

The limit shape of random permutations with polynomially growing cycle weights

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Abstract. In this work we are considering the behaviour of the limit shape of Young diagrams associated to random permutations on the set $\{1, \dots, n\}$ under a particular class of multiplicative measures with polynomial growing cycle weights. Our method is based on generating functions and complex analysis (saddle point method). We show that fluctuations near a point behave like a normal random variable and that the joint fluctuations at different points of the limiting shape have an unexpected dependence structure. We will also compare our approach with the so-called randomization of the cycle counts of permutations and we will study the convergence of the limit shape to a continuous stochastic process.

1. Introduction and main results

The aim of this paper is to study the limit shape of a random permutation under the generalised Ewens measure with polynomially growing cycle weights and the fluctuations at each point of the limit shape. The study of such objects has a long history, which started with the papers of [Temperley \(1952\)](#) and [Vershik \(1996\)](#). Later on Young diagrams have been approached under a different direction, as in the independent works of [Veršik and Kerov \(1977\)](#) and [Logan and Shepp \(1977\)](#), which first derived the limit shape when the underpinned measure on partitions is the so-called *Plancherel measure*. We will not handle this approach here, even though it presents remarkable connections with random matrix theory and random polymers, among others (see for example [Deift \(2000\)](#)).

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We first specify what we define as the limit shape of a permutation. We denote by \mathfrak{S}_n the set of permutations on n elements and write each permutation $\sigma \in \mathfrak{S}_n$ as $\sigma = \sigma_1 \cdots \sigma_\ell$ with σ_j disjoint cycles of length λ_j . Disjoint cycles commute and we thus can assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. This assigns to each permutation $\sigma \in \mathfrak{S}_n$ in a unique way a partition of n and this partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is called the *cycle type* of σ . We will indicate that λ is such a partition with the notation $\lambda \vdash n$. We define the size $|\lambda| := \sum_i \lambda_i$ (so obviously if $\lambda \vdash n$ then $|\lambda| = n$). λ features a nice geometric visualisation by its Young diagram Υ_λ . This is a left- and bottom-justified diagram of ℓ rows with the j -th row consisting of λ_j squares, see Figure 1.1(a). It is clear that the area of Υ_λ is n if $\lambda \vdash n$. After introducing a

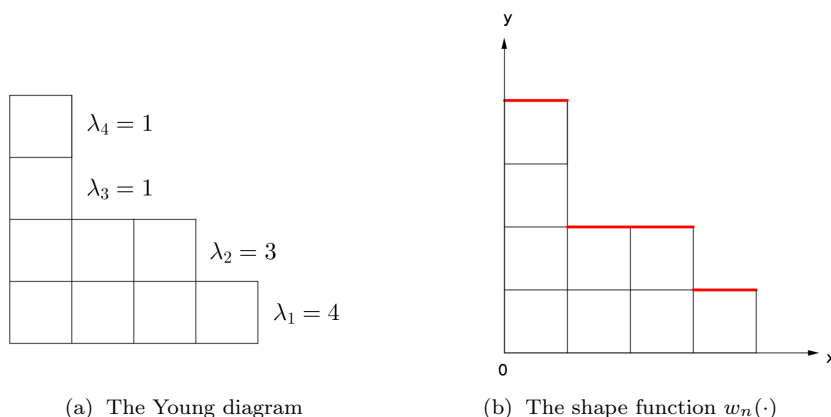


FIGURE 1.1. Illustration of the Young diagram and the shape of $\sigma = (3578)(129)(4)(6) \in \mathfrak{S}_9$

coordinate system as in Figure 1.1(b), we see that the upper boundary of a Young diagram Υ_λ is a piecewise constant and right continuous function $w_n : \mathbb{R}^+ \rightarrow \mathbb{N}$ with

$$w_n(x) := \sum_{j=1}^n \mathbb{1}_{\{\lambda_j \geq x\}} \quad (1.1)$$

The cycle type of a permutation becomes a random partition if we endow the space \mathfrak{S}_n with a probability measure \mathbb{P}_n . What we are then interested in studying is the now random shape $w_n(\cdot)$ as $n \rightarrow +\infty$, and more specifically to determine its limit shape. The limit shape with respect to a sequence of probability measures \mathbb{P}_n on \mathfrak{S}_n (and sequences of positive real numbers A_n and B_n with $A_n \cdot B_n = n$) is understood as a function $w_\infty : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for each $\epsilon, \delta > 0$

$$\lim_{n \rightarrow +\infty} \mathbb{P}_n \left[\left\{ \sigma \in \mathfrak{S}_n : \sup_{x \geq \delta} |A_n^{-1} w_n(B_n x) - w_\infty(x)| \leq \epsilon \right\} \right] = 1. \quad (1.2)$$

The assumption $A_n \cdot B_n = n$ ensures that the area under the rescaled Young diagram is 1. One of the most frequent choices is $A_n = B_n = n^{1/2}$, but we will see that it's useful to adjust the choice of A_n and B_n to the measures \mathbb{P}_n . Equation (1.2) can be viewed as a law of large numbers for the process $w_n(\cdot)$. The next natural question

is then whether fluctuations satisfy a central limit theorem, namely whether

$$A_n w_n(B_n x) - w_\infty(x)$$

converges (after centering and normalization) in distribution to a Gaussian process on the space of càdlàg functions, for example. Of course the role of the probability distribution with which we equip the set of partitions will be crucial to this end.

In this paper, we work with the following measure on \mathfrak{S}_n :

$$\mathbb{P}_n[\sigma] = \frac{1}{h_n n!} \prod_{j=1}^{\ell} \vartheta_{\lambda_j}. \tag{1.3}$$

where $(\lambda_1, \dots, \lambda_\ell)$ is the cycle type of σ , $(\vartheta_m)_{m \geq 1}$ is a sequence of non-negative weights and h_n is a normalization constant (h_0 is defined to be 1). From time to time we will also use $\vartheta_0 := 0$ introduced as convention.

This measure has recently appeared in mathematical physics for a model of the quantum gas in statistical mechanics and has a possible connection with the Bose-Einstein condensation (see e.g. [Betz et al. \(2011\)](#) and [Ercolani and Ueltschi \(2011\)](#)). Classical cases of this measure are the uniform measure ($\vartheta_m \equiv 1$) and the Ewens measure ($\vartheta_m \equiv \vartheta$). The uniform measure is well studied and has a long history (see e.g. the first chapter of [Arratia et al. \(2003\)](#) for a detailed account with references). The Ewens measure originally appeared in population genetics, see [Ewens \(1972\)](#), but has also various applications through its connection with Kingman’s coalescent process, see [Hoppe \(1987\)](#). The measure in (1.3) also has some similarities to multiplicative measure for partitions, see for instance [Bogachev \(2015\)](#). It is clear that we have to make some assumptions on the sequence $(\vartheta_m)_{m \geq 1}$ to be able to study the behaviour as $n \rightarrow +\infty$. We use in this paper the weights ϑ_m

$$\vartheta_m = (\log m)^j \frac{m^\alpha}{\Gamma(\alpha + 1)} + O(m^\beta), \quad j \in \mathbb{N} \tag{1.4}$$

with some $\alpha > 0$ and $0 \leq \beta < \alpha/2$. We would like to point out that the requirement $0 \leq \beta < \alpha/2$ and the normalisation constant $\Gamma(\alpha + 1)$ are not essential and it only simplifies the notation and the computations. In fact we can study without further mathematical problems the case $\vartheta_m \sim \text{const.}(\log m)^j m^\alpha$, see Remark 4.12. We get

Theorem 1.1. *We define*

$$n^* := (1 + \alpha)^{-j} \left(\frac{n}{(\log n)^j} \right)^{\frac{1}{\alpha+1}} \quad \text{and} \quad \bar{n} = \frac{n}{n^*} = (1 + \alpha)^j n \left(\frac{n}{(\log n)^j} \right)^{-\frac{1}{\alpha+1}}. \tag{1.5}$$

We then have

- (1) *The limit shape exists for the process $w_n(x)$ as $n \rightarrow \infty$ with respect to \mathbb{P}_n and the weights in (1.4) with the scaling $A_n = \bar{n}$ and $B_n = n^*$. The limit shape is given by*

$$w_\infty^s(x) := \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)},$$

where $\Gamma(\alpha, x)$ denotes the upper incomplete Gamma function.

(2) *The fluctuations at a point x of the limit shape behave like*

$$\tilde{w}_n^{\mathbf{s}}(x) := \frac{w_n(xn^*) - \bar{n}(w_\infty^{\mathbf{s}}(x) + z_n^{\mathbf{s}}(x))}{(\bar{n})^{1/2}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_\infty^2(x))$$

with

$$\sigma_\infty^2(x) := \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 1, x)^2}{2\Gamma(\alpha + 1)\Gamma(\alpha + 2)}$$

and $z_n^{\mathbf{s}}(x) = o(1)$.

Remark 1.2. The expectation of $w_\infty^{\mathbf{s}}$ can be expanded asymptotically also to terms of lower order with the same argument, in particular we will see it is possible to give a more explicit expression for $z_n^{\mathbf{s}}$. This will actually be important in the proof of Thm. 3.7. We will see that our computation gives $z_n^{\mathbf{s}} = O(1/\log n)$ for $j \geq 1$ (and thus $z_n^{\mathbf{s}}$ can not be dropped from the statement of the Theorem). As we do not need $z_n^{\mathbf{s}}$ explicitly for our argumentation and the expression for $z_n^{\mathbf{s}}$ is rather complicated, we decided to omit it.

Theorem 1.1 was already obtained in the special case $j = 0$, i.e. $\vartheta_m \sim m^\alpha$, by [Erlihson and Granovsky \(2008\)](#) in the context of Gibbs distributions on integer partitions and as we were writing the present paper, we were made aware of their work. To be precise, one can push forward the measure \mathbb{P}_n to a measure $\tilde{\mathbb{P}}_n$ on the set of partitions of n with

$$\tilde{\mathbb{P}}_n[\lambda] = \frac{1}{\tilde{h}_n n!} \prod_{k=1}^n \frac{1}{C_k!} \left(\frac{\vartheta_k}{k}\right)^{C_k}, \tag{1.6}$$

where λ is a partition of n and C_k is the number of parts of length k (see Section 2.1). These Gibbs distributions have been treated extensively in the literature ([Betz et al. \(2011\)](#), [Ercolani and Ueltschi \(2011\)](#) for example). One thus can work with $\tilde{\mathbb{P}}_n$ or with \mathbb{P}_n . We prefer here to use \mathbb{P}_n .

The argumentation of [Erlihson and Granovsky \(2008\)](#) is stochastic and is based on randomisation: this technique has been successfully introduced by [Fristedt \(1993\)](#) and used also in particular by [Bogachev \(2015\)](#) as a tool to investigate combinatorial structures, and later applied in many contexts. However, the approach in this paper is slightly different and based on complex analysis and uses the saddle-point method as described in Section 4. This method was used in [Ercolani and Ueltschi \(2011\)](#) and [Maples et al. \(2012\)](#) and an introduction can be found for instance in [Flajolet and Sedgewick \(2009, Section VIII\)](#). Therefore two limit shapes will appear in this paper, denoted by the superscript \mathbf{r} and \mathbf{s} respectively for randomization and saddle point method, to indicate by means of which method they were obtained. We will see in the end that they coincide. Our Ansatz enables us to reprove Theorem 1.1, but with two big advantages. First, our computations are much simpler than the one in [Erlihson and Granovsky \(2008\)](#). Second, we get almost for free large deviations estimates. More precisely

Proposition 1.3. *We have for all $a = O(\bar{n})$ and $\delta = O((\bar{n})^{-1/2})$*

$$\mathbb{P}[|\tilde{w}_n^{\mathbf{s}}(x) - a| < \epsilon] = (1 - \epsilon^{-2}(1 + \delta)) \exp(-a^2/2 + O(\delta + \epsilon a)).$$

The error terms are absolute.

