Asymptotic behaviour of exponential functionals of Lévy processes with applications to random processes in random environment

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Abstract. Let $\xi = (\xi_t, t \geq 0)$ be a real-valued Lévy process and define its associated exponential functional as follows

$$I_t(\xi) := \int_0^t \exp\{-\xi_s\}\mathrm{d}s, \qquad t \geq 0.$$

Motivated by applications to stochastic processes in random environment, we study the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where $F = (F(x), x \geq 0)$ is a function with polynomial decay at infinity and which is non increasing for large x. In particular, under some exponential moment conditions on ξ, we find five different regimes that depend on the shape of the Laplace exponent of ξ. Our proof relies on a discretization of the exponential functional $I_t(\xi)$ and is closely related to the behaviour of functionals of semi-direct products of random variables.

We apply our results to three questions associated to stochastic processes in random environment. We first consider the asymptotic behaviour of extinction and explosion for self-similar continuous state branching processes in a Lévy random environment. Secondly, we focus on the asymptotic behaviour of the mean population size in a model with competition or logistic growth which is affected by a Lévy random environment and finally, we study the tail behaviour of the maximum of a diffusion in a Lévy random environment.

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1. Introduction and main results

A one-dimensional Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost sure càdlàg paths. We write $\xi = (\xi_t : t \geq 0)$ for its trajectory and $\mathbb{P}$ for its law. The process ξ is a strong Markov process, and for each $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ its law when issued from x with the understanding that $\mathbb{P}_0 = \mathbb{P}$. The law of a Lévy process is characterized by its one-time transition probabilities. In particular there always exists a triple (δ, ρ, Π) where $\delta \in \mathbb{R}$, $\rho \in \mathbb{R}$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(\mathrm{d}x) < \infty$, such that, for all $z \in \mathbb{R}$

$$\mathbb{E}[e^{iz\xi_t}] = e^{t\psi(iz)},$$

where the Laplace exponent $\psi(z)$ is given by the Lévy-Khintchine formula

$$\psi(z) = \frac{1}{2}\rho^2 z^2 + \delta z + \int_{\mathbb{R}} (e^{zx} - 1 - zx\ell(x)) \, \Pi(\mathrm{d}x), \qquad z \in \mathbb{R}.$$

Here, $\ell(x)$ is the cutoff function which is usually taken to be $\ell(x) = \mathbf{1}_{\{|x|<1\}}$. Whenever the process ξ has finite mean, we will take $\ell(x) \equiv 1$.

In this paper, we are interested in studying the exponential functional of ξ, defined by

$$I_t(\xi) := \int_0^t e^{-\xi_s} \mathrm{d}s, \qquad t \geq 0.$$

In recent years there has been a general recognition that exponential functionals of Lévy processes play an important role in various domains of probability theory such as self-similar Markov processes, generalized Ornstein-Uhlenbeck processes, random processes in random environment, fragmentation processes, branching processes, mathematical finance, Brownian motion on hyperbolic spaces, insurance risk, queueing theory, to name but a few (see Bertoin and Yor, 2005; Carmona et al., 1997; Kyprianou and Pardo, 2008 and references therein).

There is a vast literature about exponential functionals of Lévy processes drifting to $+\infty$ or killed at an independent exponential time $\mathbf{e}_q$ with parameter $q \geq 0$, see for instance Bertoin et al. (2008); Bertoin and Yor (2005); Pardo et al. (2013). For a Lévy process ξ satisfying one of these assumptions, $I_\infty(\xi)$ or $I_{\mathbf{e}_q}(\xi)$ is finite almost surely with an absolute continuous density. According to Theorem 3.9 in Bertoin et al. (2008), there exists a density for $I_\infty(\xi)$, here denoted by h. In the case when $q > 0$, the existence of the density of $I_{\mathbf{e}_q}(\xi)$ appears in Pardo et al. (2013). Most of the known results on $I_\infty(\xi)$ and $I_{\mathbf{e}_q}(\xi)$ are related to the knowledge of their densities or the behaviour of their tail distributions.

From Theorem 2.2. in Kuznetsov et al. (2012), under the assumption that $\mathbb{E}[|\xi_1|] < \infty$, the density h is completely determined by the following integral equation: for $v > 0$,

$$\begin{aligned} \delta \int_v^\infty h(x)\mathrm{d}x + \frac{\rho^2}{2} vh(v) + \int_v^\infty \overline{\overline{\Pi}}^{(-)} \left(\ln \frac{x}{v} \right) h(x)\mathrm{d}x \\ + \int_0^v \overline{\overline{\Pi}}^{(+)} \left(\ln \frac{x}{v} \right) h(x)\mathrm{d}x + \int_v^\infty \frac{h(x)}{x}\mathrm{d}x = 0, \end{aligned} \tag{1.1}$$

where

$$\overline{\overline{\Pi}}^{(+)}(x) = \int_x^\infty \int_y^\infty \Pi(\mathrm{d}z)\mathrm{d}y \qquad \text{and} \qquad \overline{\overline{\Pi}}^{(-)}(x) = \int_x^\infty \int_{-\infty}^{-y} \Pi(\mathrm{d}z)\mathrm{d}y.$$

We refer to Bertoin et al. (2008); Bertoin and Yor (2005); Kuznetsov et al. (2012); Pardo et al. (2013), and the references therein, for more details about these facts.

In this paper, we are interested in the case when the Lévy process ξ does not satisfy such conditions, in other words when $I_t(\xi)$ does not converge almost surely to a finite random variable, as t goes to ∞. More precisely, one of our aims is to study the asymptotic behaviour of

$$\mathbb{E}\Big[F\big(I_t(\xi)\big)\Big] \qquad \text{as} \qquad t \to \infty,$$

where F is a function which, at infinity, is non-increasing and has polynomial decay, under some exponential moment conditions on ξ. We find five regimes that depend on the shape of the Laplace exponent ψ. These results will be applied in Section 2 for some particular cases such as

$$F(x) = x^{-p}, \qquad F(x) = 1 - e^{x^{-p}}, \quad F(x) = e^{-x}, \quad \text{or} \quad F(x) = \frac{a}{b+x},$$

for $a, b, p, x > 0$. Up to our knowledge, the case when the exponential functional of a Lévy process does not converge has only been studied in a few papers and not in its most general form, see for instance Bansaye et al. (2013); Böinghoff and Hutzenthaler (2012); Palau and Pardo (2016+). In all these papers, the main motivation comes from the study of some asymptotic properties of stochastic processes in random environment, that we briefly describe.

A recent manuscript authored by Li and Xu (2016), that appeared on Arxiv at the same time this manuscript was prepared, also studied the asymptotic behaviour of exponential functionals for Lévy processes. They obtained interesting results which are similar to the ones we present below using fluctuation theory for Lévy processes and the knowledge of Lévy processes conditioned to stay positive. Our approach is completely different and is based on a discretization of the exponential functional of Lévy processes and on the asymptotic behaviour of functionals of semi-direct products of random variables which was described by Guivarc'h and Liu (2001).

We would like to stress that Bansaye et al. (2013) already used a discretization technique to get the asymptotic behaviour of the survival probability for branchnig processes with catastrophes. However the discretization used in Bansaye et al. (2013) is different and only covers the case of compound Poisson processes.

Branching processes in random environment (BPREs) have been introduced and first studied by Smith and Wilkinson. This type of process has attracted considerable interest in the last decade (see for instance Afanasyev et al., 2012, 2005; Bansaye and Berestycki, 2009; Böinghoff et al., 2010 and the references therein). One of the reason is that BPREs are more realistic models than classical branching processes. And, from the mathematical point of view, they have interesting features such as a phase transition in the subcritical regime. Scaling limits for BPREs have been studied by Kurtz (1978) in the continuous case and more recently by Bansaye and Simatos (2015) in a more general setting.

Continuous state branching processes (CB-processes for short) in random environment, the continuous analogue in time and state space of BPREs, can be defined as a strong solution of a particular stochastic differential equation. They have been studied recently by several authors in different settings. Böinghoff and

Hutzenthaler (2012) and Bansaye et al. (2013) studied extinction rates for branching diffusions in a Brownian environment and branching processes in a random environment driven by a Lévy process with bounded variations, respectively. Motivated by these works, Palau and Pardo (2016+) studied the long-term behaviour (extinction, explosion, conditioned version) of branching processes in a Brownian random environment. In all these manuscripts, the existence of such processes is proved via a stochastic differential equation and it is observed that the speed of extinction is related to an exponential functional of a Lévy process which is associated to the random environment. Moreover, and similarly to the case of BPREs, a phase transition in the subcritical regime appears. Recently, branching processes in a general Lévy random environment were introduced by Palau and Pardo (2015).

