ALEA, Lat. Am. J. Probab. Math. Stat. 13, 1–31 (2016)
DOI: 10.30757/ALEA.v13-01

Path-dependent Itô formulas under (p,q)-variations

Alberto Ohashi, Evelina Shamarova and Nikolai N. Shamarov

Universidade Federal da Paraíba
Departamento de Matemática,
13560-970, João Pessoa, Paraíba, Brazil.
E-mail address: amfohashi@gmail.com

Universidade Federal da Paraíba
Departamento de Matemática,
13560-970, João Pessoa, Paraíba, Brazil.
E-mail address: evelina@mat.ufpb.br

Lomonosov Moscow State University
Faculty of Mechanics and Mathematics
119991, Vorob'evy Gory, Moscow, Russia
E-mail address: nshamarov@yandex.ru

Abstract. In this work, we establish pathwise functional Itô formulas for non-smooth functionals of real-valued continuous semimartingales. Under finite (p,q)-variation regularity assumptions in the sense of two-dimensional Young integration theory, we establish a pathwise local-time decomposition

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \int_0^t \nabla^h F_s(X_s)ds + \int_0^t \nabla^w F_s(X_s)dX(s) \\ &- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t (\nabla^w_x F_s)(^x X_s) d_{(s,x)}\ell^x(s); 0 \leqslant t \leqslant T. \end{aligned}$$

Here, $X_t = \{X(s); 0 \leqslant s \leqslant t\}$ is the continuous semimartingale path up to time $t \in [0,T]$, ∇^h is the horizontal derivative, $(\nabla^w_x F_s)(^x X_s)$ is a weak derivative of F with respect to the terminal value x of the modified path $^x X_s$ and $\nabla^w F_s(X_s) = (\nabla^w_x F_s)(^x X_s)|_{x=X(s)}$. The double integral is interpreted as a space-time 2D-Young integral with differential $d_{(s,x)}\ell^x(s)$, where ℓ is the local-time of X. Under less restrictive joint variation assumptions on $(\nabla^w_x F_t)(^x X_t)$, functional Itô formulas are established when X is a stable symmetric process. Singular cases when $x \mapsto (\nabla^w_x F_t)(^x X_t)$ is smooth off random bounded variation curves are also discussed. The results of this paper extend previous change of variable formulas in Cont and Fournié (2013) and also Peskir (2005), Feng and Zhao (2006) and Elworthy et al. (2007) in the context of path-dependent functionals. In particular, we provide a pathwise path-dependent version of the classical Föllmer-Protter-Shiryaev formula for continuous semimartingales given by Föllmer et al. (1995).

Received by the editors May 18, 2015; accepted January 5, 2016.
2010 *Mathematics Subject Classification.* 60H05, 60H20.
Key words and phrases. Itô formula, local-times, Young integral, path-dependent calculus.

1. Introduction

The celebrated Itô formula is the fundamental change of variables formula deeply connected with the concept of quadratic variation of semimartingales. It was initially conceived by Kiyosi Itô and since then many authors have been extending his formula either relaxing smoothness of the transformation or generalizing to more general stochastic processes.

After Itô, perhaps the major contribution towards a change of variables formula without C^2 assumption was due to the classical works by Tanaka, Wang and Meyer by making a beautiful use of the local time concept earlier introduced by Paul Lévy. They proved that if $F:\mathbb{R}\rightarrow\mathbb{R}$ is convex then

$$F(B(t)) = F(B(0)) + \int_0^t \nabla_- F(B(s))dB(s) + \frac{1}{2}\int_{-\infty}^{\infty} \ell^x(t)\rho(dx)$$

where B is the Brownian motion, $\ell^x(t)$ is the correspondent local time two-parameter process at $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and ρ is the Radon measure related to the generalized second-order derivative of F. A different extension to absolutely continuous functions with bounded derivatives is due to Bouleau and Yor (1981)

$$F(B(t)) = F(B(0)) + \int_0^t \nabla F(B(s))dB(s) - \frac{1}{2}\int_{-\infty}^{\infty} \nabla F(x)d_x\ell^x(t) \tag{1.1}$$

and later on extended by Föllmer et al. (1995) and Eisenbaum (2000) to functions in the Sobolev space $\mathrm{H}^{1,2}_{loc}(\mathbb{R})$ of generalized functions with weak derivatives in $L^2_{loc}(\mathbb{R})$. In this case, the correction term in (1.1) is given by an $d_x\ell^x(t)$-integral in $L^2(\mathbb{P})$-sense where $\mathbb{P}$ is the Wiener measure. See also Bardina and Rovira (2007) for the case of elliptic diffusions and Russo and Vallois (1996) for the general semimartingale case composed with C^1 functions.

Inspired by the two-dimensional Lebesgue-Stieltjes integration methodology of Elworthy et al. (2007), a different pathwise argument was introduced by Feng and Zhao (2006, 2008) based on Young/Rough Path (see e.g Friz and Victoir (2010)) integration theory. They proved that the local time curves $x\mapsto\ell^x(t)$ of any continuous semimartingale X admits p-variation ($p>2$) almost surely for any $t\geqslant 0$. In this case, the pathwise rough path integral $\int_{-\infty}^{+\infty}\nabla_- F(x)d_x\ell^x(t)$ can be used as the correction term in the change of variable formula for X as follows

$$F(X(t)) - F(X(0)) = \int_0^t \nabla_- F(X(s))dX(s) - \frac{1}{2}\int_{-\infty}^{\infty} \nabla_- F(x)d_x\ell^x(t),\ 0\leqslant t\leqslant T,$$

where $F:\mathbb{R}\rightarrow\mathbb{R}$ is an absolutely continuous function with left-continuous left derivative $\nabla_- F$ with finite p-variation where $1\leqslant p\leqslant 3$.

One important class of semimartingale transformations which cannot be recovered by the previous methods is the following one

$$X_t \mapsto F_t(X_t);\ t\geqslant 0 \tag{1.2}$$

where $X_t = \{X(u); 0\leqslant u\leqslant t\}$ is the semimartingale path up to time t and $F_t: C([0,t];\mathbb{R})\rightarrow\mathbb{R};\ t\geqslant 0$ is a functional defined on the space of real-valued continuous functions $C([0,t];\mathbb{R})$ on the intervals $[0,t]; t\geqslant 0$. Path-dependent transformations of type (1.2) have been studied in the context of the so-called functional stochastic calculus introduced by Dupire (2009) and systematically studied by Cont and Fournié (2013, 2010). In fact, this approach has been recently studied by many authors in the context of path-dependent PDEs and path-dependent optimal stochastic control problems. We refer the reader to e.g Ekren et al. (2014); Leão et al. (2015); Cosso et al. (2014); Cosso and Russo (2014); Flandoli (1996); Buckdahn et al. (2015); C. and Zhang (2016) for a detailed account on this literature. In this case, the usual space-time derivative operators are replaced by the so-called horizontal and vertical derivative operators, given by $\nabla^h F$ and $\nabla^v F$, respectively. Under suitable regularity conditions ($\mathbb{C}^{1,2}$ in the functional sense), one can show that if X is a continuous semimartingale then

$$\begin{aligned} F_t(X_t) = F_0(X_0) + \int_0^t \nabla^h F_s(X_s)ds + \int_0^t \nabla^v F_s(X_s)dX(s) \\ + \frac{1}{2}\int_0^t \nabla^{v,2}F_s(X_s)d[X,X](s) \end{aligned} \tag{1.3}$$

for $t \geqslant 0$, where $\nabla^{v,2}F$ is the second order vertical derivative and $[X,X]$ is the standard quadratic variation of X. See Cont and Fournié (2013); Dupire (2009) for further details.

Under weaker regularity assumptions, Leão et al. (2015) have extended (1.3) for functionals F which do not admit second order vertical derivatives. By means of a weaker version of functional calculus, the authors show that path dependent functionals with rough regularity in the sense of (p,q)-variation are weakly differentiable and, in particular, they satisfy

$$\begin{aligned} F_t(B_t) &= F_0(B_0) + \int_0^t \mathcal{D}F_s(B_s)dB(s) + \int_0^t \mathcal{D}^{\mathcal{F},h}F_s(B_s)ds \\ &- \frac{1}{2}\int_0^t\int_{-\infty}^{+\infty} \partial_x F_s({}^xB_s)d_{(s,x)}\ell^x(s), \end{aligned} \tag{1.4}$$

where the operators $(\mathcal{D}F, \mathcal{D}^{\mathcal{F},h}F)$ are similar in nature to $(\nabla^v F(B), \nabla^h F(B))$. The $d_{(s,x)}\ell^x(s)$-integral in (1.4) is considered in the (p,q)-variation sense based on the pathwise 2D Young integral (see Young (1938)) where ℓ is the Brownian local-time. The integrand is a suitable space derivative of F composed with a "terminal value modification" ${}^xB_t$ defined by the following pathwise operation: For a given path $\eta_t : C([0,t];\mathbb{R}) \to \mathbb{R}$, then

$${}^x\eta_t(u) := \begin{cases} \eta(u); & \text{if } 0 \leqslant u < t \\ x; & \text{if } u = t. \end{cases}$$

In this work, our goal is to study a number of path-dependent Itô formulas $F(X)$ beyond the smooth case of functionals with $\mathbb{C}^{1,2}$-regularity, where X is an arbitrary semimartingale with continuous paths. Based on the framework of pathwise functional calculus, we establish a pathwise local-time decomposition

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \int_0^t \nabla^h F_s(X_s)ds + \int_0^t \nabla^w F_s(X_s)dX(s) \\ &- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t (\nabla^w_x F_s)({}^xX_s)d_{(s,x)}\ell^x(s) \end{aligned} \tag{1.5}$$

where $(\nabla^w_x F_s)({}^xX_s)$ is a weak derivative of F with respect to the terminal value x of the modified path ${}^xX_s$ and $(\nabla^w F_s)(X_s) = (\nabla^w_x F_s)({}^xX_s)|_{X(s)=x}$. The double integral is interpreted as a space-time 2D-Young integral with differential $d_{(s,x)}\ell^x(s)$ where ℓ is the local time of X. We study differential representations of form (1.5) under a set of assumptions related to rough variations in time and space:

Two-parameter Hölder control: For each $L > 0$, there exists a constant C such that

$$|\Delta_i \Delta_j (\nabla^w_x F_{t_i})({}^{x_j}c_{t_i})| \le C|t_i - t_{i-1}|^{1/\tilde{p}}|x_j - x_{j-1}|^{1/\tilde{q}} \tag{1.6}$$

for every partition $\{t_i\}_{i=0}^N \times \{x_j\}_{j=0}^{N'}$ of $[0,T] \times [-L, L]$ and $c \in C([0,T];\mathbb{R})$. Here, Δ_j is the usual first difference operator and $\tilde{p}, \tilde{q} \geqslant 1$ are constants such that

$$\alpha + \frac{1}{\tilde{p}} > 1 \quad \text{and} \quad \frac{(1-\alpha)}{2+\delta} + \frac{1}{\tilde{q}} > 1$$

for some $\alpha \in (0,1)$ and $\delta > 0$.

General (a,b)*-variation*: In the particular case when X is a continuous symmetric stable process with index $1 < \beta \leqslant 2$, we establish formula (1.5) under general (a,b)-variation regularity assumption

$$\sup_{\pi} \left\{ \left[\sum_{j=1}^{N'} \Big[\sum_{i=1}^{N} |\Delta_i \Delta_j (\nabla^w_x F_{t_i})({}^{x_j}c_{t_i})|^a \Big]^{\frac{b}{a}} \right]^{\frac{1}{b}} \right\} < \infty; \quad c \in C([0,T];\mathbb{R}) \tag{1.7}$$

for $1 \leqslant a < \frac{2\beta}{\beta+1}$ and $1 \leqslant b < \frac{2}{3-\beta}$, where sup in (1.7) is computed over the set of partitions π of $[0,T] \times [-L, L]$ for each $L > 0$. Other types of singularities are also discussed when $x \mapsto F_t({}^x c_t)$ is smooth off path-dependent bounded variation curves.

The formulas presented in this article extend previous versions of path-dependent pathwise Itô formulas given by Cont and Fournié (2013) and Dupire (2009). In relation to non-smooth path-dependent cases, we also extend Prop. 9.3 in Leão et al. (2015) in the case when the path-dependent calculus is treated on the basis of functionals with a priori (p,q)-variation regularity rather than processes. In Leão et al. (2015), the authors show that Wiener functionals with finite (p,q)-regularity of the form (1.6) are weakly differentiable. In the present work, in the context of pathwise functional calculus, we show that this type of regularity also provides differential representations for path-dependent functionals driven by generic continuous semimartingales.

The level of regularity that we impose on the path-dependent functionals can be compared with the pioneering works of Elworthy et al. (2007), Peskir (2005) and Feng and Zhao (2006, 2008) who obtain extensions of non-path dependent change of variables formulas by means of pathwise arguments based on Lebesgue-Stieltjes/Young/rough path type integrals. Our first result (Theorem 3.2) extends the classical result due to Peskir (2005); Elworthy et al. (2007) for functionals with singularity at path-dependent bounded variation curves. Applications to some path-dependent payoffs in Mathematical Finance are briefly discussed. The change of variable formulas under (a,b)-regularity (1.7) (Proposition 5.5) extend Feng and Zhao (2006, 2008) with the restriction that the underlying noise is a continuous symmetric stable process. The general semimartingale case is treated in Theorem 4.7 under more restrictive assumptions on $\nabla^w F$ based on (1.6), One typical class of examples which fits into the assumptions of our theorems can be represented by

$$\int_{-\infty}^{X(t)} Z_t(X_t;y)dy$$

where X is the semimartingale noise which induces the underlying filtration and $Z = \{Z(\cdot;x) : C([0,t];\mathbb{R}) \to \mathbb{R}; (t,x) \in [0,T]\times\mathbb{R}\}$ is a family of functionals satisfying some two-parameter variation regularity of the forms (1.7) or (1.6). This can be seen as a pathwise path-dependent version of the classical Föllmer-Protter-Shiryaev formula (see Föllmer et al. (1995)) for continuous semimartingales.

This paper is organized as follows. Section 2 presents basic notations and some preliminary results. In Section 3, we investigate Itô formulas for path-dependent functionals which are regular off path-dependent bounded variation curves. Applications to some running maximum/minimum functionals arising in Mathematical Finance are presented. Section 4 presents Itô formulas under (p,q)-variation assumption of the particular form (1.6). Section 5 treats the general case (1.7) under the assumption that the underlying driving noise is a symmetric stable process.

2. Functional Mollification

Throughout this paper we are given a stochastic basis $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P})$. Here, the set $\Omega := \{\omega \in C([0,+\infty);\mathbb{R}); \omega(0) = z\}$ is the set of real-valued continuous paths on $\mathbb{R}_+$ which starts at a given $z \in \mathbb{R}$, X is the canonical process, $\mathbb{F} := (\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration generated by X, $\mathcal{F}$ is a sigma-algebra such that $\mathcal{F}_t \subset \mathcal{F}$ $\forall t \geqslant 0$ and $\mathbb{P}$ is the semimartingale measure on Ω. The usual quadratic variation will be denoted by $[X,X]$ and we recall the local time of X is the unique random field $\{\ell^x(t); (x,t) \in \mathbb{R}\times\mathbb{R}_+\}$ which realizes

$$\int_0^t f(X(s))d[X,X](s) = \int_{\mathbb{R}} \ell^x(t)f(x)dx; t \geqslant 0$$

for every bounded Borel measurable function $f:\mathbb{R}\to\mathbb{R}$. Throughout this article, we choose a modification of the local time $\{\ell^x(t);(x,t)\in\mathbb{R}\times\mathbb{R}_+\}$ which is jointly measurable in (ω,x,t) and right-continuous with left-hand limits (càdlàg) in the spatial variable.

