $c^0(w')$ are respectively the local times at h and 0 of this process.

Next, let us define $\tilde t_n = t_n(\tilde w)$ and $\tilde s_n = s_n(\tilde w)$ and recall that the h-cut $\tilde w_h$ is defined as

$$\tilde w_h = \tilde w - \Lambda_{0,h}(\tilde w) = -c^0(\tilde w) - c^h(\tilde w).$$

By definition of $\tilde X_n$, we have

$$\begin{aligned} \tilde X_n &= \tilde w_h(\tilde t_n) - \tilde w_h(\tilde s_n) \\ &= c^h(\tilde w)(\tilde s_n) - c^h(\tilde w)(\tilde t_n) \\ &= c^h(\tilde w)(\tilde t_{n-1}) - c^h(\tilde w)(\tilde t_n). \end{aligned} \tag{3.2}$$

The second line follows from the fact that $c^0(\tilde w)$ does not vary off the set $\Lambda_{0,h}(\tilde w)^{-1}(\{0\})$ and $\Lambda_{0,h}(\tilde w) > 0$ on $(\tilde s_n, \tilde t_n)$; the third line is a consequence of the fact that $c^h(\tilde w)$ does not vary off the set $\Lambda_{0,h}(\tilde w)^{-1}(\{h\})$ and $\Lambda_{0,h}(\tilde w) < h$ on $(\tilde t_{n-1}, \tilde s_n)$. By an analogous argument, one can prove that

$$\tilde Y_n \quad = \quad c^0(\tilde w)(\tilde t_n) - c^0(\tilde w)(\tilde t_{n-1}). \tag{3.3}$$

With those results at hand, we are now ready to prove our lemma. In the first paragraph, we already argued that $\Lambda_{0,h}(\tilde w) = \tilde w$ on $[0,\bar T_1]$. By Lemma 2.2,

$$\begin{aligned} \forall t\geq 0,\ r(t) &:= \Lambda_{0,h}(\tilde w)(t+\bar T_1) \\ &= \Lambda_{0,h}(\tilde w(\cdot+\bar T_1))(t). \end{aligned}$$

By William's decomposition of a Brownian excursion conditioned on having a height larger than 1, the process $\tilde w(\cdot+\bar T_1)$ is distributed as a standard Brownian motion starting at level 1. Hence, the discussion above implies that the path r is identical in law with a reflected Brownian motion (where the reflection is a two-sided "standard reflection") starting at level h. Further, the compensators $c^0(\tilde w)(\bar T_1+\cdot)$ and $-c^h(\tilde w)(\bar T_1+\cdot)$ are the local times at 0 and h for the process r. Using (3.2)–(3.3), we easily obtain that $\tilde X_n$ (resp., $\tilde Y_n$) is the local time accumulated at h (resp., 0), for $n\geq 1$ (resp., $n\geq 2$) on $[\tilde t_{n-1},\tilde t_n]$. This completes the proof of the second part of our lemma. Finally, by standard excursion theory, $\tilde X_1, \tilde Y_2, \tilde X_2, \tilde Y_3, \cdots$ are i.i.d. exponential random variables with mean $h/2$. (Independence follows from the strong Markov property, whereas $\tilde X_i$ and $\tilde Y_{i+1}$ are distributed as the amount of Brownian local time accumulated at 0 before occurrence of an excursion of height larger or equal to h.)

∎

Proof of Lemma 3.2: Recall that

$$\tilde{X}_n = \tilde{w}_h(\tilde{t}_n) - \tilde{w}_h(\tilde{s}_n) \text{ and } \tilde{Y}_n = \tilde{w}_h(\tilde{t}_{n-1}) - \tilde{w}_h(\tilde{s}_n),$$

where we wrote $\tilde{t}_n = t_n(\tilde{w})$, $\tilde{s}_k = s_k(\tilde{w})$. Thus,

$$\begin{aligned}
\tilde{w}_h(\tilde{s}_{n+1}) &= [\sum_{i=1}^{n}(\tilde{w}_h(\tilde{t}_i) - \tilde{w}_h(\tilde{t}_{i-1}))] \; - \; (\tilde{w}_h(\tilde{t}_n) - \tilde{w}_h(\tilde{s}_{n+1})) \\
&= [(-\tilde{Y}_1 + \tilde{X}_1) + (-\tilde{Y}_2 + \tilde{X}_2) + \cdots + (-\tilde{Y}_n + \tilde{X}_n)] \; - \tilde{Y}_{n+1} \\
&= -\tilde{Y}_1 + \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) \\
&= \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}),
\end{aligned}$$

where the last equality follows from the fact that $\tilde{Y}_1 = 0$ (see item 1 of the previous lemma).

