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Weak convergence on Wiener space: targeting the first two chaoses

Christian Krein

Mathematics Research Unit,
University of Luxembourg, Maison du Nombre,
6, avenue de la Fonte, L-4364 Esch-sur-Alzette, Grand-Duchy of Luxembourg.
E-mail address: christian.krein@education.lu

Abstract. We consider sequences of random variables living in a finite sum of Wiener chaoses. We find necessary and sufficient conditions for convergence in law to a target variable living in the sum of the first two Wiener chaoses. Our conditions hold notably for sequences of multiple Wiener integrals. Malliavin calculus and in particular the Γ-operators are used. Our results extend previous findings by Azmoodeh et al. (2015) and are applied to central and non-central convergence situations. Our methods are applied to investigate stable convergence and we exclude certain classes of random variables as target variables for sequences living in a fixed Wiener chaos.

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1. Introduction

1.1. *Overview.* The aim of this paper is to provide new criteria for non-central convergence in law for sequences of polynomial functionals of a Brownian motion W. In particular, we consider the convergence in law of a sequence of random variables $\{F_n\}_n$ to a target variable X, where:

- the random variables F_n have a representation of the form $F_n = \sum_{p=1}^{m} I_p(f_{n,p})$ for a fixed $m \geqslant 2$, where $I_p(\cdot)$ is the Wiener integral of order p with respect to the Brownian motion W;
- the target variable X lives in the sum of the first two Wiener chaoses associated with W and can be represented as:

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i],$$

  where all coefficients b_i, c_j and d_j are non-zero and N, R_i, P_j are independent standard normal variables for $1 \leqslant i \leqslant k_1$ and $1 \leqslant j \leqslant k_2$. We shall see that this representation covers in particular random variables of the form:

$$X = aU_0 + \sum_{i=1}^{n} \lambda_i(U_i^2 - 1),$$

  where all coefficients λ_i are non-zero, $U_1, \ldots, U_n$ are independent standard normal variables and U_0 is a standard normal variable which may be correlated to $U_1, \ldots, U_n$.

Our main result (Theorem 1.1) gives a necessary and sufficient criterion for the convergence in law to X:

Theorem 1.1. *Consider* $0 \leqslant k_1, k_2 < \infty$ *and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i], \tag{1.1}$$

where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. *Suppose that at least one of the parameters* a, k_1, k_2 *is non-zero. Consider a sequence* $\{F_n\}_n$ *of non-zero random variables such that* $F_n = \sum_{i=1}^{p} I_i(f_{n,i})$ *for* $p \geqslant 2$ *fixed and* $\{f_{n,i}\}_n \subset H^{\odot i}$ *for* $1 \leqslant i \leqslant p$. *Define:*

$$P(x) = x^{1+1_{[a\neq 0]}} \prod_{j=1}^{k_1}(x - b_j) \prod_{j=1}^{k_2}(x - c_j)^2.$$

As $n \to \infty$, *the following conditions (a) and (b) are equivalent:*

(a) (1) $\kappa_r(F_n) \to \kappa_r(X)$, *for* $r = 1, \ldots, deg(P)$,

(2) $$\mathbb{E}\left[\left|\mathbb{E}\left[\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]\right)\Big| F_n\right]\right|\right] \to 0,$$

(b) $F_n \overset{Law}{\to} X$, *as* $n \to \infty$.

In this paper, we consider functionals of a Brownian motion. By a standard isometry argument (see e.g. Nualart and Peccati, 2005, Section 2.2), the results immediately extend to the framework of an isonormal Gaussian process on a general real separable Hilbert space H.