We prove this by studying the cumulant generating function

$$\tilde{\Lambda}(s) := \log \mathbb{E}_n [\exp(s\tilde{w}_n^s(x))]. \tag{1.7}$$

This is an important difference to [Erlihson and Granovsky \(2008\)](#). Erlihson and Granovsky directly consider the distribution of $\tilde{w}_n^s(x)$ and studying $\tilde{\Lambda}(s)$ with their method is computationally harder. In fact, we can compute the behaviour of all cumulants.

Theorem 1.4. *Let $s^* = s(\bar{n})^{-\frac{1}{2}}$ and*

$$\Lambda(s) := \mathbb{E}_n [\exp(-s^*w_n(xn^*))] = \sum_{m \geq 1} q_m \frac{s^m}{m!}. \tag{1.8}$$

We then have for $m \geq 2$

$$q_m = \kappa_m(1 + o(1)) \tag{1.9}$$

with

$$\begin{aligned} \kappa_m &= (\bar{n})^{1-\frac{m}{2}} [s^m] \left[\left(1 - s \frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)} \right)^{-\alpha} \right. \\ &\quad \left. + (e^{-s} - 1) \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{\Gamma(\alpha + k, x)}{\Gamma(\alpha + 1)} \left(-\frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)} \right)^k \right]. \end{aligned} \tag{1.10}$$

We give the proofs of [Theorem 1.1](#), [Theorem 1.4](#) and [Proposition 1.3](#) in [Section 4.2](#). Furthermore, we introduce in [Section 3](#) the so called grand canonical ensemble (Ω_t, \mathbb{P}_t) with $\Omega_t = \dot{\cup}_{n \geq 1} \mathfrak{S}_n$ and \mathbb{P}_t a measure such that $\mathbb{P}_t[\cdot | \mathfrak{S}_n] = \mathbb{P}_n[\cdot]$ (see [\(3.2\)](#)). One might expect that the behaviour on grand canonical ensembles agrees with the behaviour on the canonical ensembles, but we will see here that this is only the case for macroscopic properties. More precisely, we will see in [Theorem 3.8](#) that $w_n(x)$ has a limit shape for the grand canonical ensemble Ω_t and this agrees with the one for the canonical ensemble in [Theorem 1.1](#). However, we will see also in [Theorem 3.8](#) that the fluctuations at the points of the limit shape follow a different central limit theorem than in [Theorem 1.1](#). Notice that we will not deduce [Theorem 1.1](#) (nor any other of our results) from the grand canonical ensemble (Ω_t, \mathbb{P}_t) .

2. Preliminaries

We introduce in this section the notation for the cycle counts and the notation for generating functions.

2.1. Cycle counts. The notation $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is very useful for the illustration of λ via its Young diagram, but in the computations it is better to work with the *cycle counts* C_k . These are defined as

$$C_k(\sigma) = C_k := \#\{j \geq 1; \lambda_j = k\} \tag{2.1}$$

for $k \geq 1$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ the cycle type of $\sigma \in \mathfrak{S}_n$. Conventionally $C_0 := 0$. We obviously have for $k \geq 1$

$$C_k \geq 0 \quad \text{and} \quad \sum_{k=1}^n kC_k = n. \tag{2.2}$$

It is also clear that the cycle type of permutation (or a partition) is uniquely determined by the vector (C_1, C_2, \dots) . The function $w_n(x)$ and the measure $\mathbb{P}_n[\cdot]$ in (1.1) and (1.3) can now be written as

$$w_n(x) = \sum_{k \geq x} C_k \quad \text{and} \quad \mathbb{P}_n[\sigma] = \frac{1}{h_n n!} \prod_{k=1}^n \vartheta_k^{C_k}. \quad (2.3)$$

Our aim is to study the behaviour of $w_n(x)$ as $n \rightarrow \infty$. It is thus natural to consider the asymptotic behaviour of C_k with respect to the measure $\mathbb{P}_n[\cdot]$.

Lemma 2.1 (Ercolani and Ueltschi (2011), Corollary 2.3). *Under the condition $h_{n-1}/h_n \rightarrow 1$ the random variables C_k converge for each $k \in \mathbb{N}$ in distribution to a Poisson distributed random variable Y_k with $\mathbb{E}[Y_k] = \frac{\vartheta_k}{k}$. More generally for all $b \in \mathbb{N}$ the following limit in distribution holds:*

$$\lim_{n \rightarrow +\infty} (C_1, C_2, \dots, C_b) = (Y_1, Y_2, \dots, Y_b)$$

with Y_k independent Poisson random variables with mean $\mathbb{E}[Y_k] = \frac{\vartheta_k}{k}$.

One might expect at this point that $w_n(x)$ is close to $\sum_{k=x}^n Y_k$. Unfortunately we will see in Section 4 that the asymptotic behaviour of $w_n(x)$ is more complicated.

2.2. Generating functions. The (ordinary) generating function of a sequence $(g_k)_{k \geq 0}$ of complex numbers is defined as the formal power series

$$g(z) := \sum_{j=0}^{\infty} g_j z^j. \quad (2.4)$$

As usual, we define the *extraction symbol* $[z^k]g(z) := g_k$, that is, as the coefficient of z^k in the power series expansion (2.4) of $g(z)$.

A generating function that plays an important role in this paper is

$$g_{\Theta}(t) := \sum_{m \geq 1} \frac{\vartheta_m}{m} t^m. \quad (2.5)$$

As mentioned in the introduction, we will use $\vartheta_m = \frac{m^\alpha (\log m)^j}{\Gamma(\alpha+1)} + O(m^\beta)$, $j \in \mathbb{N}$. We stress that generating functions of the type $(1-t)^{-\alpha}$ fall also in this category, and for them we will recover the limiting shape as previously done in Erlihson and Granovsky (2008). We will see in particular this case in Section 4.

The reason why generating functions are useful is that it is often possible to write down a generating function without knowing g_n explicitly. In this case one can try to use tools from analysis to extract information about g_n , for large n , from the generating function. It should be noted that there are several variants in the definition of generating functions. However, we will use only the ordinary generating function and thus call it ‘just’ generating function without risk of confusion.

The following well-known identity is a special case of the general *Pólya’s Enumeration Theorem* Pólya (1937, p. 17) and is the main tool in this paper to obtain generating functions.

Lemma 2.2. *Let $(a_m)_{m \in \mathbb{N}}$ be a sequence of complex numbers. We then have as formal power series in t*

$$\sum_{n \in \mathbb{N}} \frac{t^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^n a_j^{C_j} = \sum_{n \in \mathbb{N}} t^n \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{k=1}^\infty a_k^{C_k} = \exp \left(\sum_{m \geq 1} \frac{a_m}{m} t^m \right)$$

where $z_\lambda := \prod_{k=1}^n k^{C_k} C_k!$. If one series converges absolutely, so do the others.

We omit the proof of this lemma, but details can be found for instance in [Macdonald \(1995, p. 5\)](#).

2.3. Approximation of sums. We require for our argumentation the asymptotic behaviour of the generating function $g_\Theta(t)$ as t tends to the radius of convergence, which is 1 in our case.

Lemma 2.3. *Let $(v_n)_{n \in \mathbb{N}}$ a sequence of positive numbers with $v_n \downarrow 0$ as $n \rightarrow +\infty$. We have for all $\delta \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$*

$$\sum_{k=1}^\infty k^\delta e^{-kv_n} = \Gamma(\delta + 1)v_n^{-\delta-1} + \zeta(-\delta) + O(v_n). \tag{2.6}$$

$\zeta(\cdot)$ indicates the Riemann Zeta function. Furthermore, we have for $j \in \mathbb{N}$

$$\sum_{k=1}^\infty (\log k)^j k^\delta e^{-kv_n} = v_n^{-\delta-1} \left(\frac{\partial}{\partial \delta} - \log v_n \right)^j \Gamma(\delta + 1) + O(1). \tag{2.7}$$

We indicate $\left(\frac{\partial}{\partial \delta} - \log v_n \right)^j f(\delta) = \frac{\partial^j}{\partial \delta^j} f(\delta) - j \log v_n \frac{\partial^{j-1}}{\partial \delta^{j-1}} f(\delta) + \binom{j}{2} (\log v_n)^2 \frac{\partial^{j-2}}{\partial \delta^{j-2}} f(\delta) + \dots + (-\log v_n)^j f(\delta)$.

This lemma can be proven with Euler Maclaurin summation formula or with the Mellin transformation. The computations with Euler Maclaurin summation are straightforward and the details of the proof with the Mellin transformation can be found for instance in [Flajolet and Sedgewick \(2009, Chapter VI.8\)](#). We thus omit it.

We require also the behaviour of partial sum $\sum_{k=x}^\infty \frac{\theta_m}{m} t^m$ as $x \rightarrow \infty$ and as $t \rightarrow 1$. We have

Lemma 2.4 (Approximation of sums). *Let $j \in \mathbb{N}$ and v_n, z_n be given with $z_n \rightarrow +\infty$ and $z_n v_n = x(1 + \tilde{q}_n)$ for $x > 0$ and $\tilde{q}_n \rightarrow 0$. We then have for all $\delta \in \mathbb{R}$ and all $\ell \in \mathbb{N}$*

$$\begin{aligned} \sum_{k=\lfloor z_n \rfloor}^\infty (\log k)^j k^\delta e^{-kv_n} &= \left(\frac{z_n}{x} \right)^{\delta+1} \left(\sum_{k=0}^\ell \left(\frac{\partial}{\partial \delta} + \log \frac{z_n}{x} \right)^j \frac{\Gamma(\delta + k + 1, x)}{k!} (-\tilde{q}_n)^k \right) \\ &\quad + O \left(\left(\log \left(\frac{z_n}{x} \right) \right)^j \tilde{q}_n^{\ell+1} + \left(\frac{z_n}{x} \right)^\delta \left(\log \left(\frac{z_n}{x} \right) \right)^j \right) \end{aligned}$$

with $\Gamma(a, x) := \int_x^{+\infty} s^{a-1} e^{-s} ds$ the incomplete Gamma function.

Remark 2.5. One can obtain more error terms by using the Euler Maclaurin summation formula with more derivatives. We have given in [Appendix A](#) a formulation of the Euler Maclaurin summation formula with non-integer boundaries, which is more suitable for this computation than the usual one. Our primary interest is in the leading coefficient, hence we state the result only up to order z_n^δ . However, the

lower order terms can not be completely ignored. In particular they play an important role for the expectation of $\mathbb{E}_n[w_\infty^s(x)]$ in Theorem 1.1 since there are, beside the leading term $\bar{n} w_\infty^s(x)$, also other terms in the asymptotic expansion which are not $o((\bar{n})^{1/2})$.