Exponential functionals occur naturally in the study of some models of diffusions in random environment, which we now describe informally. Associated with a stochastic process $V = (V(x), x \in \mathbb{R})$ such that $V(0) = 0$, a diffusion $X_V = (X_V(t), t \geq 0)$ in the random potential V is, loosely speaking, a solution to the stochastic differential equation

$$\mathrm{d}X_V(t) = \mathrm{d}\beta_t - \frac{1}{2}V'(X_V(t))\mathrm{d}t, \qquad X_V(0) = 0,$$

where $(\beta_t, t \geq 0)$ is a standard Brownian motion independent of V. More rigorously, the process X_V should be considered as a diffusion whose conditional generator, given V, is:

$$\frac{1}{2}\exp(V(x))\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Observe that from Feller's construction of such diffusions, the potential V does not need to be differentiable. Kawazu and Tanaka (1993) studied the asymptotic behaviour of the tail of the distribution of the maximum of a diffusion in a drifted Brownian potential. Carmona et al. (1997) considered the case when the potential is a Lévy process whose jump structure is of bounded variation. More precisely, they studied the following question: How fast does $\mathbb{P}(\max_{t \geq 0} X_V(t) > x)$ decay as x go to infinity? From these works, we know that

$$\mathbb{P}\left(\max_{t \geq 0} X_V(t) > x\right) = \mathbb{E}\left[\frac{\tilde{I}}{\tilde{I} + I_x(-V)}\right]$$

where

$$\tilde{I} = \int_{-\infty}^{0} e^{V(t)}\mathrm{d}t \quad \text{and} \quad I_x(-V) = \int_{0}^{x} e^{V(t)}\mathrm{d}t$$

are independent. As a consequence, exponential functionals play an essential role in this domain.

Let us now state our main results. Assume that

$$\theta^+ = \sup\{\lambda > 0 : \psi(\lambda) < \infty\} \tag{1.2}$$

exists and is positive. In other words, the Laplace exponent of the Lévy process ξ can be defined on $[0, \theta^+)$, see for instance Lemma 26.4 in Sato (2013). Besides, ψ satisfies

$$\psi(\lambda) = \log \mathbb{E}\left[e^{\lambda \xi_1}\right], \qquad \lambda \in [0, \theta^+).$$

From Theorem 25.3 in Sato (2013), $\psi(\lambda) < \infty$ is equivalent to

$$\int_{\{|x|>1\}} e^{\lambda x}\, \Pi(\mathrm{d}x) < \infty. \tag{1.3}$$

Moreover ψ belongs to $C^\infty([0, \theta^+))$ with $\psi(0) = 0$, $\psi'(0+) \in [-\infty, \infty)$ and $\psi''(\lambda) > 0$, for $\lambda \in (0, \theta^+)$ (see Lemma 26.4 in Sato, 2013). Hence, the Laplace exponent ψ is a convex function on $[0, \theta^+)$ implying that either it is always positive or it may take negative values. In the latter scenario, ψ may have at most one global minimum on $(0, \theta^+)$. Whenever such a global minimum exists, we denote by τ the position where it is reached. As we will see below, the shape of ψ is essential to determine the asymptotic behaviour of $\mathbb{E}\left[I_t(\xi)^{-p}\right]$, for $0 < p < \theta^+$.

Let us introduce the exponential change of measure known as the Esscher transform. According to Theorem 3.9 in Kyprianou (2006), for any λ such that (1.3) is satisfied, we can perform the following change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda \xi_t - \psi(\lambda) t}, \qquad t \geq 0 \tag{1.4}$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by ξ which is naturally completed. Moreover, under $\mathbb{P}^{(\lambda)}$ the process ξ is still a Lévy process with Laplace exponent given by

$$\psi_\lambda(z) = \psi(\lambda + z) - \psi(\lambda), \qquad z \in \mathbb{R}.$$

Theorem 1.1. *Assume that $0 < p < \theta^+$.*

   i) *If $\psi'(0+) > 0$, then*

$$\lim_{t \to \infty} \mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}\left[I_\infty(\xi)^{-p}\right] > 0.$$

   ii) *If $\psi'(0+) = 0$ and $\psi''(0+) < \infty$, then there exists a positive constant c_1 such that*

$$\lim_{t \to \infty} \sqrt{t}\, \mathbb{E}\left[I_t(\xi)^{-p}\right] = c_1.$$

   iii) *Assume that $\psi'(0+) < 0$*
      a) *if $\psi'(p) < 0$, then*

$$\lim_{t \to \infty} e^{-t\psi(p)} \mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}] > 0.$$

      b) *if $\psi'(p) = 0$ and $\psi''(p) < \infty$, then there exists a positive constant c_2 such that*

$$\lim_{t \to \infty} \sqrt{t}\, e^{-t\psi(p)} \mathbb{E}\left[I_t(\xi)^{-p}\right] = c_2.$$

      c) *$\psi'(p) > 0$ and $\psi''(\tau) < \infty$ then*

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = o(t^{-1/2} e^{t\psi(\tau)}), \qquad as \quad t \to \infty.$$

      Moreover if we also assume that ξ is non-arithmetic (or non-lattice) then

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = O(t^{-3/2} e^{t\psi(\tau)}), \qquad as \quad t \to \infty.$$

It is important to note that for any $q > 0$ satisfying (1.3) for $\lambda = q$, we necessarily have that $\mathbb{E}\left[I_t(\xi)^{-q}\right]$ is finite for all $t > 0$. Indeed, since $(e^{q\xi_t - t\psi(q)}, t \geq 0)$ is a positive martingale, we deduce from L_1-Doob's inequality (see for instance Acciaio

et al., 2013) and the Esscher transform (1.4), that the following series of inequalities hold: for $t \leq 1$,

$$
\mathbb{E}\left[I_t(\xi)^{-q}\right] \leq t^{-q}\mathbb{E}\left[\sup_{0 \leq u \leq 1} e^{q\xi_u}\right] \leq t^{-q}e^{\psi(q)\vee 0}\mathbb{E}\left[\sup_{0 \leq u \leq 1} e^{q\xi_u - u\psi(q)}\right]
$$
$$
\leq t^{-q}\frac{e^{1+\psi(q)\vee 0}}{e-1}\left(1 + \mathbb{E}^{(q)}\left[q\xi_1 - \psi(q)\right]\right) \tag{1.5}
$$
$$
= t^{-q}\frac{e^{1+\psi(q)\vee 0}}{e-1}\left[1 + q\psi'(q) - \psi(q)\right],
$$

which is finite. The finiteness for $t > 1$ follows from the fact that $I_t(\xi)$ is non-decreasing.

We are now interested in extending the above result for a class of functions which have polynomial decay and are non-increasing at ∞. As we will see below such extension is not straightforward and need more conditions on the exponential moments of the Lévy process ξ.

For simplicity, we write

$$
\mathcal{E}_F(t) := \mathbb{E}\left[F(I_t(\xi))\right],
$$

where F belongs to a particular class of continuous functions on $\mathbb{R}_+$ that we will introduce below. Let $\mathbf{k}$ be a positive constant. We will consider functions F satisfying one of the following conditions: There exists $x_0 \geq 0$ such that $F(x)$ is non-increasing for $x \geq x_0$, and

(**A1**) F satisfies

$$
F(x) = \mathbf{k}(x+1)^{-p}\left[1 + (1+x)^{-\varsigma}h(x)\right], \qquad \text{for all } x > 0,
$$

where $0 < p \leq \tau$, $\varsigma \geq 1$ and h is a Lipschitz function which is bounded.

(**A2**) F is an Hölder function with index $\alpha > 0$ satisfying

$$
F(x) \leq \mathbf{k}(x+1)^{-p}, \qquad \text{for all } x > 0,
$$

with $p > \tau$.

Theorem 1.2. *Assume that $0 < p < \theta^+$. We have the following five regimes for the asymptotic behaviour of $\mathcal{E}_F(t)$ for large t.*

i) *If $\psi'(0+) > 0$ and F is a positive and continuous function which is bounded, then*
$$
\lim_{t \to \infty} \mathcal{E}_F(t) = \mathcal{E}_F(\infty).
$$

ii) *If $\psi'(0+) = 0$, F satisfies (**A2**) and $\theta^- := \inf\{\lambda < 0 : \psi(\lambda) < \infty\}$ exists and is strictly negative, then there exists a positive constant C_1 such that*
$$
\lim_{t \to \infty} \sqrt{t}\mathcal{E}_F(t) = C_1.
$$

iii) *Suppose that $\psi'(0+) < 0$:*
    a) *If F satisfies (**A1**) and $\psi'(p) < 0$, then,*
$$
\lim_{t \to \infty} e^{-t\psi(p)}\mathcal{E}_F(t) = \lim_{t \to \infty} e^{-t\psi(p)}\mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbf{k}\mathbb{E}^{(p)}\left[I_\infty(-\xi)^{-p}\right].
$$
    b) *If F satisfies (**A1**), $\psi'(p) = 0$ and $\psi''(p) < \infty$, then,*
$$
\lim_{t \to \infty} \sqrt{t}e^{-t\psi(p)}\mathcal{E}_F(t) = \lim_{t \to \infty} \sqrt{t}e^{-t\psi(p)}\mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbf{k}c_2,
$$

    where c_2 has been defined in point iii) b) of Theorem 1.1.

c) *If F satisfies (**A2**), $\psi'(p) > 0$ and $\tau + p < \theta^+$ then there exists a positive constant C_3 such that*

$$\lim_{t \to \infty} t^{3/2} e^{-t\psi(\tau)} \mathcal{E}_F(t) = C_3.$$

The remainder of the paper is structured as follows. In Section 2, we apply our results to self-similar CB-processes, a population model with competition and diffusions whose dynamics are perturbed by a Lévy random environment. In particular, we study the asymptotic behaviour of the probability of extinction and explosion for some classes of self-similar CB-processes in a Lévy random environment. For the population model with competition, we describe the asymptotic behaviour of its mean. For the diffusion in a Lévy random environment, we provide the asymptotic behaviour of the tail probability of its global maximum. Finally, Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Applications

2.1. *Self-similar CB-processes in a Lévy random environment.* A $[0, \infty]$-valued strong Markov process $Y = (Y_t, t \geq 0)$ with probabilities $(\mathbb{P}_x, x \geq 0)$ is called a CB-process if it has càdlàg paths and its law satisfies the branching property; i.e. for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$. The law of Y is completely characterized by its Laplace transform

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = e^{-xu_t(\lambda)}, \qquad \forall x > 0, \ t \geq 0,$$

where u is a differentiable function in t satisfying

$$\frac{\partial u_t(\lambda)}{\partial t} = -\Psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda.$$

The function Ψ is known as the branching mechanism of Y and satisfies the celebrated Lévy-Khintchine formula

$$\Psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x), \qquad \lambda \geq 0,$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a measure concentrated on $(0, \infty)$ such that $\int_{(0,\infty)} \left(1 \wedge x^2\right)\mu(\mathrm{d}x) < \infty$.