The notations used in this theorem are introduced in Section 2. In particular, the representation of X in Eq. (1.1) is detailed in Eq. (3.3) and (3.5), $\kappa_r(Y)$ is the r-th cumulant of a random variable Y and the sequence $\{\Gamma_i(F_n)\}_i$ is defined recursively using Malliavin operators. In Section 3 the representation of the target random variable in Eq. (1.1) is derived.
Our main result unifies, generalizes and extends previous findings. More precisely, Theorem 1.1:

- further extends to a non-central setting the seminal paper Nualart and Peccati (2005) and in particular Nourdin and Peccati (2012, Theorem 5.3.1) by dealing with central limit theorems for sequences living in a finite sum of Wiener chaoses,
- extends Nourdin and Poly (2012, Theorem 3.4) by considering a sequence $\{F_n\}_n$ of random variables which are no longer restricted to the second Wiener chaos but live in a finite sum of Wiener chaoses,
- extends Nourdin and Peccati (2009a, Theorem 1.2), Azmoodeh et al. (2015, Theorem 3.2) and Döbler and Peccati (2016, Proposition 1.7) by considering target variables involving linear combinations of independent χ^2 distributed random variables and adding a possibly correlated normal variable,
- improves Azmoodeh et al. (2015, Theorem 3.2) by replacing L^2-convergence with L^1-convergence and finding thus a necessary and sufficient criterion for convergence in law for a large class of target variables, in particular for linear combination of independent central χ^2 distributed target variables.

In addition to these applications which are discussed in Section 4, Theorem 1.1 is used to investigate stable convergence in Section 5.

1.2. *History and motivation.* The study of convergence in law for sequences of multiple Wiener integrals, by variational techniques, has been the object of an intense study in recent years. The starting point of this line of research is Nualart and Peccati (2005). In this reference and later in Nualart and Ortiz-Latorre (2008), the authors gave necessary and sufficient criteria for the convergence in law of a sequence of multiple Wiener integrals $I_p(f_{n,p})$ to a standard normal variable N: *If the functions $f_{n,p}$ are symmetric in the $p \geqslant 2$ variables with $\mathbb{E}[I_p(f_{n,p})^2] \to 1$, as $n \to \infty$, then the following conditions are equivalent:*

(i) $\mathbb{E}[I_p(f_{n,p})^4] \to 3$, *as* $n \to \infty$,
(ii) $\|f_{n,p} \otimes_l f_{n,p}\|_{H^{\otimes(2p-2l)}} \to 0$, *as* $n \to \infty$, *for every* $l = 1, \dots, p-1$,
(iii) $\|DI_p(f_{n,p})\|_H^2 \overset{L^2(\Omega)}{\to} p$, *as* $n \to \infty$,
(iv) $I_p(f_{n,p}) \overset{Law}{\to} N$, *as* $n \to \infty$.

Notice that D is the standard Malliavin derivative operator and $f_{n,p} \otimes_l f_{n,p}$ is the contraction of order l, see Section 2.2. The equivalence of (i), (ii) and (iv) has been found in Nualart and Peccati (2005), the equivalence of either one of these conditions with (iii) has been proved later in Nualart and Ortiz-Latorre (2008). Considering the proof of Nualart and Ortiz-Latorre (2008, Theorem 4), it is easy to see that condition (iii) above can be replaced by:

(iii') $\mathbb{E}\left[\|DI_p(f_{n,p})\|_H^2 \,|\, I_p(f_{n,p})\right] \overset{L^1(\Omega)}{\to} p$, *as* $n \to \infty$.

This is remarkable since conditions of this form play a crucial role in Azmoodeh et al. (2015) and in the main result of the present paper. Since this characterisation

has been published, limit theorems have been extended beyond standard normal target variables. In Nourdin and Peccati (2009a), the authors establish, for $p \geqslant 2$ even, necessary and sufficient conditions for a sequence $\{I_p(f_{n,p})\}_n$ of multiple Wiener integrals to converge in law to a Gamma random variable. The conditions found by Nourdin and Peccati use contractions, convergence of the first moments of $I_p(f_{n,p})$ and Malliavin derivatives. In particular, the results of Nourdin and Peccati (2009a) cover the convergence in law of a sequence of multiple Wiener integrals to a random variable $X_1 \overset{\text{Law}}{=} \sum_{i=1}^k (N_i^2 - 1)$ with N_i independent standard normal variables and $1 \leqslant k < \infty$. X_1 has a centered χ^2 law with k degrees of freedom. The authors also prove that the convergence is stable in this case. More results about stable convergence can be found in Peccati and Taqqu (2008) and in the more recent work Nourdin et al. (2016a).