Proof: $B_1(x) := x - \frac{1}{2}$ stands in the proof for the first Bernoulli polynomial. The proof of this lemma is based on the Euler Maclaurin summation formula, see [Apostol \(1999\)](#) or [Apostol \(1976, Theorem 3.1\)](#). We use here the following version: let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ have a continuous derivative and suppose that f and f' are integrable. Then

$$\sum_{k \geq \lfloor c \rfloor} f(k) = \int_c^{+\infty} f(s) ds - B_1(c - \lfloor c \rfloor)f(c) + \int_c^{+\infty} B_1(s - \lfloor s \rfloor)f'(s) ds. \quad (2.8)$$

We substitute $f(s) := (\log s)^j s^\delta e^{-sv_n}$, $c := z_n$ and notice that f and all its derivatives tend to zero exponentially fast as $s \rightarrow +\infty$. We begin with the first integral. Now by the change of variables $s := \frac{z_n}{x} y$

$$\begin{aligned} \int_{z_n}^{+\infty} (\log s)^j s^\delta e^{-sv_n} ds &= \left(\frac{z_n}{x}\right)^{\delta+1} \int_x^{+\infty} \left(\log y + \log \frac{z_n}{x}\right)^j y^\delta e^{-y} e^{-\tilde{q}_n y} dy = \\ &= \left(\frac{z_n}{x}\right)^{\delta+1} \left(\frac{\partial}{\partial \delta} + \log \frac{z_n}{x}\right)^j \int_x^{+\infty} y^\delta e^{-y} e^{-\tilde{q}_n y} dy = \\ &= \left(\frac{z_n}{x}\right)^{\delta+1} \left(\left(\frac{\partial}{\partial \delta} + \log \frac{z_n}{x}\right)^j \sum_{k=0}^{\ell} \frac{\Gamma(\delta + k + 1, x)}{k!} (-\tilde{q}_n)^k + O(\tilde{q}_n^{\ell+1})\right) \end{aligned} \quad (2.9)$$

where we have swapped integral and series expansion of the exponential by Fubini's theorem. This gives the behaviour of the leading term in (2.8) with $f(s) := (\log s)^j s^\delta e^{-sv_n}$. The remaining terms can be estimated with a similar computations and using that $B_1(s - \lfloor s \rfloor)$ is bounded. □

3. Randomization

We introduce in this section a probability measure $\mathbb{P}_t[\cdot]$ on $\dot{\cup}_{n \geq 1} \mathfrak{S}_n$, where $\dot{\cup}$ denotes the disjoint union, dependent on a parameter $t > 0$ with $\mathbb{P}_t[\cdot | \mathfrak{S}_n] = \mathbb{P}_n[\cdot]$ and consider the asymptotic behaviour of $w_n(x)$ with respect to $\mathbb{P}_t[\cdot]$ as $t \rightarrow 1$.

3.1. *Grand canonical ensemble.* Computations on \mathfrak{S}_n can turn out to be difficult and many formulas can not be used to study the behaviour as $n \rightarrow \infty$. A possible solution to this problem is to adopt a suitable randomization. This has been successfully introduced by [Fristedt \(1993\)](#) and used also by [Bogachev \(2015\)](#) as a tool to investigate combinatorial structures, and later applied in many contexts. The main idea of randomization is to define a one-parameter family of probability measures on $\dot{\cup}_{n \geq 1} \mathfrak{S}_n$ for which cycle counts turn out to be independent. Then one is able to study their behaviour more easily, and ultimately the parameter is tuned in such a way that randomized functionals are distributed as in the non-randomized context. Let us see how to apply this in our work. We define

$$G_\Theta(t) = \exp(g_\Theta(t)) \quad (3.1)$$

with $g_\Theta(t)$ as in (2.5). If $G_\Theta(t)$ is finite for some $t > 0$, then for each $\sigma \in \mathfrak{S}_n$ let us define the probability measure

$$\mathbb{P}_t[\sigma] := \frac{1}{G_\Theta(t)} \frac{t^n}{n!} \prod_{k=1}^n \vartheta_k^{C_k}. \tag{3.2}$$

Lemma 2.2 shows that \mathbb{P}_t is indeed a probability measure on $\dot{\cup}_{n \geq 1} \mathfrak{S}_n$. The induced distribution on cycle counts C_k can easily be determined.

Lemma 3.1. *Under $\mathbb{P}_t[\cdot]$ the C_k 's are independent and Poisson distributed with*

$$\mathbb{E}_t[C_k] = \frac{\vartheta_k}{k} t^k.$$

Proof: From Pólya's enumeration theorem (Lemma 2.2) we obtain

$$\begin{aligned} \mathbb{E}_t[e^{-sC_k}] &= \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} e^{-sC_k} \mathbb{P}_t[\sigma] = \frac{1}{G_\Theta(t)} \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} \frac{t^n}{n!} (\vartheta_k e^{-s})^{C_k} \prod_{j \leq \binom{n}{k} \neq k} (\vartheta_j)^{C_j} \\ &= \frac{1}{G_\Theta(t)} \exp\left(\sum_{j=0}^{+\infty} \frac{\vartheta_j}{j} t^j\right) \exp\left((e^{-s} - 1) \frac{\vartheta_k}{k} t^k\right) \\ &= \exp\left((e^{-s} - 1) \frac{\vartheta_k}{k} t^k\right). \end{aligned}$$

Analogously one proves the pairwise independence of cycle counts. □

Obviously the following conditioning relation holds:

$$\mathbb{P}_t[\cdot | \mathfrak{S}_n] = \mathbb{P}_n[\cdot].$$

A proof of this fact is easy and can be found for instance in Hansen (1990, Equation (1)). We note that $w_n(x)$ is \mathbb{P}_t -a.s. finite, since $\mathbb{E}_t[w_n(x)] < +\infty$. Now since the conditioning relation holds for all t with $G_\Theta(t) < +\infty$, one can try to look for t satisfying “ $\mathbb{P}_n[\cdot] \approx \mathbb{P}_t[\cdot]$ ”, which heuristically means that we choose a parameter for which permutations on \mathfrak{S}_n weigh as most of the mass of the measure \mathbb{P}_t . We have on \mathfrak{S}_n

$$n = \sum_{j=1}^{\ell} \lambda_j = \sum_{k=1}^n k C_k.$$

A natural choice for t is thus the solution of

$$n = \mathbb{E}_t \left[\sum_{k=1}^{\infty} k C_k \right] = \sum_{k=1}^{\infty} \vartheta_k t^k. \tag{3.3}$$

which is guaranteed to exist if the series on the right-hand side is divergent at the radius of convergence (we will see this holds true for our particular choice of weights). We write $t = e^{-v_n}$ and use Lemma 2.3 (in our case setting $\vartheta_k =$

$(\log k)^j \frac{k^\alpha + O(k^\beta)}{\Gamma(\alpha+1)}$) to obtain

$$\begin{aligned} n &\stackrel{!}{=} \frac{v_n^{-\alpha-1}}{\Gamma(\alpha+1)} \left(\frac{\partial}{\partial \alpha} - \log v_n \right)^j \Gamma(\alpha+1) + O(1) \\ &= \frac{v_n^{-\alpha-1}}{\Gamma(\alpha+1)} \left(\frac{\partial}{\partial \alpha} - \log v_n \right)^j \Gamma(\alpha+1) + O(1) \\ \implies v_n &= \left(\frac{(\log n)^j}{n} \right)^{1/(\alpha+1)} + o\left(\frac{n}{(\log n)^j} \right). \end{aligned} \tag{3.4}$$

We will fix this choice for the rest of the section.

3.2. Limit shape and mod-convergence. In order to derive our main results from the measure \mathbb{P}_t we will use a tool developed by [Kowalski and Nikeghbali \(2010\)](#), the *mod-Poisson convergence*.

Definition 3.2. A sequence of random variables $(Z_n)_{n \in \mathbb{N}}$ converges in the mod-Poisson sense with parameters $(\mu_n)_{n \in \mathbb{N}}$ if the following limit

$$\lim_{n \rightarrow +\infty} \exp(\mu_n(1 - e^{iu})) \mathbb{E} [e^{iuZ_n}] = \Phi(u)$$

exists for every $u \in \mathbb{R}$, and the convergence is locally uniform. The limiting function Φ is then continuous and $\Phi(0) = 1$.

This type of convergence gives stronger results than a central limit theorem, indeed it implies a CLT (and other properties we will see below). For the rest of the Section let us fix n^* and \bar{n} as in (1.5). We obtain

Proposition 3.3. *Let $x \geq 0$ be arbitrary and $x^* := xn^*$. Furthermore, let $t = e^{-v_n}$ with v_n as in (3.4). Then the random variables $(w_n(x^*))_{n \in \mathbb{N}}$ converge in the mod-Poisson sense with parameters*

$$\mu_n = \bar{n}w_\infty^{\mathbf{r}}(x) + o\left((\bar{n})^{1/2} \right),$$

where

$$w_\infty^{\mathbf{r}}(x) := \frac{\Gamma(\alpha, x)}{\Gamma(\alpha+1)}. \tag{3.5}$$

$\Gamma(\alpha, x)$ is the upper incomplete Gamma function.

Proof: We have

$$\mathbb{E}_t \left[e^{i s w_n(x^*)} \right] = \mathbb{E}_t \left[e^{i s \sum_{\ell=\lfloor x^* \rfloor}^\infty C_\ell} \right] = \exp \left((e^{i s} - 1) \sum_{\ell=\lfloor x^* \rfloor}^\infty \frac{\vartheta_\ell}{\ell} t^\ell \right). \tag{3.6}$$

This is the characteristic function of Poisson distribution. We thus obviously have mod-Poisson convergence with limiting function $\Phi(t) \equiv 1$. It remains to compute the parameter μ_n . Applying Lemma 2.3 for $x = 0$ and Lemma 2.4 for $x > 0$

together with (3.4) gives

$$\begin{aligned} & \sum_{\ell=\lfloor x^* \rfloor}^{+\infty} \frac{((\log \ell)^j \ell^{\alpha-1} + O(\ell^{\beta-1})) t^\ell}{\Gamma(\alpha + 1)} \\ &= \frac{1}{\Gamma(\alpha + 1)} (n^*)^\alpha \left(\frac{\partial}{\partial \alpha} - \log n^* \right)^j (\Gamma(\alpha, x) + o(1)) \\ &= \frac{1}{\Gamma(\alpha + 1)} (n^*)^\alpha (\log n^*)^j \left(\Gamma(\alpha, x) + o\left((n^*)^\alpha (\log n^*)^j \right) \right) \\ &= \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} \bar{n} + O(\bar{n}). \end{aligned} \tag{3.7}$$

We deduce that $\mu_n := \bar{n} w_\infty^{\mathbf{r}}(x) + o(\bar{n})$. This completes the proof. □

This yields a number of interesting consequences. In first place we can prove a CLT and detect the limit shape accordingly.

Corollary 3.4 (CLT and limit shape for randomization). *With the notation as above, we have as $n \rightarrow \infty$ with respect to \mathbb{P}_t*

$$\tilde{w}_n^{\mathbf{r}}(x) := \frac{w_n(x^*) - \bar{n} w_\infty^{\mathbf{r}}(x)}{\sqrt{\bar{n}}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, w_\infty^{\mathbf{r}}(x)). \tag{3.8}$$

Furthermore the limit shape of $w_n(x)$ is given by $w_\infty^{\mathbf{r}}(x)$ (with scaling $A_n = \bar{n}$ and $B_n = n^*$, see (1.2)). In particular, we can choose $\delta = 0$ in (1.2).

Proof: The CLT follows immediately from Kowalski and Nikeghbali (2010, Prop. 2.4), but also can be deduced easily from (3.6) by replacing s by $s(\bar{n})^{-1/2}$. It is also straightforward to show that $w_\infty^{\mathbf{r}}(x)$ is the limit shape. For a given $\epsilon > 0$, we choose $0 = x_0 < x_1 < \dots < x_\ell$ such that $w_\infty^{\mathbf{r}}(x_{j+1}) - w_\infty^{\mathbf{r}}(x_j) < \epsilon/2$ for $1 \leq j \leq \ell - 1$ and $w_\infty^{\mathbf{r}}(x_\ell) < \epsilon/2$. It is now easy to see that for each $x \in \mathbb{R}^+$

$$|(\bar{n})^{-1} w_n(x^*) - w_\infty^{\mathbf{r}}(x)| > \epsilon \implies \exists j \text{ with } |(\bar{n})^{-1} w_n(x_j^*) - w_\infty^{\mathbf{r}}(x_j)| > \epsilon/2.$$

Thus

$$\mathbb{P}_t \left[\sup_{x \geq 0} |\bar{n} w_n(x^*) - w_\infty^{\mathbf{r}}(x)| \geq \epsilon \right] \leq \sum_{j=1}^{\ell} \mathbb{P}_t [|\bar{n} w_n(x_j^*) - w_\infty^{\mathbf{r}}(x_j)| \geq \epsilon/2]. \tag{3.9}$$

It now follows from (3.8) that each summand in (3.9) tends to 0 as $n \rightarrow \infty$. This completes the proof. □

Another by-product of mod-Poisson convergence of a sequence $(Z_n)_{n \in \mathbb{N}}$ is that one can approximate Z_n with a Poisson random variable with parameter μ_n , see Kowalski and Nikeghbali (2010, Prop. 2.5). However in our situation this is trivial since $w_n(x^*)$ is already Poisson distributed.

