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Fluctuation exponents for stationary exactly solvable lattice polymer models via a Mellin transform framework

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Abstract. We develop a Mellin transform framework which allows us to simultaneously analyze the four known exactly solvable $1 + 1$ dimensional lattice polymer models: the log-gamma, strict-weak, beta, and inverse-beta models. Using this framework we prove the conjectured fluctuation exponents of the free energy and the polymer path for the stationary point-to-point versions of these four models. The fluctuation exponent for the polymer path was previously unproved for the strict-weak, beta, and inverse-beta models. An independent and concurrent work by Balázs et al. (2018) also gives the path fluctuation result for the beta model.

1. Introduction

The directed polymer in a random environment was first introduced by Huse and Henley (1985) to model the interaction between a long chain of molecules and random impurities. This was later reformulated by Imbrie and Spencer (1988) as a random walk in a random environment. See the recent lectures notes by Comets (2017) for additional historical background and a survey of techniques used to study directed polymers. In the $1 + 1$ dimensional case, a large class of

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polymer models are expected to lie in the KPZ universality class. For this class, the polymer path and free energy fluctuation exponents are conjectured to be $2/3$ and $1/3$, respectively, and the probability distribution of the rescaled free energy is conjectured to converge to the Tracy-Widom GUE distribution.

There are a few $1+1$ dimensional polymer models for which these results have been proved. Balázs et al. (2011), prove the fluctuation exponents for a Hopf-Cole solution to the KPZ equation with Brownian initial condition. This solution can be interpreted as the free energy of a stationary continuum directed polymer. Amir et al. (2011) study the Hopf-Cole solution to the KPZ equation with narrow-wedge initial condition and prove Tracy-Widom limit distribution for large time. For the semi-discrete Brownian directed polymer introduced by O'Connell and Yor (2001), the fluctuation exponents are proved by Seppäläinen and Valkó (2010), and the limit distribution is proved by Borodin and Corwin (2014) and Borodin et al. (2014).

In the setting of lattice directed polymers, there are four models for which results about the scaling exponents or limit distributions are known. The log-gamma directed polymer was introduced by Seppäläinen (2012), where the fluctuation exponents were proved. The limit distribution result was proved by Borodin et al. (2013). The strict-weak polymer model was simultaneously introduced by Corwin et al. (2015) and O'Connell and Ortmann (2015). Its limit distribution was proved through different methods in these two papers. The beta directed polymer was introduced by Barraquand and Corwin (2017), where its limit distribution was also calculated. The fourth model is the inverse-beta model, introduced by Thiery and Le Doussal (2015), in which they conjecture a formula for the Laplace transform of the polymer partition function and, contingent on this conjecture, show Tracy-Widom limit distribution for the rescaled free energy.

In this paper we provide a Mellin transform framework with which we are able to treat these four lattice polymer models simultaneously and prove the fluctuation exponents of the free energy and the polymer path for their stationary versions. While for the log-gamma model these results were previously shown by Seppäläinen (2012), for the strict-weak, beta, and inverse-beta models, the path fluctuation results are new. An independent and concurrent work by Balázs et al. (2018) also gives the path fluctuation result for the beta model.

Our methods rely upon a Burke-type stationarity property that each of these models possesses. This stationarity, along with a coupling argument, are used to prove a variance formula which is then amenable to analysis. This method was first used by Cator and Groeneboom (2006) to prove the order of the variance of the length of the longest weakly North-East path in Hammersley's process with sources and sinks. Balázs et al. (2006) adapt this method to prove the order of the fluctuations of the passage time and the fluctuations of the maximal path for last passage percolation with exponential weights. Seppäläinen (2012) used this method to prove the order of the fluctuation of the free energy and the polymer path fluctuations for the point-to-point log-gamma model with stationary boundary conditions, and upper bounds on the fluctuation exponents for the non-stationary point-to-point and point-to-line models. Seppäläinen and Valkó (2010) prove the scaling exponents for the O'Connell-Yor polymer, and Moreno Flores et al. (2014) extend the result to the intermediate disorder regime. Our paper closely follows the methods in Seppäläinen (2012); the Mellin transform framework provides a unified way to apply these methods to the four models.

In our related paper Chaumont and Noack (2017) we prove that in the setting of $1+1$ dimensional lattice directed polymers, the only four models possessing the Burke-type stationarity property are the log-gamma, strict-weak, beta, and inverse-beta models.

Notation: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, \ldots\}$, and $\mathbb{R}$ denotes the real numbers. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $\vee$ and $\wedge$ denote maximum and minimum, respectively. Given a real valued function f, let $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$ denote the support of the function f (note that we do not insist on taking the closure of this set). Given a random variable X with finite expectation, we let $\overline{X} = X - \mathbb{E}[X]$. For $A \subset \mathbb{R}$ write $-A = \{-a : a \in A\}$ and $A^{-1} = \{a^{-1} : a \in A\}$ assuming that $0 \notin A$. The symbol $\otimes$ is used to denote (independent) product distribution.

1.1. *The polymer model.* On each edge e of the $\mathbb{Z}_+^2$ lattice we place a positive random weight. The superscripts 1 and 2 will be used to denote horizontal and vertical edge weights, respectively. For $z \in \mathbb{N}^2$, let Y_z^1 and Y_z^2 denote the horizontal and vertical incoming edge weights. We assume that the collection of pairs $\{(Y_z^1, Y_z^2)\}_{z \in \mathbb{N}^2}$ is independent and identically distributed with common distribution (Y^1, Y^2), but do not insist that Y_z^1 is independent of Y_z^2. Call this collection the *bulk weights*. For $x \in \mathbb{N} \times \{0\}$, let R_x^1 denote the horizontal incoming edge weight, and for $y \in \{0\} \times \mathbb{N}$, let R_y^2 denote the vertical incoming edge weight. We assume the collections $\{R_x^1\}_{x \in \mathbb{N} \times \{0\}}$ and $\{R_y^2\}_{y \in \{0\} \times \mathbb{N}}$ are independent and identically distributed with common distributions R^1 and R^2, and refer to them as the *horizontal* and *vertical boundary weights*, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 1.1. We call

$$\omega = \{R_x^1, R_y^2, (Y_z^1, Y_z^2) : x \in \mathbb{N} \times \{0\}, y \in \{0\} \times \mathbb{N}, z \in \mathbb{N}^2\} \tag{1.1}$$

the *polymer environment.* We use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability measure and corresponding expectation of the polymer environment.

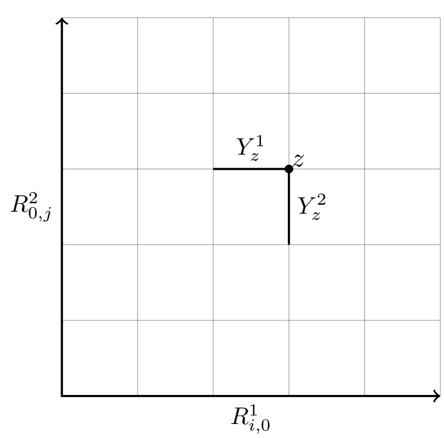

FIGURE 1.1. Assignment of edge weights.

A path is weighted according to the product of the weights along its edges. For $(m,n) \in \mathbb{Z}_+^2 \backslash \{(0,0)\}$ we define a probability measure on all up-right paths from $(0,0)$ to (m,n). See Figure 1.2 for an example of an up-right path. Let $\Pi_{m,n}$ denote the collection of all such paths. We identify paths $x_\bullet = (x_0, x_1, \dots, x_{m+n})$ by their sequence of vertices, but also associate to paths their sequence of edges $(e_1, \dots, e_{m+n})$, where $e_i = \{x_{i-1}, x_i\}$. Define the quenched polymer measure on $\Pi_{m,n}$,

$$Q_{m,n}(x_\bullet) := \frac{1}{Z_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i},$$

where ω_e is the weight associated to the edge e and

$$Z_{m,n} := \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i}$$

is the associated partition function. At the origin, define $Z_{0,0} := 1$. Taking the expectation $\mathbb{E}$ of the quenched measure with respect to the edge weights gives the annealed measure on $\Pi_{m,n}$,

$$P_{m,n}(x_\bullet) := \mathbb{E}[Q_{m,n}(x_\bullet)].$$

The annealed expectation will be denoted by $E_{m,n}$.

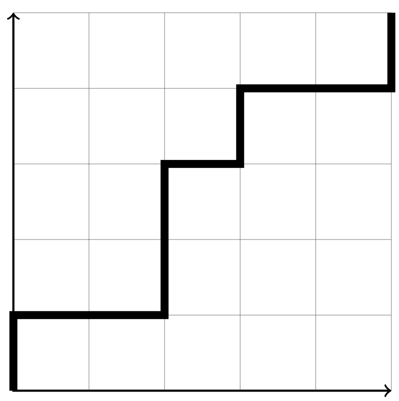

FIGURE 1.2. An up-right path from $(0,0)$ to $(5,5)$.

We specify the edge weight distributions for the four stationary polymer models. The notation $X \sim \mathrm{Ga}(\alpha,\beta)$ is used to denote that a random variable is gamma(α,β) distributed, i.e. has density $\Gamma(\alpha)^{-1}\beta^\alpha x^{\alpha-1}e^{-\beta x}$ supported on $(0,\infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function. $X \sim \mathrm{Be}(\alpha,\beta)$ is used to say that X is beta(α,β) distributed, i.e. has density $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ supported on $(0,1)$. We then use $X \sim \mathrm{Ga}^{-1}(\alpha,\beta)$ and $X \sim \mathrm{Be}^{-1}(\alpha,\beta)$ to denote that $X^{-1} \sim \mathrm{Ga}(\alpha,\beta)$ and $X^{-1} \sim \mathrm{Be}(\alpha,\beta)$, respectively. We also use $X \sim \big(\mathrm{Be}^{-1}(\alpha,\beta) - 1\big)$ to denote that $X + 1 \sim \mathrm{Be}^{-1}(\alpha,\beta)$.

- **Inverse-gamma (IG)**: This is also known as the log-gamma model. Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{aligned} R^1 &\sim \mathrm{Ga}^{-1}(\mu-\theta,\beta) \qquad R^2 \sim \mathrm{Ga}^{-1}(\theta,\beta) \\ (Y^1,Y^2) &= (X,X) \qquad \text{where} \qquad X \sim \mathrm{Ga}^{-1}(\mu,\beta). \end{aligned} \tag{1.2}$$

- **Gamma (G)**: This is also known as the strict-weak model. Assume $\theta,\ \mu,\ \beta > 0$ and

$$\begin{gathered} R^1 \sim \mathrm{Ga}(\mu+\theta,\beta) \qquad R^2 \sim \mathrm{Be}^{-1}(\theta,\mu) \\ (Y^1,Y^2) = (X,1) \qquad \text{where} \qquad X \sim \mathrm{Ga}(\mu,\beta). \end{gathered} \tag{1.3}$$

- **Beta (B)**: Assume $\theta,\ \mu,\ \beta > 0$ and

$$\begin{gathered} R^1 \sim \mathrm{Be}(\mu+\theta,\beta) \qquad R^2 \sim \mathrm{Be}^{-1}(\theta,\mu) \\ (Y^1,Y^2) = (X,1-X) \qquad \text{where} \qquad X \sim \mathrm{Be}(\mu,\beta). \end{gathered} \tag{1.4}$$

- **Inverse-beta (IB)**: Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{gathered} R^1 \sim \mathrm{Be}^{-1}(\mu-\theta,\beta) \qquad R^2 \sim \big(\mathrm{Be}^{-1}(\theta,\beta+\mu-\theta)-1\big) \\ (Y^1,Y^2) = (X,X-1) \qquad \text{where} \qquad X \sim \mathrm{Be}^{-1}(\mu,\beta). \end{gathered} \tag{1.5}$$

The name of each model refers to the distribution of the bulk weights. We call these models the **four basic beta-gamma models**.

1.2. *Results.* If X is a positive random variable with density ρ, define

$$L_X(x) := -\frac{1}{x\rho(x)}\mathbb{C}\mathrm{ov}(\log X, \mathbb{1}_{\{X\leqslant x\}}) \tag{1.6}$$

for all x such that $\rho(x) > 0$. Given a path $x_\bullet \in \Pi_{m,n}$, define the *exit points* of the path from the horizontal and vertical axes by

$$t_1 := \max\{i : (i,0) \in x_\bullet\} \qquad \text{and} \qquad t_2 := \max\{j : (0,j) \in x_\bullet\}. \tag{1.7}$$

The following proposition gives exact formulas for the expectation and variance of the free energy, which is a starting point for analysis of these four models.

Proposition 1.1. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1,Y^2)$ as in one of (1.2) through (1.5). Then for all $(m,n) \in \mathbb{Z}^2_+$,*

$$\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}[\log R^1] + n\mathbb{E}[\log R^2],$$

$$\mathbb{V}ar[\log Z_{m,n}] = -m\mathbb{V}ar[\log R^1] + n\mathbb{V}ar[\log R^2] + 2E_{m,n}\left[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})\right], \tag{1.8}$$

$$\mathbb{V}ar[\log Z_{m,n}] = m\mathbb{V}ar[\log R^1] - n\mathbb{V}ar[\log R^2] + 2E_{m,n}\left[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})\right]. \tag{1.9}$$

Using these exact formulas, we can obtain the following bounds on the variance of the free energy when (m,n) grow in a characteristic direction.

Theorem 1.2. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1,Y^2)$ as in one of (1.2) through (1.5), and let $(m,n) = (m_N,n_N)_{N=1}^\infty$ be a sequence such that*

$$|m_N - N\mathbb{V}ar[\log R^2]| \leqslant \gamma N^{2/3} \qquad \textit{and} \qquad |n_N - N\mathbb{V}ar[\log R^1]| \leqslant \gamma N^{2/3} \tag{1.10}$$

for some fixed $\gamma > 0$. Then there exist positive constants c, C, and N_0 depending only on μ,θ,β,γ such that for all $N \geqslant N_0$,

$$cN^{2/3} \leqslant \mathbb{V}ar[\log Z_{m,n}] \leqslant CN^{2/3}.$$

The same constants c, C, N_0 can be taken for all μ,θ,β,γ varying in a compact set.

Theorem 1.2 and a Borel-Cantelli argument give the following law of large numbers.

Corollary 1.3. *With assumptions as in Theorem 1.2 the following limit holds* $\mathbb{P}$-*almost surely*

$$\lim_{N\to\infty} \frac{\log Z_{m,n}}{N} = \mathbb{E}[\log R^1]\mathbb{V}ar[\log R^2] + \mathbb{E}[\log R^2]\mathbb{V}ar[\log R^1]. \tag{1.11}$$

For the four basic beta-gamma models, the right-hand side of (1.11) is given by

$$A(\mu,\theta,\beta)\Big(\frac{\partial}{\partial\theta}B(\mu,\theta)\Big) - \Big(\frac{\partial}{\partial\theta}A(\mu,\theta,\beta)\Big)B(\mu,\theta) + C(\mu,\theta),$$

where the functions A, B, and C are given in Table 1.3 and $\Psi_n(x) := \frac{\partial^{n+1}}{\partial x^{n+1}} \log\Gamma(x)$ denotes the polygamma function of order n.

Model	$A(\mu,\theta,\beta)$	$B(\mu,\theta)$	$C(\mu,\theta)$
IG	$\log\beta$	$\Psi_0(\theta)-\Psi_0(\mu-\theta)$	$-\Psi_0(\mu-\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu-\theta)$
G	$\log\beta$	$\Psi_0(\mu+\theta)-\Psi_0(\theta)$	$\Psi_0(\mu+\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu+\theta)$
B	$\Psi_0(\mu+\theta+\beta)$	$\Psi_0(\mu+\theta)-\Psi_0(\theta)$	$\Psi_0(\mu+\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu+\theta)$
IB	$-\Psi_0(\mu-\theta+\beta)$	$\Psi_0(\theta)-\Psi_0(\mu-\theta)$	$-\Psi_0(\mu-\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu-\theta)$

Figure 1.3. Functions for the limiting rescaled free energy of the four basic beta-gamma models.

The following is a result for when the sequence (m_N, n_N) does not satisfy condition (1.10). The statement is given for when the horizontal direction is too large, but an analogous result holds for the vertical direction.

Corollary 1.4. *Assume that the polymer environment has edge weight distributions* $R^1, R^2, (Y^1, Y^2)$ *as in one of* (1.2) *through* (1.5) *and that* $m, n \to \infty$. *Define* N *by* $n = N\mathbb{V}ar[\log R^1]$ *and assume*

$$N^{-\alpha}(m - N\mathbb{V}ar[\log R^2]) \to c_1 > 0$$

for some $\alpha > 2/3$. *Then as* $N \to \infty$

$$N^{-\alpha/2}\left(\log Z_{m,n} - \mathbb{E}[\log Z_{m,n}]\right)$$

converges in distribution to a centered normal with variance $c_1\mathbb{V}ar[\log R^1]$.

The variance formulas in Proposition 1.1 connect the variance of the free energy to the exit points of the path from the boundaries (1.7). This allows us to obtain bounds on the polymer path fluctuations under the annealed measure.