For the case $p = 2$, linear combinations of independent centered χ^2 distributed random variables are important since it is known that every element X_2 of the second Wiener chaos has a representation of the form $X_2 \overset{\text{Law}}{=} \sum_{i=1}^n \alpha_i (N_i^2 - 1)$, where $1 \leqslant n \leqslant \infty$ and $\{N_i\}_i$ is a sequence of independent standard normal variables, see Janson (1997, Theorem 6.1). It is proved in Nourdin and Poly (2012), that every sequence $\{I_2(f_{n,p})\}_n$ which converges in law has a limit of the form $\alpha_0 N_0 + \sum_{i=1}^n \alpha_i (N_i^2 - 1)$, where N_i are independent standard normal variables. In Nourdin and Poly (2012, Theorem 3.4) the authors use cumulants and a polynomial Q to characterise this convergence in law if $n < \infty$.

The idea of using polynomials to find necessary and sufficient conditions for the convergence in law has proved to be useful. In Azmoodeh et al. (2015), the authors consider the more general problem of finding necessary and sufficient conditions for a sequence $\{F_n\}_n$ to converge to a random variable with a representation of the form

$$X_3 \overset{\text{Law}}{=} \sum_{i=1}^k \alpha_i (N_i^2 - 1), \quad N_1, \ldots, N_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1), \tag{1.2}$$

where F_n are random variables living in a (fixed) finite sum of Wiener chaoses, see Section 2.1. For $k < \infty$, their main finding, Azmoodeh et al. (2015, Theorem 3.2) provides a necessary and a sufficient condition for $F_n \overset{\text{Law}}{\to} X_3$, as $n \to \infty$, in terms of Malliavin operators Γ_i, defined in Section 2.3.

Linear combinations of independent centered χ^2 distributed random variables as in Eq. (1.2) are of great interest because of their role within the second Wiener chaos. This class of random variables is important in stochastic geometry as well. In Marinucci et al. (2016), the authors consider the two-dimensional torus and prove the weak convergence of the normalized nodal length of the so-called 'arithmetic random waves', to a target variable $\mathcal{M}_\eta$ defined by:

$$\mathcal{M}_\eta := \frac{-1-\eta}{2\sqrt{1+\eta^2}}(X_1^2 - 1) + \frac{-1+\eta}{2\sqrt{1+\eta^2}}(X_2^2 - 1), \quad \eta \in [0,1],$$

where X_1, X_2 are independent standard normal variables. An important element of the proof is the fact that the Wiener chaos expansion of the normalised nodal length is dominated by its fourth order chaos component.

Another line of research, which is closely connected to the previous results, investigates the convergence of sequences living in a finite sum of Wiener chaoses by using distances between probability measures. In this context, the distance between

two laws F and G is defined as:

$$d_{\mathscr{H}}(F,G) = \sup_{h\in\mathscr{H}} |\,\mathbb{E}[h(F)] - \mathbb{E}[h(G)]|,$$

where $\mathscr{H}$ is a class of functions. Different classes $\mathscr{H}$ lead to distances such as the Wasserstein, total variation and Kolmogorov distance, see Nourdin and Peccati (2012, Appendix C) for details. Upper bounds for the total variation and smooth distances are proved in Nourdin and Peccati (2009b) and Nourdin and Poly (2013). For most of these results, the distance of a distribution to a centered normal distribution or a centered χ^2 distribution with ν degrees of freedom is considered. Recently, in Döbler and Peccati (2016) a new estimate is proved for the Wasserstein distance of a distribution to a centered χ^2 distribution with ν degrees of freedom. In particular, the authors find a new necessary and sufficient criterion for sequences living in a fixed Wiener chaos to converge in law to X_1 defined above.