Here we are interested in the case where the branching mechanism is stable, that is to say

$$\Psi(\lambda) = c_\beta \lambda^{\beta+1}, \qquad \lambda \in \mathbb{R},$$

for some $\beta \in (-1, 0) \cup (0, 1]$ and c_β is such that $\beta c_\beta > 0$. We call its associated CB-process a self-similar CB-process. Under this assumption, the process Y can also be defined as the unique non-negative strong solution of the following SDE (see for instance Fu and Li, 2010)

$$Y_t = Y_0 + \mathbf{1}_{\{\beta=1\}} \int_0^t \sqrt{2c_\beta Y_s}\,\mathrm{d}B_s + \mathbf{1}_{\{\beta\neq1\}} \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure independent of B with intensity

$$\frac{c_\beta \beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}}\mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

$\widetilde{N}$ is its compensated version and

$$\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \left\{ \begin{array}{ll} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (-1, 0), \\ \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (0, 1). \end{array} \right.$$

According to Palau and Pardo (2015), we can define a self-similar branching process whose dynamics are affected by a Lévy random environment (SSBLRE) as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t &= Z_0 + \mathbf{1}_{\{\beta=1\}} \int_0^t \sqrt{2c_\beta Z_s} \mathrm{d}B_s \\ &\quad + \mathbf{1}_{\{\beta \neq 1\}} \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s, \end{aligned} \tag{2.1}$$

where

$$S_t = \alpha t + \sigma W_t + \int_0^t \int_{(-1,1)} (e^v - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^v - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}v), \tag{2.2}$$

$\alpha \in \mathbb{R}, \sigma \geq 0, W = (W_t, \ t \geq 0)$ is a standard Brownian motion, $N^{(e)}$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of W with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ represents its compensated version and π is a σ-finite measure concentrated on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} (1 \wedge v^2)\pi(\mathrm{d}v) < \infty.$$

In addition, Z satisfies the strong Markov property and, conditioned on the environment, the branching property. In what follows, we introduce the auxiliary Lévy process

$$K_t = \mathbf{d}t + \sigma W_t + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v), \tag{2.3}$$

where

$$\mathbf{d} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

The Laplace transform of $Z_t e^{-K_t}$ can be computed explicitly and provides a closed formula for the probabilities of survival and non explosion of Z. Indeed according to Proposition 1 in Palau and Pardo (2015), if $(Z_t, t \geq 0)$ is a stable SSBLRE with index $\beta \in (-1, 0) \cup (0, 1]$, then for all $z, \lambda > 0$ and $t \geq 0$,

$$\mathbb{E}_z \left[\exp\left\{ -\lambda Z_t e^{-K_t} \right\} \Big| K \right] = \exp\left\{ -z \left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta K_u} \mathrm{d}u \right)^{-1/\beta} \right\}. \tag{2.4}$$

Here, we are interested in two events which are of immediate concern for the process Z, *explosion* and *extinction*. The event of *explosion* before time t, is given by $\{Z_t = \infty\}$, and the event $\{\exists\, t > 0, Z_t = 0\}$ is referred as *extinction*.

2.1.1. *Speed of explosion of SSBLRE.* Let us first study the event of *explosion* for self-similar CB-processes in a Lévy random environment. It is important to note that this event has only been studied in the case when the random environment

is driven by a Brownian motion with drift, see Palau and Pardo (2016+). From Equation (2.4) and letting λ go to 0, we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = \mathbf{1}_{\{\beta > 0\}} + \mathbf{1}_{\{\beta < 0\}} \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta K_u} \mathrm{d}u\right)^{-1/\beta}\right\} \qquad \text{a.s.}$$

(2.5)

Let us focus on the most interesting case, $\beta \in (-1, 0)$. Recall that when there is no environment, a self-similar CB-process explodes at time t with probability $1 - \exp\left\{-z(\beta c_\beta t)^{-1/\beta}\right\}$. Under the presence of a Lévy random environment three different regimes appear for the asymptotic behaviour of the non-explosion probability. We call these regimes *subcritical-explosion*, *critical-explosion* or *supercritical-explosion* depending on whether this probability stays positive, converges to zero polynomially fast or converges to zero exponentially fast.

Before stating our result, let us introduce the Laplace transform of the Lévy process K by

$$\kappa(\theta) = \log \mathbb{E}[e^{\theta K_1}], \qquad (2.6)$$

whenever it exists (see discussion on page 1238).

We assume that $\theta_K^- < 0 < \theta_K^+$,where

$$\theta_K^- := \inf\{\lambda < 0 : \kappa(\lambda) < \infty\} \quad \text{and} \quad \theta_K^+ := \sup\{\lambda > 0 : \kappa(\lambda) < \infty\}.$$

As a consequence, the Laplace exponent κ of K is well defined on the interval (θ_K^-, θ_K^+). If $\mathbf{m} := \kappa'(0+) > 0$ then κ takes only positive values on $(0, \theta_K^+)$, and some negatives values on $(\theta_K^-, 0)$. In this case, we may assume that κ has a global minimum on $(\theta_K^-, 0)$ and we denote by $\hat{\tau}$ for its position. As we see below, the asymptotic behaviour of the probability of explosion depends on the sign of $\mathbf{m}$.

Proposition 2.1. *Let $(Z_t, t \geq 0)$ be the SSBLRE with index $\beta \in (-1, 0)$ defined by the SDE (2.1) with $Z_0 = z > 0$, and recall the definition of the random environment K in (2.3). Assume that $0 < \theta_K^+$.*

i) *Subcritical-explosion. If $\mathbf{m} < 0$, then, for every $z > 0$*

$$\lim_{t \to \infty} \mathbb{P}_z\Big(Z_t < \infty\Big) = \mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta K_u} \mathrm{d}u\right)^{-1/\beta}\right\}\right] > 0.$$

ii) *Critical-explosion. If $\mathbf{m} = 0$, then for every $z > 0$ there exists $c_1(z) > 0$ such that*

$$\lim_{t \to \infty} \sqrt{t}\, \mathbb{P}_z\Big(Z_t < \infty\Big) = c_1(z).$$

iii) *Supercritical-explosion. If $\mathbf{m} > 0$, then for every $z > 0$ there exists $c_2(z) > 0$ such that*

$$\lim_{t \to \infty} t^{\frac{3}{2}} e^{t\kappa(\hat{\tau})} \mathbb{P}_z\Big(Z_t < \infty\Big) = c_2(z).$$

Proof: Observe that the function

$$F : x \in \mathbb{R}_+ \mapsto \exp(-z(\beta c_\beta x)^{-1/\beta})$$

is non-increasing, continuous, bounded, and satisfies Assumption (**A2**) for every positive p. We also note that the processes $(\beta K_t, t \geq 0)$ and $(K_t, t \geq 0)$ have oposite signs implying

$$\hat{\kappa}(\theta) := \log \mathbb{E}\left[e^{\beta \theta K_1}\right] = \kappa(\beta\theta),$$

is well defined for $\theta \in (\theta_K^+/\beta, \theta_K^-/\beta)$. Hence Proposition 2.1 is a direct application of Theorem 1.2 points i), ii) and iii) c) with $(\xi_t, t \geq 0) = (\beta K_t, t \geq 0)$. □

2.1.2. *Speed of extinction of SSBLRE.* Let us now focus on the survival probability. Throughout this section, we assume that $\beta \in (0, 1]$. From Equation (2.4) and taking λ go to ∞, we get

$$\mathbb{P}_z\left(Z_t > 0 \middle| K\right) = 1 - \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta K_u} \mathrm{d}u\right)^{-1/\beta}\right\} \qquad \text{a.s.}$$

Similarly as for the explosion probability, the asymptotic behaviour of the probability depends on the sign of $\mathbf{m} = \kappa'(0+)$. But in contrast with the explosion probability, five regimes appear, and a second parameter to take into account is the sign of

$$\mathbf{m}_1 = \kappa'(1).$$

We recall that the Laplace exponent κ of K is well defined on the interval (θ_K^-, θ_K^+) with $\theta_K^- < 0 < \theta_K^+$. In the case when $\mathbf{m} < 0$, we may also assume that κ has a global minimum on $(0, \theta_K^+)$ and we denote by τ for its position.