Given a path $x_\bullet \in \Pi_{m,n}$, for $0 \leqslant k \leqslant m$ and $0 \leqslant l \leqslant n$ define

$$\begin{aligned} v_0(l) &:= \min\{i : (i,l) \in x_\bullet\} & v_1(l) &:= \max\{i : (i,l) \in x_\bullet\} \\ w_0(k) &:= \min\{j : (k,j) \in x_\bullet\} & w_1(k) &:= \max\{j : (k,j) \in x_\bullet\}. \end{aligned} \tag{1.12}$$

This is illustrated in Figure 1.4.

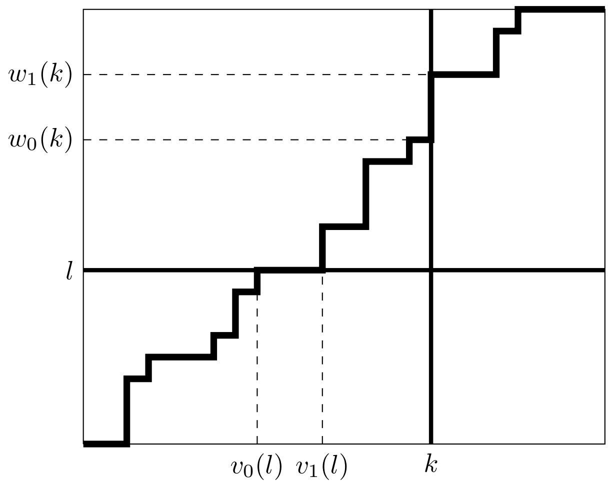

FIGURE 1.4. Example path with v_0, v_1, w_0, w_1 illustrated.

Theorem 1.5. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of (1.2) through (1.5), and let $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ be a sequence satisfying (1.10) for some fixed $\gamma > 0$. Let $0 \leqslant \tau < 1$. Then there exist positive constants b_0, C, c_0, c_1, N_0 depending only on μ, θ, β, γ, τ such that for $b \geqslant b_0$ and $N \in \mathbb{N}$,*

$$P_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3} \text{ or } v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) \leqslant \frac{C}{b^3}, \tag{1.13}$$

$$P_{m,n}(w_0(\lfloor \tau m \rfloor) \leqslant \tau n - bN^{2/3} \text{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + bN^{2/3}) \leqslant \frac{C}{b^3}, \tag{1.14}$$

and for all $N \geqslant N_0$,

$$c_0 \leqslant P_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + c_1 N^{2/3} \text{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + c_1 N^{2/3}). \tag{1.15}$$

The same constants can be taken for all μ, θ, β, γ, τ varying in a compact set.

Structure of the paper: In Section 2 we define the down-right property then state and prove consequences of this property. In Section 3 we introduce the Mellin transform framework, which allows us to treat the four basic beta-gamma models simultaneously, and prove Proposition 1.1. In Section 4 we prove the upper bound of Theorem 1.2. In Section 5 we prove bounds (1.13) and (1.14) of Theorem 1.5. In Section 6 we prove the lower bound of Theorem 1.2 and bound (1.15) of Theorem 1.5. In Appendix A we verify that each of the four basic beta-gamma models satisfies the conditions of Hypothesis 3.6. Appendix B collects technical lemmas used in Sections 3 and 4. Appendix C collects facts used in the proof of Proposition 6.1.

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2. The down-right property

Write $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. For $k = 1,2$ define ratios of partition functions

$$R_x^k := \frac{Z_x}{Z_{x-\alpha_k}} \qquad \text{for all } x \text{ such that } x - \alpha_k \in \mathbb{Z}_+^2.$$

Note that these extend the definitions of $R_{i,0}^1$ and $R_{0,j}^2$, since for example $Z_{i,0} = \prod_{k=1}^{i} R_{k,0}^1$. We say that $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$ is a down-right path in $\mathbb{Z}_+^2$ if $\pi_k \in \mathbb{Z}_+^2$ and $\pi_{k+1} - \pi_k \in \{\alpha_1, -\alpha_2\}$ for each $k \in \mathbb{Z}$. To each edge along a down-right path we associate the random variable

$$\Lambda_{\{\pi_{k-1},\pi_k\}} := \begin{cases} R_{\pi_k}^1 & \text{if } \{\pi_{k-1},\pi_k\} \text{ is horizontal,} \\ R_{\pi_{k-1}}^2 & \text{if } \{\pi_{k-1},\pi_k\} \text{ is vertical.} \end{cases}$$

The following definition is a weaker form of the Burke property (see Theorem 3.3 of Seppäläinen, 2012).

Definition 2.1. Say the polymer model has the down-right property if for all down-right paths $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$, the random variables

$$\Lambda(\pi) := \{\Lambda_{\{\pi_{k-1},\pi_k\}} : k \in \mathbb{Z}\}$$

are independent and each $R_{\pi_k}^1$ and $R_{\pi_k}^2$ appearing in the collection are respectively distributed as R^1 and R^2.

The partition functions satisfy the recurrence relation

$$Z_x = Y_x^1 Z_{x-\alpha_1} + Y_x^2 Z_{x-\alpha_2} \qquad \text{for } x \in \mathbb{N}^2. \tag{2.1}$$

This recurrence relation then implies the recursions

$$\begin{aligned} R_x^1 &= Y_x^1 + Y_x^2 \frac{R_{x-\alpha_2}^1}{R_{x-\alpha_1}^2} \\ R_x^2 &= Y_x^1 \frac{R_{x-\alpha_1}^2}{R_{x-\alpha_2}^1} + Y_x^2 \end{aligned} \qquad \text{for } x \in \mathbb{N}^2. \tag{2.2}$$

Using the recursions (2.2) we can reduce the down-right property to a simple preservation in distribution.

Lemma 2.2. *Let $R^1, R^2, (Y^1, Y^2)$ be positive random variables such that R^1, R^2 and the pair (Y^1, Y^2) are independent. Put*

$$\big(\widetilde{R}^1, \widetilde{R}^2\big) := (Y^1 + Y^2 R^1/R^2,\ Y^1 R^2/R^1 + Y^2).$$

Then the polymer model with edge weights $R^1, R^2, (Y^1, Y^2)$ has the down-right property if and only if $\big(\widetilde{R}^1, \widetilde{R}^2\big) \stackrel{d}{=} \big(R^1, R^2\big)$.

Proof of Lemma 2.2: Given a down-right path π, define its lower-left interior

$$\mathrm{Int}(\pi) := \{x \in \mathbb{Z}_+^2 \text{ such that } x + (m,n) \in \{\pi\} \text{ for some } m,n \in \mathbb{N}\}.$$

If the polymer model with edge weights $R^1, R^2, (Y^1, Y^2)$ has the down-right property, taking π to be the unique down-right path with interior $\{(0,0)\}$ implies that $(R_{1,1}^1, R_{1,1}^2) \stackrel{d}{=} (R^1, R^2)$. Then (2.2) and the fact that $\big(R_{1,0}^1, R_{0,1}^2, (Y_{1,1}^1, Y_{1,1}^2)\big) \stackrel{d}{=} \big(R^1, R^2, (Y^1, Y^2)\big)$ imply that $(\widetilde{R}^1, \widetilde{R}^2) \stackrel{d}{=} (R^1, R^2)$.

For the converse direction, we first prove the statement for π with finite interior. The case when the interior is empty is true by assumption. Assume that the down-right property holds for all paths π with $|\mathrm{Int}(\pi)| = n$. Given a path π with $|\mathrm{Int}(\pi)| = n+1$ there exists x such that π traverses the right-down corner $\{x-\alpha_1, x, x-\alpha_2\}$. Let $\widetilde{\pi}$ be the path which traverses the same points as π with the exception of instead passing through the down-right corner $\{x-\alpha_1, x-\alpha_1-\alpha_2, x-\alpha_2\}$. Then $|\mathrm{Int}(\widetilde{\pi})| = n$ and so $\left(R^1_{x-\alpha_2}, R^2_{x-\alpha_1}\right) \overset{d}{=} \left(R^1, R^2\right)$. Using (2.2), the assumption that $(\widetilde{R}^1, \widetilde{R}^2) \overset{d}{=} (R^1, R^2)$ and the independence of (Y^1_x, Y^2_x) from the collection $\Lambda(\widetilde{\pi})$ gives us that the collection $\Lambda(\pi)$ has the desired property.

To prove the statement for arbitrary π, pick a finite sub-collection F of $\Lambda(\pi)$. Then there exists $\widetilde{\pi}$ such that $\mathrm{Int}(\widetilde{\pi})$ is finite and $F \subset \Lambda(\widetilde{\pi})$. Since the statement holds for down-right paths with finite interior, we are done. □

Proposition 2.3. *Each of the four basic beta-gamma models,* (1.2) *through* (1.5)*, possesses the down-right property.*

Proof: The $(\widetilde{R}^1, \widetilde{R}^2) \overset{d}{=} (R^1, R^2)$ condition in Lemma 2.2 has been checked for the inverse-gamma, gamma, beta, and inverse-beta models by Lemma 3.2 of Seppäläinen (2012), Lemma 6.3 of Corwin et al. (2015), Lemma 3.1 of Balázs et al. (2018), and Proposition 3.1 of Thiery (2016) respectively. □

The following lemma is an immediate consequence of the down-right property and the starting point for the proof of Proposition 1.1.

Lemma 2.4. *If the polymer model with edge weights R^1, R^2, (Y^1, Y^2) possesses the down-right property and $\log R^1$, $\log R^2$ both have finite variance, then for all $(m,n) \in \mathbb{Z}^2_+$,*

(a) $\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}[\log R^1] + n\mathbb{E}[\log R^2]$,

(b) $\mathbb{V}ar[\log Z_{m,n}] = -m\mathbb{V}ar[\log R^1] + n\mathbb{V}ar[\log R^2] + 2\mathbb{C}ov(S_N, S_S)$,

(c) $\mathbb{V}ar[\log Z_{m,n}] = m\mathbb{V}ar[\log R^1] - n\mathbb{V}ar[\log R^2] + 2\mathbb{C}ov(S_E, S_W)$,

where

$$\begin{aligned} S_N &:= \log Z_{m,n} - \log Z_{0,n} = \sum_{i=1}^{m} \log R^1_{i,n}, \qquad S_S := \log Z_{m,0} = \sum_{i=1}^{m} \log R^1_{i,0}, \\ S_E &:= \log Z_{m,n} - \log Z_{m,0} = \sum_{j=1}^{n} \log R^2_{m,j}, \quad S_W := \log Z_{0,n} = \sum_{j=1}^{n} \log R^2_{0,j}. \end{aligned} \tag{2.3}$$

Proof: By the down-right property S_S is independent of S_W, S_N is independent of S_E, and

$$\mathbb{V}\mathrm{ar}[S_N] = \mathbb{V}\mathrm{ar}[S_S] = m\mathbb{V}\mathrm{ar}[\log R^1], \quad \mathbb{V}\mathrm{ar}[S_E] = \mathbb{V}\mathrm{ar}[S_W] = n\mathbb{V}\mathrm{ar}[\log R^2].$$

These facts along with the equalities $\log Z_{m,n} = S_N + S_W = S_E + S_S$ gives (a) and

$$\begin{aligned} \mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= \mathbb{V}\mathrm{ar}[S_N] + \mathbb{V}\mathrm{ar}[S_W] + 2\mathbb{C}\mathrm{ov}(S_N, S_W) \\ &= \mathbb{V}\mathrm{ar}[S_N] + \mathbb{V}\mathrm{ar}[S_W] + 2\mathbb{C}\mathrm{ov}(S_N, S_E + S_S - S_N) \\ &= -\mathbb{V}\mathrm{ar}[S_N] + \mathbb{V}\mathrm{ar}[S_W] + 2\mathbb{C}\mathrm{ov}(S_N, S_S) \\ &= -m\mathbb{V}\mathrm{ar}[\log R^1] + n\mathbb{V}\mathrm{ar}[\log R^2] + 2\mathbb{C}\mathrm{ov}(S_N, S_S). \end{aligned}$$

Similarly,

$$\mathbb{V}\mathrm{ar}[\log Z_{m,n}] = -n\mathbb{V}\mathrm{ar}[\log R^2] + m\mathbb{V}\mathrm{ar}[\log R^1] + 2\mathbb{C}\mathrm{ov}(S_E, S_W).$$

□

3. The Mellin transform framework

In this section we develop a framework which allows us to treat the four basic beta-gamma models simultaneously.

Given a function $f : (0, \infty) \to [0, \infty)$, write M_f for its Mellin transform

$$M_f(a) := \int_0^\infty x^{a-1} f(x)dx$$

for any $a \in \mathbb{R}$ such that the integral converges. Define

$$D(M_f) := \text{interior}(\{a \in \mathbb{R} : 0 < M_f(a) < \infty\}).$$

Definition 3.1. Given a function $f : (0, \infty) \to [0, \infty)$ such that $D(M_f)$ is non-empty, we define a family of densities on $(0, \infty)$ parametrized by $a \in D(M_f)$:

$$\rho_{f,a}(x) := M_f(a)^{-1} x^{a-1} f(x). \tag{3.1}$$

We write $X \sim m_f(a)$ to denote that the random variable X has density $\rho_{f,a}$.

Remark 3.2. If $f : (0, \infty) \to [0, \infty)$ is such that $D(M_f)$ is non-empty, then M_f is C^∞ throughout $D(M_f)$. Furthermore, if $X \sim m_f(a)$, then

(a) $\log X$ has finite exponential moments. That is, there exists some $\epsilon > 0$ such that

$$\mathbb{E}[e^{\epsilon |\log X|}] \leqslant \mathbb{E}[X^\epsilon] + \mathbb{E}[X^{-\epsilon}] = \frac{M_f(a+\epsilon) + M_f(a-\epsilon)}{M_f(a)} < \infty.$$

(b) For all $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial a^k} M_f(a) = M_f(a)\mathbb{E}[(\log X)^k].$$

(c) $\mathbb{E}[\log X] = \psi_0^f(a)$ and $\mathbb{V}\text{ar}[\log X] = \psi_1^f(a)$, where

$$\psi_n^f(a) := \frac{\partial^{n+1}}{\partial a^{n+1}} \log M_f(a) \text{ for } n \in \mathbb{Z}_+.$$

The following remark says that random variables with densities of the form (3.1) are closed under inversion.

Remark 3.3. If $f : (0, \infty) \to [0, \infty)$ is such that $D(M_f)$ is non-empty and $g(x) := f(\frac{1}{x})$ for $x \in (0, \infty)$, then for all $a \in D(M_f)$,

(a) $X \sim m_f(a) \Leftrightarrow X^{-1} \sim m_g(-a)$,
(b) $M_f(a) = M_g(-a)$ and therefore $D(M_g) = -D(M_f)$,
(c) $\psi_n^f(a) = (-1)^{n+1}\psi_n^g(-a)$ for all $n \in \mathbb{N}$.

Definition 3.4. Let $f^j : (0, \infty) \to [0, \infty)$ be such that $D(M_{f^j})$ is non-empty for $j = 1, 2$. We say that the polymer environment is Mellin-type with respect to (f^1, f^2) if $(R^1, R^2) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2)$ for some $a_j \in D(M_{f^j})$.

When the polymer environment is Mellin-type with parameters (a_1, a_2), we use $\mathbb{P}^{(a_1,a_2)}$, $\mathbb{E}^{(a_1,a_2)}$, $\mathbb{V}\text{ar}^{(a_1,a_2)}$, $\mathbb{C}\text{ov}^{(a_1,a_2)}$ in place of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{V}\text{ar}$, $\mathbb{C}\text{ov}$ respectively.

3.1. *The four basic beta-gamma models are Mellin-type.* We first specify functions f to obtain each of the random variables appearing in the four basic beta-gamma models. Note that the fourth column in Table 3.5 specifies the distribution of the random variable corresponding to f. We let $\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denote the beta function and recall that $\Psi_n(x) = \frac{\partial^{n+1}}{\partial x^{n+1}} \log \Gamma(x)$. For the Table 3.5 we assume $b > 0$ and $a \in D(M_f)$.

$f(x)$	$D(M_f)$	$M_f(a)$	$m_f(a)$
e^{-bx}	$(0,\infty)$	$\Gamma(a)/b^a$	$\mathrm{Ga}(a,b)$
$e^{-b/x}$	$(-\infty,0)$	$\Gamma(-a)b^a$	$\mathrm{Ga}^{-1}(-a,b)$
$(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$	$(0,\infty)$	$\mathrm{B}(a,b)$	$\mathrm{Be}(a,b)$
$(1-\frac{1}{x})^{b-1}\mathbb{1}_{\{x>1\}}$	$(-\infty,0)$	$\mathrm{B}(-a,b)$	$\mathrm{Be}^{-1}(-a,b)$
$(\frac{x}{x+1})^b$	$(-b,0)$	$\mathrm{B}(-a,b+a)$	$\mathrm{Be}^{-1}(-a,b+a)-1$

$f(x)$	$\psi_n^f(a)$
e^{-bx}	$\Psi_n(a) - \delta_{n,0}\log b$
$e^{-b/x}$	$(-1)^{n+1}(\Psi_n(-a) - \delta_{n,0}\log b)$
$(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$	$\Psi_n(a) - \Psi_n(a+b)$
$(1-\frac{1}{x})^{b-1}\mathbb{1}_{\{x>1\}}$	$(-1)^{n+1}(\Psi_n(-a) - \Psi_n(-a+b))$
$(\frac{x}{x+1})^b$	$\Psi_n(a+b) + (-1)^{n+1}\Psi_n(-a)$

FIGURE 3.5. Mellin framework data for the distributions appearing in the four basic beta-gamma models.