As we are going to do in the next section, we are also interested in the (joint) behaviour of increments.

Proposition 3.5. *For all $x, y \in \mathbb{R}$, $y > x$, set*

$$w_n(x, y) := w_n(x) - w_n(y) \quad \text{and} \quad w_\infty^{\mathbf{r}}(x, y) := \frac{\Gamma(\alpha, x) - \Gamma(\alpha, y)}{\Gamma(\alpha + 1)}.$$

Then

$$\tilde{w}_n^{\mathbf{r}}(x, y) := \frac{w_n(x^*, y^*) - \bar{n}w_\infty^{\mathbf{r}}(x, y)}{(\bar{n})^{1/2} \sqrt{w_\infty^{\mathbf{r}}(x, y)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \tag{3.10}$$

as $n \rightarrow \infty$ with $x^* := xn^*$ and with $y^* := yn^*$.

Furthermore, $\tilde{w}_n^{\mathbf{r}}(x)$ and $\tilde{w}_n^{\mathbf{r}}(x, y)$ are asymptotically independent.

Remark 3.6. As we will see, the proof of independence relies on the independence of cycles coming from Lemma 3.1. Therefore it is easy to generalize the above result to more than two points.

Proof: The proof of (3.10) is almost the same as the proof of (3.8) and we thus omit it. Since

$$w_n(x, y) = \sum_{k=x^*}^{y^*-1} C_k \quad \text{and} \quad w_n(y) = \sum_{k=y^*}^{\infty} C_k$$

and all C_k are independent, we have that $\tilde{w}_n^{\mathbf{r}}(x)$ and $\tilde{w}_n^{\mathbf{r}}(x, y)$ are independent for each $n \in \mathbb{N}$. Thus $\tilde{w}_n^{\mathbf{r}}(x)$ and $\tilde{w}_n^{\mathbf{r}}(x, y)$ are also independent in the limit. \square

3.3. Functional CLT. The topic of this section is to prove a functional CLT for the profile $w_n(x)$ of the Young diagram. Similar results were obtained in a different framework by Hansen (1990); DeLaurentis and Pittel (1985) on the number of cycle counts not exceeding $n^{\lfloor x \rfloor}$, and by Beltoft et al. (2012) for Young diagrams confined in a rectangular box. We show

Theorem 3.7. *The process $\tilde{w}_n^{\mathbf{r}} : \mathbb{R}^+ \rightarrow \mathbb{R}$ (see (3.8)) converges weakly with respect to \mathbb{P}_t as $n \rightarrow \infty$ to a continuous process $\tilde{w}_\infty^{\mathbf{r}} : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $\tilde{w}_\infty^{\mathbf{r}}(x) \sim \mathcal{N}(0, \sigma_\infty^{\mathbf{r}}(x))$ and independent increments.*

The technique we will exploit is quite standardized (see Hansen (1990)). We remark that, unlike in this paper where the Ewens measure is considered, we do not obtain here a Brownian process, as the variance of $\tilde{w}_\infty^{\mathbf{r}}(t) - \tilde{w}_\infty^{\mathbf{r}}(s)$ for $r \geq s$ is more complicated than in the case of the Wiener measure.

We know from Proposition 3.5 the finite dimensional marginals of the process. More specifically we have for $x_\ell \geq x_{\ell-1} \geq \dots \geq x_1 \geq 0$ that

$$(\bar{n})^{-\alpha/2} (w_n(x_\ell^*), w_n(x_{\ell-1}^*) - w_n(x_\ell^*), \dots, w_n(x_1^*) - w_n(x_2^*)) \sim \mathcal{N}(\mathbf{0}, \Sigma') \tag{3.11}$$

where Σ' is a diagonal matrix with

$$\Sigma'_{11} = w_\infty^{\mathbf{r}}(x_\ell) \quad \text{and} \quad \Sigma'_{jj} = w_\infty^{\mathbf{r}}(x_{\ell-j+1}, x_{\ell-j+2}) \quad \text{for } j \geq 2.$$

Now all we need to show to complete the proof of Theorem 3.7 is the tightness of the process $\tilde{w}_n^{\mathbf{r}}$. In order to do so, we will proceed similarly to Hansen (1990), namely we will show that

Lemma 3.8. *We have for $0 \leq x_1 < x \leq x_2 < K$ with K arbitrary*

$$\mathbb{E}_t [(w_n^{\mathbf{r}}(x^*) - w_n^{\mathbf{r}}(x_1^*))^2 (w_n^{\mathbf{r}}(x_2^*) - w_n^{\mathbf{r}}(x^*))^2] = O((x_2 - x_1)^2) \tag{3.12}$$

with $x^* := xn^*$, $x_1^* := x_1n^*$ and $x_2^* := x_2n^*$.

Lemma 3.8 together with Billingsley (1999, Theorem 15.6) implies that the process $\tilde{w}_n^{\mathbf{r}}$ is tight. This and the marginals in (3.11) prove Theorem 3.7.

Proof of Lemma 3.8: We define

$$E^* := \mathbb{E}_t [(\tilde{w}_n^r(x^*) - \tilde{w}_n^r(x_1^*))^2(\tilde{w}_n^r(x_2^*) - \tilde{w}_n^r(x^*))^2]. \tag{3.13}$$

The independence of the cycle counts leads us to

$$\begin{aligned} E^* &= \left(\sum_{k=x_1^*}^{x^*-1} (\bar{n})^{-1} \frac{\theta_k}{k} t^k \right) \cdot \left(\sum_{k=x^*}^{x_2^*-1} (\bar{n})^{-1} \frac{\theta_k}{k} t^k \right) \\ &\stackrel{\text{Lem. 2.4}}{\sim} \left(\frac{(\bar{n})^{-1}}{\Gamma(\alpha+1)} \int_{x_1^*}^{x^*} (\log t)^j t^{\alpha-1} e^{-t} dt \right) \\ &\quad \left(\frac{(\bar{n})^{-1}}{\Gamma(\alpha+1)} \int_{x^*}^{x_2^*} (\log t)^j t^{\alpha-1} e^{-t} dt \right) \\ &= \left(\frac{(\bar{n})^{-1} \left(\frac{\bar{n}}{n}\right)^{-\alpha} \left(-\log\left(\frac{\bar{n}}{n}\right)\right)^j \left(\Gamma(\alpha, x_1) - \Gamma(\alpha, x)\right)}{\Gamma(\alpha+1)} \right) \\ &\quad \left(\frac{(\bar{n})^{-1} \left(\frac{\bar{n}}{n}\right)^{-\alpha} \left(-\log\left(\frac{\bar{n}}{n}\right)\right)^j \left(\Gamma(\alpha, x) - \Gamma(\alpha, x_2)\right)}{\Gamma(\alpha+1)} \right) + o(1) \\ &= O((x-x_1)(x_2-x)) = O((x_2-x_1)^2). \end{aligned}$$

Here we have used the fact that $\Gamma(\alpha, \cdot)$ is a Lipschitz function and the assumption that $x_1 < x \leq x_2 < K$. Also note that $(\bar{n})^{-1} \left(\frac{\bar{n}}{n}\right)^{-\alpha} \left(-\log\left(\frac{\bar{n}}{n}\right)\right)^j = O(1)$. \square

4. Saddle point method

The aim of this section is to study the asymptotic behaviour of $w_n(x)$ with respect to $\mathbb{P}_n[\cdot]$ as $n \rightarrow \infty$ and to compare the results with the results in Section 3.

There are at least two approaches with which to tackle this problem: one is more probabilistic and was employed by [Erlihson and Granovsky \(2008\)](#) in their paper. The second one was first developed in [Maples et al. \(2012\)](#) from the standard saddle point method.

The first method to study the asymptotic statistics of $w_n(x)$ with respect to $\mathbb{P}_n[\cdot]$ as $n \rightarrow \infty$ is the so called Khintchine method. We illustrate this method briefly with the normalisation constant h_n (see (1.3)). The first step is to write down a Khintchine’s type representation for the desired quantity. For h_n this is given by

$$h_n = t^{-n} \exp\left(\sum_{k=1}^n \frac{\vartheta_k}{k} t^k\right) \mathbb{P}_t \left[\sum_{k=1}^n kC_k = n \right] \tag{4.1}$$

with $t > 0$ and $\mathbb{P}_t[\cdot]$ as in Section 3. The second step is to choose the free parameter t in such a way that $\mathbb{P}_t[\sum_{k=1}^n kC_k = n]$ gets large. Here one can choose t to be the solution of the equation $\sum_{k=1}^n \vartheta_k t^k = n$.

This argumentation is very close to the argumentation relying on complex analysis and generating functions. Indeed, it is easy to see that (4.1) is equivalent to

$$h_n = [t^n] [\exp(g_\Theta(t))] \tag{4.2}$$

with $g_\Theta(t)$ as in (2.5). Furthermore, the choice of t is (almost) the solution of the saddle point equation $tg'_\Theta(t) = n$. We have of course to justify (4.2) (or (4.1)). But this follows immediately from the definition of h_n and Lemma 2.2.

We prefer at this point to work with the second approach. We begin by writing down the generating functions of the quantities we would like to study.

Lemma 4.1. *We have for $x \geq 0$ and $s \in \mathbb{R}$*

$$\mathbb{E}_n [\exp(-sw_n(x))] = \frac{1}{h_n} [t^n] \left[\exp \left(g_\Theta(t) + (e^{-s} - 1) \sum_{k=\lfloor x \rfloor}^{\infty} \frac{\vartheta_k}{k} t^k \right) \right]. \tag{4.3}$$

Remark 4.2. Although the expressions in Lemmas 4.1 and 4.3 hold in broader generality, starting from Subsection 4.1 we will calculate moment generating functions on the positive half-line, namely we can assume all parameters s_1, \dots, s_ℓ etc to be non-negative, according to Chareka (2007, Theorem 2.2).

Proof: It follows from the definitions of $\mathbb{P}_n[\cdot]$ and $w_n(x)$ (see (2.3)) that

$$\begin{aligned} h_n \mathbb{E}_n [\exp(-sw_n(x))] &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \exp \left(-s \sum_{k=\lfloor x \rfloor}^n C_k \right) \prod_{k=1}^n \vartheta_k^{C_k} \\ &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{k=1}^{\lfloor x \rfloor - 1} \vartheta_k^{C_k} \prod_{m=\lfloor x \rfloor}^{\infty} (\vartheta_m e^{-s})^{C_m}. \end{aligned} \tag{4.4}$$

Applying now Lemma 2.2, we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n \mathbb{E}_n [\exp(-sw_n(x))] = \exp \left(\sum_{k=1}^{\lfloor x \rfloor - 1} \frac{\vartheta_k}{k} t^k + e^{-s} \sum_{k=\lfloor x \rfloor}^{\infty} \frac{\vartheta_k}{k} t^k \right) \tag{4.5}$$

$$= \exp \left(g_\Theta(t) + (e^{-s} - 1) \sum_{k=\lfloor x \rfloor}^{\infty} \frac{\vartheta_k}{k} t^k \right). \tag{4.6}$$

Equation (4.3) now follows by taking $[t^n]$ on both sides. □

We are also interested in the joint behaviour at different points of the limit shape. The results in Section 3 suggest that the increments of $w_n(x_{j+1}) - w_n(x_j)$ are independent for $x_\ell \geq x_{\ell-1} \geq \dots \geq x_1 \geq 0$. It is thus natural to consider

$$\mathbf{w}_n(\mathbf{x}) = (w_n(x_\ell), w_n(x_{\ell-1}) - w_n(x_\ell), \dots, w_n(x_1) - w_n(x_2)). \tag{4.7}$$

We obtain

Lemma 4.3. *We have for $\mathbf{x} = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$ with $x_\ell \geq x_{\ell-1} \geq \dots \geq x_1 \geq 0$ and $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{R}^\ell$*

$$\mathbb{E}_n [\exp(-\langle \mathbf{s}, \mathbf{w}_n(\mathbf{x}) \rangle)] = \frac{1}{h_n} [t^n] \left[\exp \left(g_\Theta(t) + \sum_{j=1}^{\ell} (e^{-s_j} - 1) \sum_{k=\lfloor x_j \rfloor}^{\lfloor x_{j+1} \rfloor - 1} \frac{\vartheta_k}{k} t^k \right) \right] \tag{4.8}$$

with the convention $x_{\ell+1} := +\infty$. The proof of this lemma is almost the same as for Lemma 4.1 and we thus omit it.