Frequently, localization procedures will be necessary to handle the path-dependence. For this reason, for a given $M > 0$, we set

$$T_M := \inf\{t \geqslant 0; |X(t)| > M\} \wedge T$$

where $0 < T < +\infty$ is a fixed terminal time and $a \wedge b := min\{a,b\}$. The stopped semimartingale will be denoted by $X^M(t) := X(T_M \wedge t); 0 \leqslant t \leqslant T$. We denote $D([0,t];\mathbb{R})$ $(C([0,t];\mathbb{R}))$ as the linear space of $\mathbb{R}$-valued càdlàg (continuous) paths on $[0,t]$ and we set $\Lambda := \cup_{0\leqslant t\leqslant T} D([0,t];\mathbb{R})$ and $\hat{\Lambda} := \cup_{0\leqslant t\leqslant T} C([0,t];\mathbb{R})$. In order to make clear the information encoded by a path $x \in D([0,t];\mathbb{R})$ up to a given time $0 \leqslant r \leqslant t$, we denote $x_r := \{x(s) : 0 \leqslant s \leqslant r\}$ and the value of x at time $0 \leqslant u \leqslant t$ is denoted by $x(u)$. This notation is naturally extended to processes. Throughout this paper, if f is a real-valued function defined on a metric space E, then

$$\Delta_j f(x_j) := f(x_j) - f(x_{j-1})$$

for every sequence $\{x_j\}_{j=0}^m \subset E$. In particular, if $\varphi : [0,T]\times\mathbb{R}\to\mathbb{R}$ then

$$\Delta_j\Delta_i\varphi(t_i,x_j) := \varphi(t_i,x_j) - \varphi(t_{i-1},x_j) - \big(\varphi(t_i,x_{j-1}) - \varphi(t_{i-1},x_{j-1})\big)$$

for any sequence $\{t_i\}_{i=0}^m \times \{x_k\}_{k=0}^p \subset [0,T] \times \mathbb{R}$.

For reader's convenience, let us recall some basic objects of the pathwise functional calculus. We refer the reader to Dupire (2009) and Cont and Fournié (2013, 2010) for further details. Throughout this article, if $w \in \Lambda$, then for a given $\gamma > 0$ and $h \in \mathbb{R}$, we denote

$$w_{t,\gamma}(s) := w(s \wedge t); \quad 0 \leqslant s \leqslant t + \gamma.$$

The operation $w_{t,\gamma}$ is an horizontal extension of the path w (see Figure 1). If $x \in \mathbb{R}$, we denote

$$^x w_t(s) := \begin{cases} w(s), & 0 \leqslant s < t \\ x, & s = t. \end{cases}$$

A vertical perturbation of the path w (see Figure 1) is given by

$$w_t^h(s) := {}^{w(t)+h} w_t(s).$$

Of course, $^x w_t = w_t^h$ if $x = w(t) + h$ and, in general, they may not coincide.

FIGURE 2.1. The horizontal extension $w_{t,\gamma}$ is shown in green. The vertical perturbation w_t^h is shown in blue, h is the distance between the empty ball (left limit) and the filled ball.

A natural metric on Λ is given by

$$d_\infty\big((t,w);(s,v)\big) := |t-s| + \sup_{0 \leqslant u \leqslant T} |w_{t,T-t}(u) - v_{s,T-s}(u)|;$$

for (w,v) in $\Lambda \times \Lambda$. Throughout this article, a functional $F = \{F_t; 0 \leqslant t \leqslant T\}$ is a family of mappings $F_t : D([0,t];\mathbb{R}) \to \mathbb{R}$ indexed by $t \in [0,T]$. In the sequel, continuity of functionals is defined as follows (see e.g Cont and Fournié (2013)):

Definition 2.1. A functional $F = \{F_t;\ 0 \leqslant t \leqslant T\}$ is said to be Λ-continuous if it is continuous in (t,w) under d_∞.

We recall the vertical derivative of a functional $F \in \Lambda$ is defined as

$$\nabla^v F_t(c_t) := \lim_{h \to 0} \frac{F_t(c_t^h) - F_t(c_t)}{h} \tag{2.1}$$

whenever the right-hand side of (2.1) exists for every $c \in \Lambda$. We define $\nabla^{v,(2)} F := \nabla^v(\nabla^v F)$ whenever this operation exists. The horizontal derivative is defined by the following limit

$$\nabla^h F_t(c_t) := \lim_{\gamma \to 0^+} \frac{F_{t+\gamma}(c_{t,\gamma}) - F_t(c_t)}{\gamma} \tag{2.2}$$

whenever the right-hand side of (2.2) exists for every $c \in \Lambda$.

An $\mathbb{F}$-adapted continuous process Y may be represented by the identity

$$Y(t) = \hat{F}_t(X_t);\ 0 \leqslant t \leqslant T, \tag{2.3}$$

where $\hat{F} = \{\hat{F}_t; 0 \leqslant t \leqslant T\}$ is a functional, and each $\hat{F}_t : C([0,t];\mathbb{R}) \to \mathbb{R}$ is a mapping representing the dependence of Y. We recall that we make use of the notation $X_t = \{X(s); 0 \leqslant s \leqslant t\}$ so that X_t means the whole history of X over the time interval $[0,t]$ for each $t \geqslant 0$.

Since Y is non-anticipative, $Y(\omega, t)$ only depends on the restriction of ω over $[0,t]$. In order to perform the standard pathwise functional calculus in the sense of Dupire (2009) and Cont and Fournié (2013), one has to assume there exists a functional $F = \{F_t; 0 \leqslant t \leqslant T\}$ defined on Λ which is consistent to $\hat{F}$ in the sense that

$$F_t(c_t) = \hat{F}_t(c_t) \quad \forall c \in \hat{\Lambda}.$$

Indeed, the concept of vertical derivative forces us to assume this. Throughout this article, whenever we write $Y = F(X)$ for F defined on Λ, it is implicitly assumed that F is a consistent extension of a functional representation $\hat{F}$ which realizes (2.3). This motivates the following definition.

Definition 2.2. A non-anticipative functional is a family of mappings $F = \{F_t; 0 \leqslant t \leqslant T\}$ where

$$F_t : D([0,t];\mathbb{R}) \to \mathbb{R}; \quad c \mapsto F_t(c_t)$$

is measurable w.r.t the canonical filtration $\mathcal{B}_t$ in $D([0,t];\mathbb{R})$ for each $t \in [0,T]$.

In the sequel, let $\mathbb{C}^{1,2}$ be the space of functionals F which are Λ-continuous and it has Λ-continuous derivatives $\nabla^h F, \nabla^{v,(i)} F$ for $i = 1,2$. The above notion of continuity is enough to apply the standard functional stochastic calculus techniques in the smooth case $F \in \mathbb{C}^{1,2}$. However, in order to employ mollification techniques to treat non-smooth dependence (in the sense of differentiation) of F w.r.t X, we need the following notion of continuity.

Definition 2.3. We say that a family of functionals $\{H^x : \Lambda \to \mathbb{R}; x \in \mathbb{R}\}$ is state-dependent Λ-continuous at $v \in \Lambda$ if there exists $\phi \in L^1_{loc}(\mathbb{R})$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_\infty\big((t',c);(t,v)\big) < \delta \quad \Longrightarrow \quad |H^x_{t'}(c_{t'}) - H^x_t(v_t)| \leqslant \varepsilon\phi(x); \quad \forall x \in \mathbb{R}.$$

When the family $\{H^x; x \in \mathbb{R}\}$ is state-dependent Λ-continuous for every $v \in \Lambda$, we say that it is state-dependent Λ-continuous.

Remark 2.4. If $\{H^x; x \in \mathbb{R}\}$ is state-dependent Λ-continuous,then it is Λ-continuous for each $x \in \mathbb{R}$.

Example: Let us give an example of a state-dependent Λ-continuous family of functionals. In the sequel, $(x)^+ := \max\{x; 0\}, x \in \mathbb{R}$. For a given constant K, we consider $F_t(c_t) = \Big(\sup_{0\leqslant s\leqslant t} c(s) - K\Big)^+$. Then, $F_t({}^x c_t) = \Big(\sup_{0\leqslant s\leqslant t} {}^x c(s) - K\Big)^+$ for each $x \in \mathbb{R}, c \in \Lambda$ and we readily see that the family $c \mapsto F({}^x c); x \in \mathbb{R}$ is state-dependent Λ-continuous.

For the remainder of this paper it will be convenient to use the following notation: For a given functional $F = \{F_t; 0 \leqslant t \leqslant T\}$, we define

$$\mathcal{F}^x_t(c_t) := F_t({}^x c_t) \tag{2.4}$$

for $c \in \Lambda$ and $x \in \mathbb{R}$. This notation will be useful to compute horizontal derivatives from a state-dependent Λ-continuous family of the form $\{\mathcal{F}^x; x \in \mathbb{R}\}$.

The strategy to get functional Itô formulas under non-smooth conditions will be based on path-dependent mollification techniques on the state of the functional. Indeed, in this article we are only interested in relaxing vertical smoothness of path-dependent functionals. In this case, it will be sufficient for us to deal with one parameter mollification.

For a given non-negative smooth function $\rho \in C_c^\infty(\mathbb{R})$ such that *supp* $\rho \subset (0,2)$, $\int_{\mathbb{R}} \rho(x)ds = 1$, we set $\rho_n(x) := n\rho(nx); x \in \mathbb{R}; n \geqslant 1$. If $x \mapsto \mathcal{F}_t^x(c_t) \in L^1_{loc}(\mathbb{R})$ for every $c \in \Lambda$, then we define

$$F_t^n(c_t; x) := \big(\rho_n \star \mathcal{F}_t^{\cdot}(c_t)\big)(x); x \in \mathbb{R}, c \in \Lambda, t \in [0,T], \tag{2.5}$$

where $\star$ denotes the usual convolution operation on the real line. From this convolution operator, we define the following non-anticipative functional

$$F_t^n(c_t) := \int_{\mathbb{R}} \rho_n(c(t) - y)\mathcal{F}_t^y(c_t)dy; 0 \leqslant t \leqslant T.$$

One should notice that $F_t^n({}^x c_t) = F^n(c_t; x); c \in \Lambda,\ x \in \mathbb{R}$. In the sequel, we need a notion of boundedness to treat path-dependent functionals.

Definition 2.5. We say that a functional $F = \{F_t;\, 0 \leqslant t \leqslant T\}$ is boundedness-preserving if for every compact subset K of $\mathbb{R}$, there exist $C_K > 0$ such that $|F_\cdot(c_\cdot)| \leqslant C_K$ for every $c_\cdot \in D([0,\cdot]; K)$. A family of functionals $H^x : \Lambda \to \mathbb{R};\, x \in \mathbb{R}$ is state boundedness-preserving if for every compact sets $K_1, K_2 \subset \mathbb{R}$, there exists a constant $C_{K_1,K_2} > 0$ such that

$$|H^x_\cdot(c_\cdot)| \leqslant C_{K_1,K_2} \quad \forall c \in D([0,\cdot]; K_1) \text{ and } \forall x \in K_2.$$

Let us now introduce the following hypotheses

Assumption A1:
(i) The family of functionals $\{\mathcal{F}^y; y \in \mathbb{R}\}$ is state-dependent Λ-continuous and state-boundedness-preserving.
(ii) $x \mapsto \mathcal{F}_t^x(c_t)$ is a continuous map for every $c \in \Lambda$ and $t \in [0,T]$.
(iii) $x \mapsto \mathcal{F}_t^x(c_t)$ has weak derivative for every $c \in \Lambda$ and $t \in [0,T]$.

Assumption A2: For each $y \in \mathbb{R}$, $\mathcal{F}^y$ has horizontal derivative $\nabla^h \mathcal{F}^y(c)\ \forall c \in \Lambda$. Moreover, the family $\{\nabla^h \mathcal{F}^y; y \in \mathbb{R}\}$ is state boundedness-preserving. The map $y \mapsto \nabla^h \mathcal{F}_t^y(c_t)$ is continuous for every $c \in \Lambda$. The family of functionals $\{\nabla^h \mathcal{F}^y; y \in \mathbb{R}\}$ is state-dependent Λ-continuous.

Throughout this paper, the weak derivative of $x \mapsto \mathcal{F}_t^x(c_t)$ will be denoted by $(\nabla_x^w F_t)({}^x c_t)$ and we set

$$\nabla^w F_t(c_t) := (\nabla_x^w F_t)({}^x c_t)|_{x = c(t)}; c \in \Lambda.$$

Of course, $(\nabla_x^w F_t)({}^\cdot c_t) \in L^1_{loc}(\mathbb{R})$ is uniquely specified by the property

$$\int_{\mathbb{R}} \mathcal{F}_t^x(c_t)\varphi'(x)dx = -\int_{\mathbb{R}} (\nabla_x^w F_t)({}^x c_t)\varphi(x)dx;\ c \in \Lambda,$$

for every real-valued smooth function $\varphi \in C_c^1(\mathbb{R})$.

If Assumptions A1.(iii) holds, then $F^n_t(\cdot c_t) \in C^\infty(\mathbb{R})$ $\forall c \in \Lambda, t \in [0,T], n \geqslant 1$ and integration by parts yields

$$\nabla_x F^n_t({}^x c_t) = \int_{\mathbb{R}} \rho_n(x-y)(\nabla^w_y F_t)({}^y c_t)dy.$$

Moreover, the vertical derivative of functional mollification is given by

$$\nabla^{v,i} F^n_t(c_t) = \nabla^i_x F^n_t({}^x c_t)|_{x=c(t)}$$

for $i=1,2$. To compute the horizontal derivative of mollifiers, the following simple lemma will be useful.

Lemma 2.6. *Let $\mathcal{A}$ be a parameter set, and let $f : \mathcal{A}\times\mathbb{R} \to \mathbb{R}$ be a function, continuous on the second variable and such that for each $a \in \mathcal{A}$, there exists the right derivative $\nabla^+_x f(a,x); x \in \mathbb{R}$ which is bounded on $\mathcal{A}\times\mathbb{R}$. Suppose for each $(a,x) \in \mathcal{A}\times\mathbb{R}$, there exists $a_x \in \mathcal{A}$ such that $f(a,x+h) = f(a_x,h)$. Then, the ratio $\frac{f(a,x+h)-f(a,x)}{h}$ is bounded over $\mathcal{A}\times\mathbb{R}\times\mathbb{R}_+$. The analogous result also holds for the ratio $\frac{f(a,x-h)-f(a,x)}{-h}$ under boundedness condition on $\nabla^- f(a,x)$ over $\mathcal{A}\times\mathbb{R}$.*

Proof: Let us fix an arbitrary pair $(a,x) \in \mathcal{A}\times\mathbb{R}$ and we define the set

$$H(a,x) = \{h \geqslant 0 : |f(a,x+h) - f(a,x)| \leqslant Ch\},$$

where $C = 1 + \sup_{a\in\mathcal{A},x\in\mathbb{R}} |\nabla^+_x f(a,x)|$. The set $H(a,x)$ is closed and it contains a closed interval $[0,l_{a,x}]$. Let $L_{a,x}$ be the length of the maximal interval of the form $[0,l_{a,x}]$ contained in $H(a,x)$. Suppose $L_{a,x} < \infty$. Take $h = L_{a,x}+k$, where $k \in H(a_{L_{a,x}},x)$, and $a_{L_{a,x}}$ is such that $f(a,L_{a,x}+y) = f(a_{L_{a,x}},y)$, $y \in \mathbb{R}$. We have

$$\begin{aligned}|f(a,x+h)-f(a,x)| &\leqslant |f(a,L_{a,x}+x+k)-f(a,L_{a,x}+x)|+|f(a,L_{a,x}+x)-f(a,x)|\\ &= |f(a_{L_{a,x}},x+k) - f(a_{L_{a,x}},x)| + |f(a,x+L_{a,x}) - f(a,x)| \leqslant Ck + CL_{a,x} = Ch.\end{aligned}$$

Thus, $L_{a,x}$ is not maximal and we have a contradiction. This implies that $L_{a,x} = \infty$ and therefore, the ratio $\frac{f(a,x+h)-f(a,x)}{h}$ is bounded on $\mathcal{A}\times\mathbb{R}\times\mathbb{R}_+$. The proof for the ratio related to the left-derivative is obviously the same. □

Remark 2.7. Lemma 2.6 also holds for functions $f : \mathcal{A}\times[A,B] \to \mathbb{R}$, where $[A,B] \subset \mathbb{R}$. Indeed, extend the function f to the whole real line by setting $f(a,x) = f(a,A)$ for $x < A$ and $f(a,x) = f(a,B)$ for $x > B$.