To express the distribution of the polymer environment in each of the four basic beta-gamma models given in (1.2) through (1.5) within this Mellin framework, we let

$$(R^1, R^2, X) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2) \otimes m_{f^1}(a_3), \tag{3.2}$$

where the functions f^1, f^2 and parameters a_j, $j = 1, 2, 3$ are given in Table 3.6. Recall that in each of the models, (Y^1, Y^2) are given in terms of X. For Table 3.6 we assume $\mu, \beta > 0$.

Model	$f^1(x)$	$f^2(x)$	(a_1, a_2, a_3)	
IG	$e^{-\beta/x}$	$e^{-\beta/x}$	$(\theta-\mu, -\theta, -\mu)$	$\theta \in (0,\mu)$
G	$e^{-\beta x}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta, -\theta, \mu)$	$\theta \in (0,\infty)$
B	$(1-x)^{\beta-1}\mathbb{1}_{\{0<x<1\}}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta, -\theta, \mu)$	$\theta \in (0,\infty)$
IB	$(1-\frac{1}{x})^{\beta-1}\mathbb{1}_{\{x>1\}}$	$(\frac{x}{x+1})^{(\beta+\mu)}$	$(\theta-\mu, -\theta, -\mu)$	$\theta \in (0,\mu)$

FIGURE 3.6. Functions and parameters to fit the four basic beta-gamma models into the Mellin framework.

Remark 3.5. For each fixed value of the bulk parameter a_3, we obtain a family of models with boundary parameters a_1 and a_2 satisfying $a_1 + a_2 = a_3$. For any such a_1 and a_2, by Proposition 2.3 these models will have the down-right property.

3.2. *Coupling of polymer environments.* In order to compare polymer environments with different parameters, we use a coupling to express the boundary weights as functions of i.i.d. uniform$(0,1)$ random variables.

If $f:(0,\infty)\to[0,\infty)$ is such that $D(M_f)$ is non-empty, write F^f for the CDF of the random variable $X\sim m_f(a)$. Specifically, $F^f : D(M_f)\times[0,\infty)\to[0,1]$ is given by

$$F^f(a,x)=\frac{1}{M_f(a)}\int_0^x y^{a-1}f(y)dy.$$

Define the quantile function

$$H^f(a,p):=\inf\{x : p\leqslant F^f(a,x)\}. \tag{3.3}$$

If the random variable η is uniformly distributed on the interval $(0,1)$, then $H^f(a,\eta)\sim m_f(a)$.

Suppose that a polymer environment ω is Mellin-type with respect to (f^1,f^2) with parameters (b_1,b_2). Let $\{\eta_i^1,\eta_j^2 : i,j\in\mathbb{N}\}$ be i.i.d. uniform$(0,1)$ random variables that are independent of the bulk weights $\{(Y_z^1,Y_z^2) : z\in\mathbb{N}^2\}$. Write $\widehat{\mathbb{P}}$, $\widehat{\mathbb{E}}$, and $\widehat{\mathbb{V}\mathrm{ar}}$ for the probability measure and the corresponding expectation and variance of these uniform random variables and the bulk weights. Define the coupled environment

$$\omega^{(b_1,b_2)}:=\{H^{f^1}(b_1,\eta_i^1),H^{f^2}(b_2,\eta_j^2),(Y_z^1,Y_z^2) : i\in\mathbb{N},j\in\mathbb{N},z\in\mathbb{N}^2\}. \tag{3.4}$$

Note that this environment is equal in distribution to the original environment ω.

To specifically denote weights accumulated by a path, the partition function, the quenched measure, and the annealed expectation, associated to the coupled environment $\omega^{(b_1,b_2)}$, define

$$\begin{aligned}
W(b_1,b_2)(x_\bullet) &:= \prod_{k=1}^{m+n}\omega^{(b_1,b_2)}_{(x_{k-1},x_k)} \qquad \text{for } x_\bullet\in\Pi_{m,n}\\
Z_{m,n}(b_1,b_2) &:= \sum_{x_\bullet\in\Pi_{m,n}} W(b_1,b_2)(x_\bullet)\\
Q^{(b_1,b_2)}_{m,n}(A) &:= \frac{1}{Z_{m,n}(b_1,b_2)}\sum_{x_\bullet\in A} W(b_1,b_2)(x_\bullet) \qquad \text{for } A\subset\Pi_{m,n}\\
E^{(b_1,b_2)}_{m,n}[\bullet] &:= \widehat{\mathbb{E}}\left[E^{Q^{(b_1,b_2)}_{m,n}}[\bullet]\right].
\end{aligned} \tag{3.5}$$

Recall the definition of the exit points t_j (1.7). We can decompose the weight accumulated along a path to isolate the dependence on boundary weights

$$W(b_1,b_2)(x_\bullet)=\prod_{i=1}^{t_1}H^{f^1}(b_1,\eta_i^1)\prod_{j=1}^{t_2}H^{f^2}(b_2,\eta_j^2)\prod_{k=(t_1\vee t_2)+1}^{m+n}\omega^{(b_1,b_2)}_{(x_{k-1},x_k)}. \tag{3.6}$$

Notice that one of the first two products will be empty and the third product involves only the bulk weights.

If we assume that $f:(0,\infty)\to[0,\infty)$ has open support, is continuous on its support, and $D(M_f)$ is non-empty, then F^f is continuously differentiable on the set $D(M_f)\times\mathrm{supp}(f)$. By the implicit function theorem, H^f is continuously differentiable and for all $(a,p)\in D(M_f)\times(0,1)$, we have

$$\frac{\partial H^f}{\partial a}(a,p)=\frac{-\frac{\partial F^f}{\partial a}(a,H^f(a,p))}{\frac{\partial F^f}{\partial x}(a,H^f(a,p))}=H^f(a,p)L^f(a,H^f(a,p)) \tag{3.7}$$

where L^f is given by

$$L^f(a,x) := \frac{x^{-a}}{f(x)} \int_0^x (\psi_0^f(a) - \log y) y^{a-1} f(y) dy$$
$$= -\frac{x^{-a}}{f(x)} \int_x^\infty (\psi_0^f(a) - \log y) y^{a-1} f(y) dy. \tag{3.8}$$

The second equality follows from the definition of $\psi_0^f(a)$. Notice that

$$L^f(a,x) = -\frac{1}{x\rho_{f,a}(x)} \mathbb{C}\mathrm{ov}(\log X, \mathbb{1}_{\{X \leqslant x\}}) = L_X(x) \text{ (as defined in (1.6))}$$

when $X \sim m_f(a)$, and therefore $L^f(a,x) \geqslant 0$.

The following hypothesis collects technical conditions for the function f used in the sequel.

Hypothesis 3.6. Suppose that $f : (0,\infty) \to [0,\infty)$ is such that $D(M_f)$ is non-empty, f has open support, is differentiable on its support, and for all compact $K \subset D(M_f)$ there exists a constant C depending only on K such that the following hold for all $a \in K$:

$$L^f(a,x) \leqslant C(1 + |\log x|) \qquad \text{for all } x \in \mathrm{supp}(f), \tag{3.9}$$

$$\int_0^1 \Big| \frac{\partial}{\partial a} L^f(a, H^f(a,p)) \Big| dp \leqslant C. \tag{3.10}$$

Remark 3.7. If $X \sim m_f(a)$ where f satisfies Hypothesis 3.6, then by (3.9) and Remark 3.2, $L_X(X)$ has finite exponential moments. By Lemma A.2 in the appendix, each of the functions f corresponding to the random variables appearing in the four basic beta-gamma models (see Table 3.5) satisfies Hypothesis 3.6.

Lemma 3.8. *Assume that the polymer environment is Mellin-type with respect to* (f^1, f^2)*, where* f^1 *and* f^2 *satisfy Hypothesis 3.6. Further assume that* $\log Y^1$ *and* $\log Y^2$ *have finite variance. Recall the notation (2.3). Then for all* $(m,n) \in \mathbb{Z}_+^2$,

$$\mathbb{C}ov(S_N, S_S) = E_{m,n}\Big[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})\Big], \tag{3.11}$$

$$\mathbb{C}ov(S_E, S_W) = E_{m,n}\Big[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})\Big]. \tag{3.12}$$

Proof: By assumption, there exists $(a_1, a_2) \in D(M_{f^1}) \times D(M_{f^2})$ such that $(R^1, R^2) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2)$. There exist open neighborhoods U_j about a_j contained in $D(M_{f^j})$ for $j = 1, 2$. We then show that

$$\frac{\partial}{\partial b_1} \mathbb{E}^{(b_1,a_2)}[S_N] = \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S) \text{ for all } b_1 \in U_1, \tag{3.13}$$

$$\frac{\partial}{\partial b_2} \mathbb{E}^{(a_1,b_2)}[S_E] = \mathbb{C}\mathrm{ov}^{(a_1,b_2)}(S_E, S_W) \text{ for all } b_2 \in U_2, \tag{3.14}$$

and that the mappings $b_1 \mapsto \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)$ and $b_2 \mapsto \mathbb{C}\mathrm{ov}^{(a_1,b_2)}(S_E, S_W)$ are continuous. We begin with (3.13). We will vary the parameter b_1 of the weights $R^1_{i,0}$ while keeping the parameter a_2 of the weights $R^2_{0,j}$ fixed. Let $\widetilde{\mathbb{E}}$ be the expectation over $\{R^2_{0,j}, (Y^1_x, Y^2_x)\}_{j\in\mathbb{N}, x\in\mathbb{N}\times\mathbb{N}}$. By Remark 3.2 and Lemma B.1, $\mathbb{E}^{(b_1,a_2)}[S_N^2] < \infty$ for all $b_1 \in U_1$. Then $\mathbb{E}^{(b_1,a_2)}[S_N] = \mathbb{E}^{b_1}[\widetilde{\mathbb{E}}[S_N]]$ where $\mathbb{E}^{b_1}$ denotes the expectation over $\{R^1_{i,0}\}_{i=1}^m$ when $R^1 \sim m_{f^1}(b_1)$. We now invoke Lemma B.2. Specifically, we use $r = m$, $X_k = R^1_{k,0}$, $f_k = f^1$ for all $k = 1, \dots, m$ and $A(R^1_{1,0}, \dots, R^1_{m,0}) = \widetilde{\mathbb{E}}[S_N]$ to get, for all $b_1 \in U_1$,

$$\begin{aligned}\frac{\partial}{\partial b_1}\mathbb{E}^{(b_1,a_2)}[S_N] = \frac{\partial}{\partial b_1}\mathbb{E}^{b_1}[A(X_1,\dots,X_m)] &= \mathbb{C}\mathrm{ov}^{b_1}(A(X_1,\dots,X_m), S_S)\\ &= \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)\end{aligned}$$

and $U_1 \ni b_1 \mapsto \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)$ is continuous. The third equality follows from the fact that the collection $\{R^2_{0,j}, (Y^1_x, Y^2_x)\}_{j\in\mathbb{N}, x\in\mathbb{N}\times\mathbb{N}}$ is independent of S_S. The second moment condition of Lemma B.2 is satisfied since for all $b_1 \in U_1$,

$$\mathbb{E}^{b_1}[A(X_1,\dots,X_r)^2] = \mathbb{E}^{b_1}[(\widetilde{\mathbb{E}}[S_N])^2] \leqslant \mathbb{E}^{b_1}[\widetilde{\mathbb{E}}[S_N^2]] = \mathbb{E}^{(b_1,a_2)}[S_N^2] < \infty.$$

A similar argument yields (3.14).

Using the coupling (3.4)

$$E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] = E^{(a_1,a_2)}_{m,n}[\sum_{i=1}^{t_1} L^{f^1}(a_1, H^{f^1}(a_1,\eta^1_i))]. \tag{3.15}$$

Taking the derivative of (3.6) and using (3.7), for $j = 1, 2$

$$\frac{\partial}{\partial b_j}\log(W(b_1,b_2)(x_\bullet)) = \sum_{k=1}^{t_j}\frac{\partial}{\partial b_j}\log H^{f^j}(b_j,\eta^j_k) = \sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j\eta^j_k)). \tag{3.16}$$

Therefore

$$\frac{\partial}{\partial b_j}W(b_1,b_2)(x_\bullet) = W(b_1,b_2)(x_\bullet)\sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j\eta^j_k)) \tag{3.17}$$

which implies that

$$\frac{\partial}{\partial b_j}\log Z_{m,n}(b_1,b_2) = E^{Q^{(b_1,b_2)}_{m,n}}[\sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j,\eta^j_k))]. \tag{3.18}$$

We now prove (3.11). Similar to (3.5), in the coupled environment we use $S_\bullet(b_1,b_2)$ to make explicit the dependence of $S_\bullet$ on the parameters b_1 and b_2. Recall that $\widehat{\mathbb{E}}$ is the expectation of the coupled environment. For $\epsilon > 0$ small enough such that $[a_1-\epsilon, a_1+\epsilon] \subset U_1$,

$$\begin{aligned}\int_{a_1-\epsilon}^{a_1+\epsilon}\mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N,S_S)db_1 &= \mathbb{E}^{(a_1+\epsilon,a_2)}[S_N] - \mathbb{E}^{(a_1-\epsilon,a_2)}[S_N]\\ &= \widehat{\mathbb{E}}[S_N(a_1+\epsilon,a_2) - S_N(a_1-\epsilon,a_2)]\\ &= \widehat{\mathbb{E}}[\int_{a_1-\epsilon}^{a_1+\epsilon}\frac{\partial}{\partial b_1}\log Z_{m,n}(b_1,a_2)db_1]\\ &= \int_{a_1-\epsilon}^{a_1+\epsilon}\widehat{\mathbb{E}}[\frac{\partial}{\partial b_1}\log Z_{m,n}(b_1,a_2)]db_1\end{aligned} \tag{3.19}$$

where the first equality follows from (3.13), the third equality follows because S_W does not depend on b_1 and $S_N(b_1,a_2) = \log Z_{m,n}(b_1,a_2) - S_W(a_2)$. The last equality follows from (3.18) and Tonelli's theorem (by the non-negativity of L^{f^1}).

Recall that $b_1 \mapsto \mathbb{C}\text{ov}^{(b_1,a_2)}(S_N, S_S)$ is continuous. Once we show that the mapping

$$b_1 \mapsto \widehat{\mathbb{E}}[\frac{\partial}{\partial b_1} \log Z_{m,n}(b_1, a_2)] = E_{m,n}^{(b_1,a_2)}[\sum_{i=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta_i^1))] \tag{3.20}$$

is continuous, using (3.19) and (3.15) we will have (3.11). The continuity of (3.20) follows from the continuity of $b_1 \mapsto E^{Q_{m,n}^{(b_1,a_2)}}[\sum_{k=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta_k^1))]$, the dominated convergence theorem, and the bound

$$\begin{aligned}
&\widehat{\mathbb{E}}[\sup_{|b_1-a_1|\leqslant\epsilon} E^{Q_{m,n}^{(b_1,a_2)}}[\sum_{k=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta_k^1))]] \\
&\leqslant \widehat{\mathbb{E}}[\sup_{|b_1-a_1|\leqslant\epsilon} \sum_{k=1}^{m} L^{f^1}(b_1, H^{f^1}(b_1, \eta_k^1))] \\
&\leqslant C\widehat{\mathbb{E}}[\sum_{k=1}^{m} 1 + |\log H^{f^1}(a_1 - \epsilon, \eta_k^1)| + |\log H^{f^1}(a_1 + \epsilon, \eta_k^1)|] < \infty
\end{aligned}$$

where we use the non-negativity of L^{f^1} to replace t_1 by its upper bound m, then use assumption (3.9) of Hypothesis 3.6 (with the fact that $H^{f^1}(b, x)$ is non-decreasing in b) and part (a) of Remark 3.2.

A similar argument shows that

$$\mathbb{C}\text{ov}^{(a_1,a_2)}(S_E, S_W) = E_{m,n}^{(a_1,a_2)}[\sum_{j=1}^{t_2} L^{f^2}(a_2, R_{0,j}^2)].$$

This completes the proof. □

We can now give the proof of Proposition 1.1.