Proposition 2.2. *Let $(Z_t, t \geq 0)$ be a SSBLRE with index $\beta \in (0, 1]$ defined by the SDE (2.1) with $Z_0 = z > 0$, and recall the definition of the random environment K in (2.3). Assume that $1 < \theta_K^+$.*

  i) *Supercritical case. If $\mathbf{m} > 0$, then for every $z > 0$*

$$\lim_{t \to \infty} \mathbb{P}_z\left(Z_t > 0\right) = \mathbb{E}\left[1 - \exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta K_u} \mathrm{d}u\right)^{-1/\beta}\right\}\right] > 0.$$

  ii) *Critical case. If $\mathbf{m} = 0$, then for every $z > 0$, there exists $c_1(z) > 0$ such that*

$$\lim_{t \to \infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = c_1(z).$$

 iii) *Subcritical case. Assume that $\mathbf{m} < 0$, then*

   a) *(Strongly subcritical regime). If $\mathbf{m}_1 < 0$, then there exists $c_2 > 0$ such that for every $z > 0$,*

$$\lim_{t \to \infty} e^{-t\kappa(1)}\mathbb{P}_z(Z_t > 0) = c_2 z,$$

   b) *(Intermediate subcritical regime) If $\mathbf{m}_1 = 0$, then there exists $c_3 > 0$ such that for every $z > 0$,*

$$\lim_{t \to \infty} \sqrt{t}e^{-t\kappa(1)}\mathbb{P}_z(Z_t > 0) = c_3 z,$$

   c) *(Weakly subcritical regime) If $\mathbf{m}_1 > 0$, then for every $z > 0$, there exists $c_4(z) > 0$ such that*

$$\lim_{t \to \infty} t^{3/2}e^{-t\kappa(\tau)}\mathbb{P}_z(Z_t > 0) = c_4(z).$$

Proof: This is a direct application of Theorem 1.2, with $(\xi_t, t \geq 0) = (\beta K_t, t \geq 0)$ and $F(x) = 1 - \exp(-z(\beta c_\beta x)^{-1/\beta})$. □

   In the strongly and intermediate subcritical cases $a)$ and $b)$, $\mathbb{E}[Z_t]$ provides the exponential decay factor of the survival probability which is given by $\kappa(1)$, and the probability of non-extinction is proportional to the initial state z of the population. In the weakly subcritical case $c)$, the survival probability decays exponentially with

rate $\kappa(\tau)$, which is strictly smaller than $\kappa(1)$, and c_4 may not be proportional to z (it is also the case for c_1). We refer to Bansaye (2009) for a result in this vein for discrete branching processes in random environment.

More generally, the results stated above can be compared to the results which appear in the literature of discrete (time and space) branching processes in random environment, see e.g. Afanasyev et al. (2005); Geiger et al. (2003); Guivarc'h and Liu (2001). In the continuous framework, such results have been established in Böinghoff and Hutzenthaler (2012) for the Feller diffusion case (i.e. $\beta = 1$) in a Brownian environment, in Palau and Pardo (2016+) for a general CB-process in a Brownian environment, and in Bansaye et al. (2013) for stable CB-process ($\beta \in (0, 1]$) subject to random catastrophes killing a fraction of the population.

2.2. *Population model with competition in a Lévy random environment.* We now study an extension of the competition model introduced in Evans et al. (2014) and studied by Palau and Pardo (2015). Following Palau and Pardo (2015), we define a logistic process with competition in a Lévy random environment, $(Z_t, t \geq 0)$, as the unique strong solution of the SDE

$$Z_t = Z_0 + \int_0^t Z_s(r - kZ_s)\mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}S_s$$

where $r > 0$ is the drift, $k > 0$ is the competition, and the environment S is given by the Lévy process defined in (2.2). Moreover, the process Z satisfies the Markov property and we have

$$Z_t = Z_0 e^{K_t} \left(1 + kZ_0 \int_0^t e^{K_s}\mathrm{d}s\right)^{-1}, \qquad t \geq 0,$$

where K is the Lévy process defined in (2.3).

The following result studies the asymptotic behaviour of $\mathbb{E}_z[Z_t]$, where $\mathbb{P}_z$ denotes the law of Z starting from z. Before stating our result, let us recall the definition of the Laplace transform κ of K in (2.6) and that τ is the position of its global minimum, $\mathbf{m} = \kappa'(0)$ and $\mathbf{m}_1 = \kappa'(1)$.

Proposition 2.3. *Assume that $1 < \theta_K^+$. For $z > 0$, we have the following five regimes for the asymptotic behaviour of $\mathbb{E}_z[Z_t]$.*

i) *If $\mathbf{m} > 0$, then for every $z > 0$*

$$\lim_{t\to\infty} \mathbb{E}_z[Z_t] = \frac{1}{k}\mathbb{E}\left[\frac{1}{I_\infty(K)}\right] > 0.$$

ii) *If $\mathbf{m} = 0$, then*

$$\mathbb{E}_z[Z_t] = O(t^{-1/2}).$$

iii) *Suppose that $\mathbf{m} < 0$:*
   a) *If $\mathbf{m}_1 < 0$, then,*

$$\lim_{t\to\infty} e^{-t\kappa(1)}\mathbb{E}_z[Z_t] = \mathbb{E}^{(1)}\left[\frac{z}{1 + zkI_\infty(-K)}\right] > 0,$$

   where $\mathbb{E}^{(1)}$ denotes the Esscher transform (1.4) of K with $\lambda = 1$.
   b) *If $\mathbf{m}_1 = 0$, then there exists a positive constant $c(z, k)$ that depends on z and k such that*

$$\lim_{t\to\infty} \sqrt{t}e^{-t\kappa(1)}\mathbb{E}_z[Z_t] = c(z, k).$$

c) *If $\mathbf{m}_1 > 0$ and $\tau + 1 < \theta^+$ then there exists a positive constant $c_1(z,k)$ that depends on z and k such that*

$$\lim_{t\to\infty} t^{3/2} e^{-t\kappa(\tau)} \mathbb{E}_z[Z_t] = c_1(z,k).$$

Proof: We first recall from Lemma II.2 in Bertoin (1996) that the time reversed process $(K_t - K_{(t-s)^-}, 0 \le s \le t)$ has the same law as $(K_s, 0 \le s \le t)$. Then

$$e^{-K_t} I_t(-K) = e^{-K_t} \int_0^t e^{K_{t-s}} \mathrm{d}s = \int_0^t e^{-(K_t - K_{t-s})} \mathrm{d}s \overset{(d)}{=} \int_0^t e^{-K_s} \mathrm{d}s = I_t(K), \tag{2.7}$$

and

$$(e^{-K_t}, e^{-K_t} I_t(-K)) \overset{(d)}{=} (e^{-K_t}, I_t(K)),$$

where $\overset{(d)}{=}$ stands for equality in law. The above implies that

$$\mathbb{E}_z[Z_t] = z\mathbb{E}\left[\left(e^{-K_t} + kze^{-K_t}\int_0^t e^{K_s}\mathrm{d}s\right)^{-1}\right] = z\mathbb{E}\left[\left(e^{-K_t} + kzI_t(K)\right)^{-1}\right]. \tag{2.8}$$

Let us now prove part i). Assume that $\mathbf{m} > 0$, then K drifts to ∞ and e^{-K_t} converges to 0 as t goes to ∞. By Theorem 1 in Bertoin and Yor (2005), $I_t(K)$ converges a.s. to $I_\infty(K)$, a non-negative and finite limit as t goes to ∞. We observe that the result follows from identity (2.8) and the Monotone Convergence Theorem.

Part ii) follows from the inequality

$$\mathbb{E}_z[Z_t] = z\mathbb{E}\left[\left(e^{-K_t} + kzI_t(K)\right)^{-1}\right] \le \mathbb{E}\left[\left(kI_t(K)\right)^{-1}\right],$$

and Theorem 1.1 part (ii).

Finally, we prove part iii). Observe by applying the Esscher transform (1.4) with $\lambda = 1$ that

$$\mathbb{E}_z[Z_t] = ze^{\kappa(1)t}\mathbb{E}^{(1)}\left[\left(1 + kz\int_0^t e^{K_s}\mathrm{d}s\right)^{-1}\right].$$

Part iii)-a) follows by observing that under the probability measure $\mathbb{P}^{(1)}$, the process K is a Lévy process with mean $\mathbb{E}^{(1)}[K_1] = \kappa'(1) \in (-\infty, 0)$. We then conclude as in the proof of part i) by showing that $\mathbb{E}^{(1)}[(1 + kzI_t(-K))^{-1}]$, converges to $\mathbb{E}^{(1)}[(1 + kzI_\infty(-K))^{-1}]$, as t increases.

Finally parts iii)-b) and c) follow from a direct application of Theorem (1.2) parts iii)-b) and c), respectively, with the function $F : x \in \mathbb{R}_+ \mapsto z(1 + kzx)^{-1}$. $\quad\square$

2.3. Diffusions in a Lévy random environment.

Let $(V(x), x \in \mathbb{R})$ be a stochastic process defined on $\mathbb{R}$ such that $V(0) = 0$. As presented in the introduction, a diffusion process $X = (X(t), t \ge 0)$ in a random potential V is a diffusion whose conditional generator given V is

$$\frac{1}{2}e^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

It is well known that X may be constructed from a Brownian motion through suitable changes of scale and time, see Brox (1986).

The problem that we would like to study is the following: How fast does $\mathbb{P}(\max_{t\ge 0} X(t) > x)$ decay as x go to infinity? In order to make our analysis more

tractable, we consider $(\xi_t, t \geq 0)$ and $(\eta_t, t \geq 0)$ two independent Lévy processes, and we define

$$V(x) = \begin{cases} -\xi_x & \text{if} \quad x \geq 0 \\ -\eta_{-x} & \text{if} \quad x \leq 0. \end{cases}$$

We want to determine the asymptotic behaviour of

$$\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = \mathbb{E}\left[\frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)}\right].$$

We assume that η drifts to ∞, and recall the notations of Section 1 for the Laplace exponent ψ of ξ.