Remark 2.8. Note that the horizontal derivative $\nabla^h F_t(c_t)$ can be regarded as the right derivative $\nabla^+_\gamma F_{t+\gamma}(c_{t,\gamma})$ of the function $\gamma \mapsto F_{t+\gamma}(c_{t,\gamma})$ at point $\gamma = 0$, where the pair $(t,c) \in [0,T]\times\Lambda$ is interpreted as a parameter. The assumption $f(a,x+h) = f(a_x,h)$ in Lemma 2.6 is interpreted in the setup of functional calculus as follows:

$$F_{t+\gamma+h}(c_{t,\gamma+h}) = F_{(t+\gamma)+h}(\tilde{c}_{t+\gamma,h}),$$

where $\tilde{c}_{t+\gamma} = c_{t,\gamma}$. The same remark holds for the functional $\mathcal{F}^y_t(c_t)$ with parameters $(t,c,y) \in [0,T]\times\Lambda\times U$ in some bounded open subset $U \subset \mathbb{R}$.

Lemma 2.9. *Assume that for each $y \in \mathbb{R}$, $\mathcal{F}^y$ is Λ-continuous, $\mathcal{F}^y$ has horizontal derivative and the family $\{\nabla^h\mathcal{F}^y; y \in \mathbb{R}\}$ is state boundedness-preserving. Then, for each $n \geqslant 1$, $t \in [0,T]$, and $c \in \Lambda$ taking values in a compact subset of $\mathbb{R}$, we have*

$$\nabla^h F^n_t(c_t) = \int_{\mathbb{R}} \rho_n(c(t)-y)\nabla^h\mathcal{F}^y_t(c_t)dy = \big(\rho_n \star \nabla^h\mathcal{F}^{\cdot}_t(c_t)\big)(c(t)). \tag{2.6}$$

Proof: We fix $t \in [0,T]$ and a path $c \in \Lambda$ over $[0,t]$ such that $c(u) \in K; 0 \leqslant u \leqslant t$, where K is a compact set. We also fix $n \geqslant 1$. Indeed, by the very definition

$$\frac{F^n_{t+\gamma}(c_{t,\gamma}) - F^n_t(c_t)}{\gamma} = \int_{c(t)-\frac{2}{n}}^{c(t)} \rho_n(c(t)-y)\left[\frac{\mathcal{F}^y_{t+\gamma}(c_{t,\gamma}) - \mathcal{F}^y_t(c_t)}{\gamma}\right]dy \tag{2.7}$$

for $\gamma > 0$. We claim that the ratio $\frac{\mathcal{F}^y_{t+\gamma}(c_{t,\gamma})-\mathcal{F}^y_t(c_t)}{\gamma}$ is bounded over $(\gamma, y) \in [0,T-t]\times[c(t)-\frac{2}{n}, c(t)]$. Indeed, we shall apply Lemma 2.6 to the function $\gamma \mapsto \mathcal{F}^y_{t+\gamma}(c_{t,\gamma})$ defined on $[0,T-t]$ regarding $y \in [c(t)-\frac{2}{n}, c(t)]$ as a parameter (see Remark 2.8). From the Λ-continuity of $\mathcal{F}^y$, one can easily check that

$$\gamma \mapsto \mathcal{F}^y_{t+\gamma}(c_{t,\gamma})$$

is continuous over $[0,T-t]$. Extend the function $\gamma \mapsto \mathcal{F}^y_{t+\gamma}(c_{t,\gamma})$ to $\mathbb{R}$ by the constant values that it attains at the end points of $[0,T-t]$. As we already mentioned in Remark 2.8, for each $y \in \mathbb{R}$, the right derivative $\nabla^+_\gamma \mathcal{F}^y_{t+\gamma}(c_{t,\gamma})$ at γ_0 is the horizontal derivative $\nabla^h \mathcal{F}^y_{t+\gamma_0}(c_{t,\gamma_0})$ for $\gamma_0 \in [0,T-t]$. By the state boundedness-preserving assumption, $\nabla^+_\gamma \mathcal{F}^y_{t+\gamma}(c_{t,\gamma})$ is bounded over $[0,T-t]\times[c(t)-\frac{2}{n}, c(t)]$. Again, taking into account Remark 2.8, we conclude that we are in the situation of Lemma 2.6. Bounded convergence theorem allows us to take the limit into the integral sign in (2.7) as $\gamma \to 0$ which provides (2.6). This completes the proof. $\square$

Lemma 2.10. *If F is a non-anticipative functional satisfying Assumptions A1(i) and A2, then for each positive integer $n \geqslant 1$, we have*

$$\begin{aligned} F^n_t(X_t) &= F^n_0(X_0) + \int_0^t \nabla^h F^n_s(X_s)ds + \int_0^t \nabla^v F^n_s(X_s)dX(s) \\ &+ \frac{1}{2}\int_0^t \nabla^{v,2}F^n_s(X_s)d[X,X](s)\ a.s. \end{aligned} \tag{2.8}$$

for $0 \leqslant t \leqslant T$.

Proof: Let us fix $n \geqslant 1$. By routine stopping arguments, we may assume that X is bounded. Hence, we shall assume that all paths $c \in \Lambda$ take values on a common compact subset of $\mathbb{R}$. First we show that F^n is Λ-continuous. Indeed, by the very definition

$$F^n_t(c_t) = \int_{-\infty}^{\infty} \rho_n(c(t)-y)\mathcal{F}^y_t(c_t)dy.$$

Let us fix an arbitrary $c \in \Lambda$. The Λ-continuity of F^n follows immediately from the state-dependent continuity of $\{\mathcal{F}^y; y \in \mathbb{R}\}$ and the triangle inequality:

$$\begin{aligned} |F^n_t(c_t) - F^n_{t'}(w_{t'})| &\leqslant \int_K \big|\rho_n\big(c(t)-y\big) - \rho_n\big(w(t')-y\big)\big||\mathcal{F}^y_t(c_t)|dy \\ &+ \int_K \big|\rho_n\big(w(t')-y\big)\big||\mathcal{F}^y_t(c_t) - \mathcal{F}^y_{t'}(w_{t'})|dy. \end{aligned}$$

for $w \in \Lambda$, where K is a compact set. By the very definition,

$$\nabla^{v,i}F^n_t(c_t) = n^{i+1}\int_{\mathbb{R}} \rho^{(i)}\big(n(c(t)-y)\big)\mathcal{F}^y_t(c_t)dy; 0 \leqslant t \leqslant T$$

for $i = 1,2$. Similarly, the Λ-continuity of $\nabla^{v,i}F^n$ follows immediately from the state-dependent continuity of $\{\mathcal{F}^y; y \in \mathbb{R}\}$ and the triangle inequality:

$$
\begin{aligned}
&|\nabla^{v,i}F^n_t(c_t) - \nabla^{v,i}F^n_{t'}(w_{t'})| \\
&\leqslant n^{i+1}\int_K \Big|\rho^{(i)}\big(n(c(t)-y)\big) - \rho^{(i)}\big(n(w(t')-y)\Big||\mathcal{F}^y_t(c_t)|dy \\
&+ n^{i+1}\int_K \big|\rho^{(i)}\big(n(w(t')-y)\big)\big||\mathcal{F}^y_t(c_t) - \mathcal{F}^y_{t'}(w_{t'})|dy.
\end{aligned}
$$

By Lemma 2.9 and triangle inequality,

$$
\begin{aligned}
|\nabla^h F^n_t(c_t) - \nabla^h F^n_{t'}(w_{t'})| &\leqslant \int_K \big|\rho_n\big(c(t)-y\big) - \rho_n\big(w(t')-y\big)\big||\nabla^h\mathcal{F}^y_t(c_t)|dy \\
&+ \int_K \big|\rho_n\big(w(t')-y\big)\big||\nabla^h\mathcal{F}^y_t(c_t) - \nabla^h\mathcal{F}^y_{t'}(w_{t'})|dy.
\end{aligned}
\tag{2.9}
$$

Estimate (2.9), the local integrability of $y \mapsto \nabla^h\mathcal{F}^y_t(c_t)$ and the state-dependent Λ-continuity of $\{\nabla\mathcal{F}^y; y\in\mathbb{R}\}$ yield the Λ-continuity of $\nabla^h F^n$.

Hence, F^n is $\mathbb{C}^{1,2}$. The functional Itô formula (see e.g Dupire (2009), Cont and Fournié (2013)) applied to the semimartingale X yields

$$
\begin{aligned}
F^n_t(X_t) &= F^n_0(X_0) + \int_0^t \nabla^h F^n_s(X_s)ds + \int_0^t \nabla^v F^n_s(X_s)dX(s) \\
&+ \frac{1}{2}\int_0^t \nabla^{v,2}F^n_s(X_s)d[X,X](s)
\end{aligned}
$$

for $0 \leqslant t \leqslant T$. □

3. Path-dependent Itô formula with singularity at random curves

In this section, we will investigate a path-dependent Itô formula when the function $x \mapsto (\nabla^w_x F_t)({}^x c_t)$ is smooth off path-dependent continuous bounded variation curves. The typical examples we have in mind are non-smooth functionals of the running maximum/minimum found in path-dependent payoffs arising in Mathematical Finance. Obtaining this type of Itô's formula was inspired by Elworthy et al. (2007) who derived (non-path dependent) Itô formulas where singularities are encoded by deterministic bounded variation curves. See also Peskir (2005). At first, we remark that the classical occupation time formula also holds with path-dependent functions. We omit details of the proof which can be easily checked by well-known arguments.

Lemma 3.1. *Let X be a continuous semimartingale with the local time $\{\ell^x(t); x \in \mathbb{R}, t \geqslant 0\}$. If $h : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ is bounded and measurable, then for each $\omega \in \Omega$, we have*

$$\int_0^t h(s,\omega,X(s,\omega))d[X,X](s,\omega) = \int_{-\infty}^{\infty} da \int_0^t h(s,\omega,a)d_s\ell^a(s); 0 \leqslant t \leqslant T.$$

Let $\gamma = \{\gamma_t; 0 \leqslant t \leqslant T\}$ be a family of non-anticipative functionals such that for each $c \in C([0,T];\mathbb{R})$, $t \mapsto \gamma_t(c_t)$ is a continuous bounded variation path. In the sequel, to keep notation simple, for a given $M>0$, we set $\mathcal{C}_M := C([0,T];[-M,M])$ and

$$\mathcal{G}_M := \{(t,x,c)\in[0,T]\times[-M,M]\times\mathcal{C}_M\}.$$
$$\mathcal{G}^\gamma_M := \{(t,x,c)\in\mathcal{G}_M; -M<x<\gamma_t(c_t) \text{ or } \gamma_t(c_t)<x<M\}.$$
$$\Gamma_{c,t} := (-\infty,\gamma_t(c_t))\cup(\gamma_t(c_t),+\infty);\ c\in\mathcal{C}_M,\ 0\leqslant t\leqslant T.$$

Throughout this section, for a given $c\in\mathcal{C}_M$, we write $\nabla_x F_t({}^x c_t)$ and $\nabla^-_x F_t({}^x c_t)$ to denote the usual pointwise derivative and left derivative, with respect to x, respectively. The second left derivative will be denoted by $\nabla^{-,2}_x$. Since γ is non-anticipative, then $\gamma(X)$ is an adapted bounded variation process.

Theorem 3.2. *Let us assume that A1.(i,ii) and A2 hold and for each $t\in[0,T]$, the function $x\mapsto F_t({}^x c_t)$ is C^1 on sets $\Gamma_{c,t}$ for $c\in C([0,T];\mathbb{R})$, where $\nabla_x F_t({}^x c_t)$ is bounded on the set $\mathcal{G}^\gamma_M$ for every $M>0$. We also assume that for each $t\in[0,T]$, there exist left and right limits of $\nabla_x F_t({}^x c_t)$ as $x\to\gamma_t(c_t)\pm$. Furthermore, we assume that for any $t\in[0,T]$ and $c_t\in C([0,t],\mathbb{R})$, there exists the second left derivative $\nabla^{-,2}_x F_t({}^x c_t)$ on $\Gamma_{c,t}$ which is bounded on $\Gamma_{c,t}\cap(-M,M)\times\mathcal{C}_M$ for every $M>0$. Moreover, $\nabla^{-,2}_x F_t({}^x c_t)$ has the left limit at $\gamma_t(c_t)$ for each $c\in\mathcal{C}_M$. Finally, we assume that for any $c\in C([0,T];\mathbb{R})$, $\nabla_x F_t({}^{\gamma_t(c_t)-}c_t)-\nabla_x F_t({}^{\gamma_t(c_t)+}c_t)$ is continuous in t. If X is a square-integrable continuous semimartingale, then*

$$\begin{aligned}F_t(X_t) &= F_0(X_0)+\int_0^t \nabla^h F_s(X_s)ds+\int_0^t \nabla^-_x F_s({}^{X(s)}X_s)dX(s)\\ &+\frac{1}{2}\int_0^t \nabla^{-,2}_x F_s({}^{X(s)-}X_s)d[X,X](s)\\ &+\frac{1}{2}\int_0^t(\nabla_x F_s({}^{\gamma_s(X_s)+}X_s)-\nabla_x F_s({}^{\gamma_s(X_s)-}X_s))d_s\tilde{\ell}^0(s)\end{aligned}\tag{3.1}$$

a.s for $0\leqslant t\leqslant T$, where $\{\tilde{\ell}^x(s);(x,s)\in\mathbb{R}\times\mathbb{R}_+\}$ is the local time of the semimartingale $\tilde{X}:=X-\gamma(X)$.

Proof: The proof uses some of the ideas from Corollary 2.1 of Theorem 2.3 in Elworthy et al. (2007). At first, we prove the result for the stopped process X^M where M is fixed. Let us fix $t\in[0,T]$. Let F^n be the mollifier for F according to (2.5). Since functional F satisfies the assumptions of Lemma 2.10, formula (2.8) holds for $F^n(X^M)$. In the sequel, we will study the limit of each term in (2.8) as $n\to\infty$. By A1.(ii), $F^n_t(X^M_t)\to F_t(X^M_t)$ *a.s* as $n\to\infty$.

