Concentration results for directed polymer with unbounded jumps

Shuta Nakajima

Research Institute in Mathematical Sciences, Kyoto University, Kyoto, Japan
E-mail address: `njima@kurims.kyoto-u.ac.jp`

Abstract. We study the free energy and the ground state energy for the directed polymer in random environment. The polymer is allowed to make unbounded jumps and the environment is given by i.i.d. Bernoulli variables. Our goal is to derive concentrations for the ground state energy of polymers at zero temperature and the free energy at any temperature. To do this, we control the maximum jump of any polymer nearly minimizing energy and guarantee that it is not too large with high probability.

1. Introduction

We discuss the models of directed polymer which have unbounded jumps that were introduced in Comets et al. (2015). See Comets et al. (2015) for the background and related works. That paper exhibits the following three results: (i) The continuity of the free energy with respect to inverse temperature and the appearance probability of obstacles; (ii) The asymptotic of the free energy as the appearance probability of obstacles goes to 1; (iii) The continuity of the time constant of First Passage Percolation (FPP) related to the model. In this paper, we show concentration bounds for the ground state energy, which is nothing but the minimum passage time, and the free energy. As applications, we derive the so-called "rate of convergence" results and, based on them, we are able to remove the restriction left in Comets et al. (2015) on a parameter in (i) and also give an alternative proof of (iii).

In fact, a concentration bound for the lower tail for the FPP is shown in Proposition 3.1 of Comets et al. (2015). However the upper tail is significantly more difficult as we need to control the maximum jump of the optimal path. Although such a geometric property of the minimizing path is usually hard to prove, in this paper, we succeed in controlling jumps not only of the optimal path but also of

low energy paths in the directed polymer model. As a result, we can show the concentration for the minimum passage time and the free energy. The bound on the maximum jump itself is of independent interest, see Corollary 2.7 for the precise statement.

Let us briefly explain the motivation of Comets et al. (2015) and this work. There has been revived interest on the number of open paths in oriented percolation, e.g. Fukushima and Yoshida (2012) and Garet et al. (2017). It can be regarded as a zero temperature limit of the directed polymer in random environment and hence the free energy, which corresponds to the *growth rate* of the number of open paths, is expected to be continuous at zero temperature. Unfortunately, there seems to be few techniques developed to study the zero temperature limit in this type of models. In Comets et al. (2015), we introduced a toy model where the polymer is allowed to make unbounded jumps and established the continuity at zero temperature by a combinatorial argument. This paper proposes alternative arguments for the continuity based on the concentration estimates.

1.1. *Setting of models.* Let $(\{X_n\}_{n\in\mathbb{N}\cup\{0\}}, P)$ be the random walk on $\mathbb{Z}^d$ starting from 0 and with the transition probability

$$P(X_{n+1} = z | X_n = y) = f_\alpha(|y - z|_1),$$

where $|x|_1 := \sum_{i=1}^d |x_i|$ for $x \in \mathbb{Z}^d$ and $f_\alpha : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is a function of the form

$$f_\alpha(k) = c_1 \exp\{-c_2 k^\alpha\}, \tag{1.1}$$

where $\alpha, c_2 > 0$ and c_1 is a positive constant determined as to be $\sum_{y\in\mathbb{Z}^d} f_\alpha(|y|_1) = 1$ (see Remark 1.9 for generalizations). The random environment is modelled by independent and identically distributed Bernoulli random variables $(\{\eta(j,x)\}_{(j,x)\in\mathbb{N}\cup\{0\}\times\mathbb{Z}^d}, Q)$ with parameter p:

$$Q(\eta(0,0) = 1) = p \in (0,1).$$

We introduce the Hamiltonian

$$H_n^\eta(X) = \sum_{j=1}^n \eta(j, X_j),$$

and define the partition functions by

$$Z_n^{\eta,\beta} = P[\exp\{\beta H_n^\eta\}] \text{ for } \beta \in \mathbb{R} \text{ and } Z_n^{\eta,-\infty} = P(H_n^\eta = 0),$$

where $P[\cdot]$ denotes the expectation with respect to P. Note that $Z_n^{\eta,-\infty}$ is positive for Q-almost every η, since the random walk has unbounded jumps.

An important quantity in this model is the so-called free energy defined by

$$\begin{aligned} \varphi(p,\beta) &= \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\eta,\beta} \\ &= \lim_{n\to\infty} \frac{1}{n} Q[\log Z_n^{\eta,\beta}] \end{aligned} \tag{1.2}$$

Q-almost surely, whose existence can be shown by using the subadditive ergodic theorem. In Comets et al. (2015), the continuity property and some asymptotic behavior of the free energy were studied.

Next, we introduce the FPP models related to this directed polymer model. Let us define a point process representing the sites where η takes value zero by

$$\omega_p = \sum_{(k,x) \in \mathbb{N} \times \mathbb{Z}^d} (1 - \eta(k,x)) \delta_{(k, s_p x)}, \tag{1.3}$$

with the scaling factor $s_p = (1-p)^{1/d}$. This scaling is natural since ω_p converges as $p \uparrow 1$ to the Poisson point process ω_1 on $\mathbb{N} \times \mathbb{R}^d$ whose intensity is the product of the counting measure and the Lebesgue measure. With some abuse of notation we will frequently identify ω_p, and more generally any point measure, with its support. Given a realization of ω_p, we define the minimum passage time from 0 to n by

$$T_n(\omega_p) = \min \left\{ \sum_{k=1}^{n} |x_{k-1} - x_k|_1^\alpha : x_0 = 0 \text{ and } \{(k, x_k)\}_{k=1}^n \subset \omega_p \right\}. \tag{1.4}$$

This is the directed version of Howard-Newman's Euclidean FPP model in Howard and Newman (2001). Now, a direct application of the subadditive ergodic theorem shows that the limit

$$\mu_p = \lim_{n \to \infty} \frac{1}{n} T_n(\omega_p) = \lim_{n \to \infty} \frac{1}{n} Q[T_n(\omega_p)] \tag{1.5}$$

exists Q-almost surely. The limit μ_p is non–random and called the time constant. Observe also that definition (1.4) makes perfect sense when $p = 1$, yielding a limit μ_1 in (1.5). It is again shown in Comets et al. (2015) that μ_p is continuous as $p \uparrow 1$.

This FPP is related to the *ground state at $\beta = -\infty$* of the directed polymer introduced above. Namely,

$$\sup\{P(X_i = x_i \text{ for } 1 \le i \le n) : x_i \in \mathbb{Z}^d, \ \eta(i, x_i) = 0\} = c_1^n \exp\{-c_2 s_p^{-\alpha} T_n(\omega_p)\}.$$

Furthermore, it is shown in Comets et al. (2015) that as $p \uparrow 1$, the ground state gives dominant contribution to the free energy:

$$\varphi(p, -\infty) \sim -c_2 \mu_1 (1-p)^{-\alpha/d}.$$

Remark 1.1. It is standard to show the existence of the free energy and the time constant by the subadditive ergodic theorem. However, it prevents us from getting more information about them, such as the continuity and the rate of convergence. In this paper, we will use a method of concentration of measures to get the rate of convergence in (1.2) and (1.5) and derive the continuity results as a corollary.

1.2. *Main results.* First, we shall present the results for our FPP. The first result is the concentration of the minimum passage time around the mean.

Theorem 1.2. *For any $\delta > 0$ there exist positive constants C_1, C_2, and $\lambda \in (0, 1]$ which are independent of p and n such that for any $n \in \mathbb{N}$,*

$$Q(|T_n(\omega_p) - Q[T_n(\omega_p)]| > n^{\frac{1}{2}+\delta}) < C_1 \exp\{-C_2 n^\lambda\}. \tag{1.6}$$

The next result is about the rate of convergence. It implies that the fluctuation exponent (see Krug and Spohn, 1991) is $\frac{1}{2}$ or less if exists.

Theorem 1.3. *For any $\chi > 1/2$, there exists positive constant C_1 which is independent of p such that for any $n \in \mathbb{N}$,*

$$|n\mu_p - Q[T_n(\omega_p)]| \quad < \quad C_1 n^\chi. \tag{1.7}$$

Combining (1.6) and (1.7), we have that for any $\chi > 1/2$ there exist positive constants C_0, C_1, C_2, λ which are independent of p such that for any $n \in \mathbb{N}$,

$$Q(|T_n(\omega_p) - n\mu_p| > 2C_0 n^\chi) \quad < \quad C_1 \exp\{-C_2 n^\lambda\}. \tag{1.8}$$

Corollary 1.4. *μ_p is continuous in $p \in [0,1]$.*

Corollary 1 is a slight extension of Theorem 1.5 in Comets et al. (2015) where the continuity is proved only at $p = 1$.