Proposition 2.4. *Assume that $1 < \theta^+$.*

    i) *If $\psi'(0+) > 0$, then*

$$\lim_{t \to \infty} \mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = \mathbb{E}\left[\frac{I_\infty(\eta)}{I_\infty(\eta) + I_\infty(\xi)}\right] > 0.$$

   ii) *If $\psi'(0+) = 0$, then there exists a positive constant C_1 that depends on the law of $I_\infty(\eta)$ such that*

$$\lim_{t \to \infty} \sqrt{t}\,\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = C_1.$$

  iii) *Suppose that $\psi'(0+) < 0$:*

     a) *If $\psi'(1) < 0$, then there exists a positive constant C_2 that depends on the law of $I_\infty(\eta)$ such that,*

$$\lim_{t \to \infty} e^{-t\psi(1)}\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = C_2.$$

     b) *If $\psi'(1) = 0$ and $\psi''(1) < \infty$, then there exists a positive constant C_3 that depends on the law of $I_\infty(\eta)$ such that*

$$\lim_{t \to \infty} \sqrt{t}\,e^{-t\psi(1)}\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = C_3.$$

     c) *If $\psi'(1) > 0$, and $\tau + 1 < \theta^+$, then*

$$\lim_{t \to \infty} \mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = o(t^{-1/2}e^{-t\psi(\tau)}).$$

     Moreover, if the process ξ is non-arithmetic (or non-lattice) then there exists a positive constant C_4 that depends on the law of $I_\infty(\eta)$ such that

$$\lim_{t \to \infty} t^{3/2}e^{-t\psi(\tau)}\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = C_4.$$

Furthermore, if there exists a positive ε such that

$$\mathbb{E}[I_\infty(\eta)^{1+\varepsilon}] < \infty,$$

then

$$C_i = c_i\mathbb{E}[I_\infty(\eta)], \quad i \in \{2,3\},$$

where $(c_i, i \in \{2,3\})$ do not depend on the law of $I_\infty(\eta)$.

Proof: Since η and ξ are independent, we have

$$\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = \mathbb{E}\left[I_\infty(\eta)\mathbb{E}\left[\frac{1}{I_\infty(\eta) + I_t(\xi)}\Big|\eta\right]\right] = \mathbb{E}\left[I_\infty(\eta)f(I_\infty(\eta), t)\right], \ t > 0,$$

where

$$f(a, t) = \mathbb{E}\left[(a + I_t(\xi))^{-1}\right], \qquad a, t > 0.$$

The result follows from an application of Theorems 1.1 and 1.2 with the function

$$F : x \in \mathbb{R}_+ \mapsto z(a + x)^{-1}.$$

We only prove case ii), as the others are analogous. By Theorem 1.2 there exists $c_1(a) > 0$ such that

$$\lim_{t \to \infty} t^{1/2} f(a, t) = c_1(a).$$

On the other hand, by Theorem 1, there exists c_1 such that

$$\lim_{t \to \infty} t^{1/2} f(0, t) = c_1.$$

Let us define $G_t(a) = at^{1/2} f(a, t)$, and $G_t^0(a) = at^{1/2} f(0, t)$. Observe that

$$G_t(a) \leq G_t^0(a), \qquad \text{for all } t, a \geq 0$$

and

$$\lim_{t \to \infty} \mathbb{E}\left[G_t^0(I_\infty(\eta))\right] = c_1\mathbb{E}\left[I_\infty(\eta)\right].$$

Then, by the Dominated Convergence Theorem (see for instance Dudley, 2002 problem 12 p. 145),

$$\lim_{t \to \infty} \sqrt{t}\mathbb{P}\left(\max_{s \geq 0} X(s) > t\right) = \lim_{t \to \infty} \mathbb{E}\left[G_t(I_\infty(\eta))\right] = \mathbb{E}\left[I_\infty(\eta)c_1(I_\infty(\eta))\right].$$

We complete the proof for the existence of the limits by observing that

$$0 < C_1 = \mathbb{E}\left[I_\infty(\eta)c_1(I_\infty(\eta))\right] \leq c_1\mathbb{E}\left[I_\infty(\eta)\right] < \infty.$$

The last part of the proof consists in justifying the form of the constants C_2 and C_3 under the additional condition $\mathbb{E}[I_\infty(\eta)^{1+\varepsilon}] < \infty$ for a positive ε. For every $0 \leq \varepsilon \leq 1$, we have

$$\frac{I_\infty(\eta)}{I_t(\xi)} - \frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)} = \frac{I_\infty(\eta)}{I_t(\xi)}\frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)}$$

$$\leq \frac{I_\infty(\eta)}{I_t(\xi)}\left(\frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)}\right)^\varepsilon \leq \left(\frac{I_\infty(\eta)}{I_t(\xi)}\right)^{1+\varepsilon}.$$

Hence

$$0 \leq \mathbb{E}\left[\frac{I_\infty(\eta)}{I_t(\xi)} - \frac{I_\infty(\eta)}{I_\infty(\eta) + I_t(\xi)}\right] \leq \mathbb{E}[(I_\infty(\eta))^{1+\varepsilon}]\mathbb{E}\left[\frac{1}{(I_t(\xi))^{1+\varepsilon}}\right].$$

But from point iii)-c) of Theorem 1.1 and Equation (3.10) in the proof of Theorem 1.2, we know that in the cases iii)-a) and iii)-b),

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)}\right] = o\left(\mathbb{E}\left[I_t(\xi)^{-1}\right]\right).$$

This ends the proof.   □

3. Proofs of Theorems 1.1 and 1.2.

This section is dedicated to the proofs of the main results of the paper.

We first prove Theorem 1.1. The proof of part ii) is based on the following approximation technique. Let $(N_t^{(q)}, t \geq 0)$ be a Poisson process with intensity $q > 0$, which is independent of the Lévy process ξ, and denote by $(\tau_n^q)_{n \geq 0}$ its sequence of jump times with the convention that $\tau_0^q = 0$. For simplicity, we also introduce for $n \geq 0$,

$$\xi_t^{(n)} = \xi_{\tau_n^q + t} - \xi_{\tau_n^q}, \qquad t \geq 0.$$

For $n \geq 0$, we define the following random variables

$$S_n^{(q)} := \xi_{\tau_n^q}, \qquad M_n^{(q)} := \sup_{\tau_n^q \leq t < \tau_{n+1}^q} \xi_t \qquad \text{and} \qquad I_n^{(q)} := \inf_{\tau_n^q \leq t < \tau_{n+1}^q} \xi_t.$$

Observe that $(S_n^{(q)}, n \geq 0)$ is a random walk with step distribution given by $\xi_{\tau_1^q}$ and that τ_1^q is an exponential r.v. with parameter q which is independent of ξ.

Similarly for the process $\xi^{(n)}$, we also introduce

$$m_n^{(q)} := \sup_{t < \tau_{n+1}^q - \tau_n^q} \xi_t^{(n)} \quad \text{and} \quad i_n^{(q)} := \inf_{t < \tau_{n+1}^q - \tau_n^q} \xi_t^{(n)}.$$

Lemma 3.1. *Using the above notation we have*

$$M_n^{(q)} = S_n^{(+,q)} + m_0^{(q)}, \qquad I_n^{(q)} = S_n^{(-,q)} + i_0^{(q)}, \qquad n \geq 0$$

where each of the processes $S^{(+,q)} = (S_n^{(+,q)}, n \geq 0)$ and $S^{(-,q)} = (S_n^{(-,q)}, n \geq 0)$ are random walks with the same distribution as $S^{(q)}$. Moreover $S^{(+,q)}$ and $m_0^{(q)}$ are independent, as are $S^{(-,q)}$ and $i_0^{(q)}$.

The proof of this lemma is based on the Wiener-Hopf factorisation (see Equations (4.3.3) and (4.3.4) in Doney, 2007). It follows from similar arguments as those used in the proof of Theorem IV.13 in Doney (2007), which considers the case when the exponential random variables are jump times of the process ξ restricted to $\mathbb{R} \setminus [-\eta, \eta]$, for $\eta > 0$. So, we omit its proof for the sake of brevity.