STEP 1: Let us prove the convergence

$$\int_0^t \nabla^h F^n_s(X^M_s)ds\to\int_0^t \nabla^h F_s(X^M_s)ds \quad \text{a.s.}\tag{3.2}$$

Lemma 2.9 yields

$$\nabla^h F^n_s(X^M_s)=\int_{-\infty}^{\infty}\rho_n(X^M(s)-y)\nabla^h\mathcal{F}^y_s(X^M_s)dy;0\leqslant s\leqslant T.\tag{3.3}$$

By Assumption A2, $y\mapsto\nabla^h\mathcal{F}^y_s(X^M_s)$ is continuous a.s. for each $s\in[0,T]$ and hence

$$\lim_{n\to\infty}\nabla^h F^n_s(X^M_s)=\nabla^h\mathcal{F}^{X^M(s)}(X^M_s)=\nabla^h F_s(X^M_s)\ a,s;0\leqslant s\leqslant T.$$

From Assumption A2, $\{\nabla^h \mathcal{F}^y; y \in \mathbb{R}\}$ is state boundedness-preserving. Then, bounded convergence theorem yields

$$\lim_{n\to\infty} \int_0^t \nabla^h F^n(X_s^M)ds = \int_0^t \nabla^h F(X_s^M)ds \ a.s$$

STEP 2: Next, we will prove that

$$\int_0^t \nabla^v F_s^n(X_s^M)dX^M(s) \to \int_0^t \nabla_x^- F_s({}^{X^M(s)}X_s^M)dX^M(s) \quad \text{in } L^2(\mathbb{P}) \tag{3.4}$$

as $n \to \infty$. Firstly, we will show that under the assumptions of the theorem, for each fixed $(t,c,x) \in \mathcal{G}_M$, $\nabla_x F_t^n({}^x c_t)$ converges to $\nabla_x^- F_t({}^x c_t)$ as $n \to \infty$. Fix a path $c \in C([0,t];\mathbb{R})$. At first, one should notice the left derivative $\nabla_x^- F_t({}^x c_t)$ is well defined for $x = \gamma_t(c_t)$. Indeed, we shall represent the functional $F_t({}^x c_t)$ in the following form

$$F_t({}^x c_t) = \hat{F}_t(c_t;x) + \tilde{F}_t(c_t;x) \tag{3.5}$$

where

$$\begin{aligned}\hat{F}_t(c_t;x) &:= F_t({}^x c_t) + (\nabla_x F_t({}^{\gamma_t(c_t)-}c_t) - \nabla_x F_t({}^{\gamma_t(c_t)+}c_t))(x-\gamma_t(c_t))^+,\\ \tilde{F}_t(c_t;x) &:= (\nabla_x F_t({}^{\gamma_t(c_t)+}c_t) - \nabla_x F_t({}^{\gamma_t(c_t)-}c_t))(x-\gamma_t(c_t))^+.\end{aligned}$$

It is easy to see that the function $x \mapsto \hat{F}_t(c_t;x)$ is C^1 in $x \in [-M,M]$. But on $[-M,\gamma_c(t)]$, $\hat{F}_t(c_t;x) = F_t({}^x c_t)$. Hence, $\nabla_x^- F_t({}^x c_t)$ exists at the point $x = \gamma_t(c_t)$, and therefore, everywhere on $[-M,M]$. From the assumptions of the theorem, it is also clear that $\nabla_x^- F_t({}^x c_t)$ is bounded on $\mathcal{G}_M$. Thus, we verified the assumptions of Lemma 2.6 with respect to the function $h \mapsto F_t({}^{x-h}c_t)$ with (t,c,x) being a parameter. This implies the boundedness of the ratios $(F_t({}^x c_t) - F_t({}^{x-h}c_t))/h$. Hence, Lebesgue's bounded convergence theorem yields $\nabla_x F_t^n({}^x c_t)$ is the mollifier for $\nabla_x^- F_t({}^x c_t)$:

$$\nabla_x F_t^n({}^x c_t) = \int_0^2 \rho(y)\nabla_x^- F_t({}^{x-\frac{y}{n}}c_t)dy. \tag{3.6}$$

From the assumptions of the theorem and the existence of $\nabla_x^- F_t({}^x c_t)$ at $x = \gamma_t(c_t)$, we know that $x \mapsto \nabla_x^- F_t({}^x c_t)$ is left continuous. By the boundedness of $\nabla_x^- F_t({}^x c_t)$ on $\mathcal{G}_M$ and its left continuity in x, we obtain that for each $(x,t,c) \in \mathcal{G}_M$, $\nabla_x F_t^n({}^x c_t) \to \nabla_x^- F_t({}^x c_t)$ as $n \to \infty$ by Lebesgue's theorem.

Next, since $\nabla_x^- F_t({}^x c_t)$ is bounded on $\mathcal{G}_M$, its mollifier $\nabla_x F_t^n({}^x c_t)$ is bounded on $\mathcal{G}_M$ by the same constant. In particular, there exists C such that $|\nabla^v F_s^n(X_s^M)| = |\nabla_x F_s^n({}^{X^M(s)}X_s^M)| \leqslant C$ for every $(\omega,s) \in \Omega \times [0,t]$. Now the L_2-convergence (3.4) is implied by the semimartingale decomposition, Itô's isometry, and the bounded convergence theorem.

In the sequel, to shorten notation we write $[X^M] = [X^M, X^M]$.

STEP 3: Lastly, we investigate the limit of $\frac{1}{2}\int_0^t \nabla^{v,2} F_s^n(X_s^M)d[X^M](s)$ as $n \to \infty$. By applying mollification (2.5) in (3.5), we obtain $F_t^n({}^x c_t) = \hat{F}_t^n(c_t;x) + \tilde{F}_t^n(c_t;x)$, where $\hat{F}_t^n(c_t;x) := (\rho_n \star \hat{F}_t(c_t;\cdot))(x)$ and $\tilde{F}_t^n(c_t;x) := (\rho_n \star \tilde{F}_t(c_t;\cdot))(x)$. Let us define $\hat{F}_t^n(c_t) := \hat{F}_t^n(c_t;c(t))$ and $\tilde{F}_t^n(c_t) := \tilde{F}_t^n(c_t;c(t))$. We have:

$$\frac{1}{2}\int_0^t \nabla^{v,2} F_s^n(X_s^M)d[X^M](s) \quad = \quad \frac{1}{2}\int_0^t \nabla^{v,2} \hat{F}_s^n(X_s^M)d[X^M](s)$$

$$+ \quad \frac{1}{2}\int_0^t \nabla^{v,2}\tilde{F}_s^n(X_s^M)d[X^M](s). \qquad (3.7)$$

Note that $\hat{F}_t(c_t;x)$ is C^1 in x on $\mathcal{G}_M$ and the map $x\mapsto \nabla_x\hat{F}_t(c_t;x)$ has on $\mathcal{G}_M^\gamma$ a bounded left derivative $\nabla_x^-\nabla_x\hat{F}_t(c_t;x)$. By Lemma 2.6 and Remark 2.7, $x\mapsto \nabla_x^2\hat{F}_t^n(c_t,x)$ is the mollifier for $x\mapsto\nabla_x^-\nabla_x\hat{F}_t(c_t,x)$ on $[-M,\gamma_t(c_t)-\varepsilon]$ for any sufficiently small $\varepsilon>0$, i.e.

$$\nabla_x^2\hat{F}_t^n(c_t;x) = \int_0^2 \rho(y)\nabla_x^-\nabla_x\hat{F}_t(c_t;x-\frac{y}{n})dy = \int_0^2\rho(y)\nabla_x^{-,2}F_t({}^{x-\frac{y}{n}}c_t)dy, \quad (3.8)$$

for $x\in[-M,\gamma_t(c_t)-\varepsilon]$. By assumption, $x\mapsto\nabla_x^{-,2}F_t({}^x c_t)$ is bounded on $\Gamma_{c,t}$ and its left limit exists at $x=\gamma_t(c_t)$. This implies that (3.8) holds for all $x\in[-M,\gamma_t(c_t)]$. We note also that (3.8) holds for $x\in(\gamma_t(c_t)+\frac{2}{m},M]$ whenever $n>m$ and m is fixed arbitrary. By Lebesgue's theorem, we pass to the limit in (3.8) as $n\to\infty$ while $x\in[-M,\gamma_t(c_t)]\cup(\gamma_t(c_t)+\frac{2}{m},M]$ and $(t,c)\in[0,T]\times C([0,T],\mathbb{R})$ are fixed. We obtain that for $(x,c,t)\in[-M,\gamma_t(c_t)]\cup(\gamma_t(c_t)+\frac{2}{m},M]\times C([0,T],\mathbb{R})\times[0,T]$

$$\lim_{n\to\infty}\nabla_x^2\hat{F}_t^n(c_t;x) = \nabla_x^{-,2}F_t({}^{x-}c_t).$$

Since m is fixed arbitrary, the above equality holds for all $(x,c,t)\in\mathcal{G}_M$.

Therefore, we have

$$\lim_{n\to\infty}\nabla^{v,2}\hat{F}_t^n(X_t^M) = \nabla_x^{-,2}F_t({}^{X^M(t)-}X_t^M)\ a.s,$$

and

$$\int_0^t\nabla^{v,2}\hat{F}_s^n(X_s^M)d[X^M](s)\to\int_0^t\nabla_x^{-,2}F_s({}^{X^M(s)-}X_s^M)d[X^M](s) \quad \text{a.s.}$$

by bounded convergence.

Let us investigate the convergence of the last term in (3.7). It is convenient to introduce the following notation: We define $\gamma^M(s):=\gamma_{s\wedge T_M}(X_s^M)$ and $\tilde{X}^M(s):=X^M(s)-\gamma^M(s); 0\leqslant s\leqslant T$. Let $\varphi_s^n(x)$ be the mollifier of $(x-\gamma^M(s))^+$ according to formula (2.5), and let $\varphi^n(x)$ be the mollifier of x^+. It is easy to verify that $\varphi_s^n(x)=\varphi^n(x-\gamma^M(s))$. Therefore,

$$\begin{aligned}\tilde{F}_t^n(X_t^M;x) =&(\nabla F_t({}^{\gamma^M(t)+}X_t^M)-\nabla F_t({}^{\gamma^M(t)-}X_t^M))\varphi_t^n(x)\\ =&(\nabla F_t({}^{\gamma^M(t)+}X_t^M)-\nabla F_t({}^{\gamma^M(t)-}X_t^M))\varphi^n(x-\gamma^M(t))\ a.s.\end{aligned}$$

Note that $\nabla_x\varphi^n(x)=\int_{-\infty}^{\infty}\rho_n(x-y)H(y)dy$, where H is the Heaviside function, and that $\nabla_x^2\varphi^n(x)=\int_{-\infty}^{\infty}\rho_n(x-y)dH(y)=\rho_n(x)$. Note that since $\gamma_\cdot(X_\cdot)$ has continuous bounded variation paths, then $[X^M](s)=[\tilde{X}^M](s)\ a.s; 0\leqslant s\leqslant T$.

Now let $\tilde{\ell}_M^x$ be the local time of $\tilde{X}^M$. By Lemma 3.1, we obtain:

$$\begin{aligned}&\frac{1}{2}\int_0^t\nabla^{v,2}\tilde{F}_s^n(X_s^M)d[X^M](s)=\frac{1}{2}\int_0^t\nabla_x^2\tilde{F}_s^n(X_s^M,\tilde{X}^M(s)+\gamma^M(s))d[\tilde{X}^M](s)\\ &=\frac{1}{2}\int_{-\infty}^{\infty}dx\int_0^t\nabla_x^2\tilde{F}_s^n(X_s^M;x+\gamma^M(s))d_s\tilde{\ell}_M^x(s)\\ &=\frac{1}{2}\int_{-\infty}^{\infty}\nabla_x^2\varphi^n(x)dx\int_0^t(\nabla_xF_s({}^{\gamma^M(s)+}X_s^M)-\nabla_xF_s({}^{\gamma^M(s)-}X_s^M))d_s\tilde{\ell}_M^x(s)\end{aligned}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\rho_n(x)dx\int_0^t(\nabla_x F_s({}^{\gamma^M(s)+}X^M_s) - \nabla_x F_s({}^{\gamma^M(s)-}X^M_s))d_s\tilde{\ell}^x_M(s)$$
$$\to \frac{1}{2}\int_0^t(\nabla_x F_s({}^{\gamma^M(s)+}X^M_s) - \nabla_x F_s({}^{\gamma^M(s)-}X^M_s))d_s\tilde{\ell}^0_M(s)\ a.s.\quad \text{as } n\to\infty.$$

The above computations imply formula (3.1) at time $t\wedge T_M$. Letting M go to infinity, we obtain (3.1). □

The simplest application of Theorem 3.2 is a pathwise description of the running maximum. A version of this formula appeared in Dupire (2009) but without a rigorous proof.

Example 3.3. Let us apply formula (3.1) to the running maximum

$$F_t(c_t) = \max_{s\in[0,t]} c(s); c\in\Lambda.$$

One immediately verifies that F satisfies the assumptions of Theorem 3.2. Let us compute each term of (3.1). We have: $\nabla^h F_t(X_t) = 0$, $\nabla^-_x F_t({}^xX_t) = 0$ if $x\leqslant F_t(X_t)$ and $\nabla^-_x F_t({}^xX_t) = 1$ if $x > F_t(X_t)$. In particular, $\nabla^-_x F_t({}^{X(t)}X_t) = 0$. Next, for $\gamma_t(c_t) = F_t(c_t)$, one can easily check that for each $c\in C([0,T];\mathbb{R})$, the function $x\mapsto F_t({}^xc_t)$ is C^1 for $x\in(-\infty,\gamma_t(c_t)\cup(\gamma_t(c_t),+\infty)$ and $\nabla^{-,2}_x F_t({}^xc_t) = 0$ in this open set. Finally, we notice that $\nabla_x F_t({}^{\gamma_t(X_t)+}X_t) - \nabla_x F_t({}^{\gamma_t(X_t)-}X_t) = 1$ for all $t\in[0,T]$. By formula (3.1),

$$\sup_{0\leqslant s\leqslant t} X(s) = X(0) + \frac{1}{2}\tilde{\ell}^0(t),$$

where $\tilde{\ell}$ is the local time of the semimartingale $X(t) - \sup_{0\leqslant s\leqslant t} X(s);\ 0\leqslant t\leqslant T$.

Let us now apply Theorem 3.2 to concrete path-dependent functionals arising in Mathematical Finance.

Example 3.4. Similar to example 3.3, we shall also consider the payoff decomposition of a standard lookback option with fixed strike K (see e.g Kwok (2008) for further details). For a given constant $K>0$, we consider $F_t(c_t) = \big(\sup_{0\leqslant s\leqslant t} c(s) - K\big)^+$ for $c\in\Lambda$. In this case, a straightforward application of Theorem 3.2 yields

$$\Big(\sup_{0\leqslant s\leqslant t} X(s) - K\Big)^+ = \big(X(0)-K\big)^+ + \frac{1}{2}\,\tilde{\ell}^0(t);\ 0\leqslant t\leqslant T$$

where $\tilde{\ell}$ is the local time of the semimartingale $X(t) - \max\{\sup_{0\leqslant s\leqslant t} X(s); K\};\ 0\leqslant t\leqslant T$.

