Continuous time autoregressive moving average processes with random Lévy coefficients

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Abstract. We introduce a continuous time autoregressive moving average process with random Lévy coefficients, termed RC-CARMA(p,q) process, of order p and $q < p$ via a subclass of multivariate generalized Ornstein-Uhlenbeck processes. Sufficient conditions for the existence of a strictly stationary solution and the existence of moments are obtained. We further investigate second order stationarity properties, calculate the autocovariance function and spectral density, and give sufficient conditions for their existence.

1. Introduction

Let $q < p$ be non-negative integers and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process, i.e. a process with stationary and independent increments, càdlàg sample paths and $L_0 = 0$ almost surely, which is continuous in probability. A CARMA(p,q) process $S = (S_t)_{t\in\mathbb{R}}$ driven by L is defined via

$$S_t = \mathbf{b}'\mathbf{X}_t, \quad t \in \mathbb{R}, \tag{1.1}$$

with $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ a $\mathbb{C}^p$-valued process which is a solution to the stochastic differential equation (SDE)

$$d\mathbf{X}_t = A\mathbf{X}_t \, dt + \mathbf{e} \, dL_t, \quad t \in \mathbb{R}, \tag{1.2}$$

where

$$
A = \begin{bmatrix}
0 & 1 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 \\
-a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1
\end{bmatrix}, \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}
$$

with $a_1, \dots, a_p, b_0, \dots, b_{p-1} \in \mathbb{C}$ such that $b_q \neq 0$ and $b_j = 0$ for $j > q$. For $p = 1$ the matrix A is interpreted as $A = (-a_1)$.

It is well-known that the solution of (1.2) is unique for any $\mathbf{X}_0$ and given by

$$
\mathbf{X}_t = e^{At} \left(\mathbf{X}_0 + \int_{(0,t]} e^{-As} \mathbf{e} \, dL_s \right), \quad t \geq 0.
$$

Processes of this kind were first considered for L being a Gaussian process by Doob (1944). Brockwell (2001b) gave the now commonly used definition with L being a Lévy process.

A CARMA process $S = (S_t)_{t \in \mathbb{R}}$ as defined in (1.1) and (1.2) can be interpreted as a solution of the pth-order linear differential equation

$$
a(D)S_t = b(D)DL_t,
$$

where $a(z) = z^p + a_1 z^{p-1} + \dots + a_p$, $b(z) = b_0 + b_1 z + \dots + b_{p-1}z^{p-1}$ and D denotes the differentiation operator. In this sense, CARMA processes are a natural continuous time analog of discrete time ARMA processes. Similar to ARMA processes, CARMA processes provide a tractable but rich class of stochastic processes. Their possible autocovariance functions $h \mapsto \mathbf{Cov}(S_t, S_{t+h})$ are linear combinations of (complex) exponentials and thus provide a wide variety of possible models when modeling empirical data.

In discrete time, ARMA processes with random coefficients (RC-ARMA) have attracted a lot of interest recently, in particular, AR processes with random coefficients, see e.g. Nicholls and Quinn (1982). They have applications as non-linear models for various processes, e.g. bilinear GARCH processes introduced by Storti and Vitale (2003). RC-ARMA processes also arise as a special case of conditional heteroscedastic ARMA (CHARMA) models proposed by Tsay (1987) and are used for financial volatility processes, see e.g. He and Teräsvirta (1999), to name just a few.

As CARMA processes constitute the natural continuous time analog of ARMA processes, it is, therefore, natural to ask for CARMA processes with random coefficients. The CARMA$(1,0)$ process with random (Lévy) coefficients has already been studied. It is known as the *generalized Ornstein-Uhlenbeck (GOU) process*, which is obtained as the solution to the SDE

$$
dX_t = X_{t-} \, d\xi_t + dL_t, \quad t \geq 0,
$$

where $(\xi, L) = (\xi_t, L_t)_{t \geq 0}$ is a bivariate Lévy process. It has been shown by de Haan and Karandikar (1989) that GOU processes arise as the natural continuous time analog of the AR(1) process with random i.i.d. coefficients. By choosing $(\xi_t)_{t \geq 0} = (-a_1 t)_{t \geq 0}$, the GOU process reduces to the classical Lévy-driven Ornstein-Uhlenbeck process, which is a CAR(1), i.e. CARMA(1,0) process.

Both the Ornstein-Uhlenbeck process as well as the generalized Ornstein-Uhlenbeck process have various applications in insurance and financial mathematics, see e.g. Barndorff-Nielsen and Shephard (2001) and Klüppelberg et al. (2004).

The aim of the present paper is to introduce CARMA processes with random Lévy autoregressive coefficients of higher orders, $p \geq 1$, and to study stationarity and other natural properties. The definition of our process is done in such a way that it includes the generalized Ornstein-Uhlenbeck process for order $(1,0)$ as a special case and that it reduces to the usual CARMA process when the autoregressive Lévy coefficients are chosen to be deterministic Lévy processes, i.e. pure drift and henceforth linear functions.

The paper is organized as follows. In Section 2 we give some preparative results regarding multivariate stochastic integration, the multivariate stochastic exponential, and multivariate generalized Ornstein-Uhlenbeck processes. In Section 3 we define a CARMA process with random coefficients (RC-CARMA) and present some basic properties as well as sufficient conditions for the existence of a strictly stationary solution. Similar to Brockwell and Lindner (2015) for CARMA processes, we further show that the RC-CARMA$(p,0)$ process satisfies an integral-differential equation and examine its path properties. Section 4 is concerned with the existence of moments, the autocovariance function, and the spectral density, whereby it turns out that the latter two have an interesting connection to those of CARMA processes. We end Section 4 by investigating an RC-CARMA$(2,1)$ process in more detail. Conclusively, in Section 5 we present some simulations.

2. Preliminaries

Throughout, we will always assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. By a *filtration* we mean a family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. Our filtration satisfies, if not stated otherwise, the *usual hypotheses*, i.e. $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and the filtration is right-continuous.

$\mathrm{GL}(\mathbb{R}, m)$ denotes the general linear group of order m, i.e. the set of all $m \times m$ invertible matrices associated with the ordinary matrix multiplication. If $A \in \mathrm{GL}(\mathbb{R}, m)$, we denote with A' its transpose and with A^{-1} its inverse.

For càdlàg processes $X = (X_t)_{t\geq 0}$ we denote with X_{t-} and $\Delta X_t := X_t - X_{t-}$ the left-limit and the jump at time t, respectively. A d-dimensional Lévy processes $L = (L_t)_{t\geq 0}$ can be identified by its characteristic exponent (A_L, γ_L, Π_L) due to the Lévy-Khintchine formula, i.e. if μ denotes the distribution of L_1, then its characteristic function is, for $z \in \mathbb{R}^d$, given by

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, A_L z\rangle + i\langle \gamma_L, z\rangle + \int_{\mathbb{R}^d} (\mathrm{e}^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle \mathbf{1}_{\{|x|\leq 1\}}(x))\, \Pi_L(\mathrm{d}x) \right].$$

Here, A_L is the Gaussian covariance matrix which is in one dimension denoted by σ_L^2, Π_L a measure on $\mathbb{R}^d$ which satisfies $\Pi_L(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\, \Pi_L(\mathrm{d}x) < \infty$, called the *Lévy measure*, and $\gamma_L \in \mathbb{R}^d$ a constant. Further, $|x|$ denotes the Euclidean norm of x.

For a detailed account of Lévy processes we refer to the book of Sato (2013).

Stochastic Integration

A matrix-valued stochastic process $X = (X_t)_{t \geq 0}$ is called an $\mathbb{F}$-*semimartingale* or simply a *semimartingale* if every component $(X_t^{(i,j)})_{t \geq 0}$ is a semimartingale with respect to the filtration $\mathbb{F}$.

For a semimartingale $X \in \mathbb{R}^{m \times n}$, and $H \in \mathbb{R}^{l \times m}$ and $G \in \mathbb{R}^{n \times p}$ two locally bounded predictable processes, the $\mathbb{R}^{l \times p}$-valued stochastic integral $J = \int H \mathrm{d}X G$ is defined by its components via

$$J^{(i,j)} = \sum_{k=1}^{n} \sum_{h=1}^{m} \int H^{(i,h)} G^{(k,j)} \, \mathrm{d}X^{(h,k)}.$$

It can easily be seen that also in the multivariate case the stochastic integration preserves the semimartingale property as stated, for example, in the one-dimensional case in Protter (2005).

For two semimartingales $X \in \mathbb{R}^{l \times m}$ and $Y \in \mathbb{R}^{m \times n}$ the $\mathbb{R}^{l \times n}$-valued *quadratic covariation* $[X, Y]$ is defined by its components via

$$[X, Y]^{(i,j)} = \sum_{k=1}^{m} [X^{(i,k)}, Y^{(k,j)}], \tag{2.1}$$

and similar its continuous part $[X, Y]^c$. Then

$$[X, Y]_t = [X, Y]_t^c + X_0 Y_0 + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s, \quad t \geq 0 \tag{2.2}$$

is true also for matrix-valued semimartingales.

Finally, as stated in Karandikar (1991), for two semimartingales $X, Y \in \mathbb{R}^{m \times m}$ the integration by parts formula takes the form

$$(XY)_t = \int_{0+}^{t} X_{s-} \, \mathrm{d}Y_s + \int_{0+}^{t} \mathrm{d}X_s Y_{s-} + [X., Y.]_t, \quad t \geq 0.$$

The Multivariate Stochastic Exponential

Let $X = (X_t)_{t \geq 0}$ be a semimartingale in $\mathbb{R}^{m \times m}$ with $X_0 = 0$. Then its *left sto-chastic exponential* $\overleftarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{m \times m}$-valued, adapted, càdlàg solution $(Z_t)_{t \geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} Z_{s-} \, \mathrm{d}X_s, \quad t \geq 0,$$

where $I \in \mathbb{R}^{m \times m}$ denotes the identity matrix. The *right stochastic exponential* of X, denoted as $\overrightarrow{\mathcal{E}}(X)_t$, is defined as the unique $\mathbb{R}^{m \times m}$-valued, adapted, càdlàg solution $(Z_t)_{t \geq 0}$ of the integral equation

$$Z_t = I + \int_{(0,t]} \mathrm{d}X_s Z_{s-}, \quad t \geq 0.$$

It can be shown that both the left and the right stochastic exponential are semi-martingales and that for its transpose it holds $\overleftarrow{\mathcal{E}}(X)_t' = \overrightarrow{\mathcal{E}}(X')_t$. As observed by Karandikar (1991), the right and the left stochastic exponentials of a semimartin-gale X are invertible at all times $t \geq 0$ if and only if

$$\det(I + \Delta X_t) \neq 0 \quad \forall t \geq 0. \tag{2.3}$$

We also need the following result of Karandikar (1991).