Proof of Proposition 1.1: By assumption, the polymer environment is distributed as in (3.2), where f^1 and f^2 satisfy Hypothesis 3.6 by Remark 3.7. By Remark 3.2, for each of the four models $\log u$ and $\log v$ have finite variance. Thus the conditions of Lemma 3.8 are satisfied. Combining Proposition 2.3 with Lemma 2.4, and Lemma 3.8 yields the result. □

4. Proof of the variance upper bound

The first lemma of this section allows us to compare the variance of the free energy at different parameter values.

Lemma 4.1. *Assume that the polymer environment is distributed as in* (3.2)*. Let* ϵ *be small enough such that for all* $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ *and* $a_2 - \lambda \in D(M_{f^2})$*. Then there exists a positive constant* C *depending only on* (a_1, a_2), β, *and* ϵ *such that for all* $(m, n) \in \mathbb{Z}_+^2$, *the following holds for all* $|\lambda| \leqslant \epsilon$,

$$\left|\mathbb{V}ar^{(a_1,a_2)}[\log Z_{m,n}] - \mathbb{V}ar^{(a_1+\lambda,a_2-\lambda)}[\log Z_{m,n}]\right| \leqslant C(m+n)|\lambda|$$

Proof: Let $\widetilde{a}_1 = a_1 + \lambda$ and $\widetilde{a}_2 = a_2 - \lambda$. Applying Proposition 1.1 (recalling that $\psi_1^f(a) = \mathbb{V}\text{ar}[\log X]$ when $X \sim m_f(a)$) then using the coupling (3.5) yields, for $j = 1, 2$:

$$\frac{1}{2}\Big(\mathbb{V}\mathrm{ar}^{(\tilde a_1,\tilde a_2)}[\log Z_{m,n}] - \mathbb{V}\mathrm{ar}^{(a_1,a_2)}[\log Z_{m,n}]\Big) \tag{4.1}$$

$$= \frac{(-1)^j}{2}\Big[m\big(\psi_1^{f^1}(\tilde a_1) - \psi_1^{f^1}(a_1)\big) - n\big(\psi_1^{f^2}(\tilde a_2) - \psi_1^{f^2}(a_2)\big)\Big] \tag{4.2}$$

$$+ E_{m,n}^{(\tilde a_1,\tilde a_2)}\left[\sum_{k=1}^{t_j} L^{f^j}(\tilde a_j, H^{f^j}(\tilde a_j, \eta_k^j))\right] - E_{m,n}^{(a_1,a_2)}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right]. \tag{4.3}$$

Since $\psi_1^{f^1}$ and $\psi_1^{f^2}$ are continuously differentiable, there is a constant C_1 such that line (4.2) is bounded by $C_1(m+n)|\lambda|$. Suppressing the m,n dependence, we then split line (4.3) as

$$= \widehat{\mathbb{E}}E^{Q^{(\tilde a_1,\tilde a_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(\tilde a_j, H^{f^j}(\tilde a_j, \eta_k^j))\right] - \widehat{\mathbb{E}}E^{Q^{(\tilde a_1,\tilde a_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] \tag{4.4}$$

$$+ \widehat{\mathbb{E}}E^{Q^{(\tilde a_1,\tilde a_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] - \widehat{\mathbb{E}}E^{Q^{(a_1,a_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] \tag{4.5}$$

For line (4.4), since t_j is all that is random under $E^{Q^{(\tilde a_1,\tilde a_2)}}$, we can replace t_j by $m \vee n$. Thus

$$\begin{aligned}
|\,\text{line } (4.4)| &\leqslant \widehat{\mathbb{E}}\sum_{k=1}^{m\vee n}\left|L^{f^j}(\tilde a_j, H^{f^j}(\tilde a_j, \eta_k^j)) - L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right| \\
&= (m \vee n)\int_0^1 \left|L^{f^j}(\tilde a_j, H^{f^j}(\tilde a_j, \eta)) - L^{f^j}(a_j, H^{f^j}(a_j, \eta))\right| d\eta \\
&= (m \vee n)\int_0^1 \left|\int_{a_j}^{\tilde a_j} \frac{\partial}{\partial a} L^{f^j}(a, H^{f^j}(a,\eta))da\right| d\eta \\
&\leqslant (m \vee n)\left|\int_{a_j}^{\tilde a_j}\int_0^1 \left|\frac{\partial}{\partial a} L^{f^j}(a, H^{f^j}(a,\eta))\right| d\eta da\right| \\
&\leqslant (m \vee n) C_2|\lambda|.
\end{aligned} \tag{4.6}$$

In the last step we used the fact that f^j satisfy assumption (3.10) in Hypothesis 3.6 by Remark 3.7.

We can write line (4.5) as

$$\widehat{\mathbb{E}}\Big[\sum_{k=1}^{\ell_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\big(Q^{(\tilde a_1,\tilde a_2)}(t_j \geqslant k) - Q^{(a_1,a_2)}(t_j \geqslant k)\big)\Big],$$

where $\ell_1 = m$ and $\ell_2 = n$. By Lemma B.3, $Q^{(a_1+\lambda, a_2-\lambda)}(t_1 \geqslant k)$ is stochastically non-decreasing in λ, and $Q^{(a_1+\lambda, a_2-\lambda)}(t_2 \geqslant k)$ is stochastically non-increasing in λ. Using the bound on (4.2), the bound (4.6), and the non-negativity of L^{f^j}, line (4.5) is non-negative if $j = 1$ and $\lambda > 0$ or $j = 2$ and $\lambda < 0$. This implies

$$(4.1) \geqslant -C(m+n)|\lambda|.$$

If $j = 2$ and $\lambda > 0$ or $j = 1$ and $\lambda < 0$, then (4.5) is non-positive, so

$$(4.1) \leqslant C(m+n)|\lambda|.$$

This completes the proof. □

Lemma 4.2. *Assume that the polymer environment is distributed as in (3.2). Then there exists a positive constant C depending only on (a_1, a_2) and β such that for all $(m,n) \in \mathbb{Z}_+^2$ the following two inequalities hold:*

$$E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] \leqslant C(E_{m,n}[t_1]+1), \quad E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})] \leqslant C(E_{m,n}[t_2]+1).$$

Proof: Let $L_i = L_{R^1}(R^1_{i,0})$, $\overline{L}_i = L_i - \mathbb{E}[L_i]$, and $S_k = \sum_{i=1}^k \overline{L}_i$. Note that $L_i \sim L_{R^1}(R^1)$ has finite exponential moments by Remark 3.7. Using Cauchy-Schwarz, Markov's inequality, and the bound $\mathbb{E}[S_k^8] \leqslant Ck^4$, we estimate

$$\mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right] \leqslant \left(\mathbb{P}\{S_k > k\}\right)^{1/2} \left(k\mathbb{V}\mathrm{ar}L_1\right)^{1/2} \leqslant \left(\frac{\mathbb{E}[S_k^8]}{k^8}\right)^{1/2} (kC)^{1/2} \leqslant Ck^{-3/2}.$$

Thus

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right] \leqslant C.$$

Using this, we then get

$$\begin{aligned}
E_{m,n}\left[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})\right] &= E_{m,n}\left[\sum_{i=1}^{t_1} \overline{L}_i + \mathbb{E}L_i\right] \\
&= E_{m,n}[t_1]\mathbb{E}[L_1] + E_{m,n}\left[\sum_{i=1}^{t_1} \overline{L}_i\right] \\
&= E_{m,n}[t_1]\mathbb{E}[L_1] + \sum_{k=1}^{m} \mathbb{E}\left[Q_{m,n}(t_1 = k) S_k\right] \\
&\leqslant E_{m,n}[t_1]\mathbb{E}[L_1] + \sum_{k=1}^{m} \left(k\mathbb{E}\left[Q_{m,n}(t_1 = k)\right] + \mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right]\right) \\
&\leqslant E_{m,n}[t_1]\mathbb{E}[L_1] + E_{m,n}[t_1] + C \\
&\leqslant C\left(E_{m,n}[t_1] + 1\right).
\end{aligned}$$

The proof for t_2 is analogous. □

Proposition 4.3. *Assume that the polymer environment is distributed as in (3.2). Assume that the sequence $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies*

$$|m - N\psi_1^{f^2}(a_2)| \vee |n - N\psi_1^{f^1}(a_1)| \leqslant \kappa_N$$

where $\kappa_N \leqslant \gamma N^{2/3}$ and γ is some positive constant.

Then there exist positive constants $C_1, C_2, C_3, \delta, \delta_1$ depending only on (a_1, a_2), β, and γ such that for $N \in \mathbb{N}$ and $1 \vee C_1\kappa_N \leqslant u \leqslant \delta N$,

$$\mathbb{P}\left\{Q_{m,n}(t_j \geqslant u) \geqslant e^{\frac{-\delta u^2}{N}}\right\} \leqslant C_2\left(\frac{N^2}{u^4}E_{m,n}[t_j] + \frac{N^2}{u^3}\right) \qquad \text{for } j = 1, 2,$$

while for $N \in \mathbb{N}$ and $u \geqslant 1 \vee C_1\kappa_N \vee \delta N$,

$$\mathbb{P}\left\{Q_{m,n}(t_j \geqslant u) \geqslant e^{-\delta_1 u}\right\} \leqslant 2e^{-C_3 u} \qquad \text{for } j = 1, 2.$$

Proof: Let $\epsilon > 0$ be small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1(\lambda) := a_1+\lambda \in D(M_{f^1})$ and $a_2(\lambda) := a_2 - \lambda \in D(M_{f^2})$. For the moment fix $\lambda_1 \in [0, \epsilon]$, $\lambda_2 \in [-\epsilon, 0]$, and $u \geqslant 1$. The λ_j will give the perturbation $(a_1(\lambda_j), a_2(\lambda_j))$ of the parameters (a_1, a_2) which will be used when dealing with the exit time t_j. Using the coupling in (3.5), (3.6) gives: for both $j = 1, 2$ and any path $x_\bullet$ such that $t_j(x_\bullet) \geqslant u$,

$$\frac{W(a_1,a_2)(x_\bullet)}{W(a_1(\lambda_j),a_2(\lambda_j))(x_\bullet)} = \prod_{k=1}^{t_j} \frac{H^{f^j}(a_j,\eta_k^j)}{H^{f^j}(a_j(\lambda_j),\eta_k^j)} \leqslant \prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j,\eta_k^j)}{H^{f^j}(a_j(\lambda_j),\eta_k^j)},$$

since $H^f(a, x)$ is non-decreasing in a. Therefore

$$\begin{aligned} Q_{m,n}^{(a_1,a_2)}(t_j \geqslant u) =& \frac{1}{Z_{m,n}(a_1,a_2)} \sum_{x_\bullet \in \Pi_{m,n}} \mathbb{1}_{\{x_\bullet \geqslant u\}} W(a_1,a_2)(x_\bullet) \\ \leqslant& \frac{Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))}{Z_{m,n}(a_1,a_2)} \prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j,\eta_k^j)}{H^{f^j}(a_j(\lambda_j),\eta_k^j)}. \end{aligned}$$

Then for all real numbers z, r

$$\mathbb{P}\Big\{Q_{m,n}(t_j \geqslant u) \geqslant e^{-z}\Big\} \leqslant \widehat{\mathbb{P}}\Big\{\prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j,\eta_k^j)}{H^{f^j}(a_j(\lambda_j),\eta_k^j)} \geqslant e^{-r}\Big\} \tag{4.7}$$

$$+ \widehat{\mathbb{P}}\Big\{\frac{Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))}{Z_{m,n}(a_1,a_2)} \geqslant e^{r-z}\Big\}. \tag{4.8}$$

We now split the proof into two cases.

Case 1: $1 \vee C_1\kappa_N \leqslant u \leqslant \delta N$. Let $b, \delta > 0$ be small enough such that $b\delta \leqslant \epsilon$. These constants will be determined through the course of the proof. Put $\lambda_1 = \frac{bu}{N}$ and $\lambda_2 = -\frac{bu}{N}$. The condition $u \leqslant \delta N$ guarantees that $-\epsilon \leqslant \lambda_2 < 0 < \lambda_1 \leqslant \epsilon$. Now plug in $r = \lfloor u \rfloor \Big(\psi_0^{f^j}(a_j(\lambda_j)) - \psi_0^{f^j}(a_j)\Big) - \frac{\delta u^2}{N}$ and $z = \frac{\delta u^2}{N}$ to obtain

$$\text{RHS of (4.7)} = \widehat{\mathbb{P}}\Big\{\sum_{k=1}^{\lfloor u \rfloor} \overline{\log H^{f^j}(a_j,\eta_k^j) - \log H^{f^j}(a_j(\lambda_j),\eta_k^j)} \geqslant \frac{\delta u^2}{N}\Big\} \leqslant C\frac{N^2}{u^3} \tag{4.9}$$

by Chebyshev's inequality and the fact that $H^f(a, \eta) \sim m_f(a)$. The constant C here depends only on (a_1, a_2), ϵ, and δ. We will now show how to tune b and δ as functions of (a_1, a_2) and ϵ to get a meaningful bound on

$$\begin{aligned} (4.8) =& \widehat{\mathbb{P}}\Big\{\overline{\log Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))} - \overline{\log Z_{m,n}(a_1,a_2)} \geqslant \\ & \widehat{\mathbb{E}}\big[\log Z_{m,n}(a_1,a_2) - \log Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))\big] + r - z\Big\}. \end{aligned} \tag{4.10}$$

Since the parameters satisfy $a_1(\lambda_j) + a_2(\lambda_j) = a_3$, by Remark 3.5, the down-right property is still satisfied for the perturbed model with parameters $\big(a_1(\lambda_j), a_2(\lambda_j)\big)$.

Using Proposition 1.1 we can evaluate the right-hand side inside the above probability

$$
\begin{aligned}
=&m\Big(\psi_0^{f^1}(a_1)-\psi_0^{f^1}(a_1(\lambda_j))\Big)+n\Big(\psi_0^{f^2}(a_2)-\psi_0^{f^2}(a_2(\lambda_j))\Big)\\
&+\lfloor u\rfloor\Big(\psi_0^{f^j}(a_j(\lambda_j))-\psi_0^{f^j}(a_j)\Big)-2\delta\frac{u^2}{N}\\
=&(m-N\psi_1^{f^2}(a_2))\Big(\psi_0^{f^1}(a_1)-\psi_0^{f^1}(a_1(\lambda_j))\Big)\\
&+(n-N\psi_1^{f^1}(a_1))\Big(\psi_0^{f^2}(a_2)-\psi_0^{f^2}(a_2(\lambda_j))\Big)\\
&+N\Big[\psi_1^{f^2}(a_2)\Big(\psi_0^{f^1}(a_1)-\psi_0^{f^1}(a_1(\lambda_j))\Big)+\psi_1^{f^1}(a_1)\Big(\psi_0^{f^2}(a_2)-\psi_0^{f^2}(a_2(\lambda_j))\Big)\Big]\\
&+\lfloor u\rfloor\Big(\psi_0^{f^j}(a_j(\lambda_j))-\psi_0^{f^j}(a_j)\Big)-2\delta\frac{u^2}{N}\\
\geqslant&-\kappa_N\frac{bu}{N}C'-N(\frac{bu}{N})^2C'+u(\frac{bu}{N})C''-2\delta\frac{u^2}{N}\\
=&\frac{u}{N}\Big[C''bu-C'b^2u-2\delta u-C'b\kappa_N\Big]
\end{aligned}
\tag{4.11}
$$

for some positive constants C' and C''. This can be obtained by taking a 2nd-order Taylor expansion of the functions $\psi_0^{f^j}$, keeping in mind that $\psi_1^{f^j}>0$. In the last inequality we also used $u\geqslant 1$.

Now fixing b small enough followed by then fixing δ small enough we can ensure that the entire quantity (4.11) is $\geqslant C'''\frac{u^2}{N}$ for some positive constant C''' as long as $u\geqslant C_1\kappa_N$ for some positive C_1. With these restrictions,

$$
\begin{aligned}
(4.8)\leqslant&\widehat{\mathbb{P}}\Big\{\overline{\log Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))-\log Z_{m,n}(a_1,a_2)}\geqslant C'''\frac{u^2}{N}\Big\}\\
\leqslant&\frac{N^2}{(C''')^2u^4}\widehat{\mathbb{V}\mathrm{ar}}\big[\log Z_{m,n}(a_1(\lambda_j),a_2(\lambda_j))-\log Z_{m,n}(a_1,a_2)\big]\\
\leqslant&C\frac{N^2}{u^4}\Big(\widehat{\mathbb{V}\mathrm{ar}}\big[\log Z_{m,n}(a_1,a_2)\big]+(m+n)\frac{bu}{N}\Big)\\
\leqslant&C\Big(\frac{N^2}{u^4}E_{m,n}[t_j]+\frac{N^2}{u^3}\Big).
\end{aligned}
$$

The second to last and last inequalities are applications of Lemma 4.1, Proposition 1.1, and Lemma 4.2. Combining this result with (4.9) finishes the first case.