In Arras et al. (2016a) the authors consider target variables of the form given in Eq. (1.2):

$$X_3 = \sum_{i=1}^{k} \alpha_i (N_i^2 - 1),$$

where the coefficients are not necessarily pairwise distinct and $N_1, \ldots, N_k$ are independent standard normal variables. The authors discuss Stein's method for this class of target variables and apply a new and original Fourier-based approach to derive a Stein-type characterisation. The polynomials used in Nourdin and Poly (2012); Azmoodeh et al. (2015) and Γ-operators, see Azmoodeh et al. (2015); Nourdin and Peccati (2012), are combined with the integration by parts formula of Malliavin calculus to derive a Stein operator which allows to characterise target variables as in Eq. (1.2). The authors consider a linear combination of Γ-operators which shall be generalized in the present paper and the 2-Wasserstein distance. In general, the 2-Wasserstein distance between the laws of random vectors U and V is defined as follows:

$$d_{W_2}(U,V) := \left(\inf \mathbb{E}\left[\|X-Y\|_d^2\right]\right)^{\frac{1}{2}},$$

where the infimum is taken over all joint distributions of X and Y with respective marginals U and V, and $\|\cdot\|_d$ stands for the Euclidean norm on $\mathbb{R}^d$, see Arras et al. (2017, Definition 1.1). It is shown that:

$$d_{W_2}(F_n, X_3) \leqslant C\left(\sqrt{\Delta(F_n)} + \sum_{r=2}^{k+2} |\kappa_r(F_n) - \kappa_r(X_3)|\right),$$

where C is independent of n, the quantity $\Delta(F_n)$ can be expressed in terms of cumulants κ_r and polynomials, the sequence $\{F_n\}_n$ must satisfy several conditions which hold in particular for sequences living in the second Wiener chaos. It is proved in particular that:

$$d_{W_2}(F_n, X_3) \leqslant C\sqrt{\Delta(F_n)}, \tag{1.3}$$

if $\dim_{\mathbb{Q}} \operatorname{span}\left(\alpha_1^2, \ldots, \alpha_k^2\right) = k$, see Eq. (1.2). This shows that $\Delta(F_n) \to 0$, as $n \to \infty$, is sufficient for convergence in the 2-Wasserstein metric which implies convergence in law. In Arras et al. (2017), it is shown that the convergence of the cumulants can not be omitted in the general case. In addition to the upper bound

found in Arras et al. (2016a), a lower bound for the 2-Wasserstein metric is derived in Arras et al. (2017), namely:

$$d_{W_2}(F_n, X_3) \geqslant C'\sqrt{\Delta(F_n)},$$

where $C' > 0$ is independent of n and the sequence $\{F_n\}_n$ lives in the second Wiener chaos. In other words, for sequences living in the second Wiener chaos, weak convergence to X_3 is equivalent to $\Delta(F_n) \to 0$, as $n \to \infty$, if $\dim_{\mathbb{Q}} \operatorname{span}\left(\alpha_1^2, \ldots, \alpha_k^2\right) = k$. The so-called Stein-Tikhomirov method is considered in Arras et al. (2016b). This method can be seen as a combination of Stein's method with other methods to measure the rate of convergence of a sequence of random variables. The authors consider in particular the Stein-type characterisation found in Arras et al. (2016a, Theorem 2.1) and apply their version of the Stein-Tikhomirov method. The same linear combination of Γ-operators as in Arras et al. (2016a) is used and the so-called transfer-principle allows to find upper bounds on smooth Wasserstein distances using upper bounds on the difference of the characteristic functions of the approximating sequence and the target variable. For sequences living in the sum of the first p Wiener chaoses and constants $C, \Theta > 0$ depending only on p, the following bound is proved:

$$d_{W_2}(F_n, X_3) \leqslant C\,\Delta_n\,|\log(\Delta_n)|^{\Theta},$$

where Δ_n is expressed in terms of cumulants and Γ-operators. In particular, if $k \geqslant 3$ in Eq. (1.2), we have for the Kolmogorov distance that $d_{\mathrm{Kol}}(F_n, X_3) \leqslant B\sqrt{\Delta_n}$. Finally the authors find bounds for Δ_n if $\{F_n\}_n$ lives inside a fixed Wiener chaos and $k = 1$ in Eq. (1.2). Even though the aforementioned papers present important new results for target variables living in the second Wiener chaos, none of them considers target variables living in the sum of the first two Wiener chaoses with possibly correlated first and second order components.