Proposition 2.1. (Inverse of the Stochastic Exponential)
Let $X = (X_t)_{t \geq 0}$ be a semimartingale with $X_0 = 0$ such that (2.3) holds. Define the semimartingale

$$U_t := -X_t + [X, X]_t^c + \sum_{0 < s \leq t} \left((I + \Delta X_s)^{-1} - I + \Delta X_s \right), \quad t \geq 0. \qquad (2.4)$$

Then

$$\overleftarrow{\mathcal{E}}(X)_t^{-1} = \left[\overleftarrow{\mathcal{E}}(U')_t \right]' = \overrightarrow{\mathcal{E}}(U)_t \quad \forall t \geq 0, \qquad (2.5)$$

and

$$U_t = -X_t - [X, U]_t \quad \forall t \geq 0. \qquad (2.6)$$

Further,

$$\det(I + \Delta U_t) \neq 0 \quad \forall t \geq 0,$$

and X can be represented by

$$X_t = -U_t + [U, U]_t^c + \sum_{0 < s \leq t} \left((I + \Delta U_s)^{-1} - I + \Delta U_s \right), \quad t \geq 0. \qquad (2.7)$$

Proof: For (2.5) and (2.6) see Karandikar (1991), Theorem 1. For the remaining assertions, observe that $\Delta U_t = (I + \Delta X_t)^{-1} - I$ from (2.4), so that $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$. Further, from (2.4) we obtain $[U, U]_t^c = [X, X]_t^c$. Inserting this, $\Delta U_t = (I + \Delta X_t)^{-1} - I$, and the form of U_t from (2.4) into the right-hand side of (2.7) gives X_t so that (2.7) is true. □

Multivariate Generalized Ornstein-Uhlenbeck processes

We give a short overview of results regarding multivariate generalized Ornstein-Uhlenbeck (MGOU) processes which are used throughout. MGOU processes were introduced by Behme and Lindner (2012) and further investigated in Behme (2012).

Definition 2.2. Let $(X, Y) = (X_t, Y_t)_{t \geq 0}$ be a Lévy process in $\mathbb{R}^{m \times m} \times \mathbb{R}^m$ such that X satisfies (2.3), and let V_0 be a random variable in $\mathbb{R}^m$. Then the $\mathbb{R}^m$-valued process $V = (V_t)_{t \geq 0}$ given by

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} \, dY_s \right), \quad t \geq 0,$$

is called a *multivariate generalized Ornstein-Uhlenbeck (MGOU) process* driven by (X, Y). The underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is such that it satisfies the usual hypotheses and such that (X, Y) is a semimartingale.

The MGOU process will be called *causal* or *non-anticipative*, if V_0 is independent of (X, Y), and *strictly non-causal* if V_t is independent of $(X_s, Y_s)_{0 \leq s < t}$ for all $t \geq 0$.

Remark 2.3. It follows from Behme and Lindner (2012), Theorem 3.4, that an MGOU process $V = (V_t)_{t \geq 0}$ with an $\mathcal{F}_0$-measurable V_0 solves the SDE

$$dV_t = dU_t V_{t-} + dZ_t, \quad t \geq 0, \qquad (2.8)$$

where $U = (U_t)_{t\geq 0}$ is another $\mathbb{R}^{m\times m}$-valued Lévy process defined by (2.4) so that $\overleftarrow{\mathcal{E}}(X)_t^{-1} = \overrightarrow{\mathcal{E}}(U)_t$, and $Z = (Z_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^m$ given by

$$Z_t = Y_t + \sum_{0 < s \leq t} \left((I + \Delta X_s)^{-1} - I\right)\Delta Y_s - [X, Y]_t^c, \quad t \geq 0.$$

With these U and Z the MGOU process can also be written as

$$V_t = \overrightarrow{\mathcal{E}}(U)_t \left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1}\,\mathrm{d}Y_s\right), \quad t \geq 0. \tag{2.9}$$

Conversely, if $(U, Z) = (U_t, Z_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ such that it holds $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, then for every $\mathcal{F}_0$-measurable random vector V_0 the solution to (2.8) is an MGOU process driven by (X, Y), where X is given by (2.7) and $Y_t = Z_t + [X, Z]_t$.

Convention 2.4. Observe that an MGOU process and similarly the process given by (2.9) is well-defined for any starting random vector V_0, regardless if it is $\mathcal{F}_0$-measurable or not. We shall hence speak of (2.9) as a *solution to (2.8)*, regardless if V_0 is $\mathcal{F}_0$-measurable or not. Observe that if V_0 is chosen to be independent of (U, Z) or equivalently (X, Y), then the natural augmented filtration of (U, Z) may be enlarged by $\sigma(V_0)$ such that (U, Z) still remains a semimartingale, see Protter (2005) Theorem VI.2. With this enlarged filtration, V_0 is measurable.

To investigate the strict stationarity property of RC-CARMA processes later, we introduce the property of irreducibility of a class of MGOU processes as it has been done in Section 4 of Behme and Lindner (2012).

Definition 2.5. Suppose that $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ such that X satisfies (2.3). Then an affine subspace H of $\mathbb{R}^m$ is called *invariant* for the class of MGOU processes $V = (V_t)_{t\geq 0}$ driven by (X, Y) if for all $x \in H$ the choice $V_0 = x$ implies $V_t \in H$ a.s. for all $t \geq 0$.

If there exists no proper affine subspace H such that, for all $x \in H$, $V_0 = x$ implies $V_t \in H$ a.s. for all $t \geq 0$, we call the class of MGOU processes *irreducible*.

Irreducibility is thus a property of the considered model. By abuse of language, we will call an MGOU process irreducible if the corresponding class satisfies Definition 2.5.

3. The RC-CARMA process

Let $p \in \mathbb{N}$ and $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\Pi_{M^{(1)}}(\{1\}) = 0$. Let $b_0, \ldots, b_{p-1} \in \mathbb{R}$. Let $U = (U_t)_{t\geq 0}$ be $\mathbb{R}^{p\times p}$-valued defined by

$$U_t := \begin{bmatrix} 0 & t & 0 & \ldots & 0 \\ 0 & 0 & t & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & t \\ -M_t^{(p)} & -M_t^{(p-1)} & -M_t^{(p-2)} & \ldots & -M_t^{(1)} \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}, \tag{3.1}$$

$\mathbf{e} := [0, \ldots, 0, 1]'$ the pth unit vector, and $q := \max\{j \in \{0, \ldots, p-1\}\colon b_j \neq 0\}$.

Then we call any process $R = (R_t)_{t \geq 0}$ which satisfies

$$R_t = \mathbf{b}'V_t, \quad t \geq 0, \tag{3.2}$$

where $V = (V_t)_{t \geq 0}$ is a solution to the SDE

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathbf{e}\,\mathrm{d}L_t, \quad t \geq 0, \tag{3.3}$$

an *RC-CARMA(p,q) process*, i.e. a CARMA process with random Lévy coefficients. We speak of C and $\mathbf{b}$ as the *parameters* of the RC-CARMA process.

As will be seen in Proposition 3.4 below, the assumption $\Pi_{M^{(1)}}(\{1\}) = 0$ implies $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, so that V is an MGOU process as in (2.8) and, as in Convention 2.4, by a solution of (3.3) we mean a process of the form (2.9) with starting random variable V_0 not necessarily $\mathcal{F}_0$-measurable. We shall call the process V a *state vector process* of the RC-CARMA process R.

Observe that we get a classical CARMA(p,q) process $(S_t)_{t \geq 0} = (\mathbf{b}'V_t)_{t \geq 0}$, although on the positive real line, by choosing $(M_t^{(1)}, \ldots, M_t^{(p)}) = (a_1, \ldots, a_p)t$ with $a_1, \ldots, a_p \in \mathbb{R}$. Further, we recognize that there is less sense in choosing the coefficients of the moving average side to be random since they are just defining the weights of the components of V to form S.

Recall that a CARMA(p,q) process $S = (S_t)_{t \in \mathbb{R}}$ satisfies the formal p^{th}-order linear differential equation

$$a(D)S_t = b(D)DL_t, \tag{3.4}$$

where $a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$, $b(z) = b_0 + b_1 z + \cdots + b_{p-1}z^{p-1}$, and D denotes the differentiation with respect to t. When we consider an RC-CARMA(p,0) process $(R_t)_{t \geq 0} = (b_0 V_t^1)_{t \geq 0}$ for $(V_t)_{t \geq 0} = (V_t^1, \ldots, V_t^p)_{t \geq 0}$ solving (3.3), we formally find that

$$a_M(D)R_t = a_M(D)\mathbf{b}'V_t = a_M(D)b_0 V_t^1 = b_0 DL_t = b(D)DL_t, \tag{3.5}$$

where

$$a_M(z) = z^p + \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t}z^{p-1} + \cdots + \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t}z^0.$$

Observe that $\frac{\mathrm{d}M_t^{(i)}}{\mathrm{d}t}$ is not defined in a rigorous way but just an intuitive way of writing.

Thus, it is possible to interpret also the RC-CARMA(p,0) process as a solution to a formal p^{th}-order linear differential equation with random coefficients.

To justify (3.5), look at the first $p-1$ components of V

$$\mathrm{d}V_t^i = V_t^{i+1}\,\mathrm{d}t \;\Leftrightarrow\; \frac{\mathrm{d}V_t^i}{\mathrm{d}t} = V_t^{i+1} \;\Leftrightarrow\; D^i V_t^1 = V_t^{i+1}, \quad i = 1, \ldots, p-1. \tag{3.6}$$

Formal division by $\mathrm{d}t$ yields for the pth component

$$\mathrm{d}V_t^p = -V_t^1\,\mathrm{d}M_t^{(p)} - \cdots - V_t^p\,\mathrm{d}M_t^{(1)} + \mathrm{d}L_t$$

$$\text{``}\Leftrightarrow\text{''}\quad \frac{\mathrm{d}V_t^p}{\mathrm{d}t} = -V_t^1\frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} - \cdots - V_t^p\frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} + \frac{\mathrm{d}L_t}{\mathrm{d}t}$$

$$\overset{(3.6)}{\Leftrightarrow}\quad DV_t^p = -V_t^1\frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t} - \cdots - D^{p-1}V_t^1\frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t} + DL_t$$

$$\Leftrightarrow\quad DL_t = D^pV_t^1 + \frac{\mathrm{d}M_t^{(1)}}{\mathrm{d}t}D^{p-1}V_t^1 + \cdots + \frac{\mathrm{d}M_t^{(p)}}{\mathrm{d}t}V_t^1$$

$$\Leftrightarrow\quad DL_t = a_M(D)V_t^1.$$

Brockwell and Lindner (2015) gave a rigorous interpretation of (3.4) by showing that a CARMA(p,q) process $(S_t = \mathbf{b}'\mathbf{X}_t)_{t\in\mathbb{R}}$ driven by a Lévy process L satisfies the integral equation

$$a(D)J^pS_t = b(D)J^{p-1}(L_t) + a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0), \quad t \in \mathbb{R},$$

where $a(z)$ and $b(z)$ are as before, and J denotes the integration operator which associates with any càdlàg function $f = (f_t)_{t\in\mathbb{R}}\colon \mathbb{R} \to \mathbb{C}, t \mapsto f_t$, the function $J(f)$ defined by

$$J(f)_t := \int_0^t f_s\,\mathrm{d}s.$$

Similarly, we give a rigorous interpretation of (3.5) as an integral-differential equation and show that the RC-CARMA($p, 0$) process solves this equation, hence making the formal deviation of (3.5) above thoroughly.