Let us now show that $N_h(e) = \inf\{n : \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) < 0\}$ a.s.. First, let us take $n < N_h(e)$. On the one hand, we already argued that $\tilde{s}_{n+1} \leq K(e)$. On the other hand, $\tilde{w}_h \geq 0$ on $[0, K(e)]$ since $e_h \geq 0$ (by Theorem 1.3) and that e_h and $\tilde{w}_h$ coincide up to $K(e)$. Thus,

$$\tilde{w}_h(\tilde{s}_{n+1}) = \sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_{i+1}) \geq 0.$$

Conversely, let us take $n = N_h(e)$. By Proposition 2.1, $\tilde{w}_h$ attains a minimum at $\tilde{s}_{\tilde{N}_h(e)+1}$ on the interval $[\tilde{t}_{N_h(e)}, \tilde{t}_{N_h(e)+1}]$. Since $K(e) \in [\tilde{t}_{N_h(e)}, \tilde{t}_{N_h(e)+1}]$, we get that

$$\begin{aligned}
0 = \tilde{w}_h(K(e)) &\geq \tilde{w}_h(\tilde{s}_{N_h(e)+1}) \\
&= \sum_{i=1}^{N_h(e)}(\tilde{X}_i - \tilde{Y}_{i+1}).
\end{aligned}$$

Since $\tilde{X}_i$ and $\tilde{Y}_i$ are exponential random variables, this inequality is strict almost surely. This completes the proof of the lemma. ∎

4. Proof of Theorem 1.6

Let (z^θ, w) be a weak solution of (1.11). Our proof builds on the approach of Warren (2002). In this work, it is proved that the pair (z^θ, w) can be constructed by adding some extra noise to the reflected process

$$\xi(t) := w(t) - \inf_{[0,t]} w$$

as follows. First, there exists a unique σ-finite measure – here denoted by $\mathcal{L}_\xi$ and referred to as the branch length measure – on the metric space $(\mathcal{T}_\xi, d_\xi)$ such that

$$\forall a, b \in \mathcal{T}_\xi, \quad \mathcal{L}_\xi([a,b]) \;=\; d_\xi(a,b).$$

(See Evans (2005) for more details). Conditioned on a realization of ξ, let us now consider the Poisson point process on $(\mathcal{T}_\xi, d_\xi)$ with intensity measure $2\theta\mathcal{L}_\xi$ and define the pruned tree

$$\mathcal{T}_\xi^\theta \ := \ \{z \in \mathcal{T}_\xi \ : \ [\rho_\xi, z) \text{ is unmarked}\},$$

obtained after removal of every vertex with a marked ancestor along its ancestral line. Finally, define $z^\theta(t)$ as the distance of the vertex $p_\xi(t)$ from the subset $\mathcal{T}_\xi^\theta$, i.e.,

$$z^\theta(t) \ := \ \begin{cases} 0 \text{ if } p_\xi(t) \in \mathcal{T}_\xi^\theta, \\ \xi(t) - A(t) \text{ otherwise}, \end{cases} \tag{4.1}$$

where $A(t) = 0$ if there is no mark along the ancestral line $[\rho_\xi, p_\xi(t)]$, and is equal to the height of the first mark (counted from the root) on $[\rho_\xi, p_\xi(t)]$ otherwise. Informally, $(z^\theta(t); t \geq 0)$ can be thought of as the exploration process above the pruned tree $\mathcal{T}_\xi^\theta$ – see Fig 4.5.