The results of Azmoodeh et al. (2015, Theorem 3.2) are the starting point of the present work. As anticipated, we shall consider a sequence $\{F_n\}_n$ of random variables living in a finite (fixed) sum of Wiener chaoses and provide necessary and sufficient conditions for $F_n \overset{\text{Law}}{\to} X_4$, as $n \to \infty$, where $k < \infty$ and X has the following, more general form:

$$X_4 = \sum_{i=1}^{k} \left[\alpha_i(N_i^2 - 1) + \beta_i N_i\right], \quad N_1, \ldots, N_k \overset{i.i.d.}{\sim} \mathcal{N}(0,1). \tag{1.4}$$

The representation in Eq. (1.4) is equivalent to one in Eq. (1.1), where we have dropped all vanishing coefficients and regrouped independent normal variables. Both representations (1.1) and (1.4) are useful for the discussion to follow. Random variables as in Eq. (1.4) are important since every random variable living in the sum of the first two Wiener chaoses has a representation of this form, with $k \leqslant \infty$, see Janson (1997, Theorem 6.2). Our conditions make, as in Azmoodeh et al. (2015), use of the operators Γ_i. Clearly such a result can be seen as extension of Azmoodeh et al. (2015, Theorem 3.2). For sequences of random variables living in a finite sum of Wiener chaoses, we shall apply our methods and results to derive necessary and sufficient criteria to prove stable convergence to target variables with representations as in Eq. (1.4).

1.3. *Results and plan.* The paper is organized as follows:

- In Section 2, we introduce the necessary notations and give a brief introduction to Malliavin calculus. The basic elements of this theory shall be needed in the forthcoming proofs.
- In Section 3, we prove our characterisation in Theorems 3.8 and 3.11. The main Theorem 1.1 is then a direct consequence of these theorems.
- In Section 4, we apply Theorem 1.1 to several situations, such as the convergence in law to a normal variable or a centered χ^2 distributed random variable with k_1 degrees of freedom. We shall also recover the results of Azmoodeh et al. (2015). In Theorem 4.6, we give sufficient conditions, based only on cumulants and contractions. We conclude this section by giving a criterion which excludes certain classes of target variables for sequences living in a chaos of odd order.
- In Section 5, we give criteria which can be used to determine whether a sequence converges stably.

2. Preliminaries

2.1. *Multiple Wiener integrals.* The reader is referred to Nourdin and Peccati (2012), Nualart (2006) or Di Nunno et al. (2009) for a detailed introduction to multiple Wiener integrals. Consider the real Lebesgue space $H = L^2([0,T], \lambda_T)$, where λ_T is the Lebesgue measure on $[0,T]$. The real separable Hilbert space H is endowed with the standard scalar product $\langle h, g\rangle_H := \int_0^T hg d\lambda$ for all $h, g \in H$. We write $H^{\otimes p}$ for $L^2([0,T]^p, \lambda_T^p)$, where $\lambda_T^p := \lambda^p|[0,T]^p$, and define $H^{\odot p}$ as the subspace of $H^{\otimes p}$ containing exactly the functions which are symmetric on a set of Lebesgue measure T^p. Consider a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and a standard Brownian motion $(W_t)_{t\in[0,T]}$ with respect to $\mathbb{P}$ and the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Define for every $h \in H$:

$$W(h) = I_1(h) = \int_0^T h(t)dW_t,$$

then $W(h) \in L^2(\Omega) := L^2(\Omega, \mathbb{P})$, in other words $W(h)$ is square-integrable. We have for $h, g \in H$:

$$\mathbb{E}[W(h)\, W(g)] = \mathbb{E}[I_1(h)\, I_1(g)] = \langle h, g\rangle_H.$$