4.1. *Log- n -admissibility.* The approach with which we first addressed the study of the limit shape is derived from the saddle point method for approximating integrals in the complex plane. We want to introduce the definition of log- n -admissible function, generalizing the analogous concept introduced in [Maples et al. \(2012\)](#). We stress that here, in comparison to the definition of log- (or equivalently Hayman) admissibility used there, we consider a family of functions parametrized by n for which log-admissibility holds simultaneously. The definition is therefore a natural extension.

Definition 4.4. Let $(g_n(t))_{n \in \mathbb{N}}$ with $g_n(t) = \sum_{k=0}^{\infty} g_{k,n} t^k$ be given with radius of convergence $\rho > 0$ and $g_{k,n} \geq 0$. We say that $(g_n(t))_{n \in \mathbb{N}}$ is log- n -admissible if there exist functions $a_n, b_n : [0, \rho) \rightarrow \mathbb{R}^+$, $R_n : [0, \rho) \times (-\pi/2, \pi/2) \rightarrow \mathbb{R}^+$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ s. t.

Saddle-point: For each n there exists $r_n \in [0, \rho)$ with

$$a_n(r_n) = n \tag{4.9}$$

Approximation: For all $|\varphi| \leq \delta_n$ we have the expansion

$$g_n(r_n e^{i\varphi}) = g_n(r_n) + i\varphi a_n(r_n) - \frac{\varphi^2}{2} b_n(r_n) + R_n(r_n, \varphi) \tag{4.10}$$

where $R_n(r_n, \varphi) = o(\varphi^3 \delta_n^{-3})$.

Divergence: $b_n(r_n) \rightarrow \infty$ and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Width of convergence: We have $\delta_n^2 b_n(r_n) - \log b_n(r_n) \rightarrow +\infty$ as $n \rightarrow +\infty$.

Monotonicity: For all $|\varphi| > \delta_n$, we have

$$\Re(g_n(r_n e^{i\varphi})) \leq \Re(g_n(r_n e^{\pm i\delta_n})). \tag{4.11}$$

The approximation condition allows us to compute the functions a_n and b_n exactly. We have

$$a_n(r) = r g'_n(r), \tag{4.12}$$

$$b_n(r) = r g'_n(r) + r^2 g''_n(r). \tag{4.13}$$

Clearly a_n and b_n are strictly increasing real analytic functions in $[0, \rho)$. The error in the approximation can similarly be bounded, so that

$$R_n(r, \varphi) = \varphi^3 O(r g'_n(r) + 3r^2 g''_n(r) + r^3 g'''_n(r)).$$

Having proved [Lemma 4.1](#) we are now able to write down in a more explicit way generating functions. What we are left with is trying to extract the coefficients of the expansion given therein. This is the content of

Theorem 4.5. Let $(g_n(t))_{n \in \mathbb{N}}$ be log- n -admissible with associated functions a_n, b_n and constants r_n . Call

$$G_n := [t^n] e^{g_n(t)}.$$

Then G_n has the asymptotic expansion

$$G_n = \frac{1}{\sqrt{2\pi}} (r_n)^{-n} b_n(r_n)^{-1/2} e^{g_n(r_n)} (1 + o(1)). \tag{4.14}$$

Remark 4.6. As it is explained in [Flajolet and Sedgewick \(2009, Chapter VIII\)](#) it is possible to take into account more error terms in the expansion of g_n . We could also extract here the behaviour of the coefficients h_n . However in the computations we will not need it explicitly, since these terms will always cancel out.

Proof of Theorem 4.5: The proof is exactly the same as in [Maples et al. \(2012, Prop. 2.2\)](#) and we thus give only a quick sketch of it, referring the reader to this paper for more details. As in the well-known saddle point method, we want to evaluate the integral

$$\frac{1}{2\pi i} \oint_{\gamma} \exp(g_n(z)) \frac{dz}{z^{n+1}}.$$

We choose as contour the circle $\gamma := r_n e^{i\varphi}$ with $\varphi \in [-\pi, \pi]$. On $\varphi \in [-\delta_n, \delta_n]$ after changing to polar coordinates we can expand the function g_n as

$$\frac{1}{2\pi i r_n^n} \int_{-\delta_n}^{\delta_n} \exp\left(g_n(r_n) + i\varphi a_n(r_n) - \frac{\varphi^2}{2} b_n(r_n) + o(\varphi^3 \delta_n^{-3}) - i n \varphi\right) d\varphi$$

We now choose r_n such that $a(r_n) = r_n g'_n(r_n) = n$ in order to cancel the linear terms in n . This allows us to approximate the integral on the minor arc with a Gaussian. One shows that away from the saddle point (so for $|\varphi| > \delta_n$) the contribution is exponentially smaller than on the minor arc and thus it can be neglected. \square

We would like to emphasize also that it will be not always possible to solve the saddle point equation (4.9) exactly. However it is enough to find an r_n such that

$$a_n(r_n) - n = o\left(\sqrt{b_n(r_n)}\right) \tag{4.15}$$

holds.

4.2. Calculation of the limit shape. In this section we will derive the limit shape for Young diagrams for the class of measures given by the weights. We will not go into all the details to prove the log- n -admissibility for the most general case, but will try to give a precise overview of the main steps nonetheless. One important remark we have to make is that our parameter s will not be fixed, but will be scaled and hence dependent on n . This comes from the fact that for a fixed s (4.9) becomes a fixed point equation whose solution cannot be given constructively, but has only an implicit form. We were not able to use this information for our purposes, and hence preferred to exploit a less general, but more explicit parameter to calculate asymptotics.

4.2.1. Limit shape. The main goal of this subsection is to prove that the weights (1.4) induce a sequence of log- n -admissible functions of which we can recover the asymptotics of $g_n(r_n)$. This will give us the limit shape of the Young diagram according to Theorem 4.5. Hence we pass to showing Theorem 1.1, that is the limit shape is

$$w_{\infty}^s(x) = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)},$$

and the fluctuations at a point x behave like

$$\tilde{w}_n^s(x) = \frac{w_n(x^*) - \bar{n}(w_{\infty}^s(x) + z_n^s)}{(\bar{n})^{1/2}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_{\infty}^2(x))$$

with $x^* = xn^*$ and $\sigma_{\infty}^2(x)$ as in Theorem 1.1.

Remark 4.7. We note that the limit shape matches the one obtained in [Erlihson and Granovsky \(2008, Thm. 4.8\)](#) and also the one obtained in the present paper in the randomized case (cf. the definition of $w_{\infty}^r(x)$ of Prop. 3.3).

As indicated in the introduction, we prove Theorem 1.1 by computing the Laplace transform. We now have

Proposition 4.8. *We have for $s = O(1)$ and with respect to \mathbb{P}_n as $n \rightarrow \infty$*

$$\mathbb{E}_n [\exp(-s\tilde{w}_n^s(x))] = \sigma_\infty^2(x) \frac{s^2}{2} + O\left((\bar{n})^{-\frac{1}{2}} s^3\right).$$

Obviously, Proposition 4.8 immediately implies Theorem 1.1. Moreover, knowing the behaviour of the Laplace transform enables us to compute the asymptotics of the Young diagram in the limit. More precisely, we now can prove the large deviation estimates in Prop. 1.3.

Proof of Prop. 1.3: The strategy we adopt was first exploited in Maples et al. (2012, Theorem 4.1). Specifically, let σ_n be the limit variance as in Thm. 1.1. Define the normalized log-moment generating function as (1.8). One bounds then the probability $\mathbb{P}(|\tilde{w}_n^s(x^*) - a| \leq \epsilon)$ by a random variable Y of mean a and using the second moment method. We omit the details since the computations are almost the same as in Maples et al. (2012). \square

We can also determine the behaviour of the increments of the function $w_n(\cdot)$.

Theorem 4.9. *For $\ell \geq 2$ and $x_\ell \geq x_{\ell-1} \geq \dots \geq x_1 \geq 0$, let*

$$\tilde{w}_n^s(\mathbf{x}) = (\tilde{w}_n^s(x_\ell), \tilde{w}_n^s(x_{\ell-1}) - \tilde{w}_n^s(x_\ell), \dots, \tilde{w}_n^s(x_1) - \tilde{w}_n^s(x_2)).$$

Set $x_{\ell+1} = +\infty$. For $1 \leq j < i < \ell$ we have that

$$\begin{aligned} \tilde{w}_\infty^s(x_i, x_j) &:= \lim_{n \rightarrow +\infty} \text{Cov}(\tilde{w}_n^s(x_j) - \tilde{w}_n^s(x_{j+1}), \tilde{w}_n^s(x_i) - \tilde{w}_n^s(x_{i+1})) \\ &= \frac{(\Gamma(\alpha + 1, x_i) - \Gamma(\alpha + 1, x_{i+1}))(\Gamma(\alpha + 1, x_j) - \Gamma(\alpha + 1, x_{j+1}))}{\Gamma(\alpha + 1)\Gamma(\alpha + 2)}. \end{aligned} \tag{4.16}$$

Remark 4.10. Let us comment briefly on Thm. 4.9. What we obtained in this result is most unexpected: cycle counts are asymptotically independent under very mild assumptions (see Lemma 2.1). The assumption of the lemma holds in our case as the growth of the parameters ϑ_n is algebraic. The fact that the increments depend on disjoint sets of cycles would have suggested the asymptotic independence of $w_n(y^*)$ from $w_n(x^*) - w_n(y^*)$. We are aware of the work of Babu et al. (2007) handling this issue in the case of the Ewens sampling formula, in particular showing that partial sums of cycle counts need not converge to processes with independent increments. Our result extends this idea in the sense that it shows the explicit covariance matrix for a whole category of generating functions. It would be interesting to provide a heuristic explanation for this theorem.

4.2.2. *Log-n-admissibility.* In order to determine the limit shape we would like to prove the log-n-admissibility of the function explicited in (4.3). To be more precise, what we have to prove is

Lemma 4.11. *Let ϑ_k be as in (1.4), $s, x \geq 0$ and $x^* = xn^*$, $s^* = s(\bar{n})^{1/2}$ with n^*, \bar{n} as in (1.5). Then*

$$g_\Theta(t) + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{\infty} \frac{\theta_k}{k} t^k$$

is \log - n -admissible for all $x, s \geq 0$ with

$$r_n := e^{-v_n}, v_n = p_n(1 - sq_n) \tag{4.17}$$

for some

$$p_n \sim (n^*)^{-1} \quad \text{and} \quad q_n \sim (n^*)^{-1/2} \frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)}. \tag{4.18}$$

Let us consider p_n and q_n in the case $\vartheta_k = \frac{(\log k)^j k^\alpha}{\Gamma(\alpha+1)}$ more explicitly. In this situation, p_n is the solution of the equation

$$\sum_{k=1}^{\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kp_n} = n. \tag{4.19}$$

Using Lemma 2.3, one can easily show that $p_n \sim (n^*)^{-1}$. Furthermore, we have

$$q_n = \frac{(np_n)^{-1/2} \left(\frac{\partial}{\partial \alpha} - \log p_n\right)^j \Gamma(\alpha + 1, x)}{(\alpha + 1) \left(\frac{\partial}{\partial \alpha} - \log p_n\right)^j \Gamma(\alpha + 1) + j \left(\frac{\partial}{\partial \alpha} - \log p_n\right)^{j-1} \Gamma(\alpha + 1)}. \tag{4.20}$$

The lower order terms of p_n and q_n in (4.19) and (4.20) are important for the saddle point solution in the sense that the condition $a_n(r_n) = n + o(\sqrt{b_n})$ is fulfilled (see (4.15)). However, for the computation of the limit shape, the fluctuations and the cumulants, we require only the leading term of p_n and q_n .