Next, we move on to the results for the directed polymer model.

Theorem 1.5. *For any $q \in [0,1)$, $\delta > 0$ and $\beta_0 \in \mathbb{R}$, there exist positive constants C_1, C_2, and $\lambda \in (0,1)$ such that for any $p \in [0,q)$, $\beta \in [-\infty, \beta_0]$ and $n \in \mathbb{N}$,*

$$Q(|\log Z_n^{\eta,\beta} - Q[\log Z_n^{\eta,\beta}]| > n^{\frac{1}{2}+\delta}) < C_1 \exp\{-C_2 n^\lambda\}. \tag{1.9}$$

Theorem 1.6. *For any $q \in [0,1)$, $\chi > \frac{1}{2}$ and $\beta_0 \in \mathbb{R}$, there exist positive constants ϵ, C_0–C_2, and $\lambda \in (0,1)$ such that for any $p \in [0,q)$, $\beta \in [-\infty, \beta_0]$ and $n \in \mathbb{N}$,*

$$|n\varphi(p,\beta) - Q[\log Z_n^{\eta,\beta}]| \quad < \quad C_0 n^\chi, \tag{1.10}$$
$$Q(|\log Z_n^{\eta,\beta} - n\varphi(p,\beta)| > 2C_0 n^\chi) \quad < \quad C_1 \exp\{-C_2 n^\lambda\}. \tag{1.11}$$

The following theorem is an extension of Theorem 1.2 in Comets et al. (2015) where the continuity is proved only for $\alpha < d$ where α comes from (1.1).

Corollary 1.7. *$\varphi(p,\beta)$ is jointly continuous on $[0,1) \times [-\infty, \infty)$.*

Remark 1.8. Corollary 1.4 can in fact be proved by the "coupling method" in Comets et al. (2015). Nevertheless, we think the line of the argument—proving the continuity of a limiting quantity like the time constant via a concentration bound— is of interest. In Corollary 1.7, the method of concentration indeed yields a better result than the rather bare-handed approach in Comets et al. (2015).

Remark 1.9. The method of this paper works for more general choices of (1.1). For example if $f(k) = c \exp\{-V(k)\}$ with V satisfying $0 < C_1 < V''(x) < C_2$ for any x with some positive constant C_1 and C_2 or V' being regularly varying with index $\beta > 0$. Indeed, all of the statements including the key lemma below can be shown by the essentially the same way for these choices.

1.3. *Organization of the paper.* The rest of the paper is organized as follows. In Section 2, we start with the proof of Theorems 1.2. We divide the proof into two parts; $\alpha \leq 1$ or $\alpha > 1$. We first prove the case $\alpha \leq 1$ which is relatively easy. To prove the other case, we need a key lemma in Section 2.2 which gives a control on the maximum jump of optimal paths as a special case. This lemma is the essential part of this paper. By using the same lemma, polymers with large jumps can be shown to have negligible weights under the Gibbs measure. By restricting our attention to the polymers with small jumps, we can check the condition for the entropy method (Theorem 6.7 in Boucheron et al., 2013) to show the concentration bound and obtain Theorem 1.5. In Section 3, we discuss the non-random fluctuations and prove Theorems 1.3 and 1.6 by using Theorems 1.2 and 1.5, respectively. The proofs are based on the argument in Zhang (2010). Finally, we prove Corollaries 1.4 and 1.7 in Section 4 as a consequence of the results of Section 3.

2. Proof of the concentration around the mean

For any n-path $\gamma \in (\mathbb{R}^d)^{\{1,\cdots,n\}}$, we write $\gamma(k)$ for the k–th point of γ, and $\Delta\gamma(k) := |\gamma(k) - \gamma(k-1)|_1$ with the convention $\gamma(0) = 0$.

Definition 2.1. For any n-path $\gamma \in (\mathbb{R}^d)^{\{1,\cdots,n\}}$, the passage time $t(\gamma)$ is defined by

$$t(\gamma) = \sum_{i=1}^{n} \Delta\gamma(i)^\alpha.$$

2.1. *Concentration for the FPP with $0 < \alpha \leq 1$.* We first calculate the probability that ω_p has no points inside a box of size $r > 0$.

Lemma 2.2. *There exist $C_1, C_2 > 0$ independent of p such that for any sufficiently large $r > 0$,*

$$Q(\omega_p \cap [0,r]^d = \emptyset) \leq C_1 \exp\left(-C_2 r^d\right). \tag{2.1}$$

Proof: Note that for sufficiently large $r > 0$, $[0,r]^d \cap s_p \mathbb{Z}^d$ has at least $(2^{-1} s_p^{-1} r)^d$ points, where $s_p = (1-p)^{1/d}$. Thus

$$Q(\omega_p \cap [0,r]^d = \emptyset) \leq (p^{(1-p)^{-1}})^{2^{-1} r^d}.$$

Since it is easy to check that $\sup_{p \in [0,1]} p^{(1-p)^{-1}} < 1$, we complete the proof. $\qquad\square$

Proof of (1.6) *for $0 < \alpha \leq 1$:* We prove (1.6) by using a martingale difference method. We introduce the filtration

$$\mathcal{G}_m = \sigma(\omega_p|_{[1,m] \times \mathbb{R}^d})$$

and decompose the deviation from the mean into the sum of martingale differences as

$$
\begin{aligned}
T_n(\omega_p) &- Q[T_n(\omega_p)] \\
&= \sum_{m=1}^{n} (Q[T_n(\omega_p)|\mathcal{G}_m] - Q[T_n(\omega_p)|\mathcal{G}_{m-1}]) \\
&=: \sum_{m=1}^{n} \Delta_m.
\end{aligned}
$$

We are going to prove that there exists non–random constant $c > 0$ independent of p such that

$$Q\left[\exp\left\{c|\Delta_m|^{d/\alpha}\right\} |\mathcal{G}_{m-1}\right] \leq c^{-1} \tag{2.2}$$

Q-almost surely. Then (1.6) follows by a concentration inequality for martingales, for example, Theorem 1.1 in Liu and Watbled (2009). Before proving it, let us introduce some notation. Given two configurations ω_p and ω_p' and $m \in \mathbb{N}$, we define the new configuration $[\omega_p, \omega_p']_m$ by

$$\omega_p|_{[1,m] \times \mathbb{R}^d} + \omega_p'|_{[m+1,\infty) \times \mathbb{R}^d}.$$

Let $\pi_n^{(m)} = (i, \pi_n^{(m)}(i))_{i=0}^n$ be a minimizing path for this configuration chosen by a deterministic algorithm (if not unique). Given $(n,x), (m,y) \in \mathbb{N} \times \mathbb{R}^d$, we define $|(n,x) - (m,y)|_1 = |x-y|_1$. For a time-space point $(k,x) \in \mathbb{N} \times \mathbb{R}^d$, we define $(k,x)^{*\omega_p}$ to be a point in $\omega_p|_{\{k\} \times \mathbb{R}^d}$ closest to (k,x) with a deterministic rule to break ties and sometimes use the convention that $(k,x)^{*\omega_p} = (k, x^{*\omega_p})$ with a slight

abuse of notation.

It is easy to check that the law of $(k,x)^{*\omega_p}$ is independent of (k,x) and by Lemma 2.2, there exist $C_1, C_2 > 0$ independent of p such that for any sufficiently large $r > 0$,

$$Q(|(k,x) - (k,x)^{*\omega_p}|_1 > r) \leq C_1 \exp(-C_2 r^d). \tag{2.3}$$

Now we rewrite the martingale difference as

$$\Delta_m = \int Q(\mathrm{d}\omega'_p)(T_n([\omega_p, \omega'_p]_m) - T_n([\omega_p, \omega'_p]_{m-1})).$$

We can bound $T_n([\omega_p, \omega'_p]_{m-1})$ from above by the passage time of the path

$$\pi_n^{(m)}(0), \ldots, \pi_n^{(m)}(m-1), \pi_n^{(m)}(m)^{*\omega'_p}, \pi_n^{(m)}(m+1), \ldots, \pi_n^{(m)}(n).$$

By $\alpha \leq 1$ we have

$$
\begin{aligned}
&|\pi_n^{(m)}(m \pm 1) - \pi_n^{(m)}(m)^{*\omega'_p}|_1^\alpha \\
&\quad \leq |\pi_n^{(m)}(m \pm 1) - \pi_n^{(m)}(m)|_1^\alpha + |\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1^\alpha.
\end{aligned}
\tag{2.4}
$$

These lead to

$$T_n([\omega_p, \omega'_p]_m) - T_n([\omega_p, \omega'_p]_{m-1}) \geq -2|\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1^\alpha, \tag{2.5}$$

and we get

$$\Delta_m \geq -2 \int Q(\mathrm{d}\omega'_p)|\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1^\alpha. \tag{2.6}$$