More generally the q-th Wiener chaos is defined as closed linear subspace of $L^2(\Omega)$ which is generated by the random variables of the form $H_q(W(h))$ where $h \in H$ with $\|h\|_H = 1$ and H_q is the q-th Hermite polynomial. The elements of the q-th Wiener chaos can be represented as multiple Wiener integrals. For every $f_p \in H^{\odot p}$:

$$\begin{aligned} I_p(f_p) &:= \int_{[0,T]^p} f_p(t_1, \dots, t_p) dW_{t_1} \dots dW_{t_p} \\ &:= p! \int_0^T \left(\int_0^{t_p} \dots \int_0^{t_3} \left(\int_0^{t_2} f_p(t_1, \dots, t_p) dW_{t_1} \right) dW_{t_2} \dots dW_{t_{p-1}} \right) dW_{t_p}. \end{aligned}$$

It is well known that every $F \in L^2(\Omega)$ has a representation of the form $F = \sum_{p=0}^{\infty} I_p(f_p)$, where $I_0(f_0) = f_0 = \mathbb{E}[F]$ and the right-hand side converges in $L^2(\Omega)$. We have moreover for $p, q \geqslant 1$:

$$\mathbb{E}[I_p(f_p)\, I_q(g_q)] = 1_{[p=q]}\, p!\, \langle f_p, g_q\rangle_{H^{\otimes p}} = 1_{[p=q]} \int_{[0,T]^p} f_p\, g_q\, d\lambda_T^p. \tag{2.1}$$

For two function $f_p \in H^{\odot p}$ and $g_q \in H^{\odot q}$, the contraction of r indices is defined for $1 \leqslant r \leqslant p \wedge q$ by:

$$(f_p \otimes_r g_q)(t_1, \ldots, t_{p+q-2r}) = \int_{[0,T]^r} f_p(t_1, \ldots, t_{p-r}, s) g_q(t_{p-r+1}, \ldots, t_{p+q-2r}, s)\, d\lambda^r(s).$$

We have $f_p \otimes_r g_q \in H^{p+q-2r}$. The symmetrization of $f_p \otimes_r g_q$ is $f_p \tilde{\otimes}_r g_q$. We shall also need the multiplication formula for multiple Wiener integrals. For $f_p \in H^{\odot p}$ and $g_q \in H^{\odot q}$, we have:

$$I_p(f_p) I_q(g_q) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f_p \tilde{\otimes}_r g_q).$$

2.2. *Malliavin calculus.* The reader is referred to Nourdin and Peccati (2012), Nualart (2006) or Azmoodeh et al. (2015) for a detailed introduction to Malliavin calculus. Let $k \geqslant 1$ and F be a random variable with $F = f(W(h_1), \ldots, W(h_k))$ where f is an infinitely differentiable rapidly decreasing function on $\mathbb{R}^k$ and $h_1, \ldots, h_k \in H$. Then F is called a *smooth random variable* and $\mathcal{S}$ is the set containing exactly the smooth random variables. The Malliavin derivative of F is defined by:

$$\begin{aligned} DF &:= \sum_{i=1}^{k} h_i(t)\, \partial_i f(W(h_1), \ldots, W(h_k)) \\ &= \sum_{i=1}^{k} h_i(t)\, \partial_i f\left(\int_0^T h_i(s) dW_s, \ldots, \int_0^T h_k(s) dW_s\right). \end{aligned} \tag{2.2}$$

We have that D_t is closable from $L^2(\Omega, \mathbb{P})$ to $L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda_T)$, that is (see Di Nunno et al., 2009 or Nualart, 2006, Proposition 1.2.1): *If a sequence* $\{H_n\}_{n \in \mathbb{N}} \subset L^2(\Omega, \mathbb{P})$ *converges to* 0*, that is* $\mathbb{E}\left[H_n^2\right] \to 0$ *as* $n \to \infty$*, and* $D_t H_n$ *converges in* $L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda_T)$ *as* $n \to \infty$*, then* $\lim_{n \to \infty} D_t H_n = 0$. We write Dom D for the closed domain of D. Moreover the Malliavin derivative has a closable adjoint δ (under $\mathbb{P}$). The operator δ is called the *divergence operator* or, in the white noise case, the Skorohod integral. The domain of δ is denoted by Dom δ, it is the set of square-integrable random variables $v \in L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda_T)$ with:

$$\mathbb{E}\left[\langle DF, v \rangle_H\right] \leqslant c_v \sqrt{\mathbb{E}[F^2]},$$

for a constant c_v (depending on v) and all $F \in \mathbb{D}^{1,2}$ where $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables with respect to the norm:

$$\|F\|_{1,2} := \left(\mathbb{E}\left[F^2\right] + \mathbb{E}\left[\|DF\|_H^2\right]\right)^{1/2}.$$