Recall that τ_1^q goes to 0, in probability, as q increases and that ξ has càdlàg paths. Hence, there exists an increasing sequence $(q_n)_{n \geq 0}$ such that $q_n \to \infty$ and

$$e^{\lambda i_0^{(q_n)}} \xrightarrow[n \to \infty]{} 1, \qquad \text{a.s.} \tag{3.1}$$

We also recall the following form of the Wiener-Hopf factorisation, for $q > \psi(\lambda)$

$$\frac{q}{q - \psi(\lambda)} = \mathbb{E}\left[e^{\lambda i_0^{(q)}}\right] \mathbb{E}\left[e^{\lambda m_0^{(q)}}\right]. \tag{3.2}$$

From the Dominated Convergence Theorem and identity (3.2), it follows that for $\varepsilon \in (0, 1)$, there exists $N \in \mathbb{Z}_+$ such that for all $n \geq N$

$$1 - \varepsilon \leq \mathbb{E}\left[e^{\lambda i_0^{(q_n)}}\right] \leq \mathbb{E}\left[e^{\lambda m_0^{(q_n)}}\right] \leq 1 + \varepsilon. \tag{3.3}$$

Next, we introduce the compound Poisson process

$$Y_t^{(q)} := S_{N_t^{(q)}}^{(q)}, \qquad t \geq 0,$$

whose Laplace exponent satisfies

$$\psi^{(q)}(\lambda) := \log \mathbb{E}\left[e^{\lambda Y_1^{(q)}}\right] = \frac{q\psi(\lambda)}{q - \psi(\lambda)},$$

which is well defined for λ such that $q > \psi(\lambda)$. Similarly, we define

$$\widetilde{I}_t^{(q)} = I_{N_t^{(q)}}^{(q)}, \qquad \widetilde{M}_t^{(q)} = M_{N_t^{(q)}}^{(q)}, \qquad Y_t^{(+,q)} = S_{N_t^{(q)}}^{(+,q)}, \qquad \text{and} \qquad Y_t^{(-,q)} = S_{N_t^{(q)}}^{(-,q)}.$$

We observe from the definitions of $\widetilde{M}^{(q)}$ and $\widetilde{I}^{(q)}$, and Lemma 3.1, that for all $t \geq 0$, the following inequalities are satisfied

$$e^{-m_0^{(q)}} \int_0^t e^{-Y_s^{(+,q)}} \mathrm{d}s \leq \int_0^t e^{-\xi_s} \mathrm{d}s \leq e^{-i_0^{(q)}} \int_0^t e^{-Y_s^{(-,q)}} \mathrm{d}s. \tag{3.4}$$

We have now all the tools needed to prove Theorem 1.1.

Proof of Theorem 1.1: i) Assume that $\psi'(0+) > 0$. According to Theorem 1 in Bertoin and Yor (2005), $I_t(\xi)$ converges a.s. to $I_\infty(\xi)$, a non-negative and finite limit as t goes to ∞. Then, we observe that the result follows from the Monotone Convergence Theorem.

We now prove part ii). In order to do so, we use the approximation and notation that we introduced at the beginning of this section. Let $(q_n)_{n \geq 1}$ be a sequence defined as in (3.1) and observe that for $n \geq 1$, we have $\psi^{(q_n)}(0) = 0$, $\psi'^{(q_n)}(0+) = 0$ and $\psi''^{(q_n)}(0+) < \infty$. We also observe that the processes $Y^{(+,q_n)}$ and $Y^{(-,q_n)}$ have bounded variation paths.

We take $\ell \geq N$ and $0 < \varepsilon < 1$. Hence from Lemmas 13 and 14 in Bansaye et al. (2013), we observe that there exists a positive constant $c_1(\ell)$ such that

$$(1 - \varepsilon)c_1(\ell)t^{-1/2} \leq \mathbb{E}\left[\left(\int_0^t e^{-Y_s^{(\pm,q_\ell)}} \mathrm{d}s\right)^{-p}\right] \leq (1 + \varepsilon)c_1(\ell)t^{-1/2},$$

for t sufficiently large. Therefore using (3.3) and (3.4) in the previous inequality, we obtain

$$(1 - \varepsilon)^2 c_1(\ell)t^{-1/2} \leq \mathbb{E}[I_t(\xi)^{-p}] \leq (1 + \varepsilon)^2 c_1(\ell)t^{-1/2}, \tag{3.5}$$

for t sufficiently large.

Next, we take $n, m \geq N$ and observe that the previous inequalities imply

$$\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^2 c_1(n) \leq c_1(m) \leq \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^2 c_1(n), \qquad \text{for all } n, m \geq N.$$

Thus, we deduce that $(c_1(n))_{n \geq 1}$ is a Cauchy sequence. Let us denote c_1 its limit which, by the previous inequalities is positive. Let $l \geq N$ such that

$$(1 - \varepsilon)c_1 \leq c_1(l) \leq (1 + \varepsilon)c_1.$$

Using this inequality and (3.5), we observe

$$(1 - \varepsilon)^3 c_1 t^{-1/2} \leq \mathbb{E}[I_t(\xi)^{-p}] \leq (1 + \varepsilon)^3 c_1 t^{-1/2},$$

for t sufficiently large. This completes the proof of part ii).

Now, we prove part iii)-a). Recalling (2.7) yields that

$$I_t(\xi) \stackrel{(d)}{=} e^{-\xi_t} I_t(-\xi), \qquad t \geq 0. \tag{3.6}$$

Hence using the Esscher transform (1.4), with $\lambda = p$, we have

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = \mathbb{E}\left[e^{p\xi_t}I_t(-\xi)^{-p}\right] = e^{t\psi(p)}\mathbb{E}^{(p)}\left[I_t(-\xi)^{-p}\right], \qquad t \geq 0. \qquad (3.7)$$

The inequality (1.5) with $q = p$ and the previous identity imply that the decreasing function $t \mapsto \mathbb{E}^{(p)}[I_t(-\xi)^{-p}]$ is finite for all $t > 0$. Recall that under the probability measure $\mathbb{P}^{(p)}$, the process ξ is a Lévy process with mean $\mathbb{E}^{(p)}[\xi_1] = \psi'(p) \in (-\infty, 0)$. Then, as in the proof of part i), $\mathbb{E}^{(p)}[I_t(-\xi)^{-p}]$ converges to $\mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}]$, as t increases.

Part iii)-b) follows from part ii) and the Esscher transform (1.4). More precisely, we apply the Esscher transform with $\lambda = p$ and observe that the Laplace transform of the process ξ under the probability measure $\mathbb{P}^{(p)}$, satisfies $\psi_p'(0+) = \psi'(p) = 0$ and $\psi_p''(0+) = \psi''(p) < \infty$. Therefore by applying part ii) and identity (3.7), we get the existence of a constant $c_2 > 0$ such that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = e^{t\psi(p)}\mathbb{E}^{(p)}[I_t(-\xi)^{-p}] \sim c_2 t^{-1/2}e^{t\phi(p)}.$$

Finally we prove part iii)-c). Again from the Esscher transform with $\lambda = \tau$, we see

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = e^{t\psi(\tau)}\mathbb{E}^{(\tau)}[e^{(p-\tau)\xi_t}I_t(-\xi)^{-p}], \qquad t > 0.$$

On the one hand, for $t > 0$,

$$E^{(\tau)}[e^{(p-\tau)\xi_t}I_t(-\xi)^{-p}]$$

$$= \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_t-\xi_{t/2})}\frac{(e^{-\xi_{t/2}}I_{t/2}(-\xi) + \int_{t/2}^t e^{\xi_u-\xi_{t/2}}\mathrm{d}u)^{-(p-\tau)}}{(I_{t/2}(-\xi) + e^{\xi_{t/2}}\int_{t/2}^t e^{\xi_u-\xi_{t/2}}\mathrm{d}u)^\tau}\right]$$

$$\leq \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_t-\xi_{t/2})}\frac{(\int_0^{t/2} e^{\xi_{s+t/2}-\xi_{t/2}}\mathrm{d}s)^{-(p-\tau)}}{I_{t/2}(-\xi)^\tau}\right]$$

$$= \mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_{t/2})}I_{t/2}(-\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right],$$

where we have used in the last identity the fact that $(\xi_{u+t/2} - \xi_{t/2}, u \geq 0)$ is independent of $(\xi_u, 0 \leq u \leq t/2)$ and with the same law as $(\xi_u, u \geq 0)$.

On the other hand, from (3.6) we deduce

$$\mathbb{E}^{(\tau)}\left[e^{(p-\tau)(\xi_{t/2})}I_{t/2}(-\xi)^{-(p-\tau)}\right] = \mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right], \qquad t > 0.$$

Putting all the pieces together, we get

$$\mathbb{E}^{(\tau)}[e^{(p-\tau)\xi_t}I_t(-\xi)^{-p}] \leq \mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right], \qquad t > 0$$

implying

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \leq e^{t\psi(\tau)}\mathbb{E}^{(\tau)}\left[I_{t/2}(\xi)^{-(p-\tau)}\right]\mathbb{E}^{(\tau)}\left[I_{t/2}(-\xi)^{-\tau}\right], \qquad t > 0.$$

Since $\psi'(\tau) = 0$, we have $\mathbb{E}^{(\tau)}[\xi_1] = 0$ and the process ξ oscillates under $\mathbb{P}^{(\tau)}$. Moreover since $\psi''(\tau) < \infty$, we deduce that $\psi_\tau''(0+) < \infty$. The latter condition implies from part ii) that there exists a constant $c_1(\tau) > 0$ such that

$$\mathbb{E}^{(\tau)}[I_t(\xi)^{-(p-\tau)}] \sim c_1(\tau)t^{-1/2},$$

for t sufficiently large. Since the process ξ oscillates under $\mathbb{P}^{(\tau)}$, the dual $-\xi$ also oscillates. This implies that $I_t(-\xi)$ goes to ∞ and therefore $\mathbb{E}^{(\tau)}[I_t(-\xi)^{-(p-\tau)}]$ goes to 0, as t increases. In other words, we have

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] = o(t^{-1/2}e^{t\psi(\tau)}), \qquad \text{as} \quad t \to \infty,$$

as expected.