Similarly, we have

$$\Delta_m \leq 2 \int Q(\mathrm{d}\omega'_p)|\pi_n^{(m-1)}(m) - \pi_n^{(m-1)}(m)^{*\omega_p}|_1^\alpha. \tag{2.7}$$

Note that the laws of right hand side of (2.6) and (2.7) are independent of $\mathcal{G}_{m-1}$. By Jensen's inequality, (2.6) and (2.7), if we take c sufficiently small depending only on $C_1, C_2 > 0$ of (2.3), we have

$$
\begin{aligned}
&Q\left[\exp\left\{c|\Delta_m|^{d/\alpha}\right\}|\mathcal{G}_{m-1}\right] \\
&\leq \int Q(d\omega_p) \int Q(d\omega'_p) \exp\left(2c|\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1^d\right) \\
&\quad + \int Q(d\omega_p) \int Q(d\omega'_p) \exp\left(2c|\pi_n^{(m-1)}(m) - \pi_n^{(m-1)}(m)^{*\omega_p}|_1^d\right) \\
&= 2 \int Q(d\omega_p) \exp\left(2c|(k,x) - (k,x)^{*\omega_p}|_1^d\right) \leq c^{-1}.
\end{aligned}
$$

$\square$

2.2. *Key lemma (The uniform bound for jumps).* As mentioned above, it is more difficult to prove Theorem 1.2 in the case $\alpha > 1$. Indeed, we need to estimate the change of minimum passage time when we replace the configuration on a section

by another one, but we do not have the triangular inequality (2.5). A natural alternative way is to use Taylor's theorem as follows:

$$
\begin{aligned}
&|T_n([\omega_p, \omega'_p]_m) - T_n([\omega_p, \omega'_p]_{m-1})| \\
&\leq C(|\pi_n^{(m)}(m+1) - \pi_n^{(m)}(m)|_1^{\alpha-1} |\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1 \\
&\quad + |\pi_n^{(m)}(m) - \pi_n^{(m)}(m-1)|_1^{\alpha-1} |\pi_n^{(m)}(m) - \pi_n^{(m)}(m)^{*\omega'_p}|_1),
\end{aligned}
$$

with some positive constant C. Then, the jump size of the optimal path appears in this change and we need to show that it is not too large. We shall state it in a slightly generalized way that is useful in the study of the directed polymer model.

Given a configuration ω and n-path γ, we write $\gamma \subset \omega$ if $(i, \gamma(i)) \in \omega$ for any $i = 1, \cdots, n$. We say that $\omega \subset \mathbb{N} \times \mathbb{R}^d$ has θ-property if

$$
\omega \cap (i, x + [0, n^\theta)^d) \neq \emptyset \text{ for any } i \in \mathbb{N}, \, x \in \mathbb{Z}^d.
$$

Intuitively, θ-property means that configration has no big vacant regions. As we shall see in Lemma 2.4 that the θ-property scarcely influence on the minimum passage time. The following lemma plays a key role throughout all the results in this paper. Roughly speaking, any polymer has no big jumps or we can find another polymer with a smaller passage time.

Lemma 2.3. *Suppose that $\alpha > 1$. For any $\zeta > 0$, there exist $\theta > 0$ and $N = N(\theta) \in \mathbb{N}$ such that if $n > N$, $s \in \{1, \cdots, n\}$ and a configuration $\omega' \subset \mathbb{N} \times \mathbb{R}^d$ satisfies θ-property, then for any n-path γ, one of the following holds:*

(i) *$\max\{\Delta\gamma(s), \Delta\gamma(s+1)\} \leq n^\zeta$ with the convention $\Delta\gamma(n+1) = 0$,*
(ii) *There exist an n-path γ' and $h \in \mathbb{N}$ such that $\gamma(i) = \gamma'(i)$ for any $i \notin [s, s+h-1]$, $(i, \gamma'(i)) \in \omega'$ for $i \in [s, s+h-1]$ and*

$$
t(\gamma') + (h+1)n^\theta \leq t(\gamma).
$$

The proof of Lemma 1 is not long but a bit complicated. Let us explain the idea of the proof in the case where γ is the minimizing path for ω'. Let A_0 be the point where next jump is larger than n^ζ and introduce a sequence of large numbers satisfying $n^\zeta \gg \mathcal{L}_1 \gg \mathcal{L}_2 \gg \cdots$.

We draw cones C_k with slope $\mathcal{L}_k$ starting at A_0 and let A_k be the first point where the optimal path γ touches it. Then we draw a straight line between A_0 and A_k. Thanks to the θ-property, we can find a path $\gamma'_k \subset \bar{\omega}$ close to this straight line. Due to the convexity of $|\cdot|_1^\alpha$, average jumps are better (See Figure 2.1). This yields that, if we forget about the restriction $\subset \bar{\omega}$, the straight line connecting A_0 and A_k is *the optimal path* when $\alpha > 1$. On the other hand, by definition, γ has a smaller passage time than that of γ'_k. Then it is natural to expect that most of the jumps of γ between A_0 and A_k are close to $\mathcal{L}_k$ in size. However in the first $A_0 \to A_{k-1}$ segment, most of the jumps are close to $\mathcal{L}_{k-1}$ for the same reason and hence far from $\mathcal{L}_k$. This implies that the duration of $A_0 \to A_k$ is much longer than that of $A_0 \to A_{k-1}$. Based on this observation, we can prove that A_k goes beyond the n-th section before $\mathcal{L}_k$ becomes small. This implies that the optimal path γ stays outside a cone of very large angle until n. This is very unlikely because the optimal path has too much passage time between A_0 and n-th section, and one can indeed derive a contradiction.
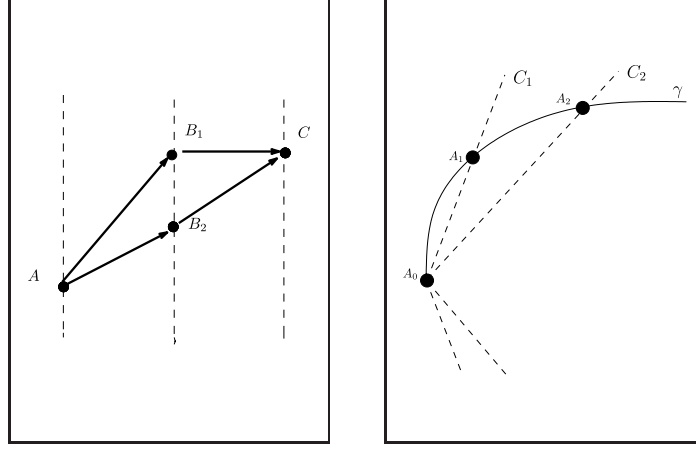
FIGURE 2.1.
Left: The passage time of AB_2C is smaller than that of AB_1C when $\alpha > 1$.
Right: Drawing cones, we investigate γ between A_0 and A_k inductively.

Proof: Let $m \in \mathbb{N}$ such that $m > 5/\zeta$, and we take $\theta > 0$ sufficiently small and N sufficiently large such that

$$2^5 m\alpha\theta < \zeta \text{ and } N^{\min\{\alpha-1,1\}\theta} > 2^{\alpha+4}\alpha^2(\alpha-1)^{-1} + (2^{\frac{1}{\alpha}}-1)^{-1}.$$

Suppose that neither (i) nor (ii) holds and we shall derive a contradiction. This means that we suppose the following two conditions:

(i') $\max\{\Delta\gamma(s), \Delta\gamma(s+1)\} > n^\zeta$,
(ii') For any n-path γ' and $h > 0$ such that $\gamma(i) = \gamma'(i)$ for $i \notin [s, s+h-1]$, and $(i, \gamma'(i)) \in \omega'$ for $i \in [s, s+h]$, $t(\gamma') + (h+1)n^\theta > t(\gamma)$.

We only treat the case $\Delta\gamma(s) > n^\zeta$ since the other case $\Delta\gamma(s+1) > \Delta\gamma(s)$ can be dealt with by simply replacing s by $s+1$ in the following proof.

<u>Step 1</u> (Preliminary bounds): For any $k \in \{1, \cdots, m\}$, let

$$\ell_k := \inf\{\ell \in [s, n] \cap \mathbb{N} : \frac{|\gamma(\ell) - \gamma(s-1)|_1}{\ell - s + 1} \le n^{\zeta-(2k-1)\theta}\}.$$

The points $(s-1, \gamma(s-1))$ and $(\ell_k, \gamma(\ell_k))$ correspond to A_0 and A_k in Figure 2.1, respectively. By the θ-property of ω', there exists an n-path γ' such that $\gamma(i) = \gamma'(i)$ for any $i \le s-1$ and $(i, \gamma'(i)) \in \omega'$, $\Delta\gamma'(i) \le dn^\theta$ for any $i \ge s$. If the range of the infimum of (2.8) is empty, then

$$t(\gamma) - t(\gamma') - (n-s+1)n^\theta \ge (n-s+1)(n^{\zeta-(2k-1)\theta} - dn^\theta - n^\theta) \ge 0,$$

and it contradicts (ii'). Hence, the range is not void for $k \le m$ and $\ell_k \le n$.