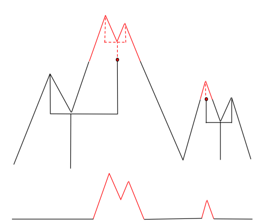

Figure 4.5. *The top panel displays a reflected random walk ξ with marking of the underlying tree $\mathcal{T}_\xi$. $\mathcal{T}_\xi^\theta$ is the black subset of the tree. The bottom panel displays the sticky path $(z^\theta(t); t \geq 0)$ obtained by concatenating the contour paths of the red subtrees attached to $\mathcal{T}_\xi^\theta$.*

Theorem 4.1 (Warren (2002))**.** *The process $(z^\theta(t), w(t);\ t \geq 0)$ is a weak solution of the SDE (1.11).*

Using this result, we proceed with the proof of Theorem 1.6. Let $\mathcal{T}_\xi^{(t)} := \mathcal{T}_\xi \cap \{p_\xi(s) :\ s \leq t\}$ be the sub-tree consisting of all the vertices in $\mathcal{T}_\xi$ visited up to time t. For every s, the set

$$\{x \in \mathcal{T}_\xi \ : \ x \preceq p_\xi(s) \text{ and } d_\xi(x, p_{\xi(s)}) \geq h\}.$$

is totally ordered. We define $a_h(s)$ as the sup of this set, with the convention that $\sup\{\emptyset\} = \rho_\xi$. Informally, $a_h(s)$ is the ancestor of $p_\xi(s)$ at a distance h. Following the construction of the pair (z^θ, ξ) described earlier, and since ρ_ξ is unmarked with probability 1, the events $\{\sup_{[0,t]} z^\theta \leq h\}$ and

$$\{\forall s \leq t, \quad [\rho_\xi, a_h(s)] \quad \text{is unmarked}\}$$

coincide a.s.. Further, the latter event is easily seen to be equivalent to not finding any mark on the h-trimming of the tree $\mathcal{T}_\xi^{(t)}$. Thus, by a standard result about Poisson point processes, we have

$$\begin{aligned} \mathbb{P}(\sup_{[0,t]} z^\theta \leq h \mid \sigma(w)) \quad &= \quad \mathbb{P}\left(\mathrm{Tr}^h(\mathcal{T}_\xi^{(t)}) \ \text{ is unmarked} \mid \sigma(w)\right) \\ &= \quad \exp\left[-2\theta \cdot \mathcal{L}_\xi(\mathrm{Tr}^h(\mathcal{T}_\xi^{(t)}))\right]. \end{aligned} \tag{4.2}$$

Let us define

$$\xi^{(t)}(s) \ := \ \begin{cases} \xi(s) \text{ if } s \leq t \\ (\xi(t) - (s-t)) \vee 0 \ \text{ otherwise.} \end{cases}$$

which is a function in $C_0^+(\mathbb{R}^+)$ from which we can construct the real rooted tree $(\mathcal{T}_{\xi^{(t)}}, d_{\xi^{(t)}})$.

Lemma 4.2. *There exists a root preserving isometry from* $(\mathcal{T}_\xi^{(t)}, d_{\mathcal{T}_\xi})$ *onto* $(\mathcal{T}_{\xi^{(t)}}, d_{\mathcal{T}_{\xi^{(t)}}})$.

Proof: For ease of notation, we write $\mathcal{T}_1 = \mathcal{T}_\xi^{(t)}$ and $\mathcal{T}_2 = \mathcal{T}_{\xi^{(t)}}$. For every $y \in \mathcal{T}_1$, define t_y to be the minimal element of the fiber $\{s \ : \ p_\xi(s) = y\}$ (i.e., the first exploration time for y) and define the mapping

$$\begin{array}{rcl} \phi \ : \ \mathcal{T}_1 & \to & \mathcal{T}_2 \\ y & \to & p_{\xi^{(t)}}(t_y). \end{array}$$