We call a function $g\colon [0, \infty) \to \mathbb{R}$ *differentiable with càdlàg derivative Dg*, if g is continuous and there exists a càdlàg function Dg such that g is at every point $t \in \mathbb{R}$ right- and left-differentiable with right-derivative Dg_t and left-derivative $Dg_{t-} = \lim_{\varepsilon\downarrow 0, \varepsilon\neq 0} Dg_{t-\varepsilon}$, respectively. In other words, g is absolutely continuous and has càdlàg density Dg (see the discussion in Brockwell and Lindner, 2015 at the beginning of Section 2).

We call a function $g\colon [0, \infty) \to \mathbb{R}$ *p-times continuously differentiable with càdlàg derivative D^pg*, if g is $p-1$-times differentiable in the usual sense and the $(p-1)^{st}$ derivative $D^{(p-1)}g$ is differentiable with càdlàg derivative $D^pg = D(D^{p-1}g)$ as defined above.

Theorem 3.1. *Let $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ and an $\mathbb{F}$-semimartingale with $\Pi_{M^{(1)}}(\{1\}) = 0$. Let $\mathbf{b}' = [b_0, \ldots, b_{p-1}] \in \mathbb{R}^p$ with $b_0 \neq 0$ and $b_1 = \cdots = b_{p-1} = 0$, and consider the RC-CARMA($p, 0$) process $R = (R_t)_{t\geq 0}$ defined by (3.2) and (3.3), where $V = (V_t)_{t\geq 0}$ is the state vector process (with V_0 not necessarily $\mathcal{F}_0$-measurable). Denote by $\mathcal{D}^{p-1}$ the set of all $\mathbb{F}$-adapted, $\mathbb{R}$-valued processes $G = (G_t)_{t\geq 0}$ which are $p-1$ times differentiable with càdlàg derivative $D^{p-1}G$.*
(a) Define $W = (W_t)_{t\geq 0}$ by

$$W_t := R_t - \mathbf{b}'\overrightarrow{\mathcal{E}}(U)_t\,V_0, \quad t \geq 0.$$

Then $W \in \mathcal{D}^{p-1}$, it is an RC-CARMA process with parameters C, $\mathbf{b}$, and initial state vector $\mathbf{0}$, and it satisfies the integral-differential equation

$$D^{p-1}W_t + \left(\sum_{i=1}^{p} \int_{(0,t]} D^{i-1}W_{s-} \, \mathrm{d}M_s^{(p-i+1)} \right) = b_0 L_t, \quad t \geq 0. \qquad (3.7)$$

If V_0 is additionally $\mathcal{F}_0$-measurable, then also $R \in \mathcal{D}^{p-1}$ and there exists an $\mathcal{F}_0$-measurable random variable Z_0 such that

$$D^{p-1}R_t + \left(\sum_{i=1}^{p} \int_{(0,t]} D^{i-1}R_{s-} \, \mathrm{d}M_s^{(p-i+1)} \right) = b_0 L_t + Z_0, \quad t \geq 0. \qquad (3.8)$$

(b) Conversely, if $\widetilde{R} = (\widetilde{R}_t)_{t \geq 0} \in \mathcal{D}^{p-1}$ satisfies

$$D^{p-1}\widetilde{R}_t + \left(\sum_{i=1}^{p} \int_{(0,t]} D^{i-1}\widetilde{R}_{s-} \, \mathrm{d}M_s^{(p-i+1)} \right) = b_0 L_t + Z_0 \qquad (3.9)$$

for some $\mathcal{F}_0$-measurable Z_0, then $\widetilde{R}$ is an RC-CARMA process with parameters C, $\mathbf{b}$, and state vector process $\widetilde{V} = (\widetilde{V}_t)_{t \geq 0} := (b_0^{-1}(\widetilde{R}_t, D\widetilde{R}_t, \ldots, D^{p-1}\widetilde{R}_t))_{t \geq 0}$. Especially, $\widetilde{V}_0$ is $\mathcal{F}_0$-measurable.

Proof: (a) As already observed (and to be shown in Proposition 3.4 (a) below), the condition $\Pi_{M^{(1)}}(\{1\}) = 0$ implies that V is an MGOU process. So

$$V_t = \overrightarrow{\mathcal{E}}(U)_t \left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} \, \mathrm{d}Y_s \right)$$

for some Lévy process Y as specified in Remark 2.3. Hence, $(V_t - \overrightarrow{\mathcal{E}}(U)_t V_0)_{t \geq 0}$ is adapted and consequently so is $(R_t - \mathbf{b}'\overrightarrow{\mathcal{E}}(U)_t V_0)_{t \geq 0}$, and it is obviously an RC-CARMA process with initial state vector $\mathbf{0}$. Hence, for the proof of (3.7) it suffices to assume that $V_0 = \mathbf{0}$.

Denote $V = (V_t)_{t \geq 0} = (V_t^1, \ldots, V_t^p)_{t \geq 0}$. By (3.6), we have $D^i V_t^1 = V_t^{i+1}$, $i = 1, \ldots, p-1$. Hence, V_t^1 is $p-1$-times differentiable with $(p-1)^{st}$ càdlàg derivative V_t^p. By the defining SDE of the RC-CARMA process (3.3) and the form of the matrix $U = (U_t)_{t \geq 0}$ given in (3.1), we also have

$$V_t^p = V_0^p - \sum_{i=1}^{p} \int_{(0,t]} V_{s-}^i \, \mathrm{d}M_s^{(p-i+1)} + L_t, \qquad (3.10)$$

and, since $V_0 = 0$, that

$$L_t = D^{p-1}V_t^1 + \sum_{i=1}^{p} \int_{(0,t]} D^{i-1}V_{s-}^1 \, \mathrm{d}M_s^{(p-i+1)}. \qquad (3.11)$$

Multiplying (3.11) by b_0 gives (3.7) when $V_0 = \mathbf{0}$ and hence (3.7) in general.

For (3.8) let

$$K_t = (K_t^1, \ldots, K_t^p)' := \overrightarrow{\mathcal{E}}(U)_t V_0.$$

When we consider the SDE which defines the right stochastic exponential $\mathrm{d}\overrightarrow{\mathcal{E}}(U)_t = \mathrm{d}U_t \overrightarrow{\mathcal{E}}(U)_{t-}$, we obtain

$$\mathrm{d}K_t = \mathrm{d}U_t K_{t-} \quad \text{with} \quad K_0 = V_0, \qquad (3.12)$$

and, due to the form of the process U,

$$\mathrm{d}K_t^i = K_t^{i+1}\,\mathrm{d}t \quad \Leftrightarrow \quad \frac{\mathrm{d}K_t^i}{\mathrm{d}t} = K_t^{i+1} \quad \Leftrightarrow \quad D^i K_t^1 = K_t^{i+1}, \quad i = 1,\ldots,p-1.$$

Further, from the last component of (3.12)

$$K_t^p = D^{p-1}K_t^1 = V_0^p - \int_{(0,t]} K_{s-}^1\,\mathrm{d}M_s^p - \cdots - \int_{(0,t]} K_{s-}^p\,\mathrm{d}M_s^1$$

$$= V_0^p - \sum_{i=1}^{p} \int_{(0,t]} D^{i-1}K_{s-}^1\,\mathrm{d}M_s^{(p-i+1)}.$$

Hence,

$$D^{p-1}K_t^1 + \sum_{i=1}^{p} \int_{(0,t]} D^{i-1}K_{s-}^1\,\mathrm{d}M_s^{(p-i+1)} = V_0^p,$$

where V_0^p is $\mathcal{F}_0$-measurable such that we obtain (3.8) via $R_t = W_t + \mathbf{b}'K_t$, where W_t satisfies (3.7).

(b) For the converse, let $\widetilde{R} \in \mathcal{D}^{p-1}$ satisfy (3.9), and denote $\widetilde{V}_t = (\widetilde{V}_t^1,\ldots,\widetilde{V}_t^p) = (b_0^{-1}(\widetilde{R}_t, D\widetilde{R}_t,\ldots,D^{p-1}\widetilde{R}_t))$, $t \geq 0$. By the fundamental theorem of calculus

$$\widetilde{V}_t^i = \widetilde{V}_0^i + \int_{(0,t]} \widetilde{V}_s^{i+1}\mathrm{d}s, \quad i = 1,\ldots,p-1, \tag{3.13}$$

and from (3.9) we obtain

$$\widetilde{V}_t^p = b_0^{-1}Z_0 - \sum_{i=1}^{p} \int_{(0,t]} D^{i-1}\widetilde{V}_{s-}^1\mathrm{d}M_s^{(p-i+1)} + L_t$$

$$= b_0^{-1}Z_0 - \sum_{i=1}^{p} \int_{(0,t]} \widetilde{V}_{s-}^i\mathrm{d}M_s^{(p-i+1)} + L_t, \quad t \geq 0. \tag{3.14}$$

But (3.13) and (3.14) mean that $\widetilde{V}_0^p = b_0^{-1}Z_0$ and that $\widetilde{V} = (\widetilde{V}_t)_{t \geq 0}$ satisfies

$$\widetilde{V}_t = \widetilde{V}_0 + \int_{(0,t]} \mathrm{d}U_s\widetilde{V}_{s-} + \mathbf{e}L_t.$$

Since obviously $\mathbf{b}'\widetilde{V}_t = b_0 b_0^{-1}\widetilde{R}_t = \widetilde{R}_t$, it follows that $\widetilde{R}$ is an RC-CARMA$(p,0)$ process with parameters C, $\mathbf{b}$, and state vector process $\widetilde{V}$. $\qquad\square$

Remark 3.2. Differentiating (3.8) formally gives

$$D^p R_t + \sum_{i=1}^{p} D^{i-1}R_{s-}DM_s^{(p-i+1)} = b_0 DL_t,$$

hence (3.5), and the RC-CARMA$(p,0)$ process can be interpreted as a solution to a formal p^{th}-order linear differential equation with random coefficients. To obtain a similar equation and hence interpretation for RC-CARMA(p,q) processes with $q > 0$ seems not so easy since it is in general not possible to interchange the stochastic integration with the differentiation operator D.

Remark 3.3. Similar as in case of CARMA(p,q) processes, we easily see for an RC-CARMA(p,q) process $R = (R_t)_{t \geq 0}$ with $q < p$, $b_q \neq 0$, and $b_j = 0$ for $j > q$ with

$$R_t = \mathbf{b}'V_t = b_0 V_t^1 + \cdots + b_q V_t^{q+1}$$

that, by (3.6), R is $(p-q-1)$-times differentiable with $(p-q-1)^{st}$ càdlàg derivative

$$D^{p-q-1}R_t = b_0 D^{p-q-1}V_t^1 + \cdots + b_q D^{p-q-1}V_t^{q+1} = b_0 V_t^{p-q} + \cdots + b_q V_t^p.$$

The following proposition shows that the state vector process $V = (V_t)_{t\geq 0}$ is an MGOU process and gives its specific driving Lévy process (X, Y). Further, V is an irreducible MGOU process if we assume that U is independent of L and L is not deterministic.