Example 3.5. For each non-negative path $c\in\Lambda$, let us consider

$$F_t(c_t) = \big(c(t) - \lambda \inf_{T_0\leqslant s\leqslant t} c(s)\big)^+$$

where $\lambda>1$ and $0\leqslant T_0<T$ are arbitrary constants. This functional is the payoff of the so-called partial lookback european call option which allows lower investments than derivative contracts based on the payoff given in Example 3.4 (see e.g Kwok (2008)). Let us now apply Theorem 3.2 to give a novel representation for this payoff. For simplicity, we set $T_0 = 0$. Indeed, A1 (i), A1(ii) and A2 hold where $\nabla^h\mathcal{F}^x_t(c_t) = 0$ for every $x\in\mathbb{R}_+$ and a non-negative path $c\in\Lambda$. By the very definition of F, it is apparent that the bounded variation functional which encodes the whole singularity is $\gamma_t(c_t) = \lambda\inf_{0\leqslant s\leqslant t} c(s); 0\leqslant t\leqslant T$. Moreover, $\nabla^-_x F_t({}^xc_t) =$

0 if $x \leqslant \gamma_t(c_t)$ and $\nabla^-_x F_t({}^x c_t) = 1$ for $\gamma_t(c_t) < x$. In particular, $\nabla^- F_t({}^{X(t)}X_t) = \mathbb{1}_{\{X(t)>\gamma_t(X_t)\}}; 0 \leqslant t \leqslant T$. Moreover, $\nabla_x F_t({}^x c_t) = 0$ if $x < \gamma_t(c_t)$ and $\nabla_x F_t({}^x c_t) = 1$ if $x > \gamma_t(c_t)$. In particular, $\nabla^-_x F_t({}^{\gamma_t(c_t)-}c_t) = 0$, $\nabla^-_x F_t({}^{\gamma_t(c_t)+}c_t) = 1$ and $\nabla^{-,2}_x F_t({}^x c_t) = 0$ over $(-\infty, \gamma_t(c_t)) \cup (\gamma_t(c_t), +\infty)$. Finally, if X is a non-negative square-integrable continuous semimartingale, then applying formula (3.1), we get

$$\big(X(t) - \lambda \inf_{0\leqslant s\leqslant t} X(s)\big)^+ = \int_0^t \mathbb{1}_{\{X(s)>\gamma_s(X_s)\}} dX(s) + \frac{1}{2}\tilde{\ell}^0(t); 0 \leqslant t \leqslant T,$$

where $\tilde{\ell}$ is the local time of the semimartingale $X(t) - \lambda \inf_{0\leqslant s\leqslant t} X(s); 0 \leqslant t \leqslant T$. and $\big((1-\lambda)X(0)\big)^+ = 0$.

4. (p,q)-bivariations and Functional Itô formulas

In this section, we provide an Itô formula in the sense of Young in the path-dependent case. We refer the reader to the seminal work by Young (1938) for a full treatment of double Lebesgue-Stieljes-type integrals for unbounded variation functions. For a more simplified presentation, see e.g Ohashi and Simas (2014).

Before presenting the main results, we recall some basic results from deterministic double integrals in the sense of Young (1938). Recall that if $f : [a,b] \to \mathbb{R}$ is a real-valued function and $p \geqslant 1$, then

$$\|f\|^p_{[a,b];p} := \sup_{\Pi} \sum_{x_i \in \Pi} |f(x_i) - f(x_{i-1})|^p < \infty$$

where sup is taken over all partitions Π of a compact set $[a,b] \subset \mathbb{R}$. The following notion is originally due to Young (1938) and it will play a key role in this section:

Definition 4.1. We say that $h : [a,b] \times [c,d] \to \mathbb{R}$ has (p,q)-bivariation for $p, q \geqslant 1$ if

$$\|h\|_{1;p} := \sup_{y_1,y_2 \in [c,d]^2} \|h(\cdot, y_1) - h(\cdot, y_2)\|_{[a,b];p} < \infty,$$

and

$$\|h\|_{2;q} := \sup_{x_1,x_2 \in [a,b]^2} \|h(x_1, \cdot) - h(x_2, \cdot)\|_{[c,d];q} < \infty.$$

The importance of (p,q)-bivariation lies in the following result, which is a particular case of Theorem 6.3 due to Young (1938).

Theorem 4.2 (Theorem 6.3 in Young (1938))**.** *Let $h, G : [a,b] \times [c,d] \to \mathbb{R}$ be two functions, where h vanishes on the lines $x = a$ and $y = c$ and has bounded (p,q)-bivariation, and G satisfies $|\Delta_i \Delta_j G(x_i, y_j)| \le C|x_i - x_{i-1}|^{1/\tilde{p}} |y_j - y_{j-1}|^{1/\tilde{q}}$, for some constant $C > 0$, and $\tilde{p}, \tilde{q} \geqslant 1$. If there exists $\alpha \in (0,1)$ such that*

$$\alpha/p + 1/\tilde{p} > 1 \quad \text{and} \quad (1-\alpha)/q + 1/\tilde{q} > 1,$$

then, the 2D Young integral $\int_a^b \int_c^d h(x,y) d_{(x,y)} G(x,y)$ exists.

Remark 4.3. We stress that there exists a related literature on $2D$-Young integral based on joint variations (see e.g Friz and Victoir (2010, 2011)) and related norms (see e.g Towghi (2002b)), rather than the bivariation concept. Indeed, one can check that $\|h\|_{1;p} \leqslant RV^{p,p}_{[a,b]\times[c,d]}(h)$ and $\|h\|_{2;q} \leqslant RV^{q,q}_{[a,b]\times[c,d]}(h)$ and these inequalities may be strict. See Section 5 for the definition of the norm RV.

Remark 4.4. In general, we only know that generic continuous semimartingales admit local times with finite $(1, 2+\delta)$-bivariation (for every $\delta > 0$) rather than joint variation (see Lemma 2.1 in Feng and Zhao (2006)). In some particular cases, the local time of a semimartingale admits joint variation. See Section 5 for details about symmetric stable processes.

4.1. *Functional Itô formula.* Throughout this section, $\delta > 0$ and $p, \tilde{p}, \tilde{q} \geqslant 1$ are constants such that $\frac{1}{p} + \frac{1}{2+\delta} > 1$ and there exists $\alpha \in (0,1)$ such that

$$\alpha + \frac{1}{\tilde{p}} > 1 \quad \text{and} \quad \frac{(1-\alpha)}{2+\delta} + \frac{1}{\tilde{q}} > 1$$

Lemma 4.5. *Let $\varphi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a stochastic process such that $(t,x) \mapsto \nabla^2_x \varphi(\omega,t,x) \in C([0,T] \times \mathbb{R}; \mathbb{R})$ for each $\omega \in \Omega$ and $\nabla^2_x \varphi$ is bounded on $\Omega \times [0,T] \times [-M,M]$ for each $M > 0$. Then,*

$$\begin{aligned} \int_0^t \nabla^2_x \varphi(s, X^M(s)) d[X^M, X^M](s) &= \int_{\mathbb{R}} \Big(\int_0^{t \wedge T_M} \nabla^2_x \varphi(s,y) d_s \ell^y(s) \Big) dy \\ &= -\int_0^{t\wedge T_M} \int_{\mathbb{R}} \nabla_x \varphi(s,x) d_{(s,x)} \ell^x(s) \ a.s \end{aligned} \tag{4.1}$$

for $0 \leqslant t \leqslant T$. In (4.1), the double integral is interpreted as a 2D Young integral in the sense of Young (1938).

Proof: Let us fix $M > 0$, $t \in [0,T]$ and $\omega \in \Omega$. In the sequel, we omit the variable ω in the computations. At first, we recall that if $\nabla^2_x \varphi : \Omega \times [0,T] \times [-M,M] \to \mathbb{R}$ is bounded, then Lemma 3.1 yields

$$\int_0^t \nabla^2_x \varphi(s, X^M(s)) d[X^M, X^M](s) = \int_{\mathbb{R}} \Big(\int_0^{t \wedge T_M} \nabla^2_x \varphi(s,y) d_s \ell^y(s) \Big) dy. \tag{4.2}$$

Let $0 = t_1 < t_2 < \ldots \leqslant t_{m+1} = t \wedge T_M$ and $-L = x_1 < x_2 < \ldots < x_{n+1} = L$ where $[-L, L]$ is a compact set. Let us fix $\omega \in \Omega$. Since the local-time has compact support, we stress that we can always add some points in the partition in such way that $\ell^{x_1}(t_j, \omega) = 0$ and $\ell^{x_{n+1}}(t_j, \omega) = 0$ for every $j = 1, \ldots, m$. To keep notation simple, we write $\varphi = \varphi(\omega)$ and $\ell = \ell(\omega)$. Mean value theorem allows us to argue just like in Remark 1 in Feng and Zhao (2006) to get the following identity

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n \nabla_x \varphi(t_j, x_i) \big(\Delta_j \ell^{x_{i+1}}(t_{j+1}) - \Delta_i \ell^{x_i}(t_{j+1}) \big) &= -\sum_{i=1}^m \sum_{j=1}^n \nabla^2_x \varphi(t_j, y_i) \\ &\times \ \Delta_j \ell^{x_{i+1}}(t_{j+1})(x_{i+1} - x_i) \end{aligned} \tag{4.3}$$

where $x_i < y_i < x_{i+1}; i = 1, \ldots, m$. Let K be the compact support of $x \mapsto \ell^x(T)$. We notice that the function $x \mapsto \sum_j \varphi(t_j, x) \Delta_j \ell^x(t_{j+1})$ is càdlàg and hence almost everywhere continuous. The boundedness assumption yields

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\nabla_x^2\varphi(t_j,y_i)\Delta_j\ell^{x_{i+1}}(t_{j+1})(x_{i+1}-x_i)\\
&=\lim_{n\to\infty}\int_K\sum_{j=1}^{n}\nabla_x^2\varphi(t_j,x)\times\Delta_j\ell^x(t_{j+1})dx\\
&=\int_{\mathbb{R}}\int_0^t\nabla_x^2\varphi(s,x)d\ell_s^x dsdx.
\end{aligned}$$

From (4.3), we conclude the proof. □

Let us now assume additional hypotheses on the functional F to shift quadratic variation to local-time integrals.

Assumption B: The spatial weak derivative $(\nabla_x^w F_t)({}^x c_t)$ satisfies: For every $L > 0$, there exists a constant C such that

$$|\Delta_i\Delta_j(\nabla_x^w F_{t_i})({}^{x_j}c_{t_i})| \le C|t_i-t_{i-1}|^{1/\tilde{p}}|x_j-x_{j-1}|^{1/\tilde{q}} \tag{4.4}$$

for every partition $\{t_i\}_{i=0}^N\times\{x_j\}_{j=0}^{N'}$ of $[0,T]\times[-L,L]$ and $c\in C([0,T];\mathbb{R})$. Moreover,

$$\sup_{0\leqslant t\leqslant T}\|(\nabla_x^w F_t)({}^{\cdot}c_t)\|_{[-L,L];p}<\infty \tag{4.5}$$

for every $c\in C([0,T];\mathbb{R})$.

In the sequel, we provide a mild hypothesis to get convergence of local-time and stochastic integrals.

Assumption C: We assume piecewise uniform left-continuity in the following sense: For every $\varepsilon>0, M>0$ and $c\in C([0,T];[-M,M])$ there exists $\{x_i\}_{i=0}^{n+1}$, $-M=x_0<x_1<\ldots<x_n<x_{n+1}=M$ such that

$$\sup_{0\leqslant t\leqslant T}|(\nabla_y^w F_t)({}^y c_t)-(\nabla_x^w F_t)({}^x c_t)|<\varepsilon$$

whenever $x_0\leqslant y\leqslant x\leqslant x_1$ or $x_i<y\leqslant x\leqslant x_{i+1}; i=1,\ldots,n$.

An immediate consequence of Lemma 4.5 is the following remark.

Corollary 4.6. *If F satisfies Assumptions A1(i) and A2, then for each $M>0$ and $n\geqslant 1$,*

$$\begin{aligned}
F_t^n(X_t^M) &= F_0^n(X_0^M)+\int_0^t\nabla^h F_s^n(X_s^M)ds+\int_0^{t\wedge T_M}\nabla^v F_s^n(X_s)dX(s)\\
&- \frac{1}{2}\int_0^{t\wedge T_M}\int_{\mathbb{R}}\nabla_x F_s^n({}^x X_s)d_{(s,x)}\ell^x(s)
\end{aligned}$$

a.s. for $0\leqslant t\leqslant T$.

Proof: Let us fix $M>0$ and $n\geqslant 1$. In one hand, $\rho^{(2)}$ has compact support and ${}^x X_{\cdot}^M\in D([0,T];[-M,M])$ a.s, then we shall use Assumption A1(i), to state that $(\omega,t,x)\mapsto\nabla_x^2F_t^n({}^xX_t^M(\omega))$ is a bounded measurable process on $\Omega\times[0,T]\times[M,M]$.

On the other hand, $\nabla^{v,2}F^n_t(X^M_t) = \nabla^2_x F^n_t({}^xX^M_t)|_{x=X^M(t)}$ so that (4.1) in Lemma 4.5 yields

$$\begin{aligned}\int_0^t \nabla^{v,2}F^n_s(X^M_s)d[X^M,X^M](s) &= -\int_0^{t\wedge T_M}\int_{\mathbb{R}} \nabla_x F^n_s({}^xX^M_s)d_{(s,x)}\ell^x(s)\\ &= -\int_0^{t\wedge T_M}\int_{\mathbb{R}} \nabla_x F^n_s({}^xX_s)d_{(s,x)}\ell^x(s)\ a.s\end{aligned}$$

for $0\leqslant t\leqslant T$. Lemma 2.10 allows us to conclude the proof. □

Now we are able to present the main result of this section. It extends Feng and Zhao (2006) in the context of path-dependent functionals as well as Th. 8.1 in Leão et al. (2015) in the context of generic semimartingales. In particular, it complements the results given in section 3 when $x\mapsto(\nabla^w_x F_t)({}^xc_t)$ has bounded variation.

Theorem 4.7. *Let F be a functional satisfying Assumptions A1, A2, B and C. Then*

$$\begin{aligned}F_t(X_t) &= F_0(X_0)+\int_0^t \nabla^h F_s(X_s)ds + \int_0^t \nabla^w F_s(X_s)dX(s)\\ &- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t (\nabla^w_x F_s)({}^xX_s)d_{(s,x)}\ell^x(s)\ a.s\end{aligned}\tag{4.6}$$

for $0\leqslant t\leqslant T$.