We require in the proof of Lemma 4.11 the observation

$$p_n \sim (n^*)^{-1} \quad \text{and} \quad p_n^{-\alpha} (-\log p_n)^j \sim \bar{n}, \tag{4.21}$$

which follows with a straightforward computation.

Proof of Lemma 4.11: We now verify the conditions in Definition 4.4.

Saddle-point and approximation: We begin with the case $\beta = 0$ and compute first the size of $b(r_n)$. We get with Lemma 2.3, (4.13) and (4.21)

$$\begin{aligned} b_n(r_n) &= \sum_{k=1}^{+\infty} \frac{(\log k)^j k^{\alpha+1}}{\Gamma(\alpha + 1)} e^{-kv_n} + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{(\log k)^j k^{\alpha+1}}{\Gamma(\alpha + 1)} e^{-kv_n} \\ &= O\left(\sum_{k=1}^{+\infty} (\log k)^j k^{\alpha+1} e^{-kv_n}\right) = O\left(v_n^{-\alpha-2} (\log v_n)^j\right). \end{aligned} \tag{4.22}$$

We thus have to show that $a_n(r_n) = n + o\left(\left(v_n^{-\alpha-2} (\log v_n)^j\right)^{1/2}\right)$ to get a suitable saddle point solution, see (4.15). Using the asymptotic behaviour of v_n , we have to show $a_n(r_n) = n + o\left(n^{\frac{\alpha+2}{2(\alpha+1)}} (\log n)^j\right)$. We use (4.12) and obtain

$$a_n(r_n) = \sum_{k=1}^{+\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kv_n} + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kv_n}. \tag{4.23}$$

We begin with the first sum. We use Lemma 2.3 and get

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kv_n} = \frac{v_n^{-\alpha-1}}{\Gamma(\alpha + 1)} \left(\frac{\partial}{\partial \alpha} - \log v_n \right)^j \Gamma(\alpha + 1) + O(1) \quad (4.24) \\ &= \frac{(p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)(1 - q_n)^{\alpha+1}} \cdot \left(\frac{\partial}{\partial \alpha} - \log p_n + \log(1 - q_n) \right)^j \Gamma(\alpha + 1) + O(1). \end{aligned}$$

We choose at this point p_n to be the solution of

$$n = \frac{(p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)(1 - q_n)^{\alpha+1}} \cdot \left(\frac{\partial}{\partial \alpha} - \log p_n \right)^j \Gamma(\alpha + 1). \quad (4.25)$$

A straightforward computation gives $p_n \sim (n^*)^{-1}$. We now expand $(1 - q_n)^{-\alpha-1}$ and $\log(1 - q_n)$. This then gives

$$\begin{aligned} & \frac{p_n^{-\alpha-1}}{\Gamma(\alpha + 1)} \left(\frac{\partial}{\partial \alpha} - \log p_n \right)^j \Gamma(\alpha + 1) \\ &+ \frac{(\alpha + 1)q_n (p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)} \left(\frac{\partial}{\partial \alpha} - \log(p_n) \right)^j \Gamma(\alpha + 1) \\ &+ \frac{q_n p_n^{-\alpha-1}}{\Gamma(\alpha + 1)} j \left(\frac{\partial}{\partial \alpha} - \log p_n \right)^{j-1} \Gamma(\alpha + 1) + O((\log p_n)^j p_n^{-\alpha-1} q_n^2) \\ &= n + \frac{(\alpha + 1)q_n (p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)} \left(\frac{\partial}{\partial \alpha} - \log p_n \right)^j \Gamma(\alpha + 1) \quad (4.26) \\ &+ \frac{q_n (p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)} j \left(\frac{\partial}{\partial \alpha} - \log(p_n) \right)^{j-1} \Gamma(\alpha + 1) + O\left(n^{\frac{1}{1+\alpha}} (\log n)^{-\frac{j}{1+\alpha}}\right). \end{aligned}$$

Since $\frac{1}{1+\alpha} < \frac{2+\alpha}{2(1+\alpha)}$, we can ignore the big- O term. We now come to the second sum. We use Lemma 2.4 with $z_n = xn^*$ and a similar estimate as in (4.22) to obtain

$$\begin{aligned} & (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kv_n} \\ &= -s^* \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{(\log k)^j k^\alpha}{\Gamma(\alpha + 1)} e^{-kv_n} + O\left((s^*)^2 v_n^{-\alpha-1} (\log v_n)^j\right) \\ &= -s^* \frac{(p_n)^{-\alpha-1}}{\Gamma(\alpha + 1)} \left(\frac{\partial}{\partial \alpha} - \log(p_n) \right)^j \Gamma(\alpha + 1, x) + O\left(n^{\frac{1}{1+\alpha}} (\log n)^{-\frac{j}{1+\alpha}}\right). \quad (4.27) \end{aligned}$$

We can ignore here also the big- O term. We thus have to check that the remaining terms in (4.26) and (4.27) combined give $a_n(r_n) = n + o\left(\sqrt{b_n(r_n)}\right)$. This follows from the definition of q_n , see (4.20).

We now come to the case $\beta < \alpha/2$. We have

$$\begin{aligned} & \sum_{k=1}^{+\infty} \frac{(\log k)^j k^\beta}{\Gamma(\alpha + 1)} e^{-kv_n} + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{(\log k)^j k^\beta}{\Gamma(\alpha + 1)} e^{-kv_n} \\ &= O\left(\sum_{k=1}^{+\infty} \frac{(\log k)^j k^\beta}{\Gamma(\alpha + 1)} e^{-kv_n}\right) = O\left(n^{\frac{\beta+1}{\alpha+1}} (\log n)^{\frac{1}{\alpha+1}}\right). \end{aligned} \tag{4.28}$$

Since $\beta < \alpha/2$, we obtain again $a_n(r_n) = n + o\left(\sqrt{b_n(r_n)}\right)$. This completes the proof of this point. For completeness, we give in Remark 4.12 some hints how to adjust this proof to the case $\vartheta_m \sim (\log m)^j m^\alpha$.

Divergence: By the above calculations we set $\delta_n := (n^*)^{-\xi}$ with $\frac{\alpha+3}{3} < \xi < \frac{\alpha+2}{2}$. This position holds also in the case $\beta > 0$.

Monotonicity: In the region $|\varphi| > \delta_n$ we wish to show (4.11). We distinguish between the cases $\varphi = o(v_n)$, $\varphi \neq o(v_n)$ and $|\varphi| = o(1)$, and finally $|\varphi| > C$. First remember that $g_n(r_n e^{\pm i\delta_n}) = O\left((\log n^*)^j (n^*)^\alpha\right)$ by Lemma 2.3.

Thus here we have:

- (1) if $\varphi = o(v_n)$, then by a change of variable $t \rightsquigarrow (v_n - i\varphi)t$

$$\begin{aligned} & \sum_{k \geq \lfloor x^* \rfloor} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha + 1)} e^{-k(v_n - i\varphi)} \\ & \sim \frac{(v_n - i\varphi)^{-\alpha}}{\Gamma(\alpha + 1)} \int_x^{+\infty} (\log t)^j t^{\alpha-1} e^{-t} dt \\ & \sim \left(\frac{\partial}{\partial \alpha} + \log n^*\right)^j \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} (v_n - i\varphi)^{-\alpha}. \end{aligned}$$

Considering the factor $e^{-s^*} - 1$ we obtain that the summand is negligible with respect to $\Re(g(r_n e^{\pm i\delta_n}))$.

- (2) If $\varphi \neq o(v_n)$ but $|\varphi| = o(1)$, then

$$\begin{aligned} & \sum_{k \geq \lfloor x^* \rfloor} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha + 1)} e^{-k(v_n - i\varphi)} \\ & \sim \frac{(v_n - i\varphi)^{-\alpha}}{\Gamma(\alpha + 1)} \int_{x - ix\varphi n^* + o(1)}^{+\infty} (\log k)^j t^{\alpha-1} e^{-t} dt \\ & \sim \left(\frac{\partial}{\partial \alpha} + \log n^*\right)^j \frac{\Gamma(\alpha, x - ix\varphi n^*)}{\Gamma(\alpha + 1)} (v_n - i\varphi)^{-\alpha} \end{aligned} \tag{4.29}$$

and afterwards use the fact that $\Gamma(\alpha, x + iy) = O(y^{\alpha-1})$ for $|y|$ large. Hence the RHS of (4.29) becomes

$$O\left((n^*)^{\alpha-1}\right) (v_n - i\varphi)^{-\alpha} = O\left((n^*)^{\alpha-1} \varphi^{-1}\right).$$

As $\varphi \neq o(v_n)$, we obtain that $O\left((n^*)^{\alpha-1} \varphi^{-1}\right) = O\left((n^*)^\alpha\right) o(1)$ which is enough to show (4.11) in this region.

- (3) To conclude we consider the case $|\varphi| > C$: the function $g_n(r_n e^{i\varphi})$ is bounded there by a constant uniform in n , and then by bounding

$g_n(r_n e^{i\varphi})$ through its modulus we have

$$\Re(g_n(r_n e^{i\varphi})) \leq \Re(g(r_n e^{\pm i\delta_n})) \left(1 + O\left((n^*)^{-\alpha/2}\right)\right). \tag{4.30}$$

□

Remark 4.12. One can use a similar argumentation as in the proof of Lemma 4.11 for the more general weights $\vartheta_m \sim (\log m)^j m^\alpha$. For this has to replace p_n to be the solution of

$$n = \sum_{k=1}^{\infty} \theta_k e^{-kp_n}. \tag{4.31}$$

In other words, e^{-p_n} is the saddle point solution for the function $g_\theta(t)$. Furthermore it is straightforward to see that the computations in (4.26) and (4.27) are similar and that one has to choose q_n to be the solution of the equation

$$\begin{aligned} & q_n (p_n)^{-\alpha-1} \left(\left(\frac{\partial}{\partial \alpha} - \log p_n \right)^j + j \left(\frac{\partial}{\partial \alpha} - \log(p_n) \right)^{j-1} \right) \Gamma(\alpha + 1) \\ &= -s^* \sum_{k=\lfloor x^* \rfloor}^{+\infty} \theta_k e^{-kp_n(1-sq_n)}. \end{aligned}$$

In order to show Thms. 1.1, 1.4 and 4.9 we need to prove first an auxiliary proposition.