Proposition 3.4. *Let* $C = (C_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq 0}$ *be an* $\mathbb{R}^{p+1}$*-valued Lévy process and an* $\mathbb{F}$*-semimartingale. Let* $\Pi_{M^{(1)}}$ *denote the Lévy measure of* $M^{(1)}$ *and let* U *be defined as in (3.1).*
(a) Then $\overleftarrow{\mathcal{E}}(U)_t \in \mathrm{GL}(\mathbb{R}, p)$ *for all* $t \geq 0$ *if and only if* $\Pi_{M^{(1)}}(\{1\}) = 0$*. In this case, for any starting random vector* V_0 *the solution to*

$$\mathrm{d}V_t = \mathrm{d}U_t V_{t-} + \mathbf{e}\,\mathrm{d}L_t$$

is an MGOU process driven by (X, Y)*, i.e.* $V = (V_t)_{t\geq 0}$ *takes the form*

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}\,\mathrm{d}Y_s\right) = \overrightarrow{\mathcal{E}}(U)_t\left(V_0 + \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1}\,\mathrm{d}Y_s\right)$$

for $t \geq 0$*. Here,* $(X, Y) = (X_t, Y_t)_{t\geq 0}$ *is a Lévy process defined by*

$$X_t = \begin{bmatrix} 0 & -t & 0 & \ldots & 0 \\ 0 & 0 & -t & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -t \\ N_t^{(p)} & N_t^{(p-1)} & N_t^{(p-2)} & \ldots & N_t^{(1)} \end{bmatrix}$$

with

$$N_t^{(i)} = M_t^{(i)} + t\,\sigma_{M^{(1)},M^{(i)}} + \sum_{0<s\leq t} \frac{\Delta M_s^{(i)}\Delta M_s^{(1)}}{1 - \Delta M_s^{(1)}}, \quad i = 1,\ldots,p,$$

where $\sigma_{M^{(1)},M^{(i)}}$ *denotes the Gaussian covariance of* $M^{(1)}$ *and* $M^{(i)}$*. X satisfies* $\det(I + \Delta X_t) \neq 0$ *for all* $t \geq 0$*, and* $Y_t = \mathbf{e}\widetilde{Y}_t$ *with*

$$\widetilde{Y}_t = L_t + t\sigma_{M^{(1)},L} + \sum_{0<s\leq t} \frac{\Delta M_s^{(1)}\Delta L_s}{1 - \Delta M_s^{(1)}}.$$

(b) Denote $M = (M_t)_{t\geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)})_{t\geq 0}$*. Assume that* M *is independent of* L *and* L *not deterministic. Then the MGOU process* V *obtained in (a) is irreducible.*

Proof: (a) Since C is a Lévy process and a semimartingale with respect to $\mathbb{F}$, it is clear that also U as defined in (3.1) is a semimartingale and therefore also $\overleftarrow{\mathcal{E}}(U)$ is a semimartingale with respect to $\mathbb{F}$. Since $\overleftarrow{\mathcal{E}}(U)_t$ is non-singular for all $t \geq 0$ if

and only if $\det(I + \Delta U_t) \neq 0$ for all $t \geq 0$, we calculate

$$\det(I + \Delta U_t) = \det \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ -\Delta M_t^{(p)} & -\Delta M_t^{(p-1)} & \ldots & -\Delta M_t^{(2)} & 1 - \Delta M_t^{(1)} \end{pmatrix}$$

$$= (1 - \Delta M_t^{(1)}) \tag{3.15}$$

which shows that $\overleftarrow{\mathcal{E}}(U)_t$ is non-singular if and only if $\Delta M_t^{(1)} \neq 1$ for all $t \geq 0$. Since $M^{(1)}$ is a Lévy process, the latter is equivalent to $\Pi_{M^{(1)}}(\{1\}) = 0$.

By Remark 2.3, V is an MGOU process driven by (X, Y), where X is given by (2.7) and satisfies $\det(I + \Delta X_t) \neq 0$ for all $t \geq 0$, and $Y_t = \mathbf{e}L_t + [X, \mathbf{e}L]_t$.

From (2.1) we obtain for the components of $[U, U]^c$ due to the form of U in (3.1)

$$([U, U]_t^c)^{(i,j)} = \sum_{k=1}^{p} [U^{(i,k)}, U^{(k,j)}]^c = \begin{cases} 0, & i = 1, \ldots, p-1, \\ t\sigma_{M^{(1)}, M^{(p-j+1)}}, & i = p. \end{cases}$$

The form of $I + \Delta U_t$ is implicitly given in (3.15) such that

$$(I + \Delta U_t)^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \frac{\Delta M_t^{(p)}}{1 - \Delta M_t^{(1)}} & \frac{\Delta M_t^{(p-1)}}{1 - \Delta M_t^{(1)}} & \ldots & \frac{\Delta M_t^{(2)}}{1 - \Delta M_t^{(1)}} & \frac{1}{1 - \Delta M_t^{(1)}} \end{pmatrix}$$

which is well-defined since $\Pi_{M^{(1)}}(\{1\}) = 0$. Summing up the terms according to (2.7) leads to the stated form of the processes $(N_t^{(1)}, \ldots, N_t^{(p)})_{t \geq 0}$ and X.

For $Y_t = \mathbf{e}L_t + [X, \mathbf{e}L]_t$ we obtain with (2.1) componentwise

$$[X, \mathbf{e}L]_t^{(i)} = \sum_{k=1}^{p} [X^{(i,k)}, (\mathbf{e}L)^{(k)}]_t = [X^{(i,p)}, L]_t = \begin{cases} 0, & i = 1, \ldots, p-1, \\ [N^{(1)}, L]_t, & i = p. \end{cases}$$

Since

$$\Delta N_t^{(1)} = \frac{\Delta M_t^{(1)}}{1 - \Delta M_t^{(1)}},$$

and $[N^{(1)}, L]_t^c = [M^{(1)}, L]_t^c = t\sigma_{M^{(1)}, L}$ we get the stated form of Y by (2.2).

(b) Suppose that $V = (V_t)_{t \geq 0} = (V_t^1, \ldots, V_t^p)_{t \geq 0}$ is not irreducible. Hence, there exists an invariant affine subspace H with $\dim H \in \{0, \ldots, p-1\}$. Then for all $x \in H$ it holds $P(V_t \in H | V_0 = x) = 1$ for all $t \geq 0$. Since V is càdlàg, we obtain $P(V_t \in H \, \forall t \geq 0 | V_0 = x) = 1$. Further, if $H' \supset H$ with $\dim H' = p - 1$, then

$$P(V_t \in H' \, \forall t \geq 0 \,|\, V_0 = x) = 1 \quad \forall x \in H. \tag{3.16}$$

We shall show that $P(V_t \in H' \, \forall t \geq 0 \,|\, V_0 = x) < 1$ for all $x \in \mathbb{R}^p$, hence contradicting (3.16). So assume w.l.o.g. that $\dim H = p - 1$. Since then H is a hyperplane, there exists $\lambda \in \mathbb{R}^p$ with $\lambda \neq 0$ and $a \in \mathbb{R}$ such that $H = \{y \in \mathbb{R}^p : \lambda' y = a\}$.

Let $V_0 = x \in H$. Let $i_1, \ldots, i_k \in \{1, \ldots, p\}$ with $i_n \neq i_m$ for all $n \neq m$, $\lambda_{i_n} \neq 0$ for all $n \in \{1, \ldots, k\}$ and $\lambda_j = 0$ for $j \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_k\}$. W.l.o.g. assume $i_1 < i_2 < \cdots < i_k$. By the existence of an invariant affine subspace, this yields

$\lambda_{i_1} V_t^{i_1} + \cdots + \lambda_{i_k} V_t^{i_k} = a$. This is, since $D^{i_n - i_1} V_t^{i_1} = V_t^{i_n}$, $n = 1, \ldots, k$, equivalent to

$$\lambda_{i_1} V_t^{i_1} + \lambda_{i_2} D^{i_2 - i_1} V_t^{i_1} + \cdots + \lambda_{i_k} D^{i_k - i_1} V_t^{i_1} = a. \tag{3.17}$$

But (3.17) is an inhomogeneous ordinary linear differential equation of order $i_k - i_1$. Hence, $V_t^{i_1}$ is a smooth deterministic function in t and so are V_t^j for all $j \in \{1, \ldots, p\}$ since $V_t^j = D^{j-1} V_t^1$. When we consider the last component of V, we obtain by (3.10) with $V_0^p = x^p$

$$L_t = V_t^p + \sum_{k=1}^p \int_{(0,t]} V_{s-}^k \mathrm{d} M_s^{(p-k+1)} - x_p. \tag{3.18}$$

But under the assumption that M and L are independent and L is not deterministic, (3.18) cannot hold (observe that V_t is deterministic by (3.17) when $V_0 = x$). This gives the wanted contradiction and therefore that V is irreducible. $\qquad\square$

Remark 3.5. We can write (3.18) using partial integration as

$$L_t = V_t^p + \sum_{k=1}^p V_t^k M_t^{(p-k+1)} - \sum_{k=1}^p \int_{(0,t]} M_{s-}^{(p-k+1)} \mathrm{d} V_s^k - x_p,$$

hence L_t is a functional of M. It would therefore be enough assuming that L is not measurable with respect to the filtration generated by M to ensure irreducibility of the state vector process V.

Remark 3.6. When the components $M^{(1)}, \ldots, M^{(p)}, L$ of the Lévy process C in Proposition 3.4 (a) are additionally independent, then the components do not jump together almost surely and the Gaussian covariances vanish for different components so that the formulas for $N^{(1)}, \ldots, N^{(p)}$ simplify to

$$N_t^{(1)} = M_t^{(1)} + t\,\sigma_{M^{(1)}}^2 + \sum_{0 < s \le t} \frac{(\Delta M_s^{(1)})^2}{1 - \Delta M_s^{(1)}} \quad \text{and} \quad N_t^{(i)} = M_t^{(i)} \text{ a.s. for } i = 2, \ldots, p.$$

Further, $Y_t = \mathbf{e} L_t$ and X are independent in that case (the latter is already true when just L is independent of $(M^{(1)}, \ldots, M^{(p)})$).

Recall that a process is strictly stationary if its finite-dimensional distributions are shift-invariant. As in the case of the MGOU process, we can find sufficient conditions for the existence of a strictly stationary solution of the RC-CARMA SDE (3.3) and therefore a strictly stationary RC-CARMA process. Assume throughout that the used norm $\|\cdot\|$ on $\mathbb{R}^{p \times p}$ is submultiplicative.