It is straightforward to show that ϕ defines a mapping from $\mathcal{T}_1$ to $\mathcal{T}_2$ preserving the root. We first show that ϕ is surjective. In order to do so, we start by showing that

$$\forall s \leq t, \ \phi(p_\xi(s)) = p_{\xi^{(t)}}(s). \tag{4.3}$$

For $s \leq t$, we have

$$\begin{aligned} t_{p_\xi(s)} &= \inf\{u \ : \ p_\xi(u) = p_\xi(s)\} \\ &= \inf\{u \leq s \ : \ \xi(u) = \xi(s) = \inf_{[u,s]} \xi\} \\ &= \inf\{u \leq s \ : \ \xi^{(t)}(u) = \xi^{(t)}(s) = \inf_{[u,s]} \xi^{(t)}\}, \end{aligned}$$

where the last equality follows from the fact that ξ and $\xi^{(t)}$ coincide before time t. The latter identity implies that $t_{p_\xi(s)} \in \{u \leq s \ : \ \xi^{(t)}(u) = \xi^{(t)}(s) = \inf_{[u,s]} \xi^{(t)}\}$ or equivalently that

$$p_{\xi^{(t)}}(t_{p_\xi(s)}) = p_{\xi^{(t)}}(s),$$

which can be rewritten as (4.3), as claimed earlier. In order to show surjectivity, let us take $v \in \mathcal{T}_2$ and s such that $v = p_{\xi^{(t)}}(s)$. We distinguish between two cases: (1) if $s \leq t$, the previous result immediately implies that $v \in \phi(\mathcal{T}_1)$, and (2) $s > t$, since $\xi^{(t)}$ is continuous and non-increasing on $[t, \infty)$, one can find $s' \leq t$ such that

$$\xi^{(t)}(s) = \xi^{(t)}(s') = \inf_{[s',s]} \xi^{(t)}$$

implying that $v = p_{\xi^{(t)}}(s')$ and we are back to case (1).

It remains to show that ϕ is an isometry. Let $x_1, x_2 \in \mathcal{T}_1$. We have $\phi(x_i) = p_{\xi^{(t)}}(t_{x_i})$ with $t_{x_i} \leq t$. Since ξ and $\xi^{(t)}$ coincide up to time t, we must have

$$\begin{aligned} d_{\mathcal{T}_2}(\phi(x_1), \phi(x_2))) &= \xi^{(t)}(t_{x_2}) \ + \ \xi^{(t)}(t_{x_2}) \ - \ 2 \inf_{[t_{x_1} \wedge t_{x_2}, t_{x_1} \vee t_{x_2}]} \xi^{(t)} \\ &= \xi(t_{x_2}) \ + \ \xi(t_{x_1}) \ - \ 2 \inf_{[t_{x_1} \wedge t_{x_2}, t_{x_1} \vee t_{x_2}]} \xi \\ &= d_{\mathcal{T}_1}(x_1, x_2). \end{aligned}$$

∎

The branch length $\mathcal{L}_\xi(\mathrm{Tr}^h(\mathcal{T}_{\xi^{(t)}}))$ is obtained by adding up all the branch lengths of the trimmed tree $\mathrm{Tr}^h(\mathcal{T}_{\xi^{(t)}})$. Following the algorithm described in Proposition 1.4, the total branch length is given by the sum of the $X_n(\xi^{(t)})$'s or equivalently

$$\begin{aligned}\mathcal{L}_\xi(\mathrm{Tr}^h(\mathcal{T}_{\xi^{(t)}})) &= \sum_{n\geq 1}\left(\xi_h^{(t)}(t_n^{(t)}) - \xi_h^{(t)}(s_n^{(t)})\right)\\ &= \sum_{n\geq 1}\left(c^h(\xi^{(t)})(t_{n-1}^{(t)}) - c^h(\xi^{(t)})(t_n^{(t)})\right)\\ &= \lim_{s\uparrow\infty} -c^h(\xi^{(t)})(s), \qquad (4.4)\end{aligned}$$

where we wrote $t_n^{(t)} := t_n(\xi^{(t)}), s_n^{(t)} := s_n(\xi^{(t)})$, and the second line can be shown as in (3.2).