With the scalar product $\langle F, G \rangle_{1,2} = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG \rangle_H]$, $\mathbb{D}^{1,2}$ is a Hilbert space. If $v \in$ Dom δ, then $\delta(v)$ is the element of $L^2(\Omega, \mathbb{P})$ characterised by

$$\mathbb{E}[F\delta(v)] = \mathbb{E}[\langle v, DF \rangle_H]. \tag{2.3}$$

This relation is often called the *integration by parts formula.* We have the more general rule (see Nourdin and Peccati, 2012, Proposition 2.5.4) for $F \in \mathbb{D}^{1,2}$, $v \in$ Dom δ such that $Fv \in$ Dom δ:

$$\delta(Fv) = F\delta(v) - \langle DF, v \rangle_H. \tag{2.4}$$

For multiple Wiener integrals and $f \in H^{\odot k}$, we have (see for instance Nualart, 2006, p.35):

$$D_{x_1} D_{x_2} \dots D_{x_l} \int_0^T \dots \int_0^T f(y_1, \dots, y_k) dW_{y_1} \dots dW_{y_k} \tag{2.5}$$
$$= \frac{k!}{(k-l)!} \int_0^T \dots \int_0^T f(y_1, \dots, y_{k-l}, x_1, \dots, x_l) dW_{y_1} \dots dW_{y_{k-l}}.$$

Formula (2.3) can be generalized for the multiple divergence (see Nourdin and Peccati, 2012, p.33): If $v \in \mathrm{Dom}\, \delta^l$ and $F \in \mathbb{D}^{l,2}$:

$$\mathbb{E}\left[F \delta^l(v)\right] = \mathbb{E}[\langle D^l F, v \rangle_{H^{\otimes l}}], \tag{2.6}$$

see Nourdin and Peccati (2012) or Nualart (2006) for details. For $m \geqslant 1$, $p \geqslant 1$ and the m-th Malliavin derivative $D^m F$, we can define $\mathbb{D}^{m,p}$ as the closure of the class of smooth random variables with respect to the norm $\| \cdot \|_{m,p}$ defined by:

$$\|F\|_{m,p} = \left(\mathbb{E}[|F|^p] + \sum_{i=1}^m \mathbb{E}\left[\|D^i F\|^p_{H^{\otimes i}}\right] \right)^{1/p}$$

We define $\mathbb{D}^\infty := \cap_{m=1}^\infty \cap_{p=1}^\infty \mathbb{D}^{m,p}$.

2.3. *Cumulants and* Γ *operators.* The reader is referred to Azmoodeh et al. (2015) or Nourdin and Peccati (2012) for a detailed introduction to cumulants, Malliavin operators and Γ-operators in particular. The r-th cumulant of a random variable F exists if the r-th moment of F exists and is defined as $\kappa_r(F) := (-i)^r \frac{d^r}{dt^r} \log \mathbb{E}[\exp(itF)]_{|t=0}$. The operator L, defined as $L = -\sum_{q=0}^\infty q J_q$, is the infinitesimal generator of the Ornstein-Uhlenbeck semi-group where J_q is the orthogonal projection operator on the q-th Wiener chaos. The domain of L is $\mathbb{D}^{2,2}$. L admits a pseudo-inverse L^{-1} and for any $F \in L^2(\Omega)$, we have $L^{-1} F = -\sum_{q=1}^\infty \frac{1}{q} J_q(F)$. For $F \in \mathbb{D}^\infty$, the sequence of random variables $\{\Gamma_i(F)\}_i \subset \mathbb{D}^\infty$ is recursively defined as follows:

$$\Gamma_i(F) = \langle DF, -DL^{-1} \Gamma_{i-1}(F) \rangle_H, \quad \text{for } i \geqslant 1,$$

and $\Gamma_0(F) = F$. For $F \in \mathbb{D}^\infty$ and $r \geqslant 0$, we define: $M_r(F) = \Gamma_r(F) - \mathbb{E}[\Gamma_r(F)]$.