Proposition 4.13. *Let v_n, \bar{n}, n^*, x^* and s^* be as in Lemma 4.11. We then have as $n \rightarrow +\infty$*

$$\begin{aligned} & (e^{-s^*} - 1) \sum_{k \geq \lfloor x^* \rfloor} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha + 1)} e^{-kv_n} \tag{4.32} \\ &= \left(-s(\bar{n})^{\frac{1}{2}} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} (1 + o(1)) + \frac{s^2}{2} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 1, x)^2}{\Gamma(\alpha + 1)\Gamma(\alpha + 2)} s^2 \right) + o(1). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & [s^m] \left[(e^{-s^*} - 1) \sum_{k \geq \lfloor x^* \rfloor} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha + 1)} e^{-kv_n} \right] \tag{4.33} \\ & \sim (\bar{n})^{1-\frac{m}{2}} [s^m] \left[(e^{-s} - 1) \sum_{k=0}^{\infty} \frac{s^k}{k!} \frac{\Gamma(\alpha + k, x)}{\Gamma(\alpha + 1)} \left(-\frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)} \right)^k \right]. \end{aligned}$$

Proof: We apply Lemma 2.4 with $z_n = x^* = xn^*$, $v_n = (\frac{\bar{n}}{n})(1 - sq_n)$, $\delta = \alpha - 1$ and $\tilde{q}_n = sq_n$. We then have for any $\ell > m$

$$\begin{aligned} \sum_{k=\lfloor x^* \rfloor}^{\infty} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha + 1)} e^{-kv_n} &= \frac{(\frac{\bar{n}}{n})^{-\alpha}}{\Gamma(\alpha + 1)} \left(\sum_{k=0}^{\ell} \left(\frac{\partial}{\partial \delta} - \log \left(\frac{\bar{n}}{n} \right) \right)^j \frac{\Gamma(\alpha + k, x)}{k!} (-sq_n)^k \right) \\ &+ O \left(\left(\log \left(\frac{\bar{n}}{n} \right) \right)^j (sq_n)^{\ell+1} + \left(\frac{\bar{n}}{n} \right)^\delta \left(\log \left(\frac{\bar{n}}{n} \right) \right)^j \right). \tag{4.34} \end{aligned}$$

We now use (4.21) and (4.17)

$$\begin{aligned}
 & [s^m] \left[\frac{\left(\frac{\bar{n}}{n}\right)^{-\alpha}}{\Gamma(\alpha+1)} \left(\sum_{k=0}^{\ell} \left(\frac{\partial}{\partial \delta} - \log \left(\frac{\bar{n}}{n} \right) \right)^j \frac{\Gamma(\alpha+k, x)}{k!} (-sq_n)^k \right) \right] \\
 &= (-q_n)^m \frac{\left(\frac{\bar{n}}{n}\right)^{-\alpha}}{\Gamma(\alpha+1)} \left(\frac{\partial}{\partial \delta} - \log \left(\frac{\bar{n}}{n} \right) \right)^j \frac{\Gamma(\alpha+m, x)}{m!} \\
 &\sim (-q_n)^m \frac{\left(\frac{\bar{n}}{n}\right)^{-\alpha}}{\Gamma(\alpha+1)} \left(-\log \left(\frac{\bar{n}}{n} \right) \right)^j \frac{\Gamma(\alpha+m, x)}{m!} \sim (-q_n)^m \frac{\bar{n}}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+m, x)}{m!} \\
 &\sim \frac{(\bar{n})^{1-\frac{m}{2}}}{m!} \frac{\Gamma(\alpha+m, x)}{\Gamma(\alpha+1)} \left(-\frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)} \right)^m.
 \end{aligned}$$

It is now easy to see the $O(\cdot)$ in (4.34) is uniform in s for s bounded. Applying Cauchy’s integral formula thus gives

$$[s^m] \left[\sum_{k=\lfloor x^* \rfloor}^{\infty} \frac{(\log k)^j k^{\alpha-1}}{\Gamma(\alpha+1)} e^{-kv_n} \right] \sim \frac{(\bar{n})^{1-\frac{m}{2}}}{m!} \frac{\Gamma(\alpha+m, x)}{\Gamma(\alpha+1)} \left(-\frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)} \right)^m. \tag{4.35}$$

One the other hand we have $[s^m] [e^{s^*} - 1] = (\bar{n})^{\frac{m}{2}} [s^m] [e^s - 1]$. Combining this observation with (4.35) then proves (4.33). Equation (4.32) then follows from (4.33) with a direct computation. \square

Remark 4.14. (4.34) and (4.35) yield the behavior of z_n^s plugging in $m = 1$ (cf. Remark 1.2).

Proof of Proposition 4.8 and Theorem 1.4: To determine the behaviour of G_n we would like to use Lemma 4.1. By (4.3)

$$\mathbb{E}_n [\exp(-s^* w_n(x^*))] = \frac{1}{h_n} [t^n] \left[\exp \left(g_{\Theta}(t) + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{\vartheta_k}{k} t^k \right) \right].$$

We have shown that $g_n(t) = g_{\Theta}(t) + (e^{-s^*} - 1) \sum_{k=\lfloor x^* \rfloor}^{+\infty} \frac{\vartheta_k}{k} t^k$ is log- n -admissible. Therefore Thm. 4.5 tells us how G_n behaves, and we have more precisely to recover three terms. In first place we collect the terms for the asymptotic of $e^{g_n(r_n)}$: We use Lemma 2.3 and get

$$g_{\Theta}(r_n) = \frac{v_n^{-\alpha}}{\Gamma(\alpha+1)} \left(\frac{\partial}{\partial \alpha} - \log \left(\frac{\bar{n}}{n} \right) \right)^j \Gamma(\alpha) + O(1). \tag{4.36}$$

Computing the coefficient of s^m gives

$$\begin{aligned}
 [s^m] [g_{\Theta}(r_n)] &\sim \frac{\left(\frac{\bar{n}}{n}\right)^{-\alpha}}{\Gamma(\alpha+1)} \left(-\log \left(\frac{\bar{n}}{n} \right) \right)^j \Gamma(\alpha) [s^m] [(1 - sq_n)^{-\alpha}] \\
 &= \frac{\bar{n}}{\alpha} [s^m] \left[\left(1 - \frac{s}{(\bar{n})^{1/2}} \frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)} \right)^{-\alpha} \right] \\
 &= \frac{(\bar{n})^{1-\frac{m}{2}}}{\alpha} [s^m] \left[\left(1 - s \frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+2)} \right)^{-\alpha} \right].
 \end{aligned}$$

We thus obtain

$$g_{\Theta}(r_n) = \frac{(\bar{n})}{\alpha} (1 + o(1)) + s(\bar{n})^{1/2} \frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)} (1 + o(1)) + \frac{s^2}{2} \frac{\Gamma(\alpha + 1, x)^2}{\Gamma(\alpha + 2)\Gamma(\alpha + 1)} (1 + o(1)) + O\left(s^3(\bar{n})^{-\frac{1}{2}}\right),$$

where $(1 + o(1))$ is independent of s . Furthermore, we get with Proposition 4.13

$$(e^{-s^*} - 1) \sum_{k \geq \lfloor x^* \rfloor} \frac{\vartheta_m}{m} e^{-kv_n} = -s(\bar{n})^{\frac{1}{2}} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} (1 + o(1)) + s^2 \left(\frac{1}{2} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 1, x)^2}{\Gamma(\alpha + 1)\Gamma(\alpha + 2)} \right) (1 + o(1)) + O\left(s^3(\bar{n})^{-\frac{1}{2}}\right).$$

Thirdly, we obviously have

$$-n \log(r_n) = \bar{n}(1 + o(1)) - s(\bar{n})^{1/2} \frac{\Gamma(\alpha + 1, x)}{\Gamma(\alpha + 2)} (1 + o(1)).$$

Finally we can use a similar computation as in Proposition 4.13 to see that

$$\log(b(r_n)) = C_1 \log(n) + C_2 \sum_{k \geq 0} s^k (\bar{n})^{-\frac{k}{2}}.$$

We thus see that the contribution of b_n to the coefficients of s^m is of lower order. We combine everything and obtain

$$\mathbb{E}_n [\exp(-s^* w_n(x^*))] = \exp\left(-s(\bar{n})^{\frac{1}{2}} \frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} (1 + o(1))\right) + \frac{s^2}{2} \left(\frac{\Gamma(\alpha, x)}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 1, x)^2}{\Gamma(\alpha + 1)\Gamma(\alpha + 2)} \right) (1 + o(1)) + O\left(s^3(\bar{n})^{-\frac{1}{2}}\right). \tag{4.37}$$

Notice that we do not require the asymptotic behaviour of the coefficient of s^0 since $\mathbb{E}_n [\exp(-s^* w_n(x^*))] = 1$ at $s = 0$. This completes the proof of Proposition 4.8 (and Theorem 1.1). The proof of Theorem 1.4 follows the same line and we thus omit it. \square

Proof of Thm. 4.9: For multiple increments, we can repeat the proof of Theorem 1.4 to compute the behaviour of the vector $\mathbf{w}_n(\mathbf{x}^*)$ with $\mathbf{w}_n(\mathbf{x}^*)$ as in (4.7) with length $\ell \geq 2$. Applying Lemma 4.3 to $\mathbf{w}_n(\mathbf{x}^*)$ shows that we have to consider the function

$$g_{\Theta}(t) + \sum_{j=1}^{\ell} (e^{-s_j^*} - 1) \sum_{k=\lfloor x_j^* \rfloor}^{\lfloor x_{j+1}^* - 1 \rfloor} \frac{\vartheta_k}{k} t^k. \tag{4.38}$$

A computation as in Lemma 4.11 shows that this function is also log- n -admissible for $r_n = e^{-v_n}$ with

$$v_n := p_n \left(1 - \frac{q_n}{\Gamma(\alpha + 2)} (s_\ell \Gamma(\alpha + 1, x_\ell) + \sum_{k=1}^{\ell-1} s_{\ell-k} (\Gamma(\alpha + 1, x_{\ell-1-k}) - \Gamma(\alpha + 1, x_{\ell-k}))) \right).$$

for some p_n and q_n with $p_n \sim (n^*)^{-1}$ and $q_n \sim (n^*)^{-1/2}$. We deduce from this as in (4.37) that

$$\begin{aligned} g_\Theta(r_n) &\sim \frac{\bar{n}}{\alpha} (1 + o(1)) - \frac{(\bar{n})^{1/2}}{\Gamma(\alpha + 2)} \\ &\left(s_\ell \Gamma(\alpha + 1, x_\ell) + \sum_{k=1}^{\ell-1} (s_{\ell-k-1} (\Gamma(\alpha + 1, x_{\ell-k-1}) - \Gamma(\alpha + 1, x_{\ell-k}))) \right) (1 + o(1)) \\ &+ \frac{1}{2\Gamma(\alpha + 2)\Gamma(\alpha + 1)} (s_\ell \Gamma(\alpha + 1, x_\ell) \\ &+ \sum_{k=1}^{\ell-1} (s_{\ell-k-1} (\Gamma(\alpha + 1, x_{\ell-k-1}) - \Gamma(\alpha + 1, x_{\ell-k}))) \right)^2 (1 + o(1)) \\ &+ o(1). \end{aligned} \tag{4.39}$$

Since the coefficients of the form $(e^{-s_j^*} - 1) \sum_{k=x_j^*}^{x_{j+1}^*-1} \frac{\vartheta_k}{k} r_n^k$ do not give a contribution to covariances, the mixed terms will stem from the expansion of the square in (4.39). In particular we see that the coefficient of $s_i s_j$, for $1 \leq j < i < \ell$, is

$$\frac{(\Gamma(\alpha + 1, x_i) - \Gamma(\alpha + 1, x_{i+1})) (\Gamma(\alpha + 1, x_j) - \Gamma(\alpha + 1, x_{j+1}))}{2\Gamma(\alpha + 1)\Gamma(\alpha + 2)}.$$

□

4.3. *Functional CLT for $w_n(\cdot)$.* As in the randomized setting, a functional CLT can be obtained here too. Unlike the previous case though we do not have the independence of cycle counts, hence we will have to show the tightness of the fluctuations as in Sec. 3.3 in two steps (cf. Hansen (1990)). The result we aim at is, precisely as before,

Theorem 4.15. *The process $\tilde{w}_n^s : \mathbb{R}^+ \rightarrow \mathbb{R}$ (see Thm. 1.1) converges weakly with respect to \mathbb{P}_n as $n \rightarrow \infty$ to a continuous process $\tilde{w}_\infty^s : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $\tilde{w}_\infty^s(x) \sim \mathcal{N}(0, (\sigma_\infty^s(x))^2)$ and whose increments are not independent. The covariance structure is given in Thm. 4.9.*

We will proceed as in the proof of Thm. 3.7. Having shown already the behaviour of the increments in Thm. 4.9 what we have to tackle now is their tightness. The proof’s goal is analogous to Lemma 3.8. However the evaluation of the corresponding expectation on the LHS of (4.43) is more difficult this time; one possible approach is present in DeLaurentis and Pittel (1985) and is based on Pólya’s enumeration lemma and the calculation of factorial moments of cycle counts. However it is easier to use the argumentation by Hansen in Hansen (1990) to obtain an expression for the corresponding generating function. We get

Lemma 4.16. For $0 \leq x_1 < x \leq x_2$ arbitrary and $x^* := xn^*$, $x_1^* := x_1n^*$ and $x_2^* := x_2n^*$

$$\begin{aligned} & (\bar{n})^2 \cdot h_n \mathbb{E}_n \left[(\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*))^2 (\tilde{w}_n^s(x_2^*) - \tilde{w}_n^s(x^*))^2 \right] \\ &= [t^n] \left[\left((g_{x_1^*}^{x^*}(t) - E_{x_1}^x)^2 + g_{x_1^*}^{x^*}(t) \right) \left((g_{x_2^*}^{x^*}(t) - E_{x_2}^x)^2 + g_{x_2^*}^{x^*}(t) \right) \exp(g_\Theta(t)) \right] \end{aligned} \tag{4.40}$$

with $g_a^b(z) := \sum_{a \leq j < b} \frac{\vartheta_j}{j} z^j$ and $E_a^b = \mathbb{E}_n [\tilde{w}_n^s(bn^*) - \tilde{w}_n^s(an^*)]$ for $a < b$.