We now assume that ξ is non-arithmetic, our arguments are similar to those used in Bansaye et al. (2013). We will prove

$$\limsup_{t\to\infty} t^{3/2}e^{-t\psi(\tau)}\mathbb{E}\left[I_t(\xi)^{-p}\right] < \infty.$$

In order to prove it, we take $t > 0$ and observe

$$I_{\lfloor t \rfloor}(\xi) = \sum_{k=0}^{\lfloor t \rfloor - 1} e^{-\xi_k} \int_0^1 e^{-(\xi_{k+u}-\xi_k)}\mathrm{d}u,$$

where $\lfloor t \rfloor$ denotes the integer part of t. Therefore

$$\mathbb{E}\left[I_{\lfloor t \rfloor}(\xi)^{-p}\right] \leq \mathbb{E}\left[\min_{k\leq\lfloor t \rfloor - 1} e^{p\xi_k}\left(\int_0^1 e^{-(\xi_{k+u}-\xi_k)}\mathrm{d}u\right)^{-p}\right].$$

Conditioning on the value when the minimum is attained, let say k', and observing that $e^{p\xi_{k'}}$ is independent of $\left(\int_0^1 e^{-(\xi_{k'+u}-\xi_{k'})}\mathrm{d}u\right)^{-p}$ and the latter has the same law as $\left(\int_0^1 e^{-\xi_u}\mathrm{d}u\right)^{-p}$, we deduce

$$\mathbb{E}\left[I_{\lfloor t \rfloor}(\xi)^{-p}\right] \leq \mathbb{E}\left[\min_{k\leq\lfloor t \rfloor - 1} e^{p\xi_k}\right]\mathbb{E}\left[\left(\int_0^1 e^{-\xi_u}\mathrm{d}u\right)^{-p}\right].$$

Finally, by Lemma 7 in Hirano (1998), there exists a $C > 0$ such that

$$\mathbb{E}\left[\min_{k\leq\lfloor t \rfloor - 1} e^{p\xi_k}\right] \sim C\lfloor t \rfloor^{-3/2}e^{\lfloor t \rfloor\psi(\tau)}, \qquad \text{for } t \text{ large.}$$

The claim follows from the monotonicity of $\mathbb{E}\left[I_{\lfloor t \rfloor}(\xi)^{-p}\right]$ and the fact that $t \in [\lfloor t \rfloor, \lfloor t \rfloor + 1)$. $\qquad\square$

The idea of the proof of Theorem 1.2 is to study the asymptotic behaviour of $\mathcal{E}_F(n/q)$ for q fixed and large n, and then to use the monotonicity of F to deduce the asymptotic behaviour of $\mathcal{E}_F(t)$ when t goes to infinity. In order to do so, we use a key result due to Guivarc'h and Liu that we state in the Appendix for the sake of completeness.

Let $q > 0$ and define the sequence $q_n = n/q$, for $n \geq 0$. For $k \geq 0$, we also define

$$\widetilde{\xi}_u^{(k)} = \xi_{q_k+u} - \xi_{q_k}, \qquad \text{for} \quad u \geq 0,$$

and

$$a_k = e^{-\widetilde{\xi}_{q_{(k+1)}-q_k}^{(k)}} \qquad \text{and} \qquad b_k = \int_0^{q_{(k+1)}-q_k} e^{-\widetilde{\xi}_u^{(k)}}\mathrm{d}u. \qquad (3.8)$$

Hence, (a_k, b_k) is a $\mathbb{R}_+^2$-valued sequence of i.i.d. random variables. Observe that

$$a_0 = e^{-\xi_{\frac{1}{q}}} \qquad \text{and} \qquad \frac{b_0}{1-a_0} = \frac{I_{\frac{1}{q}}(\xi)}{1-e^{-\xi_{\frac{1}{q}}}},$$

which are not constant a.s. as required by Theorem A.1. Moreover, we have

$$\int_{q_i}^{q_{i+1}} e^{-\xi_u} \mathrm{d}u = e^{-\xi_{q_i}} b_i = \prod_{k=0}^{i-1} a_k b_i = A_i b_i,$$

where A_k is defined as in Theorem A.1. The latter identity implies

$$I_{q_n}(\xi) = \sum_{i=0}^{n-1} \int_{q_i}^{q_{i+1}} e^{-\xi_u} \mathrm{d}u = \sum_{i=0}^{n-1} A_i b_i := B_n.$$

In other words, we have all the objects required to apply Theorem A.1.

Proof of Theorem 1.2: i) The proof uses similar arguments as those used in the proof of Theorem 1.1-i).

ii) We now assume that $\psi'(0+) = 0$. We define the sequence $(a_k, b_k)_{k \geq 0}$ as in (3.8) and follow the same notation as in Theorem A.1. We take $0 < \eta < \alpha$ and $d_p > 1$ such that $-\theta^-/d_p < p$ and $\theta^- < -\eta < \eta + p < \theta^+$, and let

$$(\eta, \kappa, \vartheta) = \left(\eta, \frac{-\theta^-}{d_p}, p \right).$$

Next, we verify the moment conditions of Theorem A.1 for the couple (a_0, b_0). From the definition of (a_0, b_0), we get

$$\mathbb{E}\left[\ln a_0\right] = \frac{\psi'(0+)}{q} = 0, \qquad \mathbb{E}\left[a_0^\kappa\right] = e^{\psi(-\kappa)/q} \quad \text{and} \quad \mathbb{E}\left[a_0^{-\eta}\right] = e^{\psi(\eta)/q},$$

which are well defined. Similarly as in (1.5), by L_1-Doob's inequality (see Acciaio et al., 2013) and the Esscher transform (1.4)

$$\mathbb{E}\left[b_0^\eta\right] \leq q^\eta \mathbb{E}\left[\sup_{0 \leq u \leq 1/q} e^{-\eta \xi_u}\right] \leq \frac{e}{e-1} q^\eta e^{\frac{\psi(-\eta)}{q}} \left(1 - \eta \psi'(-\eta) - \psi(-\eta)\right) < \infty,$$

and

$$\mathbb{E}\left[a_0^{-\eta} b_0^{-\vartheta}\right] \leq q^\vartheta \mathbb{E}\left[e^{\eta \xi_{\frac{1}{q}}} \sup_{0 \leq u \leq 1/q} e^{\vartheta \xi_u}\right] \leq q^\vartheta \mathbb{E}\left[\sup_{0 \leq u \leq 1/q} e^{(\eta+\vartheta)\xi_u}\right] < \infty.$$

Therefore the asymptotic behaviour of $\mathcal{E}_F(q_n)$ for large n follows from a direct application of Theorem A.1. In other words, there exists a positive constant $c(q)$ such that

$$\sqrt{n} \mathcal{E}_F(q_n) \sim c(q), \qquad \text{as} \quad n \to \infty.$$

In order to get our result, we take t to be a positive real number. Since the mapping $s \mapsto \mathcal{E}_F(s)$ is non-increasing, we get

$$\sqrt{t} \mathcal{E}_F(t) \leq \sqrt{t} \mathcal{E}_F(\lfloor qt \rfloor / q) = \sqrt{\frac{t}{\lfloor qt \rfloor}} \sqrt{\lfloor qt \rfloor} \mathcal{E}_F(\lfloor qt \rfloor / q).$$

Similarly

$$\sqrt{t} \mathcal{E}_F(t) \geq \sqrt{t} \mathcal{E}_F((\lfloor qt \rfloor + 1)/q) = \sqrt{\frac{t}{\lfloor qt \rfloor + 1}} \sqrt{\lfloor qt \rfloor + 1} \mathcal{E}_F((\lfloor qt \rfloor + 1)/q).$$

Therefore

$$\sqrt{t} \mathcal{E}_F(t) \sim c(q) q^{-1/2}, \qquad \text{as } t \to \infty.$$

Moreover, we deduce that $c(q)q^{-1/2}$ is positive and does not depend on q. Hence we denote this constant by c_1. This concludes the proof of point ii).

iii) For the rest of the proof, we assume that $\psi'(0) < 0$. We first prove part a). Since $\psi'(p) < 0$, from Theorem 1.1 part iii)-a) we know that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \sim e^{t\psi(p)}\mathbb{E}^{(p)}[I_\infty(-\xi)^{-p}], \qquad as \quad t \to \infty.$$

Hence the asymptotic behaviour is proven if we show that

$$\mathcal{E}_F(t) \sim \mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right], \quad as \quad t \to \infty.$$

Since $\psi'(p) < 0$, there is $\varepsilon > 0$ such that $p(1 + \varepsilon) < \theta^+$, $\psi(p(1 + \varepsilon)) < \psi(p)$ and $\psi'((1 + \varepsilon)p) < 0$. Hence, from Lemma 2 (see the Appendix), we deduce that there is a constant M such that

$$\left|F\left(I_t(\xi)\right) - \mathbf{k}I_t(\xi)^{-p}\right| \leq MI_t(\xi)^{-(1+\varepsilon)p}. \tag{3.9}$$

In other words, it is enough to prove

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = o(e^{t\psi(p)}), \quad as \quad t \to \infty. \tag{3.10}$$

From the Esscher transform (1.4) with $\lambda = (1 + \varepsilon)p$, we deduce

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = \mathbb{E}\left[e^{p(1+\varepsilon)\xi_s}I_t(-\xi)^{-(1+\varepsilon)p}\right]$$

$$= e^{t\psi(p)}e^{t\psi_p(\varepsilon p)}\mathbb{E}^{((1+\varepsilon)p)}\left[I_t(-\xi)^{-(1+\varepsilon)p}\right].$$