Theorem 3.7. *Let $C = (C_t)_{t \ge 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t \ge 0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\Pi_{M^{(1)}}(\{1\}) = 0$ and a semimartingale with respect to the given filtration $\mathbb{F}$. Let $\mathbf{b}' := [b_0, \ldots, b_{p-1}] \in \mathbb{R}^p$, $q := \max\{j \in \{0, \ldots, p-1\}: b_j \ne 0\}$ and $U = (U_t)_{t \ge 0}$ be given as in (3.1). Assume that $\mathbf{E}\left[\log^+ \|U_1\|\right] < \infty$ and $\mathbf{E}\left[\log^+ |L_1|\right] < \infty$.*

(a) Suppose there exists a $t_0 > 0$ such that

$$\mathbf{E}\left[\log\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|\right] < 0. \tag{3.19}$$

Then $\overleftarrow{\mathcal{E}}(U)_t$ *converges a.s. to* 0 *as* $t \to \infty$*, and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} \mathbf{e} \, dL_s$
converges a.s. to a finite random vector as $t \to \infty$*, denoted by* $\int_0^\infty \overleftarrow{\mathcal{E}}(U)_{s-} \mathbf{e} \, dL_s$*.*
Further, (3.3) admits a strictly stationary solution which is causal and unique in
distribution, and is achieved by choosing V_0 *to be independent of* C *and such that*
$V_0 \overset{d}{=} \int_0^\infty \overleftarrow{\mathcal{E}}(U)_{s-} \mathbf{e} \, dL_s$*.*

(b) Conversely, if V_0 *can be chosen independent of* C *such that* V *is strictly*
stationary, $M = (M_t^{(1)}, \ldots, M_t^{(p)})_{t \geq 0}$ *is independent of* L *and* L *not deterministic,*
then there exists a $t_0 > 0$ *such that (3.19) holds.*

In both cases with this choice of V_0*, the RC-CARMA process given by* $R_t = \mathbf{b}'V_t$*,*
$t \geq 0$*, is strictly stationary, too.*

Proof: (a) The assertions regarding V follow from Theorem 5.4 with Remark 5.5
(b) and Theorem 5.2 (a) in Behme and Lindner (2012).
(b) Under the assumptions made, V is irreducible by Proposition 3.4 (b). Therefore
Theorem 5.4 of Behme and Lindner (2012) applies.

That R is strictly stationary if V is, is obvious. $\square$

4. Existence of moments and second order properties

In this section, we calculate the autocovariance function (ACVF) of an RC-
CARMA process and give a connection to the autocovariance function of a specific
CARMA process obtained by choosing $A = \mathbf{E}[U_1]$. Further, we give sufficient
conditions for the existence of the ACVF and the spectral density. We end this
section with an exemplary investigation of the RC-CARMA$(2,1)$ process. But we
start with a result which guarantees the existence of higher moments. Assume that
the used norm $\|\cdot\|$ on $\mathbb{R}^{p \times p}$ is submultiplicative.

Proposition 4.1. *Let* $R = (R_t)_{t \geq 0}$ *be an RC-CARMA(p,q) process with parame-*
ters $C = (C_t)_{t \geq 0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t \geq 0}$*,* $\mathbf{b}$ *and strictly stationary state vector*
process $V = (V_t)_{t \geq 0}$ *with* V_0 *independent of* C *and* C *a semimartingale with respect*
to the given filtration $\mathbb{F}$*. Assume that for* $\kappa > 0$ *we have for some* $t_0 > 0$

$$\mathbf{E}\|C_1\|^{\max\{\kappa,1\}} < \infty \quad and \quad \mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|^\kappa < 1. \tag{4.1}$$

Then $\mathbf{E}|R_0|^\kappa < \infty$*, and if (4.1) holds for* $\kappa = 1$*,*

$$\mathbf{E}[R_0] = b_0 \frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}.$$

Remark 4.2. The assumption (4.1) in the previous proposition actually already
implies the existence of a strictly stationary state vector process $V = (V_t)_{t \geq 0}$, that is
unique in distribution, since $\mathbf{E}\|C_1\|^{\max\{\kappa,1\}} < \infty$ obviously implies $\mathbf{E}[\log^+ \|U_1\|] <$
∞ and $\mathbf{E}[\log^+ |L_1|] < \infty$, and by Jensen's inequality and (4.1) we further have

$$\kappa \mathbf{E}\left[\log \left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|\right] \leq \log \mathbf{E}\left\|\overleftarrow{\mathcal{E}}(U)_{t_0}\right\|^\kappa < 0.$$

Then Theorem 3.7 applies.

Proof of Proposition 4.1: By Remark 4.2 the strictly stationary state vector process
V is unique in distribution. By Proposition 3.3 in Behme (2012) we then have
$\mathbf{E}\|V_0\|^\kappa < \infty$ and hence $\mathbf{E}|R_0|^\kappa < \infty$.

Now let $\kappa = 1$. Again from Behme (2012), Proposition 3.3, we know that $\mathbf{E}[U_1]$ is invertible and

$$\mathbf{E}[V_0] = -\mathbf{E}[U_1]^{-1}\mathbf{e}\,\mathbf{E}[L_1].$$

Observe that $\mathbf{E}[U_1]$ is a companion matrix and it is well-known that the inverse of this is of the form

$$\mathbf{E}[U_1]^{-1} = \begin{bmatrix} -\dfrac{\mathbf{E}[M_1^{(p-1)}]}{\mathbf{E}[M_1^{(p)}]} & -\dfrac{\mathbf{E}[M_1^{(p-2)}]}{\mathbf{E}[M_1^{(p)}]} & \cdots & -\dfrac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(p)}]} & -\dfrac{1}{\mathbf{E}[M_1^{(p)}]} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{4.2}$$

such that

$$\mathbf{E}[V_0] = \mathbf{e}_1 \frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]},$$

where $\mathbf{e}_1$ denotes the first unit vector in $\mathbb{R}^p$, and

$$\mathbf{E}[R_0] = \mathbf{E}[\mathbf{b}'V_0] = \mathbf{b}'\mathbf{e}_1 \frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]} = b_0 \frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}.$$

$\square$

The following proposition gives sufficient conditions for the existence of the autocovariance function of an RC-CARMA process and states its form. We denote with $\otimes$ the Kronecker product and by vec the vectorizing operator which maps a matrix H form $\mathbb{R}^{p\times m}$ into $\mathbb{R}^{pm}$ stacking its columns one under another. $\mathrm{vec}^{-1}(H)$ means the inverse operation and yields H.

Proposition 4.3. *Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA(p,q) process with parameters $C = (C_t)_{t\geq 0}$, $\mathbf{b}$ and state vector process $V = (V_t)_{t\geq 0}$ with V_0 independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Suppose it holds $\mathbf{E}\,\|C_1\|^2 < \infty$ and $\mathbf{E}\,\|V_s\|^2 < \infty$, then, for $t \geq 0$, we have*

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\,\mathbf{Cov}(V_t)\mathbf{b} \quad \forall\, h \geq 0,$$

where $\mathbf{Cov}(V_t) = \mathbf{E}[V_t V_t'] - \mathbf{E}[V_t]\mathbf{E}[V_t']$ denotes the covariance matrix of V_t.

In particular, if V is strictly stationary, (4.1) holds for $\kappa = 2$ and we denote

$$D = \mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1] + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1], \tag{4.3}$$

then all eigenvalues of D have strictly negative real parts and the matrix

$$F = \int_0^\infty \int_0^s \mathrm{e}^{uD}(\mathrm{e}^{(s-u)(\mathbf{E}[U_1]\otimes I)} + \mathrm{e}^{(s-u)(I\otimes\mathbf{E}[U_1])})\,\mathrm{d}u\,\mathrm{d}s$$

is finite. Further, if $\mathbf{E}[L_1] = 0$, we obtain

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\,\mathrm{vec}^{-1}(-D^{-1}\mathbf{e}_{p^2})\,\mathbf{E}(L_1^2)\mathbf{b},$$

where $\mathbf{e}_{p^2}$ denotes the $(p^2)^{th}$-unit vector in $\mathbb{R}^{p^2}$, and if $M = (M_t^{(1)}, \ldots, M_t^{(p)})_{t\geq 0}$ and L are independent, we obtain

$$\mathbf{Cov}(R_{t+h}, R_t)$$

$$= \mathbf{b}'\mathrm{e}^{h\mathbf{E}[U_1]}\left(\mathrm{vec}^{-1}\left(-D^{-1}\mathbf{e}_{p^2}\mathbf{Var}(L_1) + F\mathbf{e}_{p^2}(\mathbf{E}[L_1])^2\right) - \mathbf{e}_1\mathbf{e}_1'\left(\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\right)^2\right)\mathbf{b}.$$

Proof: This follows immediately from Proposition 3.4 and the subsequent remark in Behme (2012), by observing that

$$\text{vec}^{-1}((\mathbf{E}[U_1] \otimes \mathbf{E}[U_1])^{-1}\text{vec}(\mathbf{E}[\mathbf{e}L_1]\mathbf{E}[\mathbf{e}L_1]')) = \mathbf{E}[U_1]^{-1}\mathbf{E}[\mathbf{e}L_1]\mathbf{E}[\mathbf{e}L_1]'(\mathbf{E}[U_1]^{-1})'$$
$$= \mathbf{e}_1\mathbf{e}_1'\left(\frac{\mathbf{E}[L_1]}{\mathbf{E}[M_1^{(p)}]}\right)^2$$

is obtained by (4.2), and the properties of the vectorizing and the Kronecker product operations. $\square$

The following is Remark 3.5 (a) in Behme (2012) for our purposes.

Remark 4.4. Let $C = (C_t)_{t\geq0} = (M_t^{(1)}, \ldots, M_t^{(p)}, L_t)_{t\geq0}$ be a Lévy process in $\mathbb{R}^{p+1}$ with $\mathbf{E}\|C_1\|^2 < \infty$ satisfying $\Pi_{M^{(1)}}(\{1\}) = 0$ and $U = (U_t)_{t\geq0}$ a Lévy process in $\mathbb{R}^{p\times p}$ of the form (3.1). Let D be as in (4.3). Then

$$\mathbf{E}\left\|\overset{\leftarrow}{\mathcal{E}}(U)_{t_0}\right\|^2 < 1 \quad \text{for some } t_0 > 0$$

if and only if

all eigenvalues of D have strictly negative real parts.

Therefore, that condition (4.1) holds for $\kappa = 2$ can be replaced by

$$\mathbf{E}\|C_1\|^2 < \infty \text{ and all eigenvalues of } D \text{ have strictly negative real parts.} \quad (4.4)$$

Let $R = (R_t)_{t\geq0}$ be an RC-CARMA process with parameters $\mathbf{b}$ and $C = (M^{(1)}, \ldots, M^{(p)}, L)$. If $\mathbf{E}|M^{(1)}|, \ldots, \mathbf{E}|M^{(p)}| < \infty$, we can associate to R and each vector $\mathbf{X}_0$ a CARMA process $S = (S_t)_{t\geq0}$, given by (1.1) and (1.2) with $A = \mathbf{E}[U_1]$.

Each of these processes, i.e. when $\mathbf{X}_0$ varies over all random variables, will be called a *CARMA process associated with the given RC-CARMA process* with state vector process $(\mathbf{X}_t)_{t\geq0}$. It is then interesting to compare the autocovariance function of R with that of S provided both are strictly stationary with finite variance.

We denote with $\oplus$ the Kronecker sum, i.e. $A \oplus A = A \otimes I + I \otimes A$.