Let $\bar{\gamma}_k$ be a straight line drawn between $(s-1, \gamma(s-1))$ and $(\ell_k, \gamma(\ell_k))$. We write $\mathcal{L}_k$ for a slope $\frac{|\gamma(\ell_k) - \gamma(s-1)|_1}{\ell_k - s + 1}$ of $\bar{\gamma}_k$. Since ω' has θ-property, there exists an n-path γ'_k such that $(j, \gamma'_k(j)) \in \omega'$, $|\bar{\gamma}_k(j) - \gamma'_k(j)|_1 \le dn^\theta$ for all $s \le j < \ell_k$, and $\gamma'_k(\ell) = \gamma(\ell)$ for $\ell \le s-1$ or $\ell \ge \ell_k$. Set $R_k := \ell_k - s$, $D_k := \{i \in \{0, \cdots, R_k\} : \Delta\gamma(s+i) \ge n^{\zeta-2k\theta}\}$, $D_k^c := \{0, \cdots, R_k\} \setminus D_k$. The set D_k corresponds to the time

when $\Delta\gamma$ is close to $\mathcal{L}_k$, though we put one-sided inequality in the definition.

Let $\delta_{j,k} := \Delta\gamma(s+j) - \mathcal{L}_k$ and $\delta'_{j,k} := \Delta\gamma'_k(s+j) - \mathcal{L}_k$. We note that for any $k \geq 1$, $|\delta'_{j,k}| \leq 2dn^\theta$. Moreover since γ_k and γ'_k start and end at the same points, we have

$$\sum_{i=0}^{R_k} \delta_{i,k} \geq 0, \quad \sum_{i=0}^{R_k} \delta'_{i,k} \geq 0. \tag{2.8}$$

Note that by convexity of $|\cdot|_1^\alpha$,

$$(\mathcal{L}_k + \delta_{i,k})^\alpha - \mathcal{L}_k^\alpha - \alpha\mathcal{L}_k^{\alpha-1}\delta_{i,k} \geq 0. \tag{2.9}$$

These positivities (2.8) and (2.9) will be crucial in the proof.

Step 2 (D_{k+1} is much larger than D_k): We have

$$n^{\zeta-(2k-1)\theta} - n^{\zeta-2k\theta} \leq \mathcal{L}_k \leq n^{\zeta-(2k-1)\theta}. \tag{2.10}$$

The proof is deferred to the end of this subsection.

We will show that $\#D_{k+1} \geq n^{\zeta/2}\#D_k$ for any $k < m$. Once we get this, by iteration and $D_0 \geq 1$, $\#D_m \geq n^{\zeta m/2} \geq n^{5/4}$ holds, which is a contradiction. Thus the lemma follows.

Use (2.8) to obtain that

$$\sum_{i=0}^{R_{k+1}} \{(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1}\}$$

$$= \sum_{i=0}^{R_{k+1}} \{\Delta\gamma(i+s)^\alpha - \mathcal{L}_{k+1}^\alpha\} - \alpha\mathcal{L}_{k+1}^{\alpha-1} \sum_{i=0}^{R_{k+1}} \delta_{i,k+1} \tag{2.11}$$

$$\leq \sum_{i=0}^{R_{k+1}} \{\Delta\gamma(i+s)^\alpha - \mathcal{L}_{k+1}^\alpha\}.$$

When $i \in D_{k+1}^c$, by (2.10) we have $-\delta_{i,k+1} \geq \mathcal{L}_{k+1} - n^{\zeta-2(k+1)\theta} \geq \frac{\alpha+1}{2\alpha}\mathcal{L}_{k+1}$ for sufficiently large n and hence

$$(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1} \geq \frac{\alpha-1}{2}\mathcal{L}_{k+1}^\alpha. \tag{2.12}$$

This together with (2.9) and $D_k \subset D_{k+1}$ shows that the left hand side of (2.11) is further bounded from below by

$$\sum_{i \in D_k \cup D_{k+1}^c} \{(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1}\}$$

$$\geq \sum_{i \in D_k} \{(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1}\} + \frac{\alpha-1}{2} \sum_{i \in D_{k+1}^c} \mathcal{L}_{k+1}^\alpha.$$

On the other hand, due to (ii'), the right hand side of (2.11) is bounded from above by

$$\sum_{i=0}^{R_{k+1}} \{(\mathcal{L}_{k+1} + \delta'_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha + n^\theta\}. \tag{2.13}$$

The fact that $|a+b|^\alpha - |a|^\alpha \le \alpha|b-a|(|a|+|b|)^{\alpha-1}$ for $a,b \in \mathbb{R}$ together with (2.10) and $|\delta'_{i,k+1}| \le 2dn^\theta$ leads to that for any $i \in \{0, \cdots, R_{k+1}\}$,

$$(\mathcal{L}_{k+1} + \delta'_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha + n^\theta \le 4d\alpha n^{(\alpha-1)(\zeta-(2k+1)\theta)+\theta} \le \frac{\alpha-1}{2}\mathcal{L}_{k+1}^\alpha. \quad (2.14)$$

Thus, (2.13) is further bounded from above by

$$\sum_{i \in D_{k+1}} \{(\mathcal{L}_{k+1} + \delta'_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha + n^\theta\} + \sum_{i \in D_{k+1}^c} \{(\mathcal{L}_{k+1} + \delta'_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha + n^\theta\}$$

$$\le \sum_{i \in D_{k+1}} 4d\alpha n^{(\alpha-1)(\zeta-(2k+1)\theta)+\theta} + \frac{\alpha-1}{2} \sum_{i \in D_{k+1}^c} \mathcal{L}_{k+1}^\alpha.$$

$$(2.15)$$

With these observations, since $\delta_{i,k+1}$ is much larger than $\mathcal{L}_{k+1}$ for $i \in D_k$, we have

$$4d\alpha \#D_{k+1} n^{(\alpha-1)(\zeta-(2k+1)\theta)+\theta}$$
$$\ge \sum_{i \in D_k} \{(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1}\}$$
$$\ge \#D_k \min_{i \in D_k}\{(\mathcal{L}_{k+1} + \delta_{i,k+1})^\alpha - \mathcal{L}_{k+1}^\alpha - \alpha\mathcal{L}_{k+1}^{\alpha-1}\delta_{i,k+1}\}$$
$$\ge \frac{1}{2}\#D_k n^{\alpha(\zeta-2k\theta)}.$$

Rearranging it, we obtain $\#D_{k+1} \ge n^{\zeta/2}\#D_k$ as desired. $\qquad\square$

Proof of (2.10): Our strategy is to show that the last jump $\Delta\gamma(s+R_k)$ is small or R_k is large. We choose ζ' such that $n^{\zeta'} = \Delta\gamma(s+R_k)$. By (ii') and (2.8),

$$\sum_{i=0}^{R_k}\{(\mathcal{L}_k + \delta'_{i,k})^\alpha - \mathcal{L}_k^\alpha + n^\theta\} \ge \sum_{i=0}^{R_k}\{(\mathcal{L}_k + \delta_{i,k})^\alpha - \mathcal{L}_k^\alpha - \alpha\mathcal{L}_k^{\alpha-1}\delta_{i,k}\}. \quad (2.16)$$

By arguments similar to (2.14), we get $(\mathcal{L}_k + \delta'_{i,k})^\alpha - \mathcal{L}_k^\alpha \le 4d\alpha\mathcal{L}_k^{\alpha-1}n^\theta$. Thus, the left hand side of (2.16) is bounded from above by

$$(R_k + 1)(4d\alpha\mathcal{L}_k^{\alpha-1}n^\theta + n^\theta) \le R_k n^{\zeta(\alpha-1)+2\theta}.$$

On the other hand, by (2.9) and $\delta_{0,k} = \Delta\gamma(s) - \mathcal{L}_k \ge n^\zeta/2 \gg n^{\zeta-(2k-1)\theta} \ge \mathcal{L}_k$, the right hand side of (2.16) is bounded from below by

$$\left((\mathcal{L}_k + \delta_{0,k})^\alpha - \mathcal{L}_k^\alpha - \alpha\mathcal{L}_k^{\alpha-1}\delta_{0,k}\right) + \left((\mathcal{L}_k + \delta_{R_k,k})^\alpha - \mathcal{L}_k^\alpha - \alpha\mathcal{L}_k^{\alpha-1}\delta_{R_k,k}\right)$$
$$\ge \frac{1}{2}\max\left\{n^{\zeta\alpha}, n^{\zeta'\alpha}\right\}.$$