Lemma 4.3.

$$\lim_{s\uparrow\infty} c^h(\xi^{(t)})(s) = c^h(\xi)(t).$$

Proof: Since $\xi^{(t)}$ and ξ coincide up to t, we have

$$c^h(\xi^{(t)})(t) = c^h(\xi)(t). \qquad (4.5)$$

Furthermore, by Lemma 2.2,

$$\begin{aligned}&\forall s\geq t,\ \Lambda_{0,h}(\xi^{(t)})(s) \ = \ \Lambda_{0,h}(m)(s),\\ &\text{where } m(s) = \Lambda_{0,h}(\xi)(t) + 1_{s\geq t}\left((\xi(t)-(s-t))\vee 0 \ - \ \xi(t)\right).\end{aligned}$$

The function m is non-increasing on $\mathbb{R}^+$ and $m\leq h$. From this observation, we easily get from the definition of the one-sided reflection $\Gamma^0(\cdot)$ that

$$\forall s\geq 0,\quad \Gamma^0(m)(s)\leq\Gamma^0(m)(t)\leq h.$$

Lemma 2.3 then implies that $\Lambda_{0,h}(m) = \Gamma^0(m)$ and that $c^h(m) = 0$ (in other words, no compensator is needed to keep m below level h). Finally, since $dc^h(m) = dc^h(\xi^{(t)})$ on $[t,\infty)$, we have

$$\lim_{s\uparrow\infty} c^h(\xi^{(t)})(s) \ = \ c^h(\xi^{(t)})(t).$$

The latter equation combined with (4.5) completes the proof of the lemma. ∎

Combining (4.2), (4.4) with the previous lemma, we get

$$\mathbb{P}(\sup_{[0,t]} z^\theta\leq h \mid \sigma(w)) \ = \ \exp\left[2\theta\cdot c^h(\xi)(t)\right]. \qquad (4.6)$$

In order to prove our theorem, it remains to show that $\Lambda_{0,h}(w)$ is identical in law with a Brownian motion reflected (in a “standard way”) on $[0,h]$, and that $-c^h(\xi)$ is the local time of $\Lambda_{0,h}(w)$ at h. We will need the following lemma.

Lemma 4.4. *For every continuous function f with $f(0)=0$, $c^h(\Gamma^0(f)) = c^h(f)$.*

Proof: Let g be a continuous function with $g(0)=0$. In the proof of Theorem 1.3, we showed that

$$\mathrm{Supp}\,(\Lambda_{0,h}(g))\subset\mathrm{Supp}\,(g),$$

or equivalently that every zero of the function g is also a zero of the function $\Lambda_{0,h}(g)$. On the other hand, for any continuous function f with $f(0)=0$, the definition of the one-sided Skorohod reflection at 0 (see (1.3)) implies that $\Gamma^0(f)$ can be written as $f + c$ where c is a non-decreasing continuous function, only increasing at the zeros of the reflected path $\Gamma^0(f)$. Taking $g = \Gamma^0(f)$ in the previous discussion, the set of zeros for $\Gamma^0(f)$ is included in its $\Lambda_{0,h}(\Gamma^0(f))$ counter part, and we get that the compensator $c(t) := -\inf_{[0,t]} f$ (for the one-sided reflection) only increases on the set of zeros of the doubly reflected path $\Lambda_{0,h}(\Gamma^0(f))$. Next, let $\bar{c}^0$ and $\bar{c}^h$ be the compensators associated with the function $\Gamma^0(f)$, i.e. $\Lambda_{0,h}(\Gamma^0(f)) = (f+c)+\bar{c}^0+\bar{c}^h$. where $\bar{c}^0$ and $\bar{c}^h$ satisfy the hypothesis of Theorem 1.1 for the function $f+c$. Since c and $\bar{c}^0$ only increase at the zeroes of $\Lambda_{0,h}(\Gamma^0(f))$, the functions $(c+\bar{c}^0, \bar{c}^h)$ must solve the Skorohod equation for f on the interval $[0,h]$. By uniqueness of the solution, this readily implies that $\Lambda_{0,h}(f) = \Lambda_{0,h}(\Gamma^0(f))$ and $\bar{c}^h \equiv c^h(\Gamma^0(f)) = c^h(f)$. ∎

The previous lemma and (4.6) yield

$$\mathbb{P}(\sup_{[0,t]} z^\theta \leq h \mid \sigma(w)) \quad = \quad \exp\left[2\theta \cdot c^h(w)(t)\right]. \tag{4.7}$$

As already explained in the previous section (see the proof of Lemma 3.1), $\Lambda_{0,h}(w)$ is identical in law with a Brownian motion reflected (in a "standard way") on $[0,h]$, and $-c^h(w)$ is the local time of this process at h (see again the proof of Lemma 3.1 for more details). This completes the proof of Theorem 1.6.

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