Case 2: $1\vee C_1\kappa_N\vee\delta N\leqslant u$. Take δ, ϵ fixed from the first case, let $\delta_1\in(0,\delta]$, and $\epsilon_1\in(0,\epsilon]$. The constants δ_1 and ϵ_1 will be determined throughout the course of the proof. This time, put $\lambda_1=\epsilon_1$, $\lambda_2=-\epsilon_1$, $r=\lfloor u\rfloor\big(\psi_0^{f^j}(a_j(\lambda_j))-\psi_0^{f^j}(a_j)\big)-\delta_1u$, and $z=\delta_1u$. Then

$$
(4.7)=\widehat{\mathbb{P}}\Big\{\sum_{k=1}^{\lfloor u\rfloor}\overline{\log H^{f^j}(a_j,\eta_k^j)-\log H^{f^j}(a_j(\lambda_j),\eta_k^j))}\geqslant\delta_1u\Big\}.
\tag{4.12}
$$

By Remark 3.2 the random variables in the summation have finite exponential moments. A large deviation estimate gives us the existence of a positive constant C_3 such that (4.12)$\leqslant e^{-C_3u}$.

We now consider (4.10). A similar analysis to that in Case 1 tells us that the right-hand side inside of the above probability

$$
\begin{aligned}
&\geqslant -C'\epsilon_0\kappa_N - C'\epsilon_0^2 N + C''\epsilon_0 u - 2\delta_1 u \\
&\geqslant u\Big(C''\epsilon_0 - \frac{C'\epsilon_0^2}{\delta} - 2\delta_1\Big) - C'\epsilon_0\kappa_N
\end{aligned}
\tag{4.13}
$$

for some positive constants C' and C'' (the second inequality follows from $u \geqslant \delta N$). Now fixing ϵ_0 small enough followed by then fixing δ_1 small enough we can ensure that (4.13) $\geqslant Cu$ for some positive constant C as long as $u \geqslant C_1\kappa_N$ for some positive C_1 (here we increase the previous constant C_1 found in Case 1 if necessary). With these constraints,

$$
(4.8) \leqslant \widehat{\mathbb{P}}\Big\{\overline{\log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2)} \geqslant Cu\Big\}.
$$

Since the perturbed parameters are such that the polymer environment still has the down-right property, the random variable inside the above probability can be expressed as two sums of i.i.d. random variables, each of which has entries with finite exponential moments. Therefore a large deviation estimate gives the existence of a positive constant C_3 such that (4.8) $\leqslant e^{-uC_3}$. Combining this with (4.12) completes the proof. □

Remark 4.4. If $\epsilon > 0$ is small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ and $a_2 - \lambda \in D(M_{f^2})$, then the constants in Proposition 4.3 can be chosen such that the conclusion also holds for the polymer environment with parameters $(a_1 + \lambda, a_2 - \lambda, a_3)$ for any $|\lambda| \leqslant \epsilon$.

Using the previous proposition, we can now bound the annealed expectation of the exit points of the polymer path from the axes.

Corollary 4.5. *Suppose all of the assumptions of Proposition 4.3 hold. Then there exists a positive constant C depending only on (a_1, a_2), β, and γ such that for both $j = 1, 2$,*

$$
E_{m,n}[t_j] \leqslant CN^{2/3} \qquad \textit{for all } N \in \mathbb{N}.
$$

Proof of Corollary 4.5: Since all of the constants C_1, C_2, C_3, δ, δ_1 determined by Proposition 4.3 depend only on (a_1, a_2), β, and γ, it is sufficient to show that the constant C to be determined in this proof depends only on these five constants and γ. Let $r \geqslant 1 \vee C_1\gamma$. Then $rN^{2/3} \geqslant 1 \vee C_1\kappa_N$. Suppressing the m, n dependence,

$$
\begin{aligned}
E[t_j] &= \int_0^\infty P(t_j \geqslant u)du \\
&\leqslant rN^{2/3} + \int_{rN^{2/3}}^{rN^{2/3}\vee\delta N} P(t_j \geqslant u)du + \int_{rN^{2/3}\vee\delta N}^\infty P(t_j \geqslant u)du.
\end{aligned}
\tag{4.14}
$$

We now bound the integrals in line (4.14) individually.

$$\begin{aligned}\int_{rN^{2/3}\vee\delta N}^{\infty} P(t_j \geqslant u)du &= \int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{e^{-\delta_1 u}} \mathbb{P}\{Q(t_j\geqslant u)\geqslant x\}dxdu\\ &\quad+\int_{rN^{2/3}\vee\delta N}^{\infty}\int_{e^{-\delta_1 u}}^{1} \mathbb{P}\{Q(t_j\geqslant u)\geqslant x\}dxdu.\\ &\leqslant \int_{\delta N}^{\infty} e^{-\delta_1 u}du\\ &\quad+\int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{\delta_1}\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-su}\}e^{-su}u\,dsdu\\ &\leqslant \frac{1}{\delta_1}e^{-\delta_1\delta N}+\int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{\delta_1}2e^{-(C_3+s)u}u\,dsdu\leqslant C,\end{aligned}\tag{4.15}$$

where in the first inequality we bounded the first integrand by one and made the substitution $x=e^{-su}$ for the second. For the second inequality, we apply Proposition 4.3 to get that $\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-su}\}\leqslant\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-\delta_1 u}\}\leqslant 2e^{-C_3 u}$ for all $u\geqslant rN^{2/3}\vee\delta N$ and all $0<s\leqslant\delta_1$.

We now bound the first integral of (4.14). Without loss of generality, assume that $rN^{2/3}<\delta N$. Then

$$\begin{aligned}\int_{rN^{2/3}}^{rN^{2/3}\vee\delta N} P(t_j\geqslant u)du &= \int_{rN^{2/3}}^{\delta N}\int_0^{e^{-\delta\frac{u^2}{N}}}\mathbb{P}\{Q(t_j\geqslant u)\geqslant x\}dxdu\\ &\quad+\int_{rN^{2/3}}^{\delta N}\int_{e^{-\delta\frac{u^2}{N}}}^{1}\mathbb{P}\{Q(t_j\geqslant u)\geqslant x\}dxdu\\ &\leqslant\int_{rN^{2/3}}^{\delta N}e^{-\delta\frac{u^2}{N}}du\\ &\quad+\int_{rN^{2/3}}^{\delta N}\int_0^{\delta}\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-s\frac{u^2}{N}}\}e^{-s\frac{u^2}{N}}\frac{u^2}{N}dsdu\\ &\leqslant\delta Ne^{-\delta r^2N^{1/3}}+\int_{rN^{2/3}}^{\infty}\int_{e^{-s\frac{u^2}{N}}}^{1}C_2(\frac{N^2}{u^4}E[t_j]+\frac{N^2}{u^3})dsdu\\ &\leqslant C+C_2\Big(\frac{E[t_j]}{3r^3}+\frac{N^{2/3}}{2r^2}\Big),\end{aligned}\tag{4.16}$$

where for the first inequality we bound the first integrand by one and make the substitution $x=e^{-s\frac{u^2}{N}}$ for the second. For the second inequality we apply Proposition 4.3 to get that $\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-s\frac{u^2}{N}}\}\leqslant\mathbb{P}\{Q(t_j\geqslant u)\geqslant e^{-\delta\frac{u^2}{N}}\}\leqslant C_2\Big(\frac{N^2}{u^4}E[t_j]+\frac{N^2}{u^3}\Big)$ for all $rN^{2/3}\leqslant u\leqslant\delta N$ and all $0<s\leqslant\delta$.

Combining the bounds on (4.15), (4.16) and (4.14), we have: for all $r\geqslant 1\vee C_1\gamma$,

$$E[t_j]\leqslant rN^{2/3}+C+C_2(\frac{E[t_j]}{3r^3}+\frac{N^{2/3}}{2r^2}).$$

We can now fix r large enough with respect to C and C_2 then rearrange to get the desired result. □

We can now give the proof of the upper bound of the variance of the free energy.

Proof of upper bound of Theorem 1.2: Averaging (1.8) and (1.9) of Proposition 1.1 then applying Lemma 4.2 followed by Corollary 4.5 (recalling that $\psi_1^{f^j}(a_j) = \mathbb{V}\mathrm{ar}[\log R^j]$) gives

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] + E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})] \\ &\leqslant C(E_{m,n}[t_1] + E_{m,n}[t_2] + 2) \\ &\leqslant CN^{2/3},\end{aligned}$$

which concludes the proof. □

The following corollary is obtained by combining Proposition 4.3 and Corollary 4.5.

Corollary 4.6. *Assume that the polymer environment is distributed as in (3.2) and the sequence $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies (1.10) for some positive constant γ. Then there exists positive constants b_0, C_2, C_3, δ, and δ_1 depending only on (a_1, a_2), β, and γ such that for all $N \in \mathbb{N}$ and $b_0 \leqslant b \leqslant \delta N^{1/3}$,*

$$\mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-\delta b^2 N^{1/3}}\} \leqslant \frac{2C_2}{b^3} \quad \text{for } j = 1, 2, \tag{4.17}$$

while for all $N \in \mathbb{N}$ and $b \geqslant b_0 \vee \delta N^{1/3}$,

$$\mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-\delta_1 b N^{2/3}}\} \leqslant 2e^{-C_3 b N^{2/3}} \quad \text{for } j = 1, 2. \tag{4.18}$$

Lemma 4.7. *Assume that the polymer environment is distributed as in (3.2) and the sequence $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies (1.10) for some positive constant γ. Then there exist constants $b_0 \geqslant 1$ and $C > 0$ depending only on (a_1, a_2), β, and γ such that for all $b \geqslant b_0$ and $N \in \mathbb{N}$,*

$$P_{m,n}(t_j \geqslant bN^{2/3}) \leqslant \frac{C}{b^3} \quad \text{for } j = 1, 2.$$

Therefore, for all $0 < p < 3$ there exists a positive constant C' depending on (a_1, a_2), β, γ, and p such that for all $N \in \mathbb{N}$,

$$E_{m,n}\Big[\Big(\frac{t_j}{N^{2/3}}\Big)^p\Big] \leqslant C' \quad \text{for } j = 1, 2.$$

Proof of Lemma 4.7: By Corollary (4.6) there exist positive constants b_0, C_2, C_3, δ, δ_1 with $b_0 \geqslant 1$ such that (4.17) holds for $b_0 \leqslant b \leqslant \delta N^{1/3}$ while (4.18) holds for $b \geqslant \delta N^{1/3} \vee b_0$.

We first estimate for $b \leqslant \delta N^{1/3}$,

$$\begin{aligned}P_{m,n}(t_j \geqslant bN^{2/3}) &= \int_0^1 \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant x\}dx \\ &= \int_0^{\delta} \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sb^2N^{1/3}}\}b^2N^{1/3}e^{-sb^2N^{1/3}}ds \qquad (4.19)\\ &\quad + \int_{\delta}^{\infty} \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sb^2N^{1/3}}\}b^2N^{1/3}e^{-sb^2N^{1/3}}ds \qquad (4.20)\\ &\leqslant \frac{2C_2}{b^3} + e^{-\delta b^2 N^{1/3}} \leqslant \frac{C}{b^3}\end{aligned}$$

for some positive constant C, where we made the substitution $x = e^{-sb^2 N^{1/3}}$, used (4.17) to bound the probability inside the integral of (4.19), and bounded the probability inside the integral of (4.20) by 1. For $b \geqslant \delta N^{1/3}$, we make the substitution $x = e^{-sbN^{2/3}}$ to get

$$\begin{aligned} P_{m,n}(t_j \geqslant bN^{2/3}) &= \int_0^1 \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant x\big\} dx \\ &= \int_0^{\delta_1} \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sbN^{2/3}}\big\} bN^{2/3} e^{-sbN^{2/3}} ds \quad (4.21) \\ &\quad + \int_{\delta_1}^{\infty} \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sbN^{2/3}}\big\} bN^{2/3} e^{-sbN^{2/3}} ds \quad (4.22) \\ &\leqslant 2e^{-C_3 bN^{2/3}} + e^{-\delta_1 bN^{2/3}} \leqslant \frac{C}{b^3} \end{aligned}$$

increasing the constant C if necessary, where we used (4.18) to bound the probability inside the integral of (4.21) and bounded the probability inside the integral of (4.22) by 1. ☐

Proof of Corollary 1.4: Let $m_1 = \lfloor N \mathbb{V}\mathrm{ar}[\log R^2] \rfloor$. Since $Z_{m,n} = Z_{m_1,n} \prod_{i=m_1+1}^{m} R^1_{i,n}$,

$$N^{-\alpha/2} \overline{\log Z_{m,n}} = N^{-\alpha/2} \overline{\log Z_{m_1,n}} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log R^1_{i,n}}.$$

The sequence (m_1, n) satisfies (1.10). Using Chebyshev's inequality and the upper bound of Theorem 1.2 shows that the term $N^{-\alpha/2}\overline{\log Z_{m_1,n}}$ converges to zero in probability. By the down-right property, the summands in the second term are i.i.d. with mean zero and variance $\mathbb{V}\mathrm{ar}[\log R^1]$. By the central limit theorem, $N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log R^1_{i,n}}$ converges in distribution to a centered normal with variance $c_1 \mathbb{V}\mathrm{ar}[\log R^1]$. ☐

5. Proof of the path fluctuation upper bound

Given $0 \leqslant k < m$ and $0 \leqslant l < n$, we define a partition function $Z^{(k,l)}_{m,n}$ and quenched polymer measure $Q^{(k,l)}_{m,n}$ on up-right paths from (k, l) to (m, n) by using the collections $\{R^1_{i,l} : k+1 \leqslant i \leqslant m\}$ and $\{R^2_{k,j} : l+1 \leqslant j \leqslant n\}$ as weights along the edges of the south and west boundaries of the rectangle $[k, m] \times [l, n]$ respectively, and the weights $\{(Y^1_z, Y^2_z) : z \in \{k+1, \dots, m\} \times \{l+1, \dots, n\}\}$ for the remaining edges. When the original polymer environment (1.1) has the down-right property, it follows that $Z^{(k,l)}_{m,n}$ has the same distribution as $Z_{m-k,n-l}$.

For an up-right path $x_\bullet$ from (k, l) to (m, n), define

$$t^{(k,l)}_1(x_\bullet) := \max\{i : (k+i, l) \in x_\bullet\}, \qquad t^{(k,l)}_2(x_\bullet) := \max\{j : (k, l+j) \in x_\bullet\}.$$

Recall the definition (1.12).

Lemma 5.1. *Assume that the polymer environment satisfies the down-right property. Then for all* $0 \leqslant k < m$, $0 \leqslant l < n$, *and* $u \geqslant 0$,

$$Q_{m,n}(v_1(l) \geqslant k+u) = Q_{m,n}^{(k,l)}(t_1^{(k,l)} \geqslant u) \stackrel{d}{=} Q_{m-k,n-l}(t_1 \geqslant u), \tag{5.1}$$

$$Q_{m,n}(w_1(k) \geqslant l+u) = Q_{m,n}^{(k,l)}(t_2^{(k,l)} \geqslant u) \stackrel{d}{=} Q_{m-k,n-l}(t_2 \geqslant u). \tag{5.2}$$

Proof: For $0 \leqslant i < m$ and $0 \leqslant j < n$, we let

$$Z_{(i,j),(m,n)} := \sum_{x_\bullet} \prod_{k=1}^{(m-i)+(n-j)} \omega_{(x_{k-1},x_k)}$$

denote the partition function for up-right paths from (i,j) to (m,n), where the sum is taken over all such paths. A decomposition shows that

$$\begin{aligned} Z_{m,n}^{(k,l)} &= \sum_{i=k+1}^{m} \left(\prod_{a=k+1}^{i} R_{a,l}^{1} \right) Y_{i,l+1}^{2} Z_{(i,l+1),(m,n)} \\ &\quad + \sum_{j=l+1}^{n} \left(\prod_{b=l+1}^{j} R_{k,b}^{2} \right) Y_{k+1,j}^{1} Z_{(k+1,j),(m,n)} \\ &= \sum_{i=k+1}^{m} \frac{Z_{i,l}}{Z_{k,l}} Y_{i,l+1}^{2} Z_{(i,l+1),(m,n)} + \sum_{j=l+1}^{n} \frac{Z_{k,j}}{Z_{k,l}} Y_{k+1,j}^{1} Z_{(k+1,j),(m,n)} = \frac{Z_{m,n}}{Z_{k,l}}. \end{aligned}$$

We then have that for $r \in \{0, \ldots, m-k\}$,

$$\begin{aligned} Q_{m,n}^{(k,l)}(t_1^{(k,l)} = r) &= \frac{1}{Z_{m,n}^{(k,l)}} \left(\prod_{i=1}^{r} R_{k+i,l}^{1} \right) Y_{k+r,l+1}^{2} Z_{(k+r,l+1),(m,n)} \\ &= \frac{1}{Z_{m,n}^{(k,l)}} \frac{Z_{k+r,l}}{Z_{k,l}} Y_{k+r,l+1}^{2} Z_{(k+r,l+1),(m,n)} \\ &= \frac{Z_{k+r,l} Y_{k+r,l+1}^{2} Z_{(k+r,l+1),(m,n)}}{Z_{m,n}} \\ &= Q_{m,n}(v_1(l) = k+r). \end{aligned}$$

Summing over $r \geqslant u$ gives the first equality in (5.1). The equality in distribution follows from the down-right property. An analogous argument gives (5.2). □

We can now prove the upper bound on the polymer path fluctuations under the annealed measure.