Proof: Let $M>0$ be such that *supp* $\rho\subset[-M,M]$. To keep notation simple, we set $t_M := t\wedge T_M$ and $k(dz)=\rho(z)dz$. At first, we claim that the following convergence holds

$$\int_0^{t_M}\int_{[-M,M]}\nabla_x F^n({}^xX_s)d_{(s,x)}\ell^x(s)\to\int_0^{t_M}\int_{[-M,M]}(\nabla^w_x F_s)({}^xX_s)d_{(s,x)}\ell^x(s)\tag{4.7}$$

almost surely as $n\to\infty$, for each $t\in[0,T]$. Indeed, by making a change of variable

$$\nabla_x F^n_t({}^xX_t)-(\nabla^w_x F_t)({}^xX_t)=\int_{-M}^{M}\Big((\nabla^w_x F_t)({}^{x-z/n}X_t)-(\nabla^w_x F_t)({}^xX_t)\Big)k(dz)\ a.s\tag{4.8}$$

for every $(t,x)\in[0,T]\times[-M,M]$. Let us fix $\omega\in\Omega$. By Assumption C, we then have

$$\sup_{(x,t)\in[-M,M]\times[0,T]}\big|\nabla_x F^n_t({}^xX_t(\omega))-(\nabla^w_x F_t)({}^xX_t(\omega))\big|\to 0\tag{4.9}$$

as $n\to\infty$. Moreover, for any partition $\{t_i\}_{i=0}^N\times\{x_j\}_{j=0}^{N'}$ of $[0,t]\times[-M,M]$, we have

$$|\Delta_j\Delta_i\nabla_x F^n_{t_i}({}^{x_j}X_{t_i})|\leqslant\int_{-M}^{M}|\Delta_j\Delta_i(\nabla^w_x F_{t_i})({}^{x_j-\frac{z}{n}}X_{t_i})|k(dz)\ a.s.\tag{4.10}$$

Let us now fix an arbitrary partition $\{t_i\}_{i=0}^N\times\{x_j\}_{j=0}^{N'}$ of $[0,t]\times[-M,M]$. Let $\mathcal{P}_{[-Q,Q]}$ be the set of all partitions of the interval $[-Q,Q]$ for $0<Q<\infty$. We notice that for each $z\in[0,M]$ the set $\{x_j-z/n; j=0,\ldots,N'\}$ is a partition of $[-M-z/n,M-z/n]$. In particular, $[-M-\frac{z}{n},M-\frac{z}{n}]\subset[-M-\frac{M}{n},M+\frac{M}{n}]$ for every $z \in [0, M]$ and $n \geqslant 1$. Then we shall find a compact set $[-2M, 2M]$ such that $[-M - \frac{M}{n}, M + \frac{M}{n}] \subset [-2M, 2M]$ $\forall n \geqslant 1$. More importantly. we shall add finitely many points in the set $\{x_j - z/n; j = 0, \ldots, M, z \in [0, M]\}$ in such way that this can be viewed as a subset of $\mathcal{P}_{[-2M,2M]}$. A similar argument holds for $z \in [-M, 0]$. Therefore, Assumption B and (4.10) yield the existence of a positive constant C which only depends on $M > 0$ such that

$$
\begin{aligned}
|\Delta_j \Delta_i \nabla_x F^n_{t_i}({}^{x_j}X_{t_i})| &\leqslant \int_{-M}^{M} |\Delta_j \Delta_i (\nabla^w_x F_{t_i})({}^{x_j - \frac{z}{n}}X_{t_i})| k(dz) \\
&\leqslant C|t_i - t_{i-1}|^{1/\tilde{p}} |x_j - x_{j-1}|^{1/\tilde{q}} \ a.s \qquad (4.11)
\end{aligned}
$$

for every $n \geqslant 1$. Let us fix $\omega \in \Omega^*$ and $t \in [0, T]$, where $\mathbb{P}(\Omega^*) = 1$. We may suppose that $\ell^{-M}(\cdot, \omega) = 0$ and we obviously have $\ell^{\cdot}(0, \omega) = 0$. Then, we shall apply Th 6.4 in Young (1936) to state that

$$
\begin{aligned}
\lim_{n\to\infty} \int_0^{t_M(\omega)} \int_{-M}^{M} \ell^x(\omega, s) d_{(s,x)} \nabla_x F^n_s({}^xX_s(\omega)) &= \int_0^{t_M(\omega)} \int_{-M}^{M} \ell^x(\omega, s) \qquad (4.12) \\
& \quad d_{(s,x)} (\nabla^w_x F_s)({}^xX_s(\omega)).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\Delta_j \nabla_x F^n_{t_M(\omega)}({}^{x_j}X_{t_M(\omega)}(\omega)) &= \int_{-M}^{M} \Big[(\nabla^w_x F_{t_M(\omega)})({}^{x_j - z/n}X_{t_M(\omega)}(\omega)) \\
&- (\nabla^w_x F_{t_M(\omega)})({}^{x_{j-1} - z/n}X_{t_M(\omega)}(\omega)) \Big] k(dz).
\end{aligned}
$$

Since $\int_0^2 \rho(z)dz = 1$, we shall apply Jensen inequality to get

$$
|\Delta_j \nabla_x F^n_{t_M(\omega)}({}^{x_j}X_{t_M(\omega)}(\omega))|^p \leqslant \int_{-M}^{M} |\Delta_j (\nabla^w_x F_{t_M(\omega)})({}^{x_j - z/n}X_{t_M(\omega)}(\omega))|^p k(dz). \qquad (4.13)
$$

The same argument used in (4.11) also applies here. In this case, by applying (4.5) into (4.13), we can find a compact set $[-Q, Q]$ such that

$$
\begin{aligned}
&\sum_{j=0}^{N'} |\Delta_j \nabla_x F^n_{t_M(\omega)}({}^{x_j}X_{t_M(\omega)}(\omega))|^p \\
&\leqslant \int_{-M}^{M} \sum_{j=0}^{N'} |\Delta_j (\nabla^w_x F_{t_M(\omega)})({}^{x_j - z/n}X_{t_M(\omega)}(\omega))|^p k(dz) \qquad (4.14) \\
&\leqslant \int_{-M}^{M} \|(\nabla^w_x F_{t_M(\omega)})({}^{\cdot}X_{t_M(\omega)}(\omega))\|^p_{[-Q,Q];p} k(dz) \\
&= \|(\nabla^w_x F_{t_M(\omega)})({}^{\cdot}X_{t_M(\omega)}(\omega))\|^p_{[-Q,Q];p}
\end{aligned}
$$

for every $n \geqslant 1$. Estimate (4.14) yields

$$
\|\nabla_x F^n_{t_M(\omega)}({}^{\cdot}X_{t_M(\omega)}(\omega))\|^p_{[-M,M];p} \leqslant \|(\nabla^w_x F_{t_M(\omega)})({}^{\cdot}X_{t_M(\omega)}(\omega))\|^p_{[-Q,Q];p} \qquad (4.15)
$$

for every $n \geqslant 1$. Estimate (4.15) together with (4.9) allow us to use Proposition 6.12 in e.g Friz and Victoir (2010) to get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-M}^{M}\ell^x(t_M(\omega))d_x\nabla_x F^n_{t_M(\omega)}({}^xX_{t_M(\omega)}(\omega))\\
&=\int_{-M}^{M}\ell^x(t_M(\omega))d_x(\nabla^w_x F_{t_M(\omega)})({}^xX_{t_M(\omega)}(\omega)).
\end{aligned}\tag{4.16}$$

By writing

$$\begin{aligned}
\int_0^{t_M(\omega)}\int_{-M}^{M}\nabla_x F^n_s({}^xX_s(\omega))d_{(s,x)}\ell^x(s) &= \int_0^{t_M(\omega)}\int_{-M}^{M}\ell^x(\omega,s)d_{(s,x)}\nabla_x F^n_s({}^xX_s(\omega))\\
&- \int_{-M}^{M}\ell^x(t_M(\omega))d_x\nabla_x F^n_{t_M(\omega)}({}^xX_{t_M(\omega)}(\omega))
\end{aligned}$$

and using (4.16) and (4.12), we conclude that (4.7) holds. From Assumptions A1(ii), we know that $\lim_{n\to\infty}F^n_t(X^M_t)=F_t(X^M_t)$ $a.s; 0\leqslant t\leqslant T$. From Corollary 4.6, it only remains to check that

$$\int_0^t \nabla^h F^n_s(X^M_s)ds \to \int_0^t \nabla^h F_s(X^M_s)ds \tag{4.17}$$

$$\int_0^{t_M} \nabla^v F^n_s(X_s)dX(s) \to \int_0^{t_M} \nabla^w F_s(X_s)dX(s) \tag{4.18}$$

in probability as $n\to\infty$. We have already checked that convergence (4.17) holds in the proof of Theorem 3.2.

From (4.9), we know that for each $\omega\in\Omega$

$$\sup_{0\leqslant t\leqslant T}|\nabla^v F^n_t(X_t(\omega)) - \nabla^w F_t(X_t(\omega))| \to 0$$

as $n\to\infty$ so that

$$\int_0^{t_M}|\nabla^v F^n_s(X_s)-\nabla^w F_s(X_s)|^2 d[X,X](s)\to 0$$

in probability as $n\to\infty$. This shows that (4.18) holds. Summing up the above result together with Corollary 4.6, we get

$$\begin{aligned}
F_{t_M}(X_{t_M}) &= F_0(X_0) + \int_0^t \nabla^h F_s(X^M_s)ds + \int_0^{t_M}\nabla^w F_s(X_s)dX(s)\\
&- \frac{1}{2}\int_{-M}^{M}\int_0^{t_M}(\nabla^w_x F_s)({}^xX_s)d_{(s,x)}\ell^x(s)
\end{aligned}$$

a.s for $0\leqslant t\leqslant T$. By letting $M\to\infty$ and using the fact that $(x,t)\mapsto \ell^x(t)$ has compact support a.s, then we recover (4.6). □

Example 4.8. We consider an example studied by Leão et al. (2015) given by

$$F_t(c_t)=\int_{-\infty}^{c(t)}\int_0^t \varphi(c(s),y)dsdy; 0\leqslant t\leqslant T,$$

for $c\in\Lambda$, where $\varphi:\mathbb{R}^2\to\mathbb{R}$ is a two-parameter Hölder continuous function satisfying the following hypotheses:

(i) For every compact set $K \subset \mathbb{R}$, there exist constants M_1 and M_2 such that for every $a, z \in K$,

$$|\varphi(a,x) - \varphi(a,y)| \leqslant M_1 |x-y|^{\gamma_1},$$

and

$$|\varphi(c,z) - \varphi(d,z)| \leqslant M_2 |c-d|^{\gamma_2},$$

where $\gamma_1 \in \left(\frac{1+\delta}{2+\delta}, 1\right]$, $\gamma_2 \in (0,1]$ and $\delta > 0$.

(ii) For every compact set $V_1 \subset \mathbb{R}$ there exists a compact set V_2 such that $\{x; \varphi(a,x) \neq 0\} \subset V_2$ for every $a \in V_1$.

(iii) For every continuous path $c \in C([0,T];\mathbb{R})$, $\int_{[0,T]\times\mathbb{R}} |\varphi(c(s),y)| ds dy < \infty$.

This example was studied in Leão et al. (2015) in the Brownian filtration context where the authors show that it is a weakly differentiable process. One can easily check if (**i, ii, iii**) are in force, then this functional satisfies the assumptions in Theorem 4.7. In particular, if X is a continuous semimartingale, the following decomposition holds

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \int_0^t \int_0^s \varphi(X(r), X(s)) dr dX(s) + \int_0^t \int_{-\infty}^{X(s)} \varphi(X(s), y) dy ds \\ &- \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \int_0^s \varphi(X(r), x) dr d_{(s,x)} \ell^x(s);\ a.s, 0 \leqslant t \leqslant T. \end{aligned}$$

One can also think in more general functionals of the form

$$\int_{-\infty}^{c(t)} Z_t(c_t; y) dy; c \in \Lambda$$

where $Z = \{Z_t(\cdot; x) : C([0,t];\mathbb{R}) \rightarrow \mathbb{R};\ x \in \mathbb{R};\ 0 \leqslant t \leqslant T\}$ is a family of functionals with suitable two-parameter Hölder regularity. See Example 5.8.

5. Functional Itô formula for symmetric stable processes under joint variation conditions

In this section, we investigate Itô formulas under different (and somewhat weaker) assumptions from the particular 2D-control given by (4.4) in Assumption B. In the language of rough path theory, assumption (4.4) precisely says that if $\tilde{q} = \tilde{p} = \beta$ then $\nabla_x F_t({}^x X_t)$ admits a 2D-control $\omega([t_1,t_2] \times [x_1,x_2]) = |t_1 - t_2|^{\frac{1}{\beta}} |x_1 - x_2|^{\frac{1}{\beta}}$ so that (4.4) trivially implies that $(t,x) \mapsto \nabla_x F_t({}^x X_t)$ has (β,β)-joint variation in the sense of Friz and Victoir (2010). If the semimartingale local time $\{\ell^x(t); -L \leqslant x \leqslant L, 0 \leqslant t \leqslant T\}$ admits joint variation over compact sets $[-L,L] \times [0,T]$ a.s. (see Definition 5.1), then (4.4) and (4.5) in Assumption B can be weakened to more general types of controls.

To our best knowledge, it is only known that local-times associated to general continuous semimartingales admit finite $(1, 2+\delta)$-bivariation a.s.for any $\delta > 0$. This result is due to Feng and Zhao (2006). In the sequel, we study joint variation of local-times of semimartingales in the following sense.

Definition 5.1. Let $p,q,r,s \in [1,\infty)$, $-\infty < a_1 < a_2 < +\infty$ and $-\infty < b_1 < b_2 < \infty$. A function $H : [a_1,a_2] \times [b_1,b_2] \rightarrow \mathbb{R}$ has joint right finite (p,q)-variation when

$$RV^{p,q}_{[a_1,a_2]\times[b_1,b_2]}(H) := \sup_{\pi} \left\{ \left[\sum_{i=1}^{n} \left[\sum_{j=1}^{m} |\Delta_i \Delta_j H(t_i,x_j)|^p \right]^{\frac{q}{p}} \right]^{\frac{1}{q}} \right\} < \infty.$$

It has joint left finite (r,s)-variation when

$$LV^{r,s}_{[a_1,a_2]\times[b_1,b_2]}(H) := \sup_{\pi} \left\{ \left[\sum_{j=1}^{m} \left[\sum_{i=1}^{n} |\Delta_i \Delta_j H(t_i,x_j)|^r \right]^{\frac{s}{r}} \right]^{\frac{1}{s}} \right\} < \infty$$

where sup varies over all partitions $\pi := \{a_1 = t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n = a_2\} \times \{b_1 = x_0 \leqslant x_1 \ldots \leqslant x_m = b_2\}$ of $[a_1,a_2] \times [b_1,b_2]$.

See Towghi (2002a) for more details on this variation concept. When $p = q$, this type of variation has been studied in the context of Gaussian rough paths (see e.g Cass et al. (2015) and Cass et al. (2009)). The following result is an immediate consequence of a fundamental estimate due to Marcus and Rosen (1992) in Lemma 3.3.

Lemma 5.2. *Let X be a real-valued symmetric stable process with exponent $1 < \beta \leqslant 2$. Then for every natural number $p \geqslant 1$, there exists a positive number C wich only depends on (β,p) such that*

$$\|\ell^x(t) - \ell^y(t) - \big(\ell^x(s) - \ell^y(s)\big)\|_{L^{2p}(\mathbb{P})} \leqslant C|x-y|^{\frac{\beta-1}{2}}|t-s|^{\frac{\beta-1}{2\beta}} \tag{5.1}$$

for any list of numbers $(t,s,x,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2$.

Proof: From Lemma 3.3 in Marcus and Rosen (1992), we know there exists a constant $C > 0$ which only depends on (p,β) such that

$$\Big(\mathbb{E}|\ell^x(t) - \ell^y(t)|^{2p}\Big)^{\frac{1}{2p}} \leqslant C|x-y|^{(\frac{\beta-1}{2})}t^{\frac{\beta-1}{2\beta}} \tag{5.2}$$

for every $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2$. Let $\theta_t : \Omega \rightarrow \Omega$ be the standard shift operator defined by the relation $Y \circ \theta_t := Y(\theta_t); t \geqslant 0$ for any random variable Y. Since X is a Markov process, then we know that the associated local-time process $\{\ell^x(t); (x,t) \in \mathbb{R} \times \mathbb{R}_+\}$ is an additive functional. Hence, by using the Markov property and (5.2), if $(s,t,x,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2$, then

$$\begin{aligned}\|\ell^x(t) - \ell^y(t) - \big(\ell^x(s) - \ell^y(s)\big)\|^{2p}_{L^{2p}(\mathbb{P})} &= \mathbb{E}|\ell^x(t) - \ell^y(t) - \ell^x(s) + \ell^y(s)|^{2p} \circ \theta_s \\ &= \int_{-\infty}^{+\infty} \mathbb{E}|\ell^{x-v}(t-s) - \ell^{y-v}(t-s)|^{2p}\mathbb{P}_{X(s)}(dv) \\ &\leqslant C|x-y|^{(\frac{\beta-1}{2})2p}|t-s|^{(\frac{\beta-1}{2\beta})2p}\end{aligned}$$

where $\mathbb{P}_{X(s)}$ is the law of $X(s)$. $\square$

We are now able to show the following result.