Theorem 4.5. *Let $R = (R_t)_{t\geq0}$ be an RC-CARMA(p,q) process with parameters $C = (C_t)_{t\geq0}$, $\mathbf{b}$ and strictly stationary state vector process $V = (V_t)_{t\geq0}$ with V_0 independent of C, and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that (4.1) holds for $\kappa = 2$, that $\mathbf{E}[L_1] = 0$ and denote $\widetilde{D} := \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$. Then $\mathbf{E}[U_1]$ and $\widetilde{D}$ have only eigenvalues with strictly negative real parts. Further, $\mathbf{X}_0$ can be chosen independent of C and unique in distribution such that the state vector process $(\mathbf{X}_t)_{t\geq0}$ of the associated CARMA process $S = (S_t)_{t\geq0}$ becomes strictly stationary with finite variance. Its autocovariance function can be expressed for all $t \geq 0$ as*

$$\mathbf{Cov}(S_{t+h}, S_t) = \mathbf{b}'e^{h\mathbf{E}[U_1]}\text{vec}^{-1}(-\widetilde{D}^{-1}\mathbf{e}_{p^2})\mathbf{E}(L_1^2)\mathbf{b} \quad \forall h \geq 0. \quad (4.5)$$

Then the autocovariance function of S and R differ only by a multiplicative constant. More precisely,

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{Cov}(S_{t+h}, S_t) \cdot \varrho_{RC} \quad \forall t, h \geq 0, \quad (4.6)$$

where

$$\varrho_{RC} = 1 + \mathbf{e}_{p^2}'B\widetilde{D}^{-1}\mathbf{e}_{p^2} \quad (4.7)$$

with $B = \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1]$. *Furthermore, if* $\mathbf{Var}(R_0) > 0$, *the autocorrelation functions of both* R *and* S *agree, i.e.*

$$\mathbf{Corr}[S_{t+h}, S_t] = \mathbf{Corr}[R_{t+h}, R_t] \quad \forall\, t, h \geq 0. \tag{4.8}$$

Proof: That $\mathbf{E}[U_1]$ has only eigenvalues with strictly negative real parts follows since (4.1), by Jensen's inequality, implies

$$\left\| \mathbf{E}[\overleftarrow{\mathcal{E}}\,(U)_{t_0}] \right\| \leq \mathbf{E} \left\| \overleftarrow{\mathcal{E}}\,(U)_{t_0} \right\| \leq \left(\mathbf{E} \left\| \overleftarrow{\mathcal{E}}\,(U)_{t_0} \right\|^2 \right)^{1/2} < 1.$$

Hence, since $\mathbf{E}[\overrightarrow{\mathcal{E}}\,(U)_{t_0}] = \mathrm{e}^{t_0 \mathbf{E}[U_1]}$ by Proposition 3.1 in Behme (2012) we obtain by the submultiplicativity of the norm

$$\left\| \mathrm{e}^{n t_0 \mathbf{E}[U_1]} \right\| \leq \left\| \mathrm{e}^{t_0 \mathbf{E}[U_1]} \right\|^n \leq \left(\mathbf{E} \left\| \overrightarrow{\mathcal{E}}\,(U)_{t_0} \right\|^2 \right)^{n/2} \to 0, \quad n \to \infty,$$

so that all eigenvalues of $\mathbf{E}[U_1]$ have strictly negative real parts (e.g. Proposition 11.8.2 in Bernstein, 2009). That then also $\widetilde{D}$ has only eigenvalues with strictly negative real parts follows by Fact 11.17.11 of Bernstein (2009).

That $\mathbf{X}$ admits a strictly stationary solution which is unique in distribution with finite variance and $\mathbf{X}_0$ independent of C under the given conditions, is well-known (e.g. Brockwell, 2001a) or follows alternatively from Remark 4.2. (4.5) follows from Proposition 4.3. (4.8) is clearly true as long as (4.6) holds and $\mathbf{Var}(R_0) \neq 0$.

To show that (4.6) is indeed true, we recognize first that the covariance of the CARMA process S and the RC-CARMA process R differ only in the matrices D as defined in (4.3) and $\widetilde{D}$. So, it is enough to show that

$$x \varrho_{RC} = \widetilde{x}$$

where $x := D^{-1} \mathbf{e}_{p^2}$ and $\widetilde{x} := \widetilde{D}^{-1} \mathbf{e}_{p^2}$ and ϱ_{RC} is defined by (4.7).

Since both D and $\widetilde{D}$, under the assumptions made, are invertible, x and $\widetilde{x}$ are well-defined. Further, we have that

$$D = \widetilde{D} + \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1] =: \widetilde{D} + B,$$

where the matrix $B = (b_{i,j})_{i,j=1,\ldots,p^2}$ does only have values different from zero in the last row. The latter can be seen due to the form of the matrix U_1 by

$$\mathbf{E}[U_1 \otimes U_1] = \begin{bmatrix} \mathbf{0}_p & \mathbf{E}[U_1] & \mathbf{0}_p & \cdots & \mathbf{0}_p \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{E}[U_1] & \cdots & \mathbf{0}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \cdots & \mathbf{E}[U_1] \\ -\mathbf{E}[M_1^{(p)} U_1] & -\mathbf{E}[M_1^{(p-1)} U_1] & -\mathbf{E}[M_1^{(p-2)} U_1] & \cdots & -\mathbf{E}[M_1^{(1)} U_1] \end{bmatrix},$$

and

$$\mathbf{E}[U_1] \otimes \mathbf{E}[U_1] =$$
$$\begin{bmatrix} \mathbf{0}_p & \mathbf{E}[U_1] & \mathbf{0}_p & \cdots & \mathbf{0}_p \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{E}[U_1] & \cdots & \mathbf{0}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \cdots & \mathbf{E}[U_1] \\ -\mathbf{E}[M_1^{(p)}]\mathbf{E}[U_1] & -\mathbf{E}[M_1^{(p-1)}]\mathbf{E}[U_1] & -\mathbf{E}[M_1^{(p-2)}]\mathbf{E}[U_1] & \cdots & -\mathbf{E}[M_1^{(1)}]\mathbf{E}[U_1] \end{bmatrix}.$$

Then

$$\mathbf{e}_{p^2} = \widetilde{D}\widetilde{x} = (D - B)\widetilde{x} = D\widetilde{x} - \sum_{i=1}^{p^2} b_{p^2,i}\widetilde{x}_i\mathbf{e}_{p^2}$$

such that

$$D\widetilde{x} = \left(1 + \sum_{i=1}^{p^2} b_{p^2,i}\widetilde{x}_i\right)\mathbf{e}_{p^2} = (1 + (\mathbf{e}'_{p^2}B)\widetilde{D}^{-1}\mathbf{e}_{p^2})\mathbf{e}_{p^2} = \varrho_{RC}\mathbf{e}_{p^2}.$$

Hence, $\widetilde{x} = \varrho_{RC}D^{-1}\mathbf{e}_{p^2} = \varrho_{RC}x$. $\hspace{2cm}\square$

The following two propositions give handy sufficient conditions for (4.4) and therefore also for the existence of a strictly stationary solution by Remark 4.2.

Proposition 4.6. *Suppose that* $\mathbf{E}\left\|C_1\right\|^2 < \infty$ *and that* $\mathbf{E}[U_1]$ *has only eigenvalues with strictly negative real parts. Denote* $\widetilde{D} := \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$, *and* $B := \mathbf{E}[U_1 \otimes U_1] - \mathbf{E}[U_1] \otimes \mathbf{E}[U_1] = (b_{i,j})_{i,j=1,\ldots,p^2}$ *with* $b_{i,j} = 0$ *for all* $i \neq p^2$ *and all* $j = 1, \ldots, p^2$ *and* $b_{p^2,p(k-1)+j} = \mathbf{Cov}(M_1^{(p-k+1)}, M_1^{(p-j+1)})$ *for* $k, j = 1, \ldots, p$. *Then the minimal singular value* $\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})$, *which is the square root of the minimal eigenvalue of* $(\widetilde{D} \oplus \widetilde{D})(\widetilde{D} \oplus \widetilde{D})'$, *is strictly positive, and if*

$$\sum_{i,j=1}^{p} [\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})]^2 < \frac{1}{4}\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})^2, \tag{4.9}$$

then (4.4) applies.

Proof: Assume that $\mathbf{E}[U_1]$ has only eigenvalues with strictly negative real parts. Then so does $\widetilde{D} = \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$ by Fact 11.17.11 of Bernstein (2009) and hence also $\widetilde{D} \oplus \widetilde{D}$. In particular, $\widetilde{D} \oplus \widetilde{D}$ is invertible so that its minimal singular value is strictly positive. By Fact 11.18.17 of Bernstein (2009), the sum $D = \widetilde{D} + B$ has only eigenvalues with strictly negative real parts if $\|B\|_{\mathrm{F}} < 1/2\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})$, where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Due to the form of B, we see immediately that $\|B\|_{\mathrm{F}}^2 = \sum_{j=1}^{p^2} b_{p^2,j}^2$, hence (4.9). $\hspace{1cm}\square$

Let us denote with $\|A\|_1$ the column sum and with $\|A\|_\infty$ the row sum norm of a matrix A, respectively. Further, $\kappa_1(A) = \|A\|_1\|A^{-1}\|_1$ and $\kappa_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty$ denote the condition number of an invertible A with respect to $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

Proposition 4.7. *Suppose that* $\mathbf{E}\left\|C_1\right\|^2 < \infty$ *and that* $\mathbf{E}[U_1]$ *has only pairwise distinct eigenvalues with strictly negative real parts, which we denote by* $\mu_1, \ldots, \mu_p$. *Let* $\widetilde{D}$ *and* B *be as in Proposition 4.6, denote*

$$S := \begin{bmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_p \\ \vdots & \ddots & \vdots \\ \mu_1^{p-1} & \cdots & \mu_p^{p-1} \end{bmatrix}, \quad \Lambda := \mathrm{diag}(\mu_1, \ldots, \mu_p),$$

and by $\mathrm{spec}(\widetilde{D})$ *the set of all eigenvalues of* $\widetilde{D}$. *Then* $\mathbf{E}[U_1]$ *and* $\widetilde{D}$ *are diagonalizable, more precisely* $S^{-1}\mathbf{E}[U_1]S = \Lambda$ *and* $(S^{-1} \otimes S^{-1})\widetilde{D}(S \otimes S) = \Lambda \oplus \Lambda$. *Further,*

if

(a) $\kappa_1(S)^2 \cdot \max\limits_{i,j=1,\ldots,p} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min\limits_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|, \quad or$

(b) $\kappa_\infty(S)^2 \cdot \sum\limits_{i,j=1}^{p} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min\limits_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|,$

then (4.4) applies.

Proof: That $\mathbf{E}[U_1]$ is diagonalizable and that $S^{-1}\mathbf{E}[U_1]S = \Lambda$ under the assumption of pairwise distinct eigenvalues, is well-known, see, for example, Fact 5.16.4 in Bernstein (2009). Then

$$\begin{aligned}
(S^{-1} \otimes S^{-1})\widetilde{D}(S \otimes S) &= (S^{-1} \otimes S^{-1})(\mathbf{E}[U_1] \otimes I + I \otimes \mathbf{E}[U_1])(S \otimes S) \\
&= S^{-1}\mathbf{E}[U_1]S \otimes S^{-1}S + S^{-1}S \otimes S^{-1}\mathbf{E}[U_1]S \\
&= \Lambda \otimes I + I \otimes \Lambda = \Lambda \oplus \Lambda,
\end{aligned}$$

so that $\widetilde{D}$ is diagonalizable and has only eigenvalues with strictly negative real parts. Let $r \in \{1, \infty\}$. By the Theorem of Bauer-Fike (see Theorem 7.2.2 in Golub and Van Loan, 1996) we have for μ being an eigenvalue of $D = \widetilde{D} + B$ that $\min_{\lambda \in \mathrm{spec}(\widetilde{D})} |\lambda - \mu| \leq \kappa_r(S \otimes S)\|B\|_r$. In particular, if

$$\kappa_r(S \otimes S)\|B\|_r < \min\limits_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}\lambda|,$$

then D can only have eigenvalues with strictly negative real parts. Observe that by Fact 9.9.61 in Bernstein (2009) it holds $\kappa_r(S \otimes S) = \|S \otimes S\|_r \|S^{-1} \otimes S^{-1}\|_r = \|S\|_r^2 \|S^{-1}\|_r^2 = \kappa_r(S)^2$ so that the statement follows. $\square$

Remark 4.8. (a) Proposition 4.7 can also be formulated for other natural matrix norms corresponding to the r-norms with $r \in [1, \infty]$, i.e.

$$\|A\|_r = \sup_{x \neq 0} \frac{\|Ax\|_r}{\|x\|_r}.$$

(b) For $r = \infty$ observe that Theorem 1 in Gautschi (1962) gives estimates for the condition number $\kappa_\infty(S)$ when S has the form as in Proposition 4.7 which then gives feasible conditions for (4.4) to hold.