Proof of Theorem 1.5: By assumption, the polymer environment is distributed as in (3.2). If $\tau = 0$ this reduces to Lemma 4.7. If $\tau \in (0,1)$ put $(k,l) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$. By part (c) of Remark 3.2, $\mathbb{V}\mathrm{ar}[R^i] = \psi_1^{f^i}(a_i)$ for $i = 1, 2$. Multiplying (1.10) by $(1-\tau)$, up to integer corrections the sequence $(m-k, n-l)$ satisfies

$$|m - k - M\psi_1^{f^2}(a_2)| \vee |n - l - M\psi_1^{f^1}(a_1)| \leqslant \gamma_0 M^{2/3}, \tag{5.3}$$

where $M = (1-\tau)N$ and $\gamma_0 = \gamma(1-\tau)^{1/3}$. We then apply Lemma 5.1 to get

$$\begin{aligned} Q_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) &\leqslant Q_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \lfloor \tau m \rfloor + bN^{2/3}) \\ &\stackrel{d}{=} Q_{m-k,n-l}(t_1 \geqslant bN^{2/3}). \end{aligned}$$

Applying Lemma 4.7, we get

$$P_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{5.4}$$

The same argument in the vertical direction gives us

$$P_{m,n}(w_1(\lfloor \tau m \rfloor) \geqslant \tau n + bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{5.5}$$

To prove the corresponding bounds for v_0 and w_0 we now let $k = \lfloor \tau m - bN^{2/3} \rfloor$ and $l = \lfloor \tau n - bN^{2/3}\frac{n}{m} \rfloor$. Again $(m-k, n-l)$ will satisfy (5.3) for a different constant γ_0. Since $w_1(k) \geqslant \lfloor \tau n \rfloor$ implies that $v_0(\lfloor \tau n \rfloor) \leqslant k$, it follows that

$$\begin{aligned} Q_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3}) &\leqslant Q_{m,n}(w_1(k) \geqslant \lfloor \tau n \rfloor) \\ &= Q^{(k,l)}_{m,n}(t_2^{(k,l)} \geqslant \lfloor \tau n \rfloor - l) \\ &\leqslant Q^{(k,l)}_{m,n}(t_2^{(k,l)} \geqslant CbN^{2/3}) \\ &\overset{d}{=} Q_{m-k,n-l}(t_2 \geqslant CbN^{2/3}), \end{aligned}$$

for some constant C depending on (a_1, a_2), β, and γ. Applying Lemma 4.7 gives

$$P_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{5.6}$$

An analogous argument shows that

$$P_{m,n}(w_0(\lfloor \tau m \rfloor) \leqslant \tau n - bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{5.7}$$

Combining bounds (5.4) and (5.6) gives (1.13), and (5.5) with (5.7) gives (1.14), completing the proof. □

6. Proof of the variance and path fluctuation lower bounds

Proposition 6.1. *Assume that the polymer environment is distributed as in (3.2) and the sequence $(m, n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies (1.10) for some positive constant γ. Then there exist positive constants c_0, ϵ_0, N_0 depending only on (a_1, a_2), β and γ such that for all $N \geqslant N_0$,*

$$\mathbb{P}\big(\overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big) \geqslant \epsilon_0.$$

From this proposition we can obtain the lower bound of Theorem 1.2:

$$\begin{aligned} \mathbb{V}\mathrm{ar}[\log Z_{m,n}] &\geqslant \mathbb{E}\big[(\overline{\log Z_{m,n}})^2 : \overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big] \\ &\geqslant \mathbb{P}\big(\overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big)(c_0 N^{1/3})^2 \\ &\geqslant \epsilon_0 c_0^2 N^{2/3}. \end{aligned}$$

Proof of Proposition 6.1: Let $\epsilon > 0$ be small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ and $a_2 - \lambda \in D(M_{f^2})$. Define

$$\widetilde{m} = \lfloor m \frac{\psi_1^{f^2}(a_2 - \lambda)}{\psi_1^{f^2}(a_2)} \rfloor, \qquad \widetilde{n} = \lfloor n \frac{\psi_1^{f^1}(a_1 + \lambda)}{\psi_1^{f^1}(a_1)} \rfloor.$$

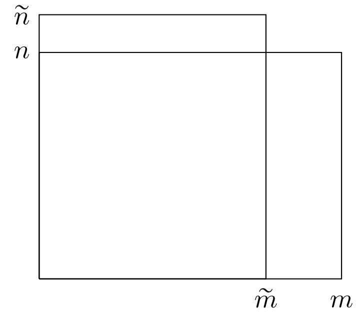

FIGURE 6.7. Case 1: $\psi_2^{f_1}$ and $\psi_2^{f_2}$ are both positive.

Taking Taylor expansions gives

$$\begin{aligned} m - \widetilde{m} &= \lambda \frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} m + o(\lambda) m \\ \widetilde{n} - n &= \lambda \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} n + o(\lambda) n. \end{aligned} \tag{6.1}$$

Let b be a large fixed constant which will be determined through the course of the proof. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$, $bN^{-1/3} \leqslant \epsilon$. Then with $\lambda = bN^{-1/3}$, the sequence $(\widetilde{m}, \widetilde{n})$ satisfies

$$|\widetilde{m} - N\psi_1^{f^2}(a_2 - \lambda)| \vee |\widetilde{n} - N\psi_1^{f^1}(a_1 + \lambda)| \leqslant \gamma_0 N^{2/3}$$

for some positive constant γ_0. By Table C.9 and (C.1) in the Appendix, in each of the four basic beta-gamma models, either $\psi_2^{f^1}(a_1)$ and $\psi_2^{f^2}(a_2)$ are both positive (inverse-beta model for certain choices of parameters and inverse-gamma model for all choices of parameters), $\psi_2^{f^1}(a_1)$ is negative and $\psi_2^{f^2}(a_2)$ is positive (gamma and beta models), or $\psi_2^{f^1}(a_1)$ is positive and $\psi_2^{f^2}(a_2)$ is non-positive (inverse-beta model with the remaining choices of parameters). By flipping the x and y axes in the second case, we only need to consider the first and third cases.

For the case where $\psi_2^{f^1}(a_1)$ and $\psi_2^{f^2}(a_2)$ are both positive define $A_N = m - \widetilde{m}$ and $B_N = \widetilde{n} - n$. This case is illustrated in Figure 6.7. By (6.1) and increasing N_0 if necessary, there exist positive constants c_1, c_2, C_1, C_2 such that for $N \geqslant N_0$,

$$\begin{aligned} c_1 b N^{2/3} &\leqslant A_N \leqslant C_1 b N^{2/3}, \\ c_2 b N^{2/3} &\leqslant B_N \leqslant C_2 b N^{2/3}. \end{aligned}$$

In the case where $\psi_2^{f^1}(a_1) > 0$ and $\psi_2^{f^2}(a_2) \leqslant 0$ we define $c := \frac{1}{2}(\frac{m}{\widetilde{m}} + \frac{n}{\widetilde{n}})$ and let $\overline{m} = c\widetilde{m}$, $\overline{n} = c\widetilde{n}$. This case is illustrated in Figure 6.8. This $(\overline{m}, \overline{n})$ will satisfy

$$|\overline{m} - M\psi_1^{f^2}(a_2 - \lambda)| \vee |\overline{n} - M\psi_1^{f^1}(a_1 + \lambda)| \leqslant \gamma_0 c^{1/3} M^{2/3}$$

where $M = cN$. A Taylor expansion gives

$$c = 1 + \left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} - \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} \right) \frac{\lambda}{2} + o(\lambda)$$

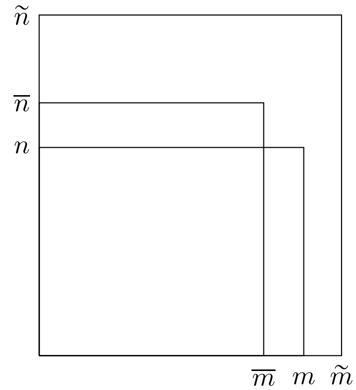

FIGURE 6.8. Case 2: $\psi_2^{f_1} > 0$ and $\psi_2^{f_2} \leqslant 0$.

and thus

$$m - \overline{m} = \frac{\lambda}{2}\left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} + \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}\right) m + o(N^{2/3}),$$
$$\overline{n} - n = \frac{\lambda}{2}\left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} + \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}\right) n + o(N^{2/3}).$$

The quantity $\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} + \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}$ is positive since $\psi_1^{f^1}$ and $\psi_1^{f^2}$ are both positive and $\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) + \psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) > 0$ by Lemma C.2 in the Appendix. Letting $\overline{A} = m - \overline{m}$ and $\overline{B} = \overline{n} - n$, there exist positive constants c_1', c_2', C_1', C_2' such that

$$c_1' bM^{2/3} \leqslant \overline{A}_M \leqslant C_1' bM^{2/3},$$
$$c_2' bM^{2/3} \leqslant \overline{B}_M \leqslant C_2' bM^{2/3}.$$

Recall that $\mathbb{P}^{(a_1,a_2)}$ is used to denote the probability measure on the polymer environment with parameters a_1 and a_2. Let $(\widetilde{a}_1, \widetilde{a}_2) = (a_1 + \lambda, a_2 - \lambda)$. Our goal is to show that

$$\mathbb{P}^{(a_1,a_2)}\big(\log Z_{m,n} \geqslant \mathbb{E}[\log Z_{m,n}] + c_0 N^{1/3}\big) \geqslant \epsilon_0.$$

We will do so by making estimates using the $(\widetilde{a}_1, \widetilde{a}_2)$ environment and then use a coupling of the two environments to transfer the results to the (a_1, a_2) environment.

We would first like to show that in the $(\widetilde{a}_1, \widetilde{a}_2)$ environment, with high probability the quenched probability gives most of the weight to paths which exit the x-axis at a point of order $bN^{2/3}$. That is: there exist positive constants C_3, C such that, given any $\varepsilon > 0$,

$$\mathbb{P}^{(\widetilde{a}_1,\widetilde{a}_2)}\big\{Q_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant 1 - \varepsilon\big\} \geqslant 1 - \frac{C}{b^3} \tag{6.2}$$

holds for all sufficiently large N.

We start by using Lemma 5.1 to relate an upper bound on t_1 to a lower bound on t_2.

$$Q_{m,n}(t_1 \leqslant A_N) \overset{d}{=} Q_{m,\widetilde{n}}(v_1(B_N) \leqslant A_N) = Q_{m,\widetilde{n}}(w_1(A_N) > B_N) \overset{d}{=} Q_{\widetilde{m},\widetilde{n}}(t_2 > B_N).$$

Using this and Corollary 4.6, there exists $\delta > 0$ such that

$$\begin{aligned}\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 > c_1 b N^{2/3}) \geqslant 1 - e^{-\frac{\delta}{N}B_N^2}\big\} \\ &\geqslant \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 > A_N) \geqslant 1 - e^{-\frac{\delta}{N}B_N^2}\big\} \\ &= \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 \leqslant A_N) \leqslant e^{-\frac{\delta}{N}B_N^2}\big\} \\ &= \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{\widetilde{m},\widetilde{n}}(t_2 > B_N) \leqslant e^{-\frac{\delta}{N}B_N^2}\big\} \\ &\geqslant 1 - Cb^{-3}.\end{aligned}$$

This implies that

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 \leqslant c_1 b N^{2/3}) \geqslant e^{-\frac{\delta}{N}B_N^2}\big\} \leqslant Cb^{-3}.$$

Applying the upper bound (4.17) directly with $C_3 > C_2$, we obtain

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 > C_3 b N^{2/3}) \geqslant e^{-\frac{\delta}{N}B_N^2}\big\} \leqslant Cb^{-3}$$

for another positive constant C. Taking a union bound we put the two bounds together and get

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant 1 - 2e^{-\frac{\delta}{N}B_N^2}\big\} \geqslant 1 - Cb^{-3}.$$

Taking N large enough, we get (6.2).

The argument for the case where we use $(\overline{m}, \overline{n})$ and $\overline{A}, \overline{B}$ is unchanged, with the exception of using the scaling parameter M rather than N. This difference can be absorbed into the constants.

In order to make use of the bound (6.2) for the system with the original (a_1, a_2) environment we create a new measure $\breve{\mathbb{P}}$ which has both a_1 and $\tilde{a}_1$ distributed weights along the x-axis and estimate the Radon-Nikodym derivative of the (a_1, a_2) environment with respect to this new environment.

Let $\breve{\omega}$ denote the environment that has the same weights as the (a_1, a_2) environment except for the weights $R^1_{i,0}$ for $1 \leqslant i \leqslant \lfloor C_3 b N^{2/3} \rfloor$, which will be distributed with parameter $\tilde{a}_1$. Let $\breve{\mathbb{P}}$ denote the probability measure of this environment. Then for each path $x_\bullet$ with $c_1 b N^{2/3} \leqslant t_1(x_\bullet) \leqslant C_3 b N^{2/3}$, the weight of the path in the $(\tilde{a}_1, \tilde{a}_2)$ environment and the weight of the path in the $\breve{\omega}$ environment agree. Thus, defining $Z_{m,n}(A) := \sum_{x_\bullet \in A} \prod_{k=1}^{m+n} \omega_{(x_{k-1},x_k)}$,

$$Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \tag{6.3}$$

is the same in distribution under $\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}$ and $\breve{\mathbb{P}}$. We can now make use of the bound (6.2).

Using a third-order Taylor expansion, the same series of calculations which leads to inequality (4.11) in the proof of Proposition 4.3 gives the existence of a constant $C' > 0$ such that:

$$\begin{aligned}\mathbb{E}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] &= m\left(\Psi_0^{f_1}(\tilde{a}_1) - \Psi_0^{f_1}(a_1)\right) \\ &\quad + n\left(\Psi_0^{f_2}(\tilde{a}_2) - \Psi_0^{f_2}(a_2)\right) \\ &\geqslant -\gamma b N^{1/3} C' + 4c_4 b^2 N^{1/3} - b^3 C' \\ &\geqslant c_4 b^2 N^{1/3}\end{aligned} \tag{6.4}$$

where $c_4 := \frac{1}{8}\Big(\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) + \psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)\Big)$ is positive by Lemma C.2 in the Appendix. The last inequality is obtained by first fixing b large enough then increasing N_0 if necessary.

We now split the probability

$$\begin{aligned}
&\mathbb{P}^{(\tilde a_1,\tilde a_2)}\big\{Q_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant 1-\varepsilon\big\}\\
&= \mathbb{P}^{(\tilde a_1,\tilde a_2)}\left\{\frac{1}{Z_{m,n}} Z_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant 1-\varepsilon\right\}\\
&\leqslant \check{\mathbb{P}}\left\{Z_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant (1-\varepsilon)e^{\mathbb{E}^{(\tilde a_1,\tilde a_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\} \qquad (6.5)\\
&\qquad + \mathbb{P}^{(\tilde a_1,\tilde a_2)}\left\{Z_{m,n} \leqslant e^{\mathbb{E}^{(\tilde a_1,\tilde a_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\}\\
&\leqslant \check{\mathbb{P}}\left\{Z_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant (1-\varepsilon)e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2}c_4 b^2 N^{1/3}}\right\} \qquad (6.6)\\
&\qquad + \mathbb{P}^{(\tilde a_1,\tilde a_2)}\left\{Z_{m,n} \leqslant e^{\mathbb{E}^{(\tilde a_1,\tilde a_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\}. \qquad (6.7)
\end{aligned}$$

The transition from $\mathbb{P}^{(\tilde a_1,\tilde a_2)}$ to $\check{\mathbb{P}}$ in (6.5) is due to the equality in distribution of (6.3) under these measures. Inequality (6.6) comes from (6.4).