Consequently, we have

$$R_k \ge \frac{1}{2}\max\{n^{\zeta-2\theta}, n^{\zeta'\alpha-(\alpha-1)\zeta-2\theta}\} \ge \max\{n^{\zeta-3\theta}, n^{\zeta'-3\theta}\}.$$

By the definition of R_k, we have $|\gamma(s + R_k - 1) - \gamma(s - 1)|_1 \geq R_k n^{\zeta - (2k-1)\theta}$. This yields

$$
\begin{aligned}
\mathcal{L}_k &\geq \frac{1}{1 + R_k}(|\gamma(s + R_k - 1) - \gamma(s - 1)|_1 - \Delta\gamma(s + R_k)) \\
&\geq n^{\zeta - (2k-1)\theta}(1 - \frac{1}{1 + R_k}) - n^{3\theta} \\
&\geq n^{\zeta - (2k-1)\theta} - n^{\zeta - 2k\theta}.
\end{aligned}
$$

We have $\mathcal{L}_k \leq n^{\zeta - (2k-1)\theta}$ by the construction and (2.10) follows. $\qquad\square$

2.3. *Concentration for the FPP with $\alpha > 1$.* In this subsection, we prove Theorem 1.2 for $\alpha > 1$. We fix a small $\theta > 0$. Given a configuration $\omega_p \subset \mathbb{N} \times \mathbb{R}^d$, we define a new configuration $\bar{\omega}_p$ as

$$
\bar{\omega}_p = \omega_p + \sum_{(k,x) \in \mathbb{N} \times n^\theta \mathbb{Z}^d} 1_{\{\omega(\{k\} \times (x + [0, n^\theta)^d)) = 0\}} \delta_{(k,x)}, \tag{2.17}
$$

i.e. when we find a large vacant box, we artificially add an ω-point at a corner. We first recall Lemma 3.3 in Comets et al. (2015) which shows that $T_n(\omega_p)$ and $T_n(\bar{\omega}_p)$ are essentially the same.

Lemma 2.4. *There exists $c > 0$ independent of p such that for sufficiently large $n \in \mathbb{N}$,*

$$
\begin{aligned}
\max\{Q(T_n(\omega_p) \neq T_n(\bar{\omega}_p)), Q[|T_n(\omega_p) - T_n(\bar{\omega}_p)|]\} \\
\leq \exp\{-cn^{d\theta}\}.
\end{aligned}
$$

Proof: This is proved only for ω_1 in Comets et al. (2015) but the same proof works for general ω_p. In fact, it is proved that the probability that the set of all optimal paths for ω_p coincides with that for $\bar{\omega}_p$ is larger than $1 - \exp(-cn^{d\theta})$. The proof of Lemma 2.9 below contains a very similar argument. $\qquad\square$

Proof of (1.6) *for $\alpha > 1$:* Fix $\delta > 0$. Let us denote by $\bar{\omega}_p^{(m)}$ the point process obtained by replacing its $\{m\} \times \mathbb{R}^d$-section by another configuration $\bar{\omega}_p'$. We are going to use the so-called entropy method and it requires a bound on

$$
\sum_{m=1}^n \left(\sup_{\omega_p'} |T_n(\bar{\omega}_p^{(m)}) - T_n(\bar{\omega}_p)| \right)^2, \tag{2.18}
$$

where the supremum is taken over all configrations ω_p'.

Lemma 2.5. *Suppose that $\alpha > 1$. For any $\zeta > 0$, there exist $\theta \in (0, \zeta)$ and $N = N(\theta) \in \mathbb{N}$ such that for any $n > N$, $1 \leq i \leq n$, $\omega_p \subset \mathbb{N} \times \mathbb{R}^d$, and a minimizing path π_n for $\bar{\omega}_p$,*

$$
\Delta\pi_n(i) \leq n^\zeta.
$$

Proof: Let $\omega' = \bar{\omega}_p$ and $\gamma = \pi_n$ in Lemma 2.3. Then for any $s \in \{1, \cdots, n\}$, (i) or (ii) holds. If (ii) holds, it contradicts that π_n is a minimizing path. It follows that (i) holds and we get the desired conclusion. $\qquad\square$

By using Lemma 2.5, we can bound the summands of (2.18) from above by $n^{o(1)}$ and consequently (2.18) itself by $n^{1+o(1)}$.

Lemma 2.6. *For any $\zeta > 0$, there exist $\theta, N > 0$ such that for all $n \geq N$ and $\omega_p \subset \mathbb{N} \times \mathbb{R}^d$,*

$$|\Delta_m| \leq 4n^{\zeta\alpha},$$

where $\Delta_m := \sup_{\omega'} |T_n(\bar{\omega}_p^{(m)}) - T_n(\bar{\omega}_p)|$.

Proof: We prove the lower bound for Δ_m. The upper bound can be proved similarly. Fix some ω_p and ω_p'. Let π_n denote a minimizing path for $T_n(\bar{\omega}_p)$, chosen by a deterministic algorithm if not unique. We define a new point $\tilde{\pi}_n^{(m)}(m)$ as a point in $\bar{\omega}_p'|_{\{m\} \times \mathbb{R}^d}$ satisfying

$$|\pi_n(m) - \tilde{\pi}_n^{(m)}(m)|_1 \leq dn^\theta. \tag{2.19}$$

Then we can bound $T_n(\bar{\omega}_p^{(m)})$ from above by the passage time of the path

$$\tilde{\pi}_n^{(m)} = (\pi_n(1), \ldots, \pi_n(m-1), \tilde{\pi}_n^{(m)}(m), \pi_n(m+1), \ldots, \pi_n(n)).$$

By using Lemma 2.5 together with (2.19), for sufficiently small θ and large n, we get

$$\begin{aligned} T_n(\bar{\omega}_p^{(m)}) - T_n(\bar{\omega}_p) &\leq t(\tilde{\pi}_n^{(m)}) - t(\pi_n) \\ &\leq |\tilde{\pi}_n^{(m)}(m-1) - \tilde{\pi}_n^{(m)}(m)|_1^\alpha + |\tilde{\pi}_n^{(m)}(m) - \tilde{\pi}_n^{(m)}(m+1)|_1^\alpha \\ &\leq 2(n^\zeta + dn^\theta)^\alpha < 4n^{\zeta\alpha} \end{aligned} \tag{2.20}$$

as desired. The reverse inequality can be proved by a similar way. $\square$

If we take ζ sufficiently small, (2.18) is less than $Cn^{1+\delta}$. Then, Theorem 6.7 in Boucheron et al. (2013) yields that there exists $c > 0$ independent of p such that

$$Q\left(|T_n(\bar{\omega}_p) - Q[T_n(\bar{\omega}_p)]| > n^{\frac{1}{2}+\delta}\right) \leq \exp\{-cn^{1-\delta}\}.$$

By Lemma 2.4, there exists $c' > 0$ independent of p such that

$$\begin{aligned} &Q\left(|T_n(\omega_p) - Q[T_n(\omega_p)]| > 2n^{\frac{1}{2}+\delta}\right) \\ &\leq Q\left(|T_n(\bar{\omega}_p) - Q[T_n(\bar{\omega}_p)]| > n^{\frac{1}{2}+\delta}\right) + Q\left(T_n(\omega_p) \neq T_n(\bar{\omega}_p)\right) \\ &\leq \exp\{-c'n^{1-\delta}\} + \exp\{-c'n^{d\theta}\}, \end{aligned}$$

which implies (1.6). $\square$

By the proof of Lemma 2.4, Lemma 2.6 and the Borel–Cantelli Lemma, we also have the following corollary.

Corollary 2.7. *Suppose that $\alpha > 1$. For any $\zeta > 0$, the following happens Q-almost surely: there exists $N = N(\omega_p)$ such that for any $n \geq N$ and any minimizing path π_n for $T_n(\omega_p)$,*

$$\max_{1 \leq i \leq n} \Delta\pi_n(i) \leq n^\zeta.$$

2.4. *Concentration of the free energy.* In this subsection, we prove Theorem 1.5. By the relation

$$\log Z_n^{\eta,\beta} = \beta n + \log Z_n^{1-\eta,-\beta}, \tag{2.21}$$

we have only to consider the case $\beta \in [-\infty, 0]$. Note that, in this case, $Z_n^{\eta,\beta} \leq 1$ Q-a.s. For simplicity of notation, we write $Z_n(\omega)$ instead of $Z_n^{\beta,\eta}$ where ω is defined as

$$\omega := \sum_{(k,x)\in\mathbb{N}\times\mathbb{Z}^d} (1 - \eta(k,x))\delta_{(k,x)}.$$

We write $M := 1 + \alpha^{-1}$. For a given configuration η, or equivalently ω, and an n-path γ, we denote the free energy per path by

$$F_n(\gamma; \omega) := c_2 \sum_{i=1}^n \Delta\gamma_i^\alpha - \beta \sum_{i=1}^n \eta(i, \gamma(i)) = c_2 t(\gamma) - \beta H_n^\eta(\gamma),$$

where the constant c_2 comes from (1.1).