(c) Both Proposition 4.6 and 4.7 state that, if a strictly stationary CARMA process with finite second moments and matrix A whose eigenvalues have only strictly negative real parts is given, then an RC-CARMA process with $\mathbf{E}[U_1] = A$ can be chosen to be strictly stationary with finite second moments, provided the variances of the $M^{(i)}$ are sufficiently small. In other words, the CARMA matrix may be slightly perturbed and still give a strictly stationary RC-CARMA process with finite second moments.

Example 4.9. Consider an RC-CARMA$(2,1)$ process under the assumption of Proposition 4.7. Denote with $\lambda_1 \neq \lambda_2$ the eigenvalues of $\mathbf{E}[U_1]$ with strictly negative real parts. If

$$\kappa \cdot \max\limits_{i,j=1,2} |\mathbf{Cov}(M_1^{(i)}, M_1^{(j)})| < \min\limits_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)|,$$

where
$$\kappa = \left(\frac{(1 + \max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|} \right)^2,$$
then $\widetilde{D} + B$ has only eigenvalues with strictly negative real parts.

Proof: Since $\mathbf{E}[U_1]$ is a companion matrix, we have
$$S = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} \frac{\lambda_2}{\lambda_2 - \lambda_1} & -\frac{1}{\lambda_2 - \lambda_1} \\ -\frac{\lambda_1}{\lambda_2 - \lambda_1} & \frac{1}{\lambda_2 - \lambda_1} \end{bmatrix}.$$
Hence, straightforward calculations yield
$$\kappa_1(S) = \frac{(1 + \max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|}.$$
Observe that $\kappa_\infty(S) = \kappa_1(S)$ such that Proposition 4.7 (b) gives a weaker sufficient condition. $\qquad\square$

Let $X = (X_t)_{t \in \mathbb{R}}$ be a weakly stationary real-valued stochastic process with $\mathbf{E}|X_t|^2 < \infty$ for each $t \in \mathbb{R}$, then the *autocovariance function* of X with lag h is defined by
$$\gamma_X(h) := \mathbf{Cov}(X_{t+h}, X_t) = \mathbf{E}[(X_{t+h} - \mathbf{E}[X_{t+h}])(X_t - \mathbf{E}[X_t])], \quad h \in \mathbb{R}.$$
and the *autocorrelation function* of X is
$$\rho_X(h) := \frac{\gamma_X(h)}{\gamma_X(0)} = \mathbf{Corr}(X_{t+h}, X_t), \quad h \in \mathbb{R}.$$

If $\gamma_X : \mathbb{R} \to \mathbb{R}$ is the autocovariance function of such a process $X = (X_t)_{t \in \mathbb{R}}$ with $\int_{\mathbb{R}} |\gamma_X(h)| \, dh < \infty$, then its Fourier transform
$$f_X(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega h} \gamma_X(h) \, dh, \quad \omega \in \mathbb{R},$$
is called the *spectral density* if the integral exists.

It is well-known (e.g. Brockwell, 2001a) that the spectral density of a CARMA (p,q)-process $S = (S_t)_{t \geq 0}$ of order $q < p$ is given by
$$f_S(\omega) = \frac{\sigma^2}{2\pi} \frac{|b(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R},$$
with σ^2 being the variance of the driving Lévy process. Under the stated conditions of Theorem 4.5 we see that the spectral density of an RC-CARMA(p,q) process $R = (R_t)_{t \geq 0}$ with parameters C and $\mathbf{b}$ is given by
$$f_R(\omega) = f_S(\omega) \varrho_{RC}, \quad \omega \in \mathbb{R},$$
where $f_S(\omega)$ denotes the autocovariance function of the associated CARMA process with $A = \mathbf{E}[U_1]$ as in Theorem 4.5 and the constant ϱ_{RC} is as in (4.7).

Remark 4.10. It is well-known that if $S = (S_t)_{t \in \mathbb{R}}$ is a weakly stationary CARMA (p, q) process, then the equidistantly sampled process $S^\Delta = (S_{n\Delta})_{n \in \mathbb{N}_0}$, $\Delta > 0$, is a weak ARMA(p, q') process for some $q' < p$, see e.g. Section 3 in Brockwell (2009).

Since under the conditions of Theorem 4.5 the autocovariance function of an RC-CARMA(p, q) process $R = (R_t)_{t \geq 0}$ differs from that of the associated CARMA(p, q)

process only by a multiplicative constant, it follows immediately that also $R^\Delta = (R_{\Delta n})_{n \in \mathbb{N}_0}$ is a weakly stationary ARMA(p, q') process for some $q' < p$.

Next, we evaluate exemplarily the covariance structure of an RC-CARMA$(2, 1)$ process under the assumption $\mathbf{E}[L_1] = 0$.

Example 4.11. (**Covariance of RC-CARMA(2,1)**)
Let $R = (R_t)_{t \geq 0}$ be an RC-CARMA$(2, 1)$ process with parameters $C = (C_t)_{t \geq 0} = (M_t^{(1)}, M_t^{(2)}, L_t)_{t \geq 0}$, $\mathbf{b}$ and strictly stationary state vector process $V = (V_t)_{t \geq 0}$. Let V_0 be independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that $\mathbf{E}[L_1] = 0$, that (4.1) holds for $\kappa = 2$, and denote with $S = (S_t)_{t \geq 0}$ the associated CARMA process characterized by Theorem 4.5. Then, for all $t \geq 0$,

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{Cov}(S_{t+h}, S_t) \varrho_{RC} = \mathbf{b}' e^{h\mathbf{E}[U_1]} \begin{bmatrix} \frac{b_0}{\mathbf{E}[M_1^{(2)}]} \\ b_1 \end{bmatrix} \frac{\mathbf{E}[L_1^2]}{2\mathbf{E}[M_1^{(1)}]} \varrho_{RC} \quad (4.10)$$

for all $h \geq 0$, where

$$\varrho_{RC} = \frac{2\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}]}{(2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]}.$$

Proof: Under the assumptions made, an application of Theorem 4.5 yields an associated CARMA process S and the first equality in (4.10). Clearly,

$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -\mathbf{E}[M_1^{(2)}] & -\mathbf{E}[M_1^{(1)}] \end{bmatrix},$$

and the general form of $\mathbf{Cov}(R_{t+h}, R_t)$ is obtained from Proposition 4.3, i.e.

$$\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}' e^{h\mathbf{E}[U_1]} \text{vec}^{-1}(-D^{-1}\mathbf{e}_4) \mathbf{E}(L_1^2)\mathbf{b}. \quad (4.11)$$

Easy calculations show that

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -\mathbf{E}[M_1^{(2)}] & -\mathbf{E}[M_1^{(1)}] & 0 & 1 \\ -\mathbf{E}[M_1^{(2)}] & 0 & -\mathbf{E}[M_1^{(1)}] & 1 \\ \mathbf{Var}[M_1^{(2)}] & \eta & \eta & \mathbf{Var}[M_1^{(1)}] - 2\mathbf{E}[M_1^{(1)}] \end{bmatrix},$$

where $\eta := \mathbf{Cov}[M_1^{(2)}, M_1^{(1)}] - \mathbf{E}[M_1^{(2)}]$. Let

$$\varrho = \frac{1}{(2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]},$$

and denote $y = \begin{bmatrix} -\varrho & 0 & 0 & -\varrho\mathbf{E}[M_1^{(2)}] \end{bmatrix}'$. Then $Dy = \mathbf{e}_4$ such that

$$\text{vec}^{-1}(-D^{-1}\mathbf{e}_4) = \text{vec}^{-1}\left(-\begin{bmatrix} -\varrho \\ 0 \\ 0 \\ -\varrho\mathbf{E}[M_1^{(2)}] \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{E}[M_1^{(2)}] \end{bmatrix} \varrho.$$

Summarizing,

$$
\text{vec}^{-1}(-D^{-1}\mathbf{e}_4)\mathbf{b} = \begin{bmatrix} \frac{1}{\mathbf{E}[M_1^{(2)}]} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \frac{2\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}]}{2\mathbf{E}[M_1^{(1)}]} \varrho
$$

$$
= \begin{bmatrix} \frac{b_0}{\mathbf{E}[M_1^{(2)}]} \\ b_1 \end{bmatrix} \frac{1}{2\mathbf{E}[M_1^{(1)}]} \varrho_{RC},
$$

and with (4.11) we get the stated shape of (4.10). $\qquad\square$

Observe that for $M^{(1)}$ and $M^{(2)}$ being deterministic, $\varrho_{RC} = 1$, hence dividing (4.10) by ϱ_{RC} gives the autocovariance function of a CARMA$(2,1)$ process.

Example 4.12. **(Covariance of RC-CARMA(3,2))**
Let $R = (R_t)_{t\geq 0}$ be an RC-CARMA$(3,2)$ process with parameters $C = (C_t)_{t\geq 0}$, **b** and strictly stationary state vector process $V = (V_t)_{t\geq 0}$. Let V_0 be independent of C and C a semimartingale with respect to the given filtration $\mathbb{F}$. Assume that $\mathbf{E}[L_1] = 0$ and (4.1) hold for $\kappa = 2$. Then, for all $t, h \geq 0$,

$$
\mathbf{Cov}(R_{t+h}, R_t) = \mathbf{b}'e^{h\mathbf{E}[U_1]} \begin{bmatrix} \frac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(2)}]}b_0 - b_2 \\ b_1 \\ \mathbf{E}[M_1^{(2)}]b_2 - b_0 \end{bmatrix} \frac{\mathbf{E}[L_1^2]}{2(\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}] - \mathbf{E}[M_1^{(3)}])} \cdot \varrho_{RC},
$$

where

$$
\varrho_{RC} = \frac{2(\mathbf{E}[M_1^{(1)}]\mathbf{E}[M_1^{(2)}] - \mathbf{E}[M_1^{(3)}])}{\xi + 2(\mathbf{Cov}(M_1^{(3)}, M_1^{(1)}) - \mathbf{E}[M_1^{(3)}]) - \frac{\mathbf{E}[M_1^{(1)}]}{\mathbf{E}[M_1^{(3)}]}\mathbf{Var}[M_1^{(3)}]}
$$

with $\xi = (2\mathbf{E}[M_1^{(1)}] - \mathbf{Var}[M_1^{(1)}])\mathbf{E}[M_1^{(2)}] - \mathbf{Var}[M_1^{(2)}]$.