For (6.7) we can use Chebyshev's inequality then the upper bound of the variance to get

$$(6.7) \leqslant \frac{C}{b^3}.$$

Thus (6.6) $\geqslant 1 - \frac{C}{b^3}$ for some new positive constant C. Let g be the Radon-Nikodym derivative $d\check{\mathbb{P}}/d\mathbb{P}^{(a_1,a_2)}$. Recall that the distributions differ only on the weights along the x-axis up until site $\lfloor C_3 bN^{2/3}\rfloor$. Thus

$$g(\omega) = \left(\frac{M_{f^1}(a_1)}{M_{f^1}(\tilde a_1)}\right)^{\lfloor C_3 bN^{2/3}\rfloor} \prod_{i=1}^{\lfloor C_3 bN^{2/3}\rfloor} \omega_{i,0}^{\lambda}.$$

We can evaluate $\mathbb{E}^{(a_1,a_2)}[g^2]$ explicitly. Increasing N_0, if necessary, so that $2\lambda \leqslant \epsilon$,

$$\mathbb{E}^{(a_1,a_2)}[\omega_{i,0}^{2\lambda}] = \frac{1}{M_{f^1}(a_1)}\int_0^\infty x^{2\lambda}x^{a_1-1}f^1(x)dx = \frac{M_{f^1}(a_1+2\lambda)}{M_{f^1}(a_1)}.$$

Now

$$\begin{aligned}
\mathbb{E}^{(a_1,a_2)}[g^2] &= \left(\frac{M_{f^1}(a_1)}{M_{f^1}(\tilde a_1)}\right)^{2\lfloor C_3 bN^{2/3}\rfloor} \prod_{i=1}^{\lfloor C_3 bN^{2/3}\rfloor} \mathbb{E}^{(a_1,a_2)}[\omega_{i,0}^{2\lambda}]\\
&= \left(\frac{M_{f^1}(a_1)M_{f^1}(a_1+2\lambda)}{M_{f^1}(a_1+\lambda)^2}\right)^{\lfloor C_3 bN^{2/3}\rfloor}.
\end{aligned}$$

Taking logarithms of both sides,

$$\begin{aligned}
&\log \mathbb{E}^{(a_1,a_2)}[g^2]\\
&\quad= \lfloor C_3 bN^{2/3}\rfloor\Big(\log M_{f^1}(a_1) + \log M_{f^1}(a_1+2bN^{-1/3}) - 2\log M_{f^1}(a_1+bN^{-1/3})\Big)
\end{aligned}$$

Recall that $\frac{\partial^2}{\partial a^2}\log M_{f^1}(a) = \psi_1^{f^1}(a) > 0$. Then

$$\begin{aligned}
&\lim_{N\to\infty} \log \mathbb{E}^{(a_1,a_2)}[g^2] \\
&\qquad = C_3 b \lim_{N\to\infty} \frac{\log M_{f^1}(a_1) + \log M_{f^1}(a_1+2bN^{-1/3}) - 2\log M_{f^1}(a_1+bN^{-1/3})}{N^{-2/3}} \\
&\qquad = C_3 b^2 \lim_{N\to\infty} \frac{\psi_0^{f^1}(a_1+2bN^{-1/3}) - \psi_0^{f^1}(a_1+bN^{-1/3})}{N^{-1/3}} \\
&\qquad = C_3 b^3 \psi_1^{f^1}(a_1) > 0
\end{aligned}$$

Increase N_0 if necessary so that for all $N \geqslant N_0$,

$$\mathbb{E}^{(a_1,a_2)}[g^2] \leqslant e^{2C_3 b^3}.$$

Defining the event

$$D = \left\{ Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant (1-\epsilon) e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2}c_4 b^2 N^{1/3}} \right\},$$

we get

$$\begin{aligned}
1 - \frac{C}{b^3} \leqslant (6.6) = \breve{\mathbb{P}}(D) & \\
&= \mathbb{E}^{(a_1,a_2)}[g \mathbb{1}_D] \\
&\leqslant \left(\mathbb{E}^{(a_1,a_2)}[g^2]\right)^{1/2} \left(\mathbb{P}^{(a_1,a_2)}(D)\right)^{1/2} \\
&\leqslant e^{C_3 b^3} \left(\mathbb{P}^{(a_1,a_2)}(D)\right)^{1/2}.
\end{aligned}$$

Thus

$$\epsilon_0 := (1 - \frac{C}{b^3})^2 e^{-2C_3 b^3} \leqslant \mathbb{P}^{(a_1,a_2)}(D).$$

Finally we have that

$$\begin{aligned}
\epsilon_0 \leqslant \mathbb{P}^{(a_1,a_2)}(D) &\leqslant \mathbb{P}^{(a_1,a_2)} \left(Z_{m,n} \geqslant (1-\varepsilon) e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2}c_4 b^2 N^{1/3}} \right) \\
&= \mathbb{P}^{(a_1,a_2)} \left(\log Z_{m,n} \geqslant \log(1-\varepsilon) + \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \tfrac{c_4 b^2 N^{1/3}}{2} \right) \\
&\leqslant \mathbb{P}^{(a_1,a_2)} \left(\log Z_{m,n} \geqslant \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + c_0 N^{1/3} \right).
\end{aligned}$$

Increasing N_0 if necessary and taking $c_0 = \frac{1}{4}c_4 b^2$ the final inequality holds for all $N \geqslant N_0$. This concludes the proof. □

We can use the variance lower bound to obtain a lower bound on the exit points of the path from the horizontal and vertical axes.

Corollary 6.2. *Assume that the polymer environment is distributed as in (3.2) and the sequence $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ satisfies (1.10) for some positive constant γ. Then there exist positive constants c_0, c_1, N_0 depending only on (a_1,a_2), β and γ such that for all $N \geqslant N_0$,*

$$c_0 \leqslant P_{m,n}(t_1 > c_1 N^{2/3} \text{ or } t_2 > c_1 N^{2/3}).$$

Proof: Averaging (1.8) and (1.9) of Proposition 1.1 then applying Lemma 4.2 followed by the lower bound of Theorem 1.2 gives the existence of positive constants c, C, N_0 such that for all $N \geqslant N_0$

$$cN^{2/3} \leqslant \mathbb{V}\text{ar}[\log Z_{m,n}] = E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] + E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})]$$
$$\leqslant C(E_{m,n}[t_1 + t_2] + 2).$$

Letting $c_1 := c/6C$ and increasing N_0 if necessary followed by an application of the Cauchy-Schwartz inequality along with Lemma 4.7 gives

$$3c_1 \leqslant E_{m,n}[\frac{t_1 + t_2}{N^{2/3}}] \leqslant 2c_1 + E_{m,n}[\frac{t_1 + t_2}{N^{2/3}} : t_1 + t_2 > 2c_1 N^{2/3}]$$
$$\leqslant 2c_1 + C' P_{m,n}(t_1 + t_2 > 2c_1 N^{2/3})^{\frac{1}{2}}$$

for some positive constant C'. Thus

$$c_0 := (\frac{c_1}{C'})^2 \leqslant P_{m,n}(t_1 + t_2 > 2c_1 N^{2/3}) \leqslant P_{m,n}(t_1 > c_1 N^{2/3} \text{ or } t_2 > c_1 N^{2/3}),$$

which completes the proof. □

We now prove the path fluctuation lower bound.

Proof of (1.15)*:* If $\tau = 0$, this reduces to Corollary 6.2. If $\tau \in (0,1)$ put $(k,l) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$. Then the sequence $(m-k, n-l)$ satisfies (1.10) with a new scaling parameter $M = (1-\tau)N$. By the down-right property and Lemma 5.1

$$Q_{m-k,n-l}(t_1 > u \text{ or } t_2 > u) \overset{d}{=} Q^{(k,l)}_{m,n}(t_1^{(k,l)} > u \text{ or } t_2^{(k,l)} > u)$$
$$= Q_{m,n}(v_1(l) > k + u \text{ or } w_1(k) > l + u)$$
$$\leqslant Q_{m,n}(v_1(l) > \tau m + \frac{u}{2} \text{ or } w_1(k) > \tau n + \frac{u}{2})$$

provided that $u \geqslant 2$. Corollary 6.2 applied to the sequence $(m-k, n-l)$ completes the proof. □

Appendix Appendix A Verification of Hypothesis 3.6

Lemma A.1. *If the function f satisfies the conditions of Hypothesis 3.6 and $g(x) := f(\frac{1}{x})$ for $x \in (0,\infty)$, then g also satisfies the conditions of Hypothesis 3.6.*

Proof: Note that $\text{supp}(g) = \text{supp}(f)^{-1}$. Fix a compact $K \subset D(M_g)$ and let $a \in K$. By parts (c) and (b) of Remark 3.3, $\psi_0^g(a) = -\psi_0^f(-a)$ and $-K \subset D(M_f)$. Thus there exists a positive constant C depending only $-K$ such that for all $b \in -K$, (3.9) and (3.10) hold. It therefore suffices to show the following two relations hold:

$$L^g(a,x) = L^f(-a, \frac{1}{x}) \qquad \text{for all } x \in \text{supp}(g) \tag{A.1}$$

$$\int_0^1 \Big|\frac{\partial}{\partial a} L^g(a, H^g(a,p))\Big| dp = \int_0^1 \Big|\frac{\partial}{\partial b} L^f(b, H^f(b,p))\Big| dp \tag{A.2}$$

where the right hand side of (A.2) is evaluated at $b = -a$.

(A.1) can be proven by using $\psi_0^g(a) = -\psi_0^f(-a)$ and making the substitution $y \mapsto \frac{1}{y}$ in the first integral appearing in (3.8).

(A.2) will now follow from (A.1) and

$$H^g(a, 1-p) = \frac{1}{H^f(-a,p)} \qquad \text{for all } p \in (0,1).$$

To see that this equality holds, let $X \sim m_g(a)$ and $x > 0$. Using part (a) of Remark 3.3

$$F^g(a,x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X^{-1} \geqslant x^{-1}) = 1 - \mathbb{P}(X^{-1} < x^{-1}) = 1 - F^f(-a, x^{-1}). \tag{A.3}$$

Fix $p \in (0,1)$ and recall the definition of $H^\bullet$, (3.3). Note that $H^f(-a,p)$ and $H^g(a,1-p)$ lie in $\operatorname{supp}(f)$ and $\operatorname{supp}(g) = \operatorname{supp}(f)^{-1}$ respectively. Plugging $x = H^g(a,1-p)$ into (A.3) gives

$$1 - p = F^g(a, H^g(a,1-p)) = 1 - F^f\big(-a, \frac{1}{H^g(a,1-p)}\big).$$

Rearranging yields

$$F^f\big(-a, \frac{1}{H^g(a,1-p)}\big) = p = F^f(-a, H^f(-a,p)).$$

Since $x \mapsto F^f(-a,x)$ is one-to-one on $\operatorname{supp}(f)$ we have the desired result. □

Lemma A.2. *Each of the functions f in Table 3.5 satisfy Hypothesis 3.6.*

Proof: Fix $b > 0$. By Lemma A.1 it suffices to show the three functions

$$f(x) = e^{-bx}, \qquad f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}, \qquad f(x) = \big(\frac{x}{x+1}\big)^b$$

satisfy the conditions of Hypothesis 3.6. In Seppäläinen (2012) (equation 3.30 and the computation following equation 4.7), Seppäläinen showed that the function $f(x) = e^{-x}$ satisfies these conditions. A simple rescaling then shows that these conditions are also satisfied for $f(x) = e^{-bx}$.

We will write $C_0(a), C_1(a), \ldots$ to indicate the positive constants $C_k(a)$ have a continuous dependence on a. We claim it is sufficient to show that if $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$ or $f(x) = (\frac{x}{x+1})^b$, then for all $x \in \operatorname{supp}(f)$ the following three bounds hold:

$$L^f(a,x) \leqslant C_0(a)(1 + |\log x|) \tag{A.4}$$

$$|x\frac{f'(x)}{f(x)}|L^f(a,x) \leqslant C_1(a)(1 + |\log x|) \tag{A.5}$$

$$|G^f(a,x)| \leqslant C_2(a)(1 + (\log x)^2) \tag{A.6}$$

where

$$\begin{aligned} G^f(a,x) :&= \frac{x^{-a}}{f(x)} \int_0^x (\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2) y^{a-1} f(y)dy \\ &= -\frac{x^{-a}}{f(x)} \int_x^\infty (\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2) y^{a-1} f(y)dy. \end{aligned} \tag{A.7}$$

Note that the second equality in the definition of $G^f(a,x)$ follows from the definitions of $\psi_0^f(a)$ and $\psi_1^f(a)$ in part (c) of Remark 3.2. (A.4) clearly implies (3.9). To show (3.10) is satisfied, using (3.7), we calculate

$$\begin{aligned}\frac{\partial}{\partial a}L^f(a,H^f(a,p)) &= \frac{\partial L^f}{\partial a}(a,H^f(a,p)) + \frac{\partial}{\partial a}H^f(a,p)\frac{\partial L^f}{\partial x}(a,H^f(a,p)) \\ &= \left(\frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x)\right)\Big|_{x=H^f(a,p)}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x) =& (\psi_0^f(a) - 2\log x)L^f(a,x) - aL^f(a,x)^2 \\ &+ G^f(a,x) - x\frac{f'(x)}{f(x)}L^f(a,x)^2,\end{aligned}$$

the conditions (A.4), (A.5), and (A.6) imply the existence of a positive constant $C_3(a)$ such that for all $x \in \mathrm{supp}(f)$,

$$\left|\frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x)\right| \leqslant C_3(a)\big(1 + (\log x)^2\big).$$

Condition (3.10) now follows from

$$\begin{aligned}\int_0^1 \left|\frac{\partial}{\partial a}L^f(a,H^f(a,p))\right| dp &\leqslant C_3(a)\int_0^1 \big(1 + (\log H^f(a,p))^2\big)dp \\ &= C_3(a)\big(1 + \psi_1^f(a) + (\psi_0^f(a))^2\big) < \infty.\end{aligned}$$

The last equality is justified by parts (a) and (c) of Remark 3.2 along with the fact that $H^f(a,\eta) \sim m_f(a)$ when η is uniformly distributed on $(0,1)$.

We first show (A.4), (A.5) and (A.6) for the case $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$. Let $a \in D(M_f) = (0,\infty)$. Then there exists some positive constant $C_4(a)$ such that the following two inequalities hold:

$$\big|\psi_0^f(a) - \log y\big|y^{a-1}f(y) \leqslant \begin{cases} C_4(a)(1-\log y)y^{a-1} & \text{if } 0 < y < \frac{1}{2} \\ C_4(a)(1-y)^{b-1} & \text{if } \frac{1}{2} \leqslant y < 1 \end{cases}$$

$$\big|\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2\big|y^{a-1}f(y) \leqslant \begin{cases} C_4(a)\big(1 + (\log y)^2\big)y^{a-1} & \text{if } 0 < y < \frac{1}{2} \\ C_4(a)(1-y)^{b-1} & \text{if } \frac{1}{2} \leqslant y < 1. \end{cases}$$

Since $a > 0$, (3.8) and (A.7) give: for $0 < x < \frac{1}{2}$,

$$L^f(a,x) \leqslant \frac{2^bC_4(a)}{x^a}\int_0^x (1-\log y)y^{a-1}dy \leqslant C_0(a)(1 + |\log x|) \tag{A.8}$$

$$\big|G^f(a,x)\big| \leqslant \frac{2^bC_4(a)}{x^a}\int_0^x \big(1 + (\log y)^2\big)y^{a-1}dy \leqslant C_2(a)\big(1 + (\log x)^2\big).$$

Similarly, the secondary expressions in (3.8) and (A.7) give: for $1/2 \leqslant x < 1$,

$$L^f(a,x) \leqslant \frac{2^aC_4(a)}{(1-x)^{b-1}}\int_x^1 (1-y)^{b-1}dy \leqslant C_0(a)(1-x) \tag{A.9}$$

$$\big|G^f(a,x)\big| \leqslant \frac{2^aC_4(a)}{(1-x)^{b-1}}\int_x^1 (1-y)^{b-1}dy \leqslant C_2(a)(1-x)$$

where we increased $C_0(a)$ and $C_2(a)$ if necessary. Thus the bounds (A.4) and (A.6) hold. Moreover, by (A.8) and (A.9),

$$\left|x\frac{f'(x)}{f(x)}\right|L^f(a,x) = |b-1|\frac{x}{1-x}L^f(a,x) \leqslant \begin{cases} C_1(a)(1+|\log x|) & \text{if } 0 \leqslant x < \frac{1}{2} \\ C_1(a) & \text{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

proving the bound (A.5).

We now consider the case $f(x) = (\frac{x}{x+1})^b$. Let $a \in D(M_f) = (-b, 0)$. Then

$$\left|\psi_0^f(a) - \log y\right|y^{a-1}f(y) \leqslant \begin{cases} C_4(a)(1-\log y)y^{a+b-1} & \text{if } 0 < y < 1 \\ C_4(a)(1+\log y)y^{a-1} & \text{if } y \geqslant 1 \end{cases}$$

$$\left|\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2\right|y^{a-1}f(y) \leqslant \begin{cases} C_4(a)\big(1+(\log y)^2\big)y^{a+b-1} & \text{if } 0 < y < 1 \\ C_4(a)\big(1+(\log y)^2\big)y^{a-1} & \text{if } y \geqslant 1. \end{cases}$$

Since $a+b > 0$, (3.8) and (A.7) give: for $0 < x < 1$,

$$L^f(a,x) \leqslant \frac{2^bC_4(a)}{x^{a+b}}\int_0^x (1-\log y)y^{a+b-1}dy \leqslant C_0(a)(1+|\log x|)$$
$$\left|G^f(a,x)\right| \leqslant \frac{2^bC_4(a)}{x^{a+b}}\int_0^x \big(1+(\log y)^2\big)y^{a+b-1}dy \leqslant C_2(a)\big(1+(\log x)^2\big).$$

Similarly, since $a < 0$, the secondary expressions in (3.8) and (A.7) give: for $x \geqslant 1$,

$$L^f(a,x) \leqslant \frac{2^bC_4(a)}{x^{a}}\int_x^\infty (1+\log y)y^{a-1}dy \leqslant C_0(a)(1+|\log x|)$$
$$\left|G^f(a,x)\right| \leqslant \frac{2^bC_4(a)}{x^{a}}\int_x^\infty (1+(\log y)^2)y^{a-1}dy \leqslant C_2(a)(1+(\log x)^2)$$

where we increased $C_0(a)$ and $C_2(a)$ if necessary. Thus the bounds (A.4) and (A.6) hold. Since $|x\frac{f'(x)}{f(x)}| = b\frac{1}{x+1} \leqslant b$, (A.4) implies (A.5) completing the proof. □

Appendix Appendix B Lemmas used in Section 3 and Section 4

Lemma B.1. *Assume the polymer environment is such that* $\log R^1$, $\log R^2$, $\log Y^1$, *and* $\log Y^2$ *have finite second moments. Then* $\mathbb{E}[(\log Z_x)^2] < \infty$ *for any* $x \in \mathbb{Z}_+^2$.