We assume that γ starts at the origin throughout this subsection. Then we can write the partition function as

$$Z_n(\omega) = c_1^n \sum_\gamma e^{-F_n(\gamma;\omega)}.$$

We define $\bar{\omega}$ by (2.17) and the restricted partition function by

$$\tilde{Z}_n(\omega) := c_1^n \sum_{\gamma: t(\gamma) \leq n^{1+2\alpha\theta}} e^{-F_n(\gamma;\omega)}.$$

First, we bound the difference of partition functions $Z_n(\omega)$ and $Z_n(\bar{\omega})$. We begin with the following tail bounds.

Lemma 2.8. *The following results hold:*

(i) *There exists $C_0 > 0$ independent of $p \in (0, 1]$ such that for all $n \in \mathbb{N}$ and $m > C_0 n$,*

$$Q(T_n(\omega_p) > m) \leq \exp\{-m^{1\wedge\frac{d}{\alpha}}/C_0\}. \tag{2.22}$$

(ii) *There exist $c > 0$ and $N \in \mathbb{N}$ such that for all $n > N$,*

$$Z_n(\omega) - \tilde{Z}_n(\omega) \leq e^{-cn^{1+2\alpha\theta}}.$$

Proof: (i) The proof is exactly the same as in Lemma 3.2 in Comets et al. (2015). We skip the details.
(ii) We bound $c_1^{-n}(Z_n(\omega) - \tilde{Z}_n(\omega))$ from above as follows:

$$
\begin{aligned}
c_1^{-n}(Z_n(\omega) - \tilde{Z}_n(\omega)) &= \sum_{t(\gamma)>n^{1+2\alpha\theta}} e^{-F_n(\gamma;\omega)} \\
&= \sum_{k\geq 2} \sum_{\gamma: n^{1+k\alpha\theta}<t(\gamma)\leq n^{1+(k+1)\alpha\theta}} e^{-c_2 t(\gamma)} \\
&\leq \sum_{k\geq 2} \sum_{\gamma: t(\gamma)\leq n^{1+(k+1)\alpha\theta}} e^{-c_2 n^{1+k\alpha\theta}}.
\end{aligned}
\tag{2.23}
$$

If $t(\gamma) \leq n^{1+(k+1)\alpha\theta}$, then γ has no jump larger than $n^{\frac{1}{\alpha}+(k+1)\theta}$ and thus

$$\max_{1 \leq i \leq n} |\gamma(i)|_1 \leq n^{M+(k+1)\theta}.$$

This yields the bound

$$\#\{\gamma : t(\gamma) \leq n^{1+(k+1)\alpha\theta}\} \leq n^{d(M+(k+1)\theta)n}.$$

Substituting this into (2.23), for sufficiently large n, we obtain

$$Z_n(\omega) - \tilde{Z}_n(\omega) \leq c_1^n \sum_{k \geq 2} n^{d(M+(k+1)\theta)n} e^{-c_2 n^{1+k\alpha\theta}} \leq e^{-cn^{1+2\alpha\theta}}$$

with some small constant $c > 0$. $\hfill\square$

We define $\bar{\omega}$ as in (2.17).

Lemma 2.9. *For any $q \in [0,1)$ and $\theta > 0$, there exist $\lambda, c > 0$ and $N \in \mathbb{N}$ such that for any $n > N$ and $p \in [0,q)$,*

    (i) $Q(|\log Z_n(\omega) - \log Z_n(\bar{\omega})| > \log 2) \leq e^{-cn^\lambda}$,
    (ii) $0 \leq Q[\log Z_n(\bar{\omega})] - Q[\log Z_n(\omega)] \leq 1$.

Proof: Since $Z_n(\omega) \geq c_1^n e^{-c_2 s_p^{-\alpha} T_n(\omega_p)}$, Lemma 2.8-(i) and $Z_n(\omega) \leq 1$ imply that there exist $C_1, C_2 > 0$ depending on q such that

$$Q(Z_n(\omega) \leq c_1^n e^{-c_2 n^{1+\alpha\theta}}) \leq e^{-C_1 n^{(1 \wedge \frac{d}{\alpha})(1+2\alpha\theta)}}, \tag{2.24}$$
$$Q[|\log Z_n(\omega)|^2] \leq C_2 n^2.$$

Also it is plain to see (from Lemma 3.3 in Comets et al., 2015) that there exists $C_3 > 0$ such that

$$Q(\omega = \bar{\omega} \text{ on } [-n^{M+2\theta}, n^{M+2\theta}]^d \times [0,n]) \leq e^{-C_3 n^{d\theta}}. \tag{2.25}$$

Thus to prove (i), it suffices to show $Z_n(\bar{\omega}) \leq 2Z_n(\omega)$ under the two conditions: $Z_n(\omega) \geq c_1^n e^{-c_2 n^{1+\alpha\theta}}$ and $\omega = \bar{\omega}$ on $[-n^{M+2\theta}, n^{M+2\theta}]^d \times [0,n]$. Observe that $Z_n(\omega) \leq Z_n(\bar{\omega})$ and that if γ exits from $[-n^{M+2\theta}, n^{M+2\theta}]^d \times [0,n]$, then $t(\gamma) > n^{1+2\alpha\theta}$ as it must contain a jump larger than $n^{\frac{1}{\alpha}+2\theta}$. Therefore under the above conditions, by Lemma 2.8-(ii), there exists $C_4 > 0$ such that

$$Z_n(\bar{\omega}) - Z_n(\omega) \leq c_1^n \sum_{\gamma : \max_{1 \leq k \leq n} |\gamma(k)|_\infty > n^{M+2\theta}} e^{-F_n(\gamma;\omega)}$$
$$\leq Z_n(\omega) - \tilde{Z}_n(\omega)$$
$$\leq e^{-C_4 n^{1+2\alpha\theta}} \leq Z_n(\omega).$$

This in turn implies

$Q[\log Z_n(\bar{\omega})] - Q[\log Z_n(\omega)]$
$\quad \leq Q(Z_n(\omega) \geq c_1^n e^{-c_2 n^{1+\alpha\theta}} \text{ and } \omega = \bar{\omega} \text{ on } [-n^{M+2\theta}, n^{M+2\theta}]^d \times [0,n]) \log 2$
$\quad\quad + Q[|\log Z_n(\omega)|; Z_n(\omega) < c_1^n e^{-c_2 n^{1+\alpha\theta}} \text{ or } \omega \neq \bar{\omega} \text{ on } [-n^{M+2\theta}, n^{M+2\theta}]^d \times [0,n]]$
$\quad \leq 1,$

where in the last line we have used (2.24), (2.25) and the Schwarz inequality to bound the second term. $\hfill\square$

From Lemma 2.9, once we show the concentration for $\bar\omega$, i.e. for any $\delta > 0$ there exist $\lambda \in (0,1)$ and $C_0, C_1 > 0$ such that

$$Q(|\log Z_n(\bar\omega) - Q[\log Z_n(\bar\omega)]| > n^{\frac{1}{2}+\delta}) \le C_0 e^{-C_1 n^\lambda}, \qquad (2.26)$$

the concentration for ω immediately follows. We will use the entropy method as in the FPP case and the following two lemmas correspond to Lemma 2.5 and 2.6.

Lemma 2.10. *Let $s \in \{1, \cdots, n\}$. We take configurations ω, ω' such that $\omega|_{\{\ell\}\times\mathbb{R}^d} = \omega'|_{\{\ell\}\times\mathbb{R}^d}$ for any $\ell \ne s$. Then, for any n-path γ, there exists γ' such that one of the following holds:*

(I) *for any $i \ne s$, $\gamma(i) = \gamma'(i)$, and $F_n(\gamma'; \bar\omega') \le c_2 n^{\zeta(\alpha-1)_+ + 3\theta} + F_n(\gamma; \bar\omega)$ or*
(II) *there exists $k > 0$ such that for any $i \notin [s, s+k]$, $\gamma(i) = \gamma'(i)$, and*

$$F_n(\gamma'; \bar\omega') + c_2(k+1)n^\theta \le F_n(\gamma; \bar\omega),$$

where the constant c_2 comes from (1.1) and $(\alpha-1)_+ = (\alpha-1) \vee 0$.