Lemma 5.3. *Let X be a stable symmetric process with exponent $1 < \beta \leqslant 2$. Then for every compact subset $[-L,L] \subset \mathbb{R}$, the associated local time process ℓ of X satisfies $RV^{\alpha_1,\alpha_2}_{[0,T]\times[-L,L]}(\ell) + LV^{\alpha_2,\alpha_1}_{[0,T]\times[-L,L]}(\ell) < \infty$ a.s. for any $\alpha_1 > \frac{2}{\beta-1}$ and $\alpha_2 > \frac{2\beta}{\beta-1}$.*

Proof: Let us fix a compact set $[-L,L]\subset\mathbb{R}$ and let $p\geqslant 1$ be an arbitrary positive integer. Theorem 3.1 from Hu and Le (2013) and Lemma 5.2 imply that for every γ_1 and γ_2 satisfying

$$\gamma_1<\frac{\beta-1}{2}-\frac{1}{2p}\quad\text{and}\quad\gamma_2<\frac{\beta-1}{2\beta}-\frac{1}{2p}\tag{5.3}$$

there exists a non-negative random variable $C_p(\omega)$, which depends on p, such that

$$|\ell^x(\omega,t)-\ell^y(\omega,t)-(\ell^x(\omega,s)-\ell^y(\omega,s))|\leqslant C_p(\omega)|x-y|^{\gamma_1}|t-s|^{\gamma_2}\tag{5.4}$$

for every $s,t\in[0,T]$ and almost all $\omega\in\Omega$. In other words, for each pair of positive constants γ_1 and γ_2 satisfying

$$\gamma_1<\frac{\beta-1}{2}\quad\text{and}\quad\gamma_2<\frac{\beta-1}{2\beta}$$

there exists $p\geqslant 1$ which realizes (5.3) and a non-negative random variable $C_p(\omega)$, depending on p, such that (5.4) holds.

Now let (α_1,α_2) be any pair of numbers satisfying $\alpha_1>\frac{2}{\beta-1}$ and $\alpha_2>\frac{2\beta}{\beta-1}$. Inequality (5.4) is fulfilled for $\gamma_1=\alpha_1^{-1}$ and $\gamma_2=\alpha_2^{-1}$ and for a non-negative random variable $C_p(\omega)$. For a given partition, $\pi=\{-L=x_0\leqslant x_1\ldots\leqslant x_m=L\}\times\{0=t_1\leqslant t_2\leqslant\ldots\leqslant t_n=T\}$ of $[-L,L]\times[0,T]$, we then have

$$\sum_{j=1}^m|\Delta_j\Delta_i\ell^{x_j}(\omega,t_i)|^{\alpha_1}\leqslant 2LC_p(\omega)^{\alpha_1}|t_i-t_{i-1}|^{\gamma_2\alpha_1},$$

and hence,

$$\Big[\sum_{i=1}^n\Big[\sum_{j=1}^m|\Delta_j\Delta_i\ell^{x_j}(\omega,t_i)|^{\alpha_1}\Big]^{\frac{\alpha_2}{\alpha_1}}\Big]^{\frac{1}{\alpha_2}}\leqslant(2L)^{\frac{1}{\alpha_1}}T^{\frac{1}{\alpha_2}}C_p(\omega)\text{ for almost all }\omega\in\Omega.$$

This shows that $RV^{\alpha_1,\alpha_2}_{[0,T]\times[-L,L]}(\ell)<\infty$ a.s. for any $\alpha_1>\frac{2}{\beta-1}$ and $\alpha_2>\frac{2\beta}{\beta-1}$. The above argument also shows that $LV^{\alpha_2,\alpha_1}_{[0,T]\times[-L,L]}(\ell)<\infty$ *a.s.* This allows us to conclude the proof. □

In the sequel, we denote $\Delta f(t,s;x,y):=f(t,x)-f(t,y)-\big(f(s,x)-f(s,y)\big)$ for $(t,s,x,y)\in[0,T]^2\times\mathbb{R}^2$. A routine manipulation yields the following interpolation result. We omit the details of the proof.

Lemma 5.4. *Let $f:[0,T]\times[-M,M]\to\mathbb{R}$ be a function such that, for $a,b\geqslant 1$, $LV^{a,b}_{[0,T]\times[-M,M]}(f)<\infty$. If $a<a'$ and $b'=\frac{a'}{a}b$, then*

$$LV^{a',b'}_{[0,T]\times[-M,M]}(f)\leqslant\sup_{\substack{t,s\in[0,T];\\x,y\in[-M,M]}}|\Delta f(t,s;x,y)|^{\frac{a'-a}{a'}}\sup_\pi\left[\sum_{j=1}^m\Big[\sum_{i=1}^n|\Delta_i\Delta_jf(t_i,x_j)|^a\Big]^{\frac{b}{a}}\right]^{\frac{1}{b'}}.$$

Similarly, if $RV^{p,q}_{[0,T]\times[-M,M]}(f)<\infty$ for $p,q\geqslant 1$ and $p<p'$ and $q'=\frac{p'}{p}q$, then

$$RV^{p',q'}_{[0,T]\times[-M,M]}(f)\leqslant\sup_{\substack{t,s\in[0,T];\\x,y\in[-M,M]}}|\Delta f(t,s;x,y)|^{\frac{p'-p}{p'}}\sup_\pi\left[\sum_{i=1}^n\Big[\sum_{j=1}^m|\Delta_i\Delta_jf(t_i,x_j)|^p\Big]^{\frac{q}{p}}\right]^{\frac{1}{q'}},$$

where sup varies over all partitions $\pi:=\{0=t_1\leqslant t_2\leqslant\ldots\leqslant t_n=T\}\times\{-M=x_0\leqslant x_1\ldots\leqslant x_m=M\}$ of $[0,T]\times[-M,M]$.

In the sequel, for a compact set $[0,T]\times[-M,M]$, we denote

$$\|f\|_{a,b;[0,T]\times[-M,M]}$$
$$:= LV^{a,b}_{[0,T]\times[-M,M]}(f) + \|f(0,\cdot))\|_{[-M,M];b} + \|f(\cdot,-M)\|_{[0,T];a} + |f(0,-M)|$$

where $a,b\geqslant 1$. We define $LW_{a,b}([0,T]\times[-M,M])$ as the set of all functions $f:[0,T]\times[-M,M]\to\mathbb{R}$ such that $\|f\|_{a,b;[0,T]\times[-M,M]}<\infty$.

For $p,q\geqslant 1$, we also denote

$$|f|_{p,q;[0,T]\times[-M,M]}$$
$$:= RV^{p,q}_{[0,T]\times[-M,M]}(f) + \|f(0,\cdot))\|_{[-M,M];q} + \|f(\cdot,-M)\|_{[0,T];p} + |f(0,-M)|$$

and $RW_{p,q}([0,T]\times[-M,M])$ is the set of all functions $f:[0,T]\times[-M,M]\to\mathbb{R}$ such that $|f|_{p,q;[0,T]\times[-M,M]}<\infty$. We refer the reader to Towghi (2002a) for details on this joint variation concept.

Assumption D(i) There exists $1\leqslant a<\frac{2\beta}{\beta+1}$ such that $\sup_{x\in K}\|\nabla^w F_\cdot({}^x c_\cdot)\|_{a;[0,T]}<\infty$ for every $c\in C([0,T];\mathbb{R})$ and a compact subset $K\subset\mathbb{R}$.

Assumption D(ii) There exists $1\leqslant b<\frac{2}{3-\beta}$ such that $\sup_{0\leqslant t\leqslant T}\|\nabla^w F_t({}^\cdot c_t)\|_{b;[-M,M]}<\infty$ for every $c\in C([0,T];\mathbb{R})$ and $M>0$.

Proposition 5.5. *Let X be a stable symmetric process with index $1<\beta\leqslant 2$. Assume that F is a functional which satisfies Assumptions A1, A2, C and D(i). If for each $c\in C([0,T];\mathbb{R})$, $(t,x)\mapsto(\nabla^w_x F_t)({}^x c_t)\in LW_{a,b}([0,T]\times[-M,M])$ for every $M>0$ with $1\leqslant a<\frac{2\beta}{\beta+1}$, $1\leqslant b<\frac{2}{3-\beta}$ and $1\leqslant a\leqslant b$, then*

$$\begin{aligned}F_t(X_t) &= F_0(X_0)+\int_0^t\nabla^h F_s(X_s)ds+\int_0^t\nabla^w F_s(X_s)dX(s)\\ &- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t(\nabla^w_x F_s)({}^x X_s)d_{(s,x)}\ell^x(s)\ a.s\end{aligned}\tag{5.5}$$

for $0\leqslant t\leqslant T$.

Proof: In the sequel, we fix $M>0$ and to shorten notation, we omit $[0,T]\times[-M,M]$ and we write $\|\cdot\|_{a,b}$ and $LW_{a,b}$. We also write $\|\cdot\|_\gamma$ for the one-parameter Hölder norm over a compact set. Throughout this section, C is a generic constant which may differ from line to line. From Boylan (1964), we know that $\{\ell^x(s);(s,x)\in\mathbb{R}_+\times\mathbb{R}\}$ has jointly continuous paths a.s. From Lemma 5.3 and Th 1.2 (b) in Towghi (2002a), we know that the following integral process

$$\int_0^t\int_{-M}^{M}(\nabla^w F_s)({}^x X_s)d_{(s,x)}\ell^x(s);0\leqslant t\leqslant T,\tag{5.6}$$

exists if for any $c\in C([0,T];\mathbb{R})$, $(t,x)\mapsto(\nabla^w_x F_t)({}^x c_t)\in LW_{a,b}$ where

$$1\leqslant a<\frac{\alpha_2}{\alpha_2-1},1\leqslant b<\frac{\alpha_1}{\alpha_1-1}\quad\text{and}\quad\alpha_1>\frac{2}{\beta-1},\alpha_2>\frac{2\beta}{\beta-1}.\tag{5.7}$$

Since $\frac{2}{3-\beta}=\sup\{\frac{\alpha_1}{\alpha_1-1};\alpha_1>\frac{2}{\beta-1}\}$ and $\frac{2\beta}{\beta+1}=\sup\{\frac{\alpha_2}{\alpha_2-1};\alpha_2>\frac{2\beta}{\beta-1}\}$, then (5.6) exists whenever $\nabla^w F(c)\in LW_{a,b}$ for any $a<\frac{2\beta}{\beta+1}$ and $b<\frac{2}{3-\beta}$.

From Assumptions A1-A2 and Corollary 4.6, the following decomposition holds

$$\begin{aligned} F^n_t(X^M_t) &= F^n_0(X^M_0) + \int_0^t \nabla^h F^n_s(X^M_s)ds + \int_0^{t\wedge T_M} \nabla^v F^n_s(X_s)dX(s) \\ &- \int_0^{t\wedge T_M}\int_{-M}^M \nabla_x F^n({}^xX_s)d_{(s,x)}\ell^x(s) \end{aligned}$$

a.s for $0\leqslant t\leqslant T, n\geqslant 1$. From Assumptions A1, A2 and C, we have already proved (See convergence in (4.17) and (4.18)) that $\lim_{n\to\infty} F^n_t(X^M_t) = F_t(X^M_t)$ $a.s$ and

$$\begin{aligned} &\lim_{n\to\infty}\Big(\int_0^t \nabla^h F^n_s(X^M_s)ds + \int_0^{t\wedge T_M}\nabla^v F^n_s(X_s)dX(s)\Big) \\ &= \int_0^t \nabla^h F_s(X^M_s)ds + \int_0^{t\wedge T_M}\nabla^w F_s(X_s)dX(s) \end{aligned} \tag{5.8}$$

in probability for each $t\in[0,T]$. It only remains to check

$$\int_0^{t\wedge T_M}\int_{-M}^M \nabla_x F^n_s({}^xX_s)d_{(s,x)}\ell^x(s)\to\int_0^{t\wedge T_M}\int_{-M}^M(\nabla^w_x F_s)({}^xX_s)d_{(s,x)}\ell^x(s) \tag{5.9}$$

a.s. as $n\to\infty$ for every $t\in[0,T]$. To shorten notation, let us denote $\Phi^n_s(x) := \nabla_x F^n_s({}^xX_s) - (\nabla^w_x F_s)({}^xX_s); (s,x)\in[0,T]\times[-M,M]$. Let us fix an arbitrary $t\in[0,T]$. In the sequel, we take $\varepsilon>0$ small enough such that $a' = a+\varepsilon$ and $b' = \frac{a'}{a}b$ satisfy $a' < \frac{2\beta}{\beta+1}$ and $b' < \frac{2}{3-\beta}$. We claim that

$$\|\Phi^n\|_{a',b'}\to 0 \ a.s \text{ as } n\to\infty. \tag{5.10}$$

A simple one parameter interpolation estimate (similar to Lemma 5.4) yields

$$\|\Phi^n_0\|_{b'}\leqslant \sup_{x,y\in[-M,M]^2}|\Phi^n_0(x)-\Phi^n_0(y)|^{1-\frac{b}{b'}}\|\Phi^n_0\|_b^{\frac{b}{b'}} \ a.s \tag{5.11}$$

and

$$\|\Phi^n_\cdot(-M)\|_{a'}\leqslant \sup_{s,t\in[0,T]^2}|\Phi^n_t(-M)-\Phi^n_s(-M)|^{1-\frac{a}{a'}}\|\Phi^n_0\|_a^{\frac{a}{a'}} \ a.s \tag{5.12}$$

where (4.9) yields $\sup_{x,y\in[M,M]^2}|\Phi^n_0(x)-\Phi^n_0(y)|^{1-\frac{b}{b'}}\to 0$ $a.s$ as $n\to\infty$. Moreover,

$$\sum_{j=1}^m|\Delta_j\nabla_x F^n_0({}^{x_j}X_0)|^b\leqslant\int_0^2\rho(z)\sum_{j=1}^m|\nabla_j\nabla^w F_0\big({}^{x_j-\frac{z}{n}}X_0\big)|^b dz\leqslant C\|\nabla^w F_0({}^\cdot X_0)\|_b^b$$

so that $\sup_{n\geqslant1}\|\nabla_x F^n_0({}^\cdot X_0)\|_b^b\leqslant C\|\nabla^w F_0({}^\cdot X_0)\|_b^b$ $a.s.$ Triangle inequality then allows us to conclude that $\sup_{n\geqslant 1}\|\Phi^n_0\|_b^{\frac{b}{b'}}\leqslant C\|\nabla^w F_0({}^\cdot X_0)\|_b^{\frac{b}{b'}}$ $a.s.$ Then (5.11) yields

$$\lim_{n\to+\infty}\|\Phi_0\|_{b'} = 0 \quad \text{a.s.} \tag{5.13}$$

Similarly, by D(i),

$$\begin{aligned}\sum_{i=1}^k|\Delta_i\nabla_x F^n_{t_i}({}^{-M}X_{t_i})|^a &\leqslant \int_0^2\rho(z)\sum_{i=1}^k|\Delta_i\nabla^w F_{t_i}\big({}^{-M-\frac{z}{n}}X_{t_i}\big)|^a dz \\ &\leqslant C\sup_{-2M\leqslant x\leqslant 0}\|\nabla^w F_\cdot({}^xX_\cdot)\|_a^a\end{aligned}$$

so that $\sup_{n\geqslant 1}\|F^n_\cdot(^{-M}X_\cdot)\|_a^a \leqslant \sup_{-2M\leqslant x\leqslant 0}\|\nabla^w F_\cdot(^xX_\cdot)\|^a_a$ *a.s.* Triangle inequality, (5.12) and (4.9) yield

$$\lim_{n\to+\infty}\|\Phi^n_\cdot(-M)\|_{a'} = 0 \quad \text{a.s.} \tag{5.14}$$

Summing up (5.13) and (5.14) and invoking again (4.9), we conclude that

$$\lim_{n\to+\infty}\Big(|\Phi^n_0(-M)| + \|\Phi^n_0\|_{b'} + \|\Phi^n(-M)\|_{a'}\Big) = 0 \quad \text{a.s.} \tag{5.15}$$