This and Equation (1.5) with $\lambda = (1 + \varepsilon)p$ imply that $\mathbb{E}^{((1+\varepsilon)p)}[I_t(-\xi)^{-(1+\varepsilon)p}]$ is finite for all $t > 0$. Similarly as in the proof of Theorem 1.1 iii)-a), we can deduce that $\mathbb{E}^{((1+\varepsilon)p)}[I_t(-\xi)^{-(1+\varepsilon)p}]$ has a finite limit, as t goes to ∞. We conclude by observing that $\psi_p(\varepsilon p)$ is negative implying that (3.10) holds. We complete the proof of point iii)-a) by observing that (3.9) and (3.10) yield

$$\mathbb{E}[F\left(I_t(\xi)\right)] \sim \mathbf{k}\mathbb{E}[I_t(\xi)^{-p}], \quad t \to \infty.$$

We now prove part b). Since $\psi'(p) = 0$ and $\psi''(p) < \infty$, from Theorem 1.1 part iii)-b) we know that there exists a positive constant c_2 such that

$$\mathbb{E}\left[I_t(\xi)^{-p}\right] \sim c_2 t^{-1/2}e^{t\psi(p)}, \qquad as \quad t \to \infty.$$

Similarly as in the proof of part a), the asymptotic behaviour is proven if we show that

$$\mathcal{E}_F(t) \sim \mathbf{k}\mathbb{E}\left[I_t(\xi)^{-p}\right], \quad as \quad t \to \infty,$$

which amounts to showing that

$$\mathbb{E}\left[I_t(\xi)^{-(1+\varepsilon)p}\right] = o(t^{-1/2}e^{t\psi(p)}), \quad as \quad t \to \infty$$

for ε small enough. The latter follows from Theorem 1.1 iii)-c).

Finally, we prove part c). Similarly as in the proof of part ii), we define the sequence $(a_k, b_k)_{k\geq 0}$ as in (3.8) and follow the same notation as in Theorem A.1. Let us choose $0 < \eta < \alpha$ such that $0 < \tau - \eta < \tau + p + \eta < \theta^+$ and take

$$(\eta, \kappa, \vartheta) = (\eta, \tau, p).$$

Next, we apply the Esscher transform (1.4) with $\lambda = \tau$ and observe

$$\mathbb{E}[F(I(q_n))]e^{-q_n\psi(\tau)} = \mathbb{E}^{(\tau)}[e^{-\tau\xi_{q_n}}F(I(q_n))] = \mathbb{E}^{(\tau)}[A_n^\tau F(B_n)]. \qquad (3.11)$$

Hence in order to apply Theorem A.1, we need the moment conditions on (a_0, b_0) to be satisfied under the probability measure $\mathbb{P}^{(\tau)}$. We first observe,

$$\mathbb{E}^{(\tau)}[\ln a_0] = \mathbb{E}^{(\tau)}[\xi_{1/q}] = e^{-\psi(\tau)/q}\mathbb{E}[\xi_{1/q}e^{\tau\xi_{1/q}}] = \frac{\psi'(\tau)}{q} = 0.$$

Similarly, we get

$$\mathbb{E}^{(\tau)}\left[a_0^\kappa\right] = \mathbb{E}^{(\tau)}[e^{-\kappa\xi_{1/q}}] = e^{-\psi(\tau)/q} \quad \text{and} \quad \mathbb{E}^{(\tau)}\left[a_0^{-\eta}\right] = \mathbb{E}^{(\tau)}[e^{\eta\xi_{1/q}}] = e^{\psi_\tau(\eta)/q},$$

where $\psi_\tau(\lambda) = \psi(\tau+\lambda) - \psi(\tau)$. From our assumptions both expectations are finite.

Again, we use similar arguments as those used in (1.5) to deduce

$$\mathbb{E}^{(\tau)}\left[b_0^\eta\right] \leq q^{-\eta}\mathbb{E}^{(\tau)}\left[\sup_{0\leq u\leq 1/q} e^{-\eta\xi_u}\right] \leq q^{-\eta}e^{-\psi(\tau)/q}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{(\tau-\eta)\xi_u}\right] < \infty,$$

and

$$\mathbb{E}^{(\tau)}\left[a_0^{-\eta}b_0^{-p}\right] \leq q^p\mathbb{E}^{(\tau)}\left[e^{\eta\xi_{\frac{1}{q}}}\sup_{0\leq u\leq 1/q} e^{p\xi_u}\right]$$
$$\leq q^p e^{-\psi(\tau)/q}\mathbb{E}\left[\sup_{0\leq u\leq 1} e^{(\tau+\eta+p)\xi_u}\right] < \infty.$$

Therefore the asymptotic behaviour of $\mathbb{E}^{(\tau)}[A_n^\tau F(B_n)]$ follows from a direct application of Theorem A.1 with the functions $\tilde{\psi}(x) = F(x)$ and $\tilde{\phi}(x) = x^\tau$. In other words, we conclude that there exists a positive constant $c(q)$ such that

$$n^{3/2}\mathbb{E}^{(\tau)}[A_n^\tau F(B_n)] \sim c(q), \quad n \to \infty.$$

In particular from (3.11), we deduce

$$\mathcal{E}_F(q_n) \sim c(q)e^{-n\psi(\tau)/q}n^{-3/2}, \quad n \to \infty.$$

Then using the monotonicity of F at infinity as in the proof of part ii), we get that for n large enough,

$$c(q)q^{-3/2}e^{-\psi(\tau)/q} \leq n^{3/2}e^{n\psi(\tau)}\mathcal{E}_F(n) \leq c(q)q^{-3/2}. \qquad (3.12)$$

A direct application of Lemma A.3 then yields the existence of a non-negative constant c_4 such that

$$\lim_{q\to\infty} c(q)q^{-3/2} = C_3.$$

Moreover, (3.12) yields that C_3 is positive. This ends the proof. $\qquad \square$

A. Appendix

We recall in this section a Theorem due to Guivarc'h and Liu (see Theorem 2.1 in Guivarc'h and Liu, 2001) and two technical Lemmas stated in Bansaye et al. (2013):

Theorem A.1 (Guivarc'h and Liu, 2001). *Let $(a_n, b_n)_{n \geq 0}$ be a $\mathbb{R}_+^2$-valued sequence of i.i.d. random variables such that $\mathbb{E}[\ln a_0] = 0$. Assume that $b_0/(1-a_0)$ is not constant a.s. and define*

$$A_0 := 1, \quad A_n := \prod_{k=0}^{n-1} a_k \quad and \quad B_n := \sum_{k=0}^{n-1} A_k b_k, \quad for \quad n \geq 1.$$

Let η, κ, ϑ be three positive numbers such that $\kappa < \vartheta$, and $\tilde{\phi}$ and $\tilde{\psi}$ be two positive continuous functions on $\mathbb{R}_+$ such that they do not vanish and for a constant $C > 0$ and for every $a > 0$, $b \geq 0$, $b' \geq 0$, we have

$$\tilde{\phi}(a) \leq Ca^\kappa, \quad \tilde{\psi}(b) \leq \frac{C}{(1+b)^\vartheta}, \quad and \quad |\tilde{\psi}(b) - \tilde{\psi}(b')| \leq C|b - b'|^\eta.$$

Moreover, assume that

$$\mathbb{E}\left[a_0^\kappa\right] < \infty, \quad \mathbb{E}\left[a_0^{-\eta}\right] < \infty, \quad \mathbb{E}\left[b_0^\eta\right] < \infty \quad and \quad \mathbb{E}\left[a_0^{-\eta} b_0^{-\vartheta}\right] < \infty.$$

Then, there exist two positive constants $c(\tilde{\phi}, \tilde{\psi})$ and $c(\tilde{\psi})$ such that

$$\lim_{n \to \infty} n^{3/2} \mathbb{E}\left[\tilde{\phi}(A_n)\tilde{\psi}(B_n)\right] = c(\tilde{\phi}, \tilde{\psi}) \quad and \quad \lim_{n \to \infty} n^{1/2} \mathbb{E}\left[\tilde{\psi}(B_n)\right] = c(\tilde{\psi}).$$

Lemma A.2. *Assume that F satisfies one of the Assumptions (**A1**) or (**A2**). Then there exist two positive finite constants η and M such that for all (x, y) in $\mathbb{R}_+^2$ and ε in $[0, \eta]$,*

$$\left|F(x) - \mathbf{k}x^{-p}\right| \leq Mx^{-(1+\varepsilon)p},$$
$$\left|F(x) - F(y)\right| \leq M\left|x^{-p} - y^{-p}\right|.$$

Lemma A.3. *Assume that the non-negative sequences $(a_{n,q})_{(n,q) \in \mathbb{Z}_+^2}$, $(a'_{n,q})_{(n,q) \in \mathbb{Z}_+^2}$ and $(b_n)_{n \in \mathbb{Z}_+}$ satisfy for every $(n, q) \in \mathbb{Z}_+^2$:*

$$a_{n,q} \leq b_n \leq a'_{n,q},$$

and that there exist three sequences $(a(q))_{q \in \mathbb{Z}_+}$, $(c^-(q))_{q \in \mathbb{Z}_+}$ and $(c^+(q))_{q \in \mathbb{Z}_+}$ such that

$$\lim_{n \to \infty} a_{n,q} = c^-(q)a(q), \quad \lim_{n \to \infty} a'_{n,q} = c^+(q)a(q), \quad and \quad \lim_{q \to \infty} c^-(q) = \lim_{q \to \infty} c^+(q) = 1.$$

Then there exists a non-negative constant a such that

$$\lim_{q \to \infty} a(q) = \lim_{n \to \infty} b_n = a.$$

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