Proof: When $\alpha \le 1$, it is easy to prove that (I) holds for any γ. Suppose that $\alpha > 1$ and we will apply the Lemma 2.3 with $\omega' = \bar\omega'$. It suffices to show that (i) and (ii) in Lemma 2.3 imply (I) and (II) in Lemma 2.10, respectively. When (i) holds, we construct a polymer γ' such that $\gamma'(i) = \gamma(i)$ for $i \ne s$, $(s, \gamma'(s)) \in \bar\omega'|_{\{m\}\times\mathbb{R}^d}$, and $|\gamma'(s) - \gamma(s)|_1 \le dn^\theta$. Then, it is trivial that γ' satisfies condition (I). On the other hand, we can easily check that (ii) leads to (II) because of the assumption of $\beta \in [-\infty, 0]$, which is declared at the beginning of this subsection. $\qquad \square$

Lemma 2.11. *Let ω and ω' be the configurations as in Lemma 2.10. For any $\delta > 0$, there exist $\theta > 0$ and $C > 0$ such that for any $n \in \mathbb{N}$,*

$$|\log Z_n(\bar\omega) - \log Z_n(\bar\omega')| \le Cn^\delta.$$

Proof: Note first that for any ω,

$$\tilde Z_n(\bar\omega) \ge c_1^n e^{-c_2 T_n(\bar\omega)} \ge c_1^n e^{-c_2 d^\alpha n^{1+\alpha\theta}}$$

since $\bar\omega$ has θ-property. This together with Lemma 2.8-(ii) allows us to replace Z_n by $\tilde Z_n$ in the claim. For a path γ' and $k \in \mathbb{N}$, we define the following sets:

$$\Phi_1(\gamma') := \{\gamma : t(\gamma) \le c_2 n^{1+2\alpha\theta}, (\gamma, \gamma') \text{ satisfies condition (I)}\},$$

$$\Phi_{2,k}(\gamma') := \{\gamma : t(\gamma) \le c_2 n^{1+2\alpha\theta}, (\gamma, \gamma') \text{ satisfies condition (II) with } k\},$$

where (I) and (II) are those in Lemma 2.10. Then any n-path γ lies in one of the above sets and

$$\tilde Z_n(\bar\omega) \le c_1^n \sum_{\gamma'} \left\{ \sum_{\gamma \in \Phi_1(\gamma')} e^{-F_n(\gamma;\bar\omega)} + \sum_{k\ge 1}\sum_{\gamma\in\Phi_{2,k}(\gamma')} e^{-F_n(\gamma;\bar\omega)} \right\}$$

$$\le c_1^n \sum_{\gamma'} e^{-F_n(\gamma';\bar\omega')} \left\{ \sum_{\gamma\in\Phi_1(\gamma')} e^{c_2 n^{\zeta(\alpha-1)_+ + 3\theta}} + \sum_{k\ge 1}\sum_{\gamma\in\Phi_{2,k}(\gamma')} e^{-c_2 k n^\theta} \right\}$$

$$\le c_1^n \sum_{\gamma'} e^{-F_n(\gamma';\bar\omega')} \left\{ |\Phi_1(\gamma')| e^{c_2 n^{\zeta(\alpha-1)_+ + 3\theta}} + \sum_{k\ge 1} |\Phi_{2,k}(\gamma')| e^{-c_2 k n^\theta} \right\}.$$

Since for any γ', $|\Phi_1(\gamma')| \leq n^{2d(M+2\theta)}$ and $|\Phi_{2,k}(\gamma')| \leq n^{2d(M+2\theta)k}$ (recall the argument (2.23)), this is further bounded from above by

$$c_1^n \sum_{\gamma'} e^{-F_n(\gamma';\bar{\omega}')} \left\{ n^{2d(M+2\theta)} e^{c_2 n^{\zeta(\alpha-1)_+ +3\theta}} + \sum_{k \geq 1} n^{2dk(M+2\theta)} e^{-c_2 kn^\theta} \right\}$$

$$\leq \quad c_1^n \sum_{\gamma'} e^{-F_n(\gamma';\bar{\omega}')} \left\{ n^{2d(M+2\theta)} e^{c_2 n^{\zeta(\alpha-1)_+ +3\theta}} + \sum_{k \geq 1} e^{-c_2 kn^\theta/2} \right\}$$

$$\leq \quad c_1^n e^{2c_2 n^{\zeta(\alpha-1)_+ +3\theta}} \sum_{\gamma'} e^{-F_n(\gamma';\bar{\omega}')} = e^{2c_2 n^{\zeta(\alpha-1)_+ +3\theta}} Z_n(\bar{\omega}').$$

With the symmetry between ω and ω', this implies

$$\begin{aligned} |\log Z_n(\bar{\omega}) - \log Z_n(\bar{\omega}')| &\leq & \log\{2e^{2c_2 n^{\zeta(\alpha-1)_+ +3\theta}}\} \\ &\leq & 3c_2 n^{\zeta(\alpha-1)_+ +3\theta}. \end{aligned}$$

If we take $\zeta > 0$ sufficiently small so that $3c_2 n^{\zeta(\alpha-1)_+ +3\theta} < n^\delta$, the proof is completed. $\qquad\square$

Thanks to Lemma 2.11, we can use Theorem 6.7 in Boucheron et al. (2013) and we get the desired concentration (2.26) of $\log Z_n(\bar{\omega})$.

3. Non-random fluctuation

3.1. *FPP Case.* In this subsection, we deduce the so-called non-random fluctuation bound (1.7) from the concentration bound (1.6). This is a well-studied subject in the theory of first passage percolation and we shall adapt the argument of Zhang (2010) to our setting.

Proof of (1.7)*:* Let $\chi > 1/2$, $M = 1 + \alpha^{-1}$ and $\pi_n = \pi_n^{(n)}$, i.e. a minimizing path for $T_n(\omega_p)$. We define a face to face passage time

$$\Phi_n(k,l;\omega_p) = \inf\left\{ \sum_{i=k+1}^{l} |x_{i-1} - x_i|_1^\alpha : |x_k|_\infty < n^M \text{ and } \{(i,x_i)\}_{k+1}^l \subset \omega_p \right\}$$

and introduce the events

$$\begin{aligned} \mathcal{A}_1^\theta(n) &= \{\omega_p = \bar{\omega}_p \text{ on } [0,2n] \times [-n^M, n^M]^d\}, \\ \mathcal{A}_2(n) &= \left\{ \begin{array}{c} \text{There exist } |x|_\infty \leq 1/2 \text{ and a minimizing path for} \\ \Phi_n(n,2n;\omega_p) \text{ starting at } (n,x). \end{array} \right\}. \end{aligned}$$

Lemma 3.1. *We fix a sufficiently small constant $\theta > 0$. Then, there exists $c > 0$ such that for all sufficiently large n,*
   (i) $Q(\mathcal{A}_1^\theta(n)) \geq 1 - \exp\{-cn^\theta\}$,
   (ii) $Q(\mathcal{A}_2(n)) \geq 2^{-d} n^{-dM}$.

Proof: The first statement follows from the proof of Lemma 3.3 in Comets et al. (2015). Since the law of ω_p is invariant under time-space shift, the probabilities

$$Q(\text{There exists a minimizing path for } \Phi_n(0,n;\omega_p)$$

$$\text{starting at } (0,x) \text{ with } |x-y|_\infty \leq 1/2)$$

for $y \in \mathbb{Z}^d \cap (-n^M, n^M)^d$ are the same as $Q(\mathcal{A}_2(n))$. Since the union of the above events has probability one and the number of possible y's is less than $(2n^M)^d$, the second statement is completed. □