The proof follows by similar calculations as in Example 4.11 and is left to the reader.

5. Simulations

We compare in this section two simulations of an RC-CARMA$(2,1)$ process, one when $\mathbf{E}[U_1]$ has only real strictly negative eigenvalues and the other when $\mathbf{E}[U_1]$ has complex eigenvalues with strictly negative real parts.

For our simulations we have chosen as random coefficient processes $M^{(1)}$, $M^{(2)}$ two independent compound poisson processes, i.e. with depiction

$$
M_t^{(1)} = \sum_{i=1}^{N_t^{(1)}} X_i^{(1)} \quad \text{and} \quad M_t^{(2)} = \sum_{i=1}^{N_t^{(2)}} X_i^{(2)}, \quad t \geq 0.
$$

In the case of real eigenvalues we have chosen $\mathbf{E}[N_1^{(1)}] = 1.5$, $\mathbf{E}[N_1^{(2)}] = 2$, $X_i^{(1)} \sim \mathcal{N}(1, 0.3^2)$, and $X_i^{(2)} \sim \mathcal{N}(0.25, 0.2^2)$, and as driving process a standard Brownian motion $B = (B_t)_{t\geq 0}$. Hence, for $\widetilde{D} = \mathbf{E}[U_1] \oplus \mathbf{E}[U_1]$ and $D = \widetilde{D} + \mathbf{E}[U_1 \otimes U_1] -$

$\mathbf{E}[U_1] \otimes \mathbf{E}[U_1]$ we have

$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -1/2 & -3/2 \end{bmatrix}, \quad \widetilde{D} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1/2 & -3/2 & 0 & 1 \\ -1/2 & 0 & -3/2 & 1 \\ 0 & -1/2 & -1/2 & -3 \end{bmatrix}, \quad \text{and}$$

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1/2 & -3/2 & 0 & 1 \\ -1/2 & 0 & -3/2 & 1 \\ 0.205 & -1/2 & -1/2 & -1.365 \end{bmatrix}.$$

Consequently, we have for $\mathbf{E}[U_1]$ the eigenvalues $\mu_1 = -1/2$ and $\mu_2 = -1$, and for D the eigenvalues $\lambda_1 \approx -0.29$, $\lambda_2 \approx -1.29 + 1.28i$, $\lambda_3 = -1.29 - 1.28i$, and $\lambda_4 = -1.5$. Since all eigenvalues of D have strictly negative real parts, we obtain the existence of a strictly stationary solution by Remark 4.2 and Remark 4.4.

Nevertheless, observe that

$$\mathbf{Var}(M_1^{(1)}) + \mathbf{Var}(M_1^{(2)}) = 1.84 \not< \frac{1}{4}\sigma_{\min}(\widetilde{D} \oplus \widetilde{D})^2 \approx 0.0799$$

showing that the condition in Proposition 4.6 is not necessary. Moreover,

$$\kappa = \left(\frac{(1 + \max\{|\lambda_1|, |\lambda_2|\}) \cdot \max\{2, |\lambda_1| + |\lambda_2|\}}{|\lambda_2 - \lambda_1|} \right)^2 = 25$$

so that

$$\kappa \cdot \max_{i=1,2} |\mathbf{Var}(M_1^{(i)})| = 40.875 \not< \min_{\lambda \in \mathrm{spec}(\widetilde{D})} |\mathrm{Re}(\lambda)| = 1,$$

showing that also the condition in Example 4.11 and hence in Proposition 4.7 is not necessary.

In case of complex eigenvalues we have chosen $\mathbf{E}[N_1^{(1)}] = 2$, $\mathbf{E}[N_1^{(2)}] = 7.5$, $X_i^{(1)} \sim \mathcal{N}(0.5, 0.1^2)$, and $X_i^{(2)} \sim \mathcal{N}(0.4, 0.05^2)$, and have left the driving process unchanged. Then

$$\mathbf{E}[U_1] = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}$$

gives two complex-valued eigenvalues $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$. Also, D has just eigenvalues with strictly negative real parts. Furthermore, an observation similar to the one above can be made showing non-necessity of the conditions in Proposition 4.6 and 4.7.

For both the complex and the real eigenvalues case, we have simulated $10,000,000$ observations with a mesh size $k = 0.01$, i.e. $R_{0.01}, R_{0.02}, \ldots, R_{100,000}$. In Figure 5.1(a) and (c) we see the corresponding plots until time 300. Plots 5.1(b) and (d) show the corresponding autocovariance functions (ACVF).

The solid line corresponds to the model autocovariance function, the dashed one to the sampled ACVF based on the data $R_{0.01}, \ldots, R_{100,000}$, and the dotted shows the model ACVF of the corresponding CARMA$(2,1)$ process. The dashed-dotted line shows the sample ACVF based on $R_{0.01}, \ldots, R_{100}$.

We see that using $10,000,000$ observations to calculate the sample autocovariance function, it nearly agrees in both cases with the model autocovariance function. Plot 1(d) shows a sinusoidal oscillation which is also in the CARMA case characteristic for allowing complex eigenvalues and visualizes the variety of possible autocovariance functions.
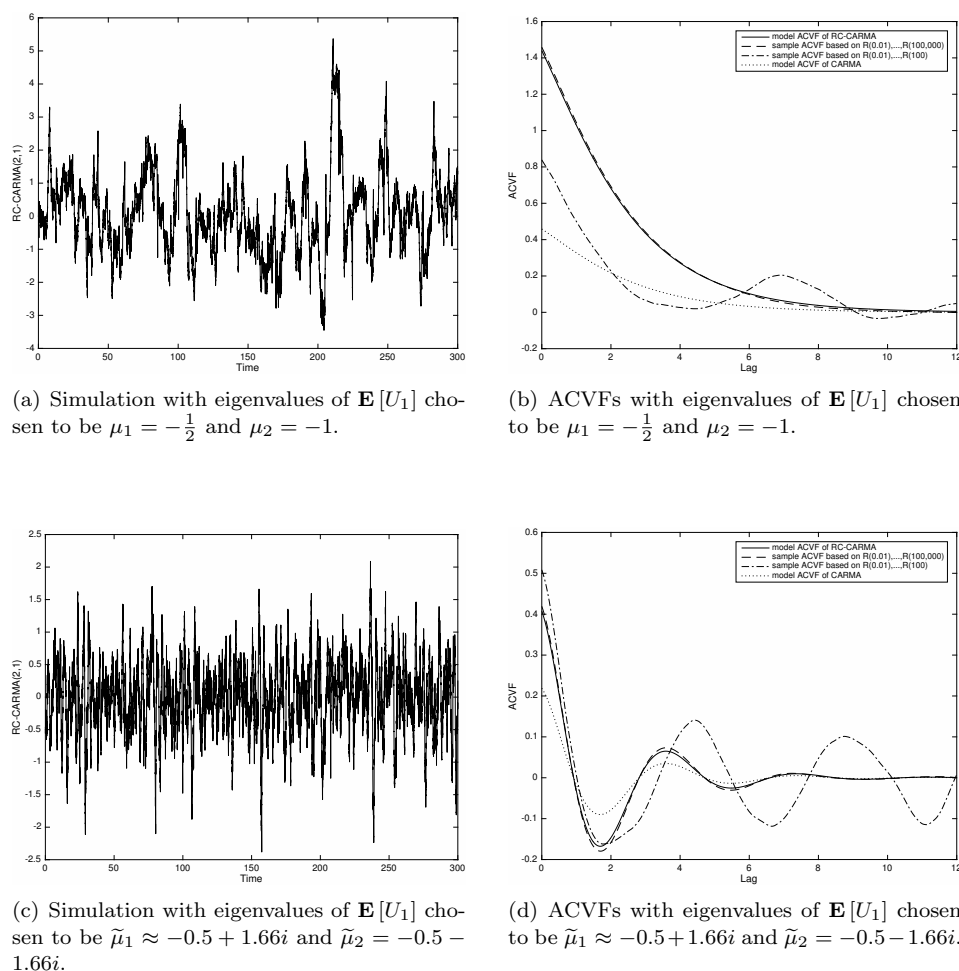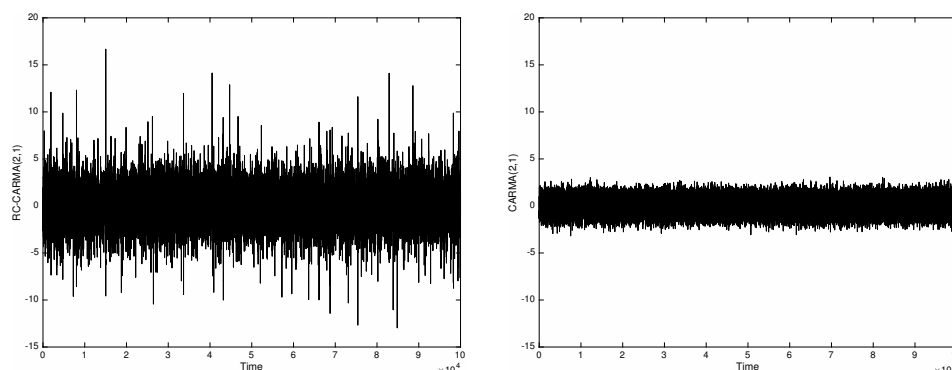
(a) Simulation with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.

(b) ACVFs with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.

(c) Simulation with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$.

(d) ACVFs with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\widetilde{\mu}_1 \approx -0.5 + 1.66i$ and $\widetilde{\mu}_2 = -0.5 - 1.66i$.

FIGURE 5.1. Simulated RC-CARMA(2,1) process and ACVFs.

Figure 5.2(a) shows all simulated observations of the RC-CARMA$(2,1)$ process where $\mathbf{E}[U_1]$ has real eigenvalues. On the other hand, Figure 5.2(b) shows an equally sized simulation of the associated CARMA$(2,1)$ process, i.e. with the choice $A = \mathbf{E}[U_1]$. It can be seen that the RC-CARMA$(2,1)$ process provides larger outliers around the between -5 and 5 concentrated band than the CARMA$(2,1)$ process around its band. This may indicate possible heavy tails of RC-CARMA processes.

We justify these observations intuitively in the following remark recalling results on random recurrence equations.

Remark 5.1. Observe that a stationary CARMA process driven by a Brownian motion has a normal marginal stationary distribution, in particular, it has light tails.

(a) Simulation of RC-CARMA$(2,1)$ with eigenvalues of $\mathbf{E}[U_1]$ chosen to be $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -1$.

(b) Simulation of associated CARMA$(2,1)$ with $A = \mathbf{E}[U_1]$.

FIGURE 5.2. Simulated RC-CARMA(2,1) and CARMA(2,1) processes.

On the other hand, as a consequence of results of Kesten (1973) and Goldie (1991), it is known that a generalized Ornstein-Uhlenbeck process and so an RC-CARMA$(1,0)$ process will have Pareto tails under wide conditions, even if the driving process is a Brownian motion, see e.g. Lindner and Maller (2005) (Theorem 4.5) or Behme (2011) (Theorem 4.1).

We henceforth expect using the multivariate results of Kesten (1973) that under wide conditions the RC-CARMA process will also have Pareto tails for higher orders. However, we leave a thorough investigation of this for forthcoming research.

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