Proof: We define for $0 < t < 1$ the measure \mathbb{P}_t on $\mathfrak{S} := \cup_n \mathfrak{S}_n$ as in Section 3. By repeating the proof of Hansen (1990, Lemma 2.1) we see that

$$\mathbb{P}_t \left[\sum k C_k = n \right] = t^n h_n e^{g_\Theta(t)}.$$

Let now $\Psi : \mathfrak{S} \rightarrow \mathbb{C}$ with $\mathbb{E}_t [|\Psi|] < \infty$ and Ψ only depending on the cycles counts, i.e. $\Psi = \Psi(C_1, C_2, \dots)$. Mimicking Hansen’s strategy, one can prove for such a Ψ that

$$\mathbb{E}_t [\Psi] e^{g_\Theta(t)} = \sum_{n \geq 1} t^n h_n \mathbb{E}_n [\Psi_n] + \Psi(0, 0, 0, \dots), \tag{4.41}$$

where $\Psi_n : \mathfrak{S}_n \rightarrow \mathbb{C}$ is defined by $\Psi_n = \Psi(C_1, \dots, C_n, 0, 0, \dots)$. The proof of (4.41) is almost the same as in Hansen (1990) and we thus omit it. We now use

$$\Psi(C_1, C_2, \dots) := (\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*))^2 (\tilde{w}_n^s(x_2^*) - \tilde{w}_n^s(x^*))^2.$$

We use the definition of w_n^s in Theorem 1.1 and that the C_k are independent with respect to \mathbb{P}_t and get

$$\begin{aligned} & (\bar{n}) \mathbb{E}_t \left[(\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*))^2 \right] = \mathbb{E}_t \left[\left(\sum_{k=x_1^*}^{x_1^*} C_k - E_{x_1}^x \right)^2 \right] \\ &= \mathbb{E}_t \left[\sum_{k=x_1^*}^{x_1^*} C_k^2 + \sum_{\substack{x_1^* \leq k, k' < x_1^* \\ k \neq k'}} C_k C_{k'} - 2E_{x_1}^x \sum_{k=x_1^*}^{x_1^*} C_k + (E_{x_1}^x)^2 \right] \\ &= \mathbb{E}_t \left[\sum_{k=x_1^*}^{x_1^*} C_k(C_k - 1) + \sum_{\substack{x_1^* \leq k, k' < x_1^* \\ k \neq k'}} C_k C_{k'} - 2E_{x_1}^x \sum_{k=x_1^*}^{x_1^*} C_k + (E_{x_1}^x)^2 + \sum_{k=x_1^*}^{x_1^*} C_k \right] \\ &= \sum_{k=x_1^*}^{x_1^*} \left(\frac{\vartheta_k}{k} t^k \right)^2 + \sum_{\substack{x_1^* \leq k, k' < x_1^* \\ k \neq k'}} \frac{\vartheta_k}{k} t^k \frac{\vartheta_{k'}}{k'} t^{k'} - 2E_{x_1}^x \sum_{k=x_1^*}^{x_1^*} \frac{\vartheta_k}{k} t^k + (E_{x_1}^x)^2 + \sum_{k=x_1^*}^{x_1^*} \frac{\vartheta_k}{k} t^k \\ &= \left(\sum_{k=x_1^*}^{x_1^*} \frac{\vartheta_k}{k} t^k - E_{x_1}^x \right)^2 + \sum_{k=x_1^*}^{x_1^*} \frac{\vartheta_k}{k} t^k. \end{aligned} \tag{4.42}$$

This completes the proof since the cycle counts in both factors are independent. \square

We can now prove the tightness of the process $\tilde{w}_n^s(x^*)$. We have

Lemma 4.17. *We have for $0 \leq x_1 < x \leq x_2 < K$ with K arbitrary*

$$\mathbb{E}_n [(\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*))^2(\tilde{w}_n^s(x_2^*) - \tilde{w}_n^s(x^*))^2] = O((x_2 - x_1)^2). \tag{4.43}$$

Proof: We want to apply the saddle-point method to the sequence of functions

$$g_n(t) := e^{g_\Theta(t) + \log((g_{x_1^*}^{x^*}(t) - E_{x_1}^x)^2 + g_{x_1^*}^{x^*}(t)) + ((g_{x_2^*}^{x^*}(t) - E_{x_1}^x)^2 + g_{x_2^*}^{x^*}(t))}$$

to extract coefficients. Our first target is to show the log- n -admissibility for $r_n = e^{-p_n}$ with p_n as in Lemma 4.11. Note that the order of magnitude of

$$\log(g_{x_1^*}^{x^*}(r_n) - E_{x_1}^x)^2 + g_{x_1^*}^{x^*}(r_n)$$

(and similarly for x_2) is $O(\log p_n)$, which is lower than that of g_Θ as of (4.36). This tells us that the proof of log- n -admissibility goes through without major modifications. Hence we can safely use (4.40). It tells us that

$$\begin{aligned} & \mathbb{E}_n [(\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*))^2(\tilde{w}_n^s(x_2^*) - \tilde{w}_n^s(x^*))^2] \\ &= \frac{1}{h_n} \left(\frac{1}{\bar{n}}\right)^2 [t^n] \left[(g_{x_1^*}^{x^*}(t) - E_{x_1}^x)^2 + g_{x_1^*}^{x^*}(t) \right] \cdot \\ & \quad \cdot \left[(g_{x_2^*}^{x^*}(t) - E_{x_2}^x)^2 + g_{x_2^*}^{x^*}(t) \right] e^{(g_\Theta(t))}. \end{aligned} \tag{4.44}$$

Differentiating (4.8) with respect to s_1 and substituting $s_1 = 0$ shows that

$$E_{x_1}^x = \mathbb{E}_n [\tilde{w}_n^s(x^*) - \tilde{w}_n^s(x_1^*)] = \frac{1}{h_n} [t^n] \left[g_{x_2^*}^{x^*}(t) \exp(g_\Theta(t)) \right]. \tag{4.45}$$

This function is again log- n -admissibility with the same r_n . It is thus straightforward to show that $g_{x_1^*}^{x^*}(r_n) - E_{x_1}^x = o(x - x_1)$. Therefore

$$\left((g_{x_1^*}^{x^*}(r_n) - E_{x_1}^x)^2 + g_{x_1^*}^{x^*}(r_n) \right) (\bar{n})^{-1} = O\left(g_{x_1^*}^{x^*}(r_n) (\bar{n})^{-1} \right).$$

As it was shown in the proof of Lemma 3.8 we have that

$$g_{x_1^*}^{x^*}(r_n) (\bar{n})^{-1} = O((x - x_1)).$$

Similar considerations apply for x_2 . Hence we can say that the RHS of (4.44) yields

$$h_n^{-1} [t^n] [\exp(g_\Theta(t))] O((x_2 - x_1)^2) = O((x_2 - x_1)^2).$$

□

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Appendix A. Euler Maclaurin formula with non integer boundaries

We prove in this section a slight extension of Euler Maclaurin formula, which allows to deal also with non-integer summation limits.

Theorem A.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, $B_k(x)$ be the Bernoulli polynomials and $c < d$ with $c, d \in \mathbb{R}$. We then have for $p \in \mathbb{N}$*

$$\begin{aligned} \sum_{\lfloor c \rfloor \leq k < d} f(k) &= \int_c^d f(x) dx - B_1(d - \lfloor d \rfloor)f(d) - B_1(c - \lfloor c \rfloor)f(c) \\ &+ \sum_{k=1}^p (-1)^{k+1} \frac{B_{k+1}(d - \lfloor d \rfloor)f^{(k)}(d) - B_{k+1}(c - \lfloor c \rfloor)f^{(k)}(c)}{k!} \\ &+ \frac{(-1)^{p+1}}{(p+1)!} \int_c^d B_{p+1}(x - \lfloor x \rfloor)f^{(p+1)}(x) dx. \end{aligned} \tag{A.1}$$

Proof: The proof of this theorem follows the same lines as the proof of the Euler-Maclaurin summation formula with integer summation limits, see for instance [Apostol \(1976, Theorem 3.1\)](#). We give it here though for completeness. Our proof considers only the case $d \notin \mathbb{Z}$. The argumentation for $d \in \mathbb{Z}$ is completely similar. One possible definition of the Bernoulli polynomials is by induction:

$$B_0(y) \equiv 1, \tag{A.2}$$

$$B'_k(y) = kB_{k-1}(y) \text{ and } \int_0^1 B_k(y) dy = 1 \text{ for } k \geq 1. \tag{A.3}$$

In particular, we have $B_1(y) = y - \frac{1}{2}$. We now have for $m \in \mathbb{Z}$

$$\begin{aligned} \int_m^{m+1} f(y) dy &= \int_m^{m+1} B_0(y - m)f(y) dy \\ &= [B_1(y - m)f(y)]_{y=m}^{m+1} - \int_m^{m+1} B_1(y - m)f'(y) dy \\ &= \frac{1}{2}f(m) + \frac{1}{2}f(m + 1) - \int_m^{m+1} B_1(y - \lfloor y \rfloor)f'(y) dy. \end{aligned}$$

since $B_1(0) = -\frac{1}{2}$ and $B_1(1) = \frac{1}{2}$. We obtain

$$\sum_{k=\lfloor c \rfloor}^{\lfloor d \rfloor} f(k) = \int_{\lfloor c \rfloor}^{\lfloor d \rfloor} f(x) dx + \frac{1}{2}f(\lfloor c \rfloor) + \frac{1}{2}f(\lfloor d \rfloor) + \int_{\lfloor c \rfloor}^{\lfloor d \rfloor} B_1(y - \lfloor y \rfloor)f'(y) dy.$$

Furthermore, we use

$$\int_{\lfloor d \rfloor}^d f(y) dy = \frac{1}{2}f(\lfloor d \rfloor) + B_1(d - \lfloor d \rfloor)f(d) - \int_{\lfloor d \rfloor}^d B_1(y - \lfloor y \rfloor)f'(y) dy$$

and get

$$\sum_{k=\lfloor c \rfloor}^{\lfloor d \rfloor} f(k) = \int_{\lfloor c \rfloor}^d f(x) dx + \frac{1}{2}f(\lfloor c \rfloor) - B_1(d - \lfloor d \rfloor)f(d) + \int_{\lfloor c \rfloor}^d B_1(y - \lfloor y \rfloor)f'(y) dy.$$

The argumentation for replacing $\lfloor c \rfloor$ by c is similar. One gets

$$\begin{aligned} \sum_{\lfloor c \rfloor \leq k < d} f(k) &= \int_c^d f(x) dx - B_1(c - \lfloor c \rfloor)f(c) - B_1(d - \lfloor d \rfloor)f(d) \\ &+ \int_c^d B_1(y - \lfloor y \rfloor)f'(y) dy. \end{aligned}$$

The theorem now follows by successive partial integrations of $\int_c^d B_1(y - \lfloor y \rfloor) f'(y) dy$. \square

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