With this lemma and (1.6), we can complete the proof of (1.7). Note first that on the event $\mathcal{A}_1^\theta(n)$, we have $\pi_{2n}(n) \in (-n^M, n^M)^d$ as the displacement of π_{2n} until time n is at most $2n \cdot n^\theta \vee (2n)^{\theta+\alpha^{-1}} < n^M$ for any sufficiently small θ. As a result, one has

$$T_{2n}(\omega_p) \geq T_n(\omega_p) + \Phi_n(n, 2n; \omega_p)$$

since the second half of π_{2n} is candidates of the face to face minimizing paths. On the other hand, from Lemma 2.5, for any $\zeta > 0$, if we take $\theta > 0$ sufficiently small, on the event on $\mathcal{A} = \mathcal{A}_1^\theta(n) \cap \mathcal{A}_2(n)$,

$$\Phi_n(n, 2n; \omega_p) \geq T(n, 2n; \omega_p) - Cn^{\zeta(\alpha-1)_+ + \theta}$$

with some $C > 0$ independent of p since only possible differences come from the starting points, which can be controlled by using the mean value theorem. Therefore on $\mathcal{A}$, for any $\chi > 1/2$ and sufficiently large $n \in \mathbb{N}$, we have the following almost super-additivity:

$$T_{2n}(\omega_p) \geq T_n(\omega_p) + T(n, 2n; \omega_p) - n^\chi. \tag{3.1}$$

Now we use (1.6) to obtain

$$Q\left(|T_n(\omega_p) - Q[T_n(\omega_p)]| > n^\chi\right) \leq C_1 \exp\left\{-C_2 n^\lambda\right\}$$

and the same bound for $T_{2n}(\omega_p)$ and $T_{2n}(\omega_p)$. These bounds and Lemma 3.1 show that for all sufficiently large n,

$$\mathcal{A} \cap \bigcap_{(k,l) \in \{(0,2n),(0,n),(n,2n)\}} \{|T(k, l; \omega_p) - Q[T(k, l; \omega_p)]| \leq n^\chi\}$$

has positive probability and in particular non-empty. Hence we can replace the passage times in (3.1) by their expectation at the cost of extra $-3n^\chi$ on the right-hand side to obtain

$$\frac{1}{2n}Q[T_{2n}(\omega_p)] \geq \frac{1}{n}Q[T_n(\omega_p)] - 4n^{\chi-1}.$$

Iterating this, we arrive at

$$\frac{1}{n}Q[T_n(\omega_p)] \leq \frac{1}{2^k n}Q[T_{2^k n}(\omega_p)] + 4n^{\chi-1} \sum_{j=1}^{k-1} 2^{(\chi-1)j}.$$

and letting $k \to \infty$ proves $Q[T_n(\omega_p)] \leq n\mu_p + Cn^\chi$. Moreover, due to the usual subadditivity, we get the converse inequality $n\mu_p \leq Q[T_n(\omega_p)]$. Thus we have (1.7). From (1.6) and (1.7), we obtain (1.8). □

3.2. *Free energy Case.* The proof is almost the same as that for the FPP, so we only mention the outline here. To prove Theorem 1.3, we again use the argument of Zhang. Let

$$Z(k, \ell, x, \omega) := c_1^n \sum_{\gamma: \gamma(k)=x} e^{\beta \sum_{i=k+1}^\ell \eta(i, \gamma(i))} e^{-c_2 \sum_{i=k+1}^\ell \Delta\gamma(i)^\alpha}.$$

Let $M = 1 + \frac{1}{\alpha}$ and we define an event $\mathcal{A}$ by

$$\mathcal{A} := \{Z(n, 2n, 0, \omega) = \sup_{x \in [-n^M/2, n^M/2]^d} Z(n, 2n, x, \omega), Z_{2n}(\omega) \leq 2\tilde{Z}_{2n}(\omega)\}.$$

One can show, as in the case of the FPP, that for sufficiently large n,

$$Q(\mathcal{A}) \geq \frac{1}{2n^{Md}}.$$

On this event,

$$\log Z_{2n} - \log 2 \leq \log \tilde{Z}_{2n}$$

$$\leq \log \left\{ c_1^{2n} \sum_{\substack{\gamma: \max_{1 \leq i \leq n} |\gamma(i)|_\infty \leq n^M/2 \\ \gamma(0)=0}} e^{-F_{2n}(\gamma;\omega)} \right\}$$

$$\leq \log \left\{ c_1^n \sum_{x \in [-n^M/2, n^M/2]^d} \sum_{\substack{\gamma: \gamma(n)=x \\ \gamma(0)=0}} e^{-F_n(\gamma;\omega)} Z(n, 2n, x, \omega) \right\},$$

where we have used the argument below (2.23) in the second inequality and divided paths at its n-th point in the third ineqality. Thanks to the choice of $\mathcal{A}$, this is further bounded from above by

$$\log \left\{ c_1^n \sum_{x \in [-n^M/2, n^M/2]^d} \sum_{\substack{\gamma: \gamma(n)=x \\ \gamma(0)=0}} e^{-F_n(\gamma;\omega)} Z(n, 2n, 0, \omega) \right\}$$

$$\leq \log\left(Z_n(\omega) Z(n, 2n, 0, \omega)\right) = \log Z_n(\omega) + \log Z(n, 2n, 0, \omega).$$

From the concentration (1.9) and this, for any $\chi > 1/2$, there exists constant $C > 0$ independent of n such that,

$$Q[\log Z_{2n}] \leq 2Q[\log Z_n] + Cn^\chi.$$

In a similar way to the proof of Theorem 1.3, we have that for any k,

$$\frac{Q[\log Z_{2^k n}]}{2^k n} \leq \frac{Q[\log Z_n]}{n} + C' n^{\chi-1},$$

where $C' > 0$ is a constant independent of n. As $k \to \infty$, we have

$$\varphi(p, \beta) \leq \frac{1}{n} Q[\log Z_n] + C' n^{\chi-1}.$$

Finally, we shall derive the converse estimate by a similar way. Indeed, we replace $\mathcal{A}$ by $\{Z(n, 2n, 0, \omega) = \min_{x \in [-n^M/2, n^M/2]^d} Z(n, 2n, x, \omega), Z_n(\omega) \leq 2\tilde{Z}_n(\omega)\}$. On this event, we have

$$\log Z_{2n} \geq \log \tilde{Z}_n + \log Z(n, 2n, 0, \omega) \geq \log 2 + \log Z_n + \log Z(n, 2n, 0, \omega),$$

which implies

$$\varphi(p, \beta) \geq \frac{1}{n} Q[\log Z_n] - C' n^{\chi-1}.$$

This yields (1.10). From (1.9) and (1.10), it is immediate to prove (1.11).

4. Proof of continuity results

Thanks to the non-random fluctuation results, it suffices to show the continuities of $Q[T_n]$ and $Q[\log Z_n]$ for fixed $n \in \mathbb{N}$.

Lemma 4.1. *For any* $n \in \mathbb{N}$, $Q[T_n(\omega_p)]$ *is a continuous function of* $p \in [0, 1]$.

Proof: Note that $T_n(\cdot)$ is continuous on the set of locally finite point configurations with respect to the vague topology. Indeed for any ω, the definition of passage time tells us that only points inside a compact set matter. By local finiteness we can choose the compact set in such a way that its boundary contains no points of ω. Now if $\omega_N \to \omega$ vaguely as $N \to \infty$, then the points of ω_N inside the compact set converge to those of ω in the Hausdorff metric and then it easily follows that $T_n(\omega_N) \to T_n(\omega)$. By using the Skorohod representation theorem, we may assume that Q-almost surely, $\omega_p \to \omega'_p$ vaguely as $p \uparrow p'$. Then by continuity, $T_n(\omega_p) \to T_n(\omega'_p)$ as $p \uparrow p'$, Q-almost surely. From this, $T_n(\omega_p)$ is continuous in $p \in [0, 1]$. Since the uniform integrability of $\{T_n(\omega_p)\}_{p \in (0,1]}$ follows from Lemma 2.8, $L^1(Q)$ convergence follows. $\square$

The following lemma can be proved by a similar way to Lemma 4.1.

Lemma 4.2. $Q[\log Z_n]$ *is jointly continuous on* $(p, \beta) \in [0, 1) \times [-\infty, \infty)$.

From Lemmas 4.1 and 4.2, we obtain Corollaries 1.4 and 1.7, respectively.

Acknowledgements

References

S. Boucheron, G. Lugosi and P. Massart. *Concentration inequalities*. Oxford University Press, Oxford (2013). ISBN 978-0-19-953525-5. MR3185193.

F. Comets, R. Fukushima, S. Nakajima and N. Yoshida. Limiting results for the free energy of directed polymers in random environment with unbounded jumps. *J. Stat. Phys.* **161** (3), 577–597 (2015). MR3406700.

R. Fukushima and N. Yoshida. On exponential growth for a certain class of linear systems. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** (2), 323–336 (2012). MR3069368.

O. Garet, J.-B. Gouéré and R. Marchand. The number of open paths in oriented percolation. *Ann. Probab.* **45** (6A), 4071–4100 (2017). MR3729623.

C. D. Howard and C. M. Newman. Geodesics and spanning trees for Euclidean first-passage percolation. *Ann. Probab.* **29** (2), 577–623 (2001). MR1849171.

H. Krug and H. Spohn. Kinetic roughening of growing surfaces. In *Solids Far From Equilibrium*, pages 479–582. Cambridge University Press (1991). MR1163829.

Q. Liu and F. Watbled. Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. *Stochastic Process. Appl.* **119** (10), 3101–3132 (2009). MR2568267.

Y. Zhang. On the concentration and the convergence rate with a moment condition in first passage percolation. *Stochastic Process. Appl.* **120** (7), 1317–1341 (2010). MR2639748.