Proof: Since $\log Z_{k,0} = \sum_{i=1}^k R_{i,0}^1$ and $\log Z_{0,\ell} = \sum_{j=1}^\ell \log R_{0,j}^2$, $\log Z_x$ has finite second moment for each $x \in \mathbb{Z}_+^2 \backslash \mathbb{N}^2$. If $x \in \mathbb{N}^2$, the recursion (2.1) implies that

$$(\log Y_x^1 + \log Z_{x-\alpha_1}) \wedge (\log Y_x^2 + \log Z_{x-\alpha_2}) \leqslant \frac{\log Z_x}{2} \leqslant (\log Y_x^1 + \log Z_{x-\alpha_1}) \vee (\log Y_x^2 + \log Z_{x-\alpha_2}).$$

Thus

$$(\log Z_x)^2 \leqslant 4(\log Y_x^1 + \log Z_{x-\alpha_1})^2 + 4(\log Y_x^2 + \log Z_{x-\alpha_2})^2.$$

Since $\log Y^1$ and $\log Y^2$ have finite second moments, an inductive argument finishes the proof. □

Lemma B.2. *Suppose* $f_k : (0,\infty) \to [0,\infty)$ *for* $k = 1,\dots,r$ *and* $a_0 < a < a_1$ *are real numbers such that* $[a_0,a_1] \subset \bigcap_{k=1}^r D(M_{f_k})$. *Suppose we have a collection of independent random variables* $\{X_k\}_{k=1}^r$ *where* $X_k \sim m_{f_k}(a)$ *for all* $1 \leqslant k \leqslant r$. *Let* $\mathbb{E}^a$ *be the expectation corresponding to the product measure induced by* $\{X_k\}_{k=1}^r$.

Let $S = \sum_{k=1}^{r} \log X_k$ *and* $A : \mathbb{R}^r \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}^a[A(X_1, \ldots, X_r)^2] < \infty$ *for all* $a \in [a_0, a_1]$. *Then*

$$\frac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)] = \mathbb{C}ov^a(A(X_1, \ldots, X_r), S) \quad \textit{for all} \quad a \in (a_0, a_1)$$

and $(a_0, a_1) \ni a \mapsto \frac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)]$ *is continuous.*

Proof: The joint density of $(\log X_1, \log X_2, \ldots, \log X_r)$ is given by

$$g(x_1, \ldots, x_r) = \frac{e^{a\sum_{k=1}^{r} x_k}}{\prod_{k=1}^{r} M_{f_k}(a)} \prod_{k=1}^{r} f_k(e^{x_k}).$$

Thus the density of S is given by

$$h_a(s) = \frac{e^{as}}{\prod_{k=1}^{r} M_{f_k}(a)} \int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2-x_1}) \ldots f_r(e^{s-x_{r-1}}) dx_1, \ldots, x_{r-1} \quad \text{(B.1)}$$

Therefore the joint density of $(\log X_1, \log X_2, \ldots, \log X_r)$ given that $S = s$ is

$$\frac{g(x_1, \ldots, x_r)\mathbb{1}_{\{\sum_{k=1}^{r} x_k = s\}}}{h_a(s)} = \frac{\prod_{k=1}^{r} f_k(e^{x_k})\mathbb{1}_{\{\sum_{k=1}^{r} x_k = s\}}}{\int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2-x_1}) \ldots f_r(e^{s-x_{r-1}}) dx_1, \ldots, x_{r-1}},$$

which has no a dependence. Thus

$$\begin{aligned}
\tfrac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)] &= \tfrac{\partial}{\partial a}\int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] h_a(s) ds \\
&= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] \tfrac{\partial}{\partial a} h_a(s) ds \\
&= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] h_a(s)\big(s - \sum_{k=1}^{r} \tfrac{\partial}{\partial a} \log M_{f_k}(a)\big) ds \\
&= \mathrm{Cov}^a(A(X_1, \ldots, X_r), S).
\end{aligned}$$

The last equality comes from $\mathbb{E}[S] = \sum_{k=1}^{r} \mathbb{E}[\log X_k] = \sum_{k=1}^{r} \frac{\partial}{\partial a} \log M_{f_k}(a)$, by part (a) of Remark 3.2. The interchanging of the derivative and the integral is justified by the bound

$$\int_{\mathbb{R}} \mathbb{E}[|A(X_1, \ldots, X_r)| \, |S = s] \sup_{a \in [a_0, a_1]} \left|\frac{\partial}{\partial a} h_a(s)\right| ds < \infty. \quad \text{(B.2)}$$

Once we show that there is a constant C depending only on a_0 and a_1 such that

$$\sup_{a \in [a_0, a_1]} \left|\frac{\partial}{\partial a} h_a(s)\right| \leqslant C(1 + |s|)(h_{a_0}(s) + h_{a_1}(s)) \quad \text{(B.3)}$$

we will have the bound (B.2) since

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}[|A(X_1, \ldots, X_r)| \, |S = s](1 + |s|) h_{a_j}(s) ds &= \mathbb{E}^{a_j}[|A(X_1, \ldots, X_r)|(1 + |S|)] \\
&\leqslant \mathbb{E}^{a_j}[A(X_1, \ldots, X_r)^2]^{\frac{1}{2}} \mathbb{E}^{a_j}[(1 + |S|)^2]^{\frac{1}{2}}.
\end{aligned}$$

The last expression is finite since $\mathbb{E}^{a_j}[A(X_1, \ldots, X_r)^2] < \infty$ by assumption, and S is a finite sum of independent random variables each of which has finite exponential moments, by part (a) of Remark 3.2. Notice that the bound (B.2) also implies that $a \mapsto \frac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)]$ is continuous. All that is left to do is verify the bound

(B.3). To accomplish this, notice that equation (B.1) implies that $\frac{\partial}{\partial a}\log h_a(s) = s - \mathbb{E}^a[S]$. So

$$\sup_{a\in[a_0,a_1]}\left|\frac{\partial}{\partial a}h_a(s)\right| \leqslant C_1(1+|s|)\sup_{a\in[a_0,a_1]}h_a(s)$$

where $C_1 := 1 \vee \sup_{a\in[a_0,a_1]}|E^a[S]|$. Thus it suffices to show that $\sup_{a\in[a_0,a_1]}h_a(s) \leqslant C_2\Big(h_{a_0}(s)+h_{a_1}(s)\Big)$ for some constant C_2 independent of s. By part (c) of Remark 3.2, $a\mapsto\mathbb{E}^a[S]$ is an increasing function. Therefore, for all $s\leqslant\mathbb{E}^{a_0}[S]$ the function $a\mapsto h_a(s)$ is non-increasing on $[a_0,a_1]$. Thus

$$\sup_{a\in[a_0,a_1]}h_a(s)\leqslant h_{a_0}(s) \text{ for all } s\leqslant\mathbb{E}^{a_0}[S].$$

On the other hand, if $s>\mathbb{E}^{a_0}[S]$, then $\frac{\partial}{\partial a}\log\Big(h_a(s)\exp\left(a(\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S])\right)\Big) = s-\mathbb{E}^a[S]+\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S]>0$ for all $a\in[a_0,a_1]$. Thus for all $s>\mathbb{E}^{a_0}[S]$, $a\mapsto h_a(s)\exp\Big(a(\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S])\Big)$ is increasing on the interval $[a_0,a_1]$. Therefore,

$$\sup_{a\in[a_0,a_1]}h_a(s)\leqslant C_3h_{a_1}(s) \text{ for all } s>\mathbb{E}^{a_0}[S]$$

where $C_3=\exp\Big((a_1-a_0)(\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S])\Big)$. We now get the desired result with $C_2=1+C_3$. □

Lemma B.3. *Assume that the polymer environment is distributed as in (3.2) and let ϵ be small enough such that for all $|\lambda|\leqslant\epsilon$, $a_1+\lambda\in D(M_{f^1})$ and $a_2-\lambda\in D(M_{f^2})$. Let $(m,n)\in\mathbb{N}^2$ and $k\in\mathbb{N}$. Then, with notation as in (3.5), $Q_{m,n}^{(a_1+\lambda,a_2-\lambda)}(t_1\geqslant k)$ is stochastically non-decreasing in λ and $Q_{m,n}^{(a_1+\lambda,a_2-\lambda)}(t_2\geqslant k)$ is stochastically non-increasing in λ.*

Proof:

$$\frac{\partial}{\partial b_i}Q_{m,n}^{(b_1,b_2)}(t_j\geqslant k) = \frac{\partial}{\partial b_i}\left(\frac{1}{Z_{m,n}(b_1,b_2)}\sum_{x_\bullet\in\Pi_{m,n}}\mathbb{1}_{\{t_j\geqslant k\}}W(b_1,b_2)(x_\bullet)\right). \tag{B.4}$$

If $i\neq j$, the sum in (B.4) has no b_i dependence, so

$$\frac{\partial}{\partial b_i}Q_{m,n}^{(b_1,b_2)}(t_j\geqslant k) = \frac{-1}{(Z_{m,n}(b_1,b_2))^2}\left(\frac{\partial}{\partial b_i}Z_{m,n}(b_1,b_2)\right)\sum_{x_\bullet\in\Pi_{m,n}}\mathbb{1}_{\{t_j\geqslant k\}}W(b_1,b_2)(x_\bullet),$$

which is non-positive by (3.18). If $i=j$, then by (3.17) and (3.18),

$$\begin{aligned}\frac{\partial}{\partial b_i}Q_{m,n}^{(b_1,b_2)}(t_i\geqslant k) &= \frac{\sum_{x_\bullet\in\Pi_{m,n}}\mathbb{1}_{\{t_i\geqslant k\}}\frac{\partial}{\partial b_i}W(b_1,b_2)(x_\bullet)}{Z_{m,n}(b_1,b_2)}\\ &\quad-\left(\frac{\partial}{\partial b_i}\log Z_{m,n}(b_1,b_2)\right)\frac{\sum_{x_\bullet\in\Pi_{m,n}}\mathbb{1}_{\{t_i\geqslant k\}}W(b_1,b_2)(x_\bullet)}{Z_{m,n}(b_1,b_2)}\\ &= \mathbb{C}\mathrm{ov}^{Q_{m,n}^{(b_1,b_2)}}\big(\sum_{k=1}^{t_i}L^{f^i}(b_i,H^{f^i}(b_i,\eta_k^i)),\mathbb{1}_{\{t_i\geqslant k\}}\big),\end{aligned}$$

which is non-negative. □

Appendix Appendix C Properties of ψ_n^f

Model	$\psi_n^{f^1}(a_1)$	$\psi_n^{f^2}(a_2)$
IG	$(-1)^{n+1}(\Psi_n(\mu-\theta)-\delta_{n,0}\log\beta)$	$(-1)^{n+1}(\Psi_n(\theta)-\delta_{n,0}\log\beta)$
G	$\Psi_n(\mu+\theta)-\delta_{n,0}\log\beta$	$(-1)^{n+1}(\Psi_n(\theta)-\Psi_n(\mu+\theta))$
B	$\Psi_n(\mu+\theta)-\Psi_n(\mu+\theta+\beta)$	$(-1)^{n+1}(\Psi_n(\theta)-\Psi_n(\mu+\theta))$
IB	$(-1)^{n+1}(\Psi_n(\mu-\theta)-\Psi_n(\mu-\theta+\beta))$	$\Psi_n(\mu-\theta+\beta)+(-1)^{n+1}\Psi_n(\theta)$

FIGURE C.9. ψ_n^f functions for each of the four basic beta-gamma models.

By Abramowitz and Stegun (1964) (p.260 line 6.4.1) the polygamma function of order n, $\Psi_n(x)=\frac{\partial^{n+1}}{\partial x^{n+1}}\log\Gamma(x)$, has integral representation

$$\Psi_n(x)=(-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt. \tag{C.1}$$

Lemma C.1. *For any $n\in\mathbb{N}$, the map $a\mapsto\frac{\Psi_{n+1}(a)}{\Psi_n(a)}$ is strictly increasing on $(0,\infty)$.*

Proof: Fix $n\in\mathbb{N}$ and $a\in(0,\infty)$. We will show that $\frac{\partial^2}{\partial a^2}\log|\Psi_n(a)|>0$.

After substituting $y=e^{-t}$ in (C.1) we get

$$|\Psi_n(a)|=\int_0^\infty y^{a-1}f(y)dy=M_f(a)$$

where $f(y):=\frac{(-\log y)^n}{1-y}\mathbb{1}_{\{0<y<1\}}$. Note that $D(M_f)=(0,\infty)$. Now given a random variable $X\sim m_f(a)$, by part (c) of Remark 3.2,

$$\frac{\partial^2}{\partial a^2}\log|\Psi_n(a)|=\frac{\partial^2}{\partial a^2}\log M_f(a)=\mathbb{V}\text{ar}[\log X]>0,$$

since X is non-degenerate. □

Lemma C.2. *Assume the polymer environment is distributed as in (3.2). Then*

$$\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)+\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1)>0.$$

Proof: Recall that $\psi_1^{f^j}$ are always positive and by (C.1) Ψ_n has sign $(-1)^{n+1}$ throughout $(0,\infty)$.

For the inverse-gamma model (1.2) with fixed constants $\beta>0$ and $\mu>\theta>0$, Table C.9 implies that $\psi_2^{f^j}(a_j)>0$ for $j=1,2$. The conclusion follows immediately.

For the gamma model (1.3) with fixed positive constants β, μ, and θ, by Table C.9

$$\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)+\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1)=-\Psi_1(\theta+\mu)\Psi_2(\theta)+\Psi_1(\theta)\Psi_2(\theta+\mu).$$

The quantity on the right hand side is positive if and only if

$$\frac{\Psi_2(\theta+\mu)}{\Psi_1(\theta+\mu)}>\frac{\Psi_2(\theta)}{\Psi_1(\theta)}$$

which holds true by Lemma C.1 with $n=1$.

For the beta model (1.4) with fixed positive constants β, μ, and θ, using Table C.9

$$\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) + \psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) > 0 \quad \Leftrightarrow$$

$$\frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} > -\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} \quad \Leftrightarrow$$

$$\frac{\Psi_2(\theta+\mu+\beta) - \Psi_2(\theta+\mu)}{\Psi_1(\theta+\mu+\beta) - \Psi_1(\theta+\mu)} > \frac{\Psi_2(\theta+\mu) - \Psi_2(\theta)}{\Psi_1(\theta+\mu) - \Psi_1(\theta)}. \tag{C.2}$$

By Cauchy's mean value theorem there exist constants $\theta < \xi_1 < \theta + \mu < \xi_2 < \theta + \mu + \beta$ such that the left and right-hand sides of (C.2) equal $\frac{\Psi_3(\xi_2)}{\Psi_2(\xi_2)}$ and $\frac{\Psi_3(\xi_1)}{\Psi_2(\xi_1)}$ respectively. Lemma C.1 with $n = 2$ now gives (C.2).

For the inverse-beta model (1.5) with fixed constants $\beta > 0$ and $\mu > \theta > 0$, by Table C.9, $\psi_2^{f^1}(a_1) > 0$, $\psi_1^{f^2}(a_2) > \Psi_1(-\theta+\mu+\beta)$, and $\psi_2^{f^2}(a_2) > \Psi_2(-\theta+\mu+\beta)$. Therefore

$$\begin{aligned}\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)+&\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1)\\ &> \psi_1^{f^1}(a_1)\Psi_2(-\theta+\mu+\beta) + \Psi_1(-\theta+\mu+\beta)\psi_2^{f^1}(a_1)\\ &= \Psi_1(-\theta+\mu)\Psi_2(-\theta+\mu+\beta) - \Psi_1(-\theta+\mu+\beta)\Psi_2(-\theta+\mu).\end{aligned}$$

Letting $x = -\theta + \mu$, the last line is positive if and only if

$$\frac{\Psi_2(x+\beta)}{\Psi_1(x+\beta)} > \frac{\Psi_2(x)}{\Psi_1(x)}$$

which holds true by Lemma C.1 with $n = 1$. □

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