Now, we take $\frac{b}{a}\geqslant 1$ and Jensen inequality yields

$$\begin{aligned}&\sum_{j=1}^m\Big(\sum_{i=1}^k|\Delta_i\Delta_j\nabla_x F^n_{t_i}(^{x_j}X_{t_i})|^a\Big)^{\frac{b}{a}}\\ &\leqslant \int_0^2\sum_{j=1}^m\Big(\sum_{i=1}^k|\Delta_i\Delta_j(\nabla^w F_{t_i})(^{x_j-\frac{z}{n}}X_{t_i})|^a\Big)^{\frac{b}{a}}\rho(z)dz\end{aligned} \tag{5.16}$$

a.s. for every $n\geqslant 1$ and partition π of $[0,T]\times[-M,M]$. Lemma 5.4 yields

$$\begin{aligned}LV^{a',b'}(\Phi^n) &\leqslant \sup_{\substack{t,s\in[0,T]\\x,y\in[-M,M]}}|\Delta\Phi^n(t,s;x,y)|^{\frac{a'-a}{a'}}\times\sup_\pi\left\{\left[\sum_{j=1}^m\Big[\sum_{i=1}^k|\Delta_i\Delta_j\Phi^n_{t_i}(x_j)|^a\Big]^{\frac{b}{a}}\right]^{\frac{1}{b'}}\right\}\\
&\leqslant C\sup_{\substack{t,s\in[0,T]\\x,y\in[-M,M]}}|\Delta\Phi^n(t,s;x,y)|^{\frac{a'-a}{a'}}\times\sup_\pi\left\{\left[\sum_{j=1}^m\Big[\sum_{i=1}^k|\Delta_i\Delta_j F^n_{t_i}(^{x_j}X_{t_i})|^a\Big]^{\frac{b}{a}}\right]^{\frac{1}{b'}}\right\}\\
&+C\sup_{\substack{t,s\in[0,T]\\x,y\in[-M,M]}}|\Delta\Phi^n(t,s;x,y)|^{\frac{a'-a}{a'}}\times\sup_\pi\left\{\left[\sum_{j=1}^m\Big[\sum_{i=1}^k|\Delta_i\Delta_j(\nabla^w F_{t_i})(^{x_j}X_{t_i})|^a\Big]^{\frac{b}{a}}\right]^{\frac{1}{b'}}\right\}\end{aligned}$$

a.s. for every $n\geqslant 1$. Then (4.9), (5.15) and (5.16) allow us to state that (5.10) holds true. Lastly, we take (α_1,α_2) such that $a'<\frac{\alpha_2}{\alpha_2-1}, b'<\frac{\alpha_1}{\alpha_1-1}$ for $\alpha_1>\frac{2}{\beta-1}$ and $\alpha_2>\frac{2\beta}{\beta-1}$. By Th. 1.2 in Towghi (2002a), we know there exists a constant C such that

$$\Big|\int_0^{t\wedge T_M}\int_{-M}^M\Phi^n_s(x)d_{(s,x)}\ell^x(s)\Big|\leqslant C\|\Phi^n\|_{(a',b')}\times LV^{\alpha_2,\alpha_1}(\ell) \tag{5.17}$$

a.s. for every $n\geqslant 1$ and hence Lemma 5.3, (5.17) and (5.10) allow us to conclude that decomposition (5.5) holds over the stochastic set $[0,t\wedge T_M]$. By taking $M\to\infty$, we may conclude the proof. □

A complete similar proof also yields the symmetric result of Corollary 5.5 as follows.

Corollary 5.6. *Let X be a stable symmetric process with index $1<\beta\leqslant 2$. Assume that F is a functional which satisfies Assumptions A1, A2, C and D(ii). If for each $c\in C([0,T];\mathbb{R})$, $(t,x)\mapsto(\nabla^w_x F_t)(^xc_t)\in RW_{p,q}([0,T]\times[-M,M])$ for every $M>0$ with $1\leqslant p<\frac{2}{3-\beta}$, $1\leqslant q<\frac{2\beta}{\beta+1}$ and $1\leqslant p\leqslant q$, then*

$$F_t(X_t) = F_0(X_0)+\int_0^t\nabla^h F_s(X_s)ds+\int_0^t\nabla^w F_s(X_s)dX(s) \tag{5.18}$$

$$- \quad \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t (\nabla_x^w F_s)(^x X_s) d_{(s,x)}\ell^x(s) \ a.s$$

for $0 \leqslant t \leqslant T$.

Example 5.7 (Path-dependent cylindrical functionals). Let $\{0 = t_0 < t_1 < t_2 < \cdots < t_n = T\}$ be a partition of $[0,T]$. Consider a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ weakly differentiable in each variable. Let us assume that for each k and for each $i > k$, the ith weak partial derivative

$$x \mapsto \nabla_i^w f(c(t_1-), c(t_2-), \ldots, c(t_k-), x, \ldots, x_i, \ldots, x)\big|_{x_i = x}, \tag{5.19}$$

evaluated at x, is left continuous and is of bounded q-variation on $[-M, M]$ for each $M > 0$ and for some $q \in [1, \frac{2}{3-\beta})$. For every $c \in \Lambda$, define the functional F_t by the formulas:

$$F(c) = f(c(t_1-), c(t_2-), \ldots, c(t_n-)) \quad \text{and} \quad F_t(c_t) = F(c_{t,T-t}). \tag{5.20}$$

Let us prove that Itô's formula (5.18) holds the functional F_t. Let us notice that the functional $\mathcal{F}^x$, defined by (2.4), takes the form:

$$\mathcal{F}_t^x(c_t) = \sum_{k=0}^{n-1} f(c(t_1-), c(t_2-), \ldots, c(t_k-), x, \ldots, x)\, \mathbb{I}_{\{t_k \leqslant t < t_{k+1}\}}. \tag{5.21}$$

From this formula one immediately verifies that the family $\mathcal{F}_t^x(c_t)$ is state boundedness preserving and that $\nabla^h \mathcal{F}_t^x(c_t) = 0$. For the weak derivative we obtain:

$$\begin{aligned}
&\nabla_x^w \mathcal{F}_t^x(c(t_1-), c(t_2-), \ldots, c(t_k-), x, \ldots, x) \\
&= \sum_{k=0}^{n-1} \mathbb{I}_{\{t_k \leqslant t < t_{k+1}\}} \sum_{i=k+1}^{n} \nabla_i^w f(c(t_1-), c(t_2-), \ldots, c(t_k-), x, \ldots, x_i, \ldots, x)\big|_{x_i = x}.
\end{aligned} \tag{5.22}$$

This immediately implies that Assumptions C and D(ii) are fulfilled. We also remark that $(t,x) \mapsto (\nabla_x^w \mathcal{F}_t^x)(c_t) \in LW_{p,q}([0,T] \times [-M, M])$, where q is the same number as of the q-variation of (5.19), and p is arbitrary.

We further note that the family $\mathcal{F}^x$ fails to be state-dependent Λ-continuous. However, one immediately verifies that it is state-dependent Λ-continuous on each interval $[t_{i-1}, t_i - \varepsilon]$ for any sufficiently small ε.

Therefore, on the interval $[0, t_1 - \varepsilon]$ all assumptions of Proposition 5.5 are fulfilled, and therefore,

$$\begin{aligned}
F_t(X_t) = F_0(X_0) + \int_0^{t_1-\varepsilon} \nabla^h F_s(X_s) ds + \int_0^{t_1-\varepsilon} \nabla^w F_s(X_s) dX(s) \\
- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{t_1-\varepsilon} (\nabla_x^w F_s)(^x X_s) d_{(s,x)}\ell^x(s).
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, we obtain (5.5) for any $t \in [0, t_1]$. By the same argument, (5.5) holds on each interval $[t_{i-1}, t_i]$ with the initial condition $F_{t_{i-1}}(X_{t_{i-1}})$. This implies (5.5) for every $t \in [0,T]$.

Example 5.8. Let us now summarize Theorem 4.7, Proposition 5.5 and Corollary 5.6. One typical class of examples which can be treated by using the results of

Sections 4.1 and 5 is the following pathwise path-dependent version of the classical formula given by Föllmer et al. (1995)

$$F_t(X_t) = \int_{-\infty}^{X(t)} Z_t(X_t; y)dy$$

where $Z = \{Z_t(\cdot; x) : C([0,t];\mathbb{R}) \to \mathbb{R}; 0 \leqslant t \leqslant T, x \in \mathbb{R}\}$ can be chosen in such way that

$$\nabla^w F_t(^x X_t) = Z_t(X_t; x) \quad \text{and} \quad \nabla^h F_t(X_t) = \int_{-\infty}^{X(t)} \nabla^h Z_s(X_s; y)dy$$

satisfy the set of assumptions (A1, A2, C, D(i)) or (A1, A2, B). For a concrete case, see Example 4.8. In this case, the following formula holds

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \int_0^t \int_{-\infty}^{X(s)} \nabla^h Z_s(X_s; y)dyds + \int_0^t Z_s(X_s; X(s))dX(s) \\ &- \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^t Z_s(X_s; x)d_{(s,x)}\ell^x(s) \end{aligned}$$

a.s. for $0 \leqslant t \leqslant T$.

Acknowledgements

The authors would like to thank Francesco Russo, Dorival Leão and Estevão Rosalino for stimulating discussions on the topic of this paper.

References

X. Bardina and C. Rovira. On Itô's formula for elliptic diffusion processes. *Bernoulli* **13** (3), 820–830 (2007). MR2348752.

N. Bouleau and M. Yor. Sur la variation quadratique des temps locaux de certaines semimartingales. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (9), 491–494 (1981). MR612544.

E. S. Boylan. Local times for a class of Markoff processes. *Illinois J. Math.* **8**, 19–39 (1964). MR0158434.

R. Buckdahn, J. Ma and J. Zhang. Pathwise Taylor expansions for random fields on multiple dimensional paths. *Stochastic Process. Appl.* **125** (7), 2820–2855 (2015). MR3332856.

Keller C. and J. Zhang. Pathwise Itô calculus for rough paths and rough PDEs with path dependent coefficients (2016). To appear in Stochastic Process. Appl.

T. Cass, P. Friz and N. Victoir. Non-degeneracy of Wiener functionals arising from rough differential equations. *Trans. Amer. Math. Soc.* **361** (6), 3359–3371 (2009). MR2485431.

T. Cass, M. Hairer, C. Litterer and S. Tindel. Smoothness of the density for solutions to Gaussian rough differential equations. *Ann. Probab.* **43** (1), 188–239 (2015). MR3298472.

R. Cont and D. Fournié. Change of variable formulas for non-anticipative functionals on path space. *J. Funct. Anal.* **259** (4), 1043–1072 (2010). MR2652181.

R. Cont and D. Fournié. Functional Itô calculus and stochastic integral representation of martingales. *Ann. Probab.* **41** (1), 109–133 (2013). MR3059194.

A. Cosso, C. Di Girolami and F. Russo. Calculus via regularization in Banach spaces and Kolmogorov-type path dependent equations. *ArXiv Mathematics e-prints* (2014). arXiv: 1411.8000.

A. Cosso and F. Russo. A regularization approach to functional Itô calculus and strong-viscosity solutions to path-dependent PDEs. *ArXiv Mathematics e-prints* (2014). arXiv: 1401.5034.

B. Dupire. Functional Itô calculus. *Bloomberg Portfolio Research Paper* **04**, 1–25 (2009). DOI: 10.2139/ssrn.1435551.

N. Eisenbaum. Integration with respect to local time. *Potential Anal.* **13** (4), 303–328 (2000). MR1804175.

I. Ekren, C. Keller, N. Touzi and J. Zhang. On viscosity solutions of path dependent PDEs. *Ann. Probab.* **42** (1), 204–236 (2014). MR3161485.

K. D. Elworthy, A. Truman and H. Zhao. Generalized Itô formulae and space-time Lebesgue-Stieltjes integrals of local times. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 117–136. Springer, Berlin (2007). DOI: 10.1007/978-3-540-71189-6_5.

C. Feng and H. Zhao. Two-parameter p, q-variation paths and integrations of local times. *Potential Anal.* **25** (2), 165–204 (2006). MR2238942.

C. Feng and H. Zhao. Rough path integral of local time. *C. R. Math. Acad. Sci. Paris* **346** (7-8), 431–434 (2008). MR2417564.

F. Flandoli. Stochastic flows for nonlinear second-order parabolic SPDE. *Ann. Probab.* **24** (2), 547–558 (1996). MR1404520.

H. Föllmer, P. Protter and A. N. Shiryayev. Quadratic covariation and an extension of Itô's formula. *Bernoulli* **1** (1-2), 149–169 (1995). MR1354459.

P. Friz and N. Victoir. A note on higher dimensional p-variation. *Electron. J. Probab.* **16**, no. 68, 1880–1899 (2011). MR2842090.

P. K. Friz and N. B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2010). ISBN 978-0-521-87607-0. MR2604669.

Y. Hu and K. Le. A multiparameter Garsia-Rodemich-Rumsey inequality and some applications. *Stochastic Process. Appl.* **123** (9), 3359–3377 (2013). MR3071383.

Y. Kwok. *Mathematical models of financial derivatives*. Springer Finance. Springer, Berlin, second edition (2008). ISBN 978-3-540-42288-4.

D. Leão, A. Ohashi and A. B. Simas. Weak functional itô calculus and applications. *ArXiv Mathematics e-prints* (2015). arXiv: 1408.1423v3.

M. B. Marcus and J. Rosen. p-variation of the local times of symmetric stable processes and of Gaussian processes with stationary increments. *Ann. Probab.* **20** (4), 1685–1713 (1992). MR1188038.

A. Ohashi and A. B. Simas. A maximal inequality of the 2d young integral based on bivariations. *ArXiv Mathematics e-prints* (2014). arXiv: 1408.1428.

G. Peskir. A change-of-variable formula with local time on curves. *J. Theoret. Probab.* **18** (3), 499–535 (2005). MR2167640.

F. Russo and P. Vallois. Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields* **104** (1), 27–41 (1996). MR1367665.

N. Towghi. Littlewood's inequality for p-bimeasures. *JIPAM. J. Inequal. Pure Appl. Math.* **3** (2), Article 19, 7 pp. (electronic) (2002a). MR1906388.

N. Towghi. Multidimensional extension of L. C. Young's inequality. *JIPAM. J. Inequal. Pure Appl. Math.* **3** (2), Article 22, 13 pp. (electronic) (2002b).

MR1906391.

L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** (1), 251–282 (1936). MR1555421.

L. C. Young. General inequalities for Stieltjes integrals and the convergence of Fourier series. *Math. Ann.* **115** (1), 581–612 (1938). MR1513204.