2.4. *Stable convergence.* The concept of stable convergence is used in Section 5. The reader can find an extensive discussion of this topic in Jacod and Shiryaev (2003) or Häusler and Luschgy (2015), the basic facts are resumed in Nourdin et al. (2016a). Consider a sequence $\{F_n\}_n$ of real random variables on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see Section 2.1. Let F be a real random variable defined on some extended probability space $\left(\Omega', \mathcal{F}', \mathbb{P}'\right)$. We say that F_n converges stably to F, written $F_n \xrightarrow{\text{st.}} F$, as $n \to \infty$, if:

$$\lim_n \mathbb{E}\left[Z \exp\left(i\lambda F_n\right)\right] = \mathbb{E}'\left[Z \exp\left(i\lambda F\right)\right],$$

for every $\lambda \in \mathbb{R}$ and every bounded $\mathcal{F}$-measurable random variable Z. Obviously, stable convergence implies convergence in law, whereas the converse does not hold in general. We notice that the $\mathbb{P}$-completion of the σ-field generated by the set $\{I_1(f) : f \in H \text{ with } \|f\|_H = 1\}$ is $\mathcal{F}$. We have thus the following useful characterisation of stable convergence:

$F_n \xrightarrow{\text{st.}} F$ if and only if $(F_n, I_1(f))^\top \xrightarrow{\text{Law}} (F, I_1(f))^\top$ for every $f \in H$ with $\|f\|_H = 1$.

3. Main results

We start with an example in order to motivate the reader.

Example 3.1. Consider sequences $\{h_{n,2}\}_n, \left\{f_{n,2}^{(i)}\right\}_n, \left\{g_{n,2}^{(j)}\right\}_n \subset H^{\odot 2}$ for $1 \leqslant i \leqslant k_1$ and $1 \leqslant j \leqslant k_2$ for $k_1 > 0$ and $k_2 > 0$. Suppose that the sequences $\left\{I_2(f_{n,2}^{(i)})\right\}_n$, $\left\{I_2(g_{n,2}^{(j)})\right\}_n$ and $\{I_2(h_{n,2})\}_n$ converge in law to a standard normal variable and that $\langle k_{n,2}, k'_{n,2}\rangle_{H\otimes H} \to 0$, for any distinct sequences $\{k_{2,n}\}_n$ and $\left\{k'_{2,n}\right\}_n$ chosen among $\{h_{n,2}\}_n, \left\{f_{n,2}^{(i)}\right\}_n, \left\{g_{n,2}^{(j)}\right\}_n$ for $1 \leqslant i \leqslant k_1$ and $1 \leqslant j \leqslant k_2$. We have then, see Peccati and Tudor (2005, Theorem 1), as $n \to \infty$:

$$(I_2(h_{n,2}), I_2(f_{n,2}^{(1)}), \ldots, I_2(f_{n,2}^{(k_1)}), I_2(g_{n,2}^{(1)}), \ldots, I_2(g_{n,2}^{(k_2)}))^\top \overset{\text{Law}}{\to} Z^\top,$$

where $Z^\top := (N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2})^\top$ has a $k_1 + k_2 + 1$-dimensional standard normal distribution. The continuous mapping theorem yields that, as $n \to \infty$:

$$I_2(h_{n,2}) + \sum_{i=1}^{k_1} b_i \left(I_2(f_{n,2}^{(i)})^2 - 1\right) + \sum_{j=1}^{k_2} \left[c_j \left(I_2(g_{n,2}^{(j)})^2 - 1\right) + d_j I_2(g_{n,2}^{(j)})\right] \tag{3.1}$$

$$\overset{\text{Law}}{\to} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{j=1}^{k_2} \left[c_j(P_j^2 - 1) + d_j P_j\right], \tag{3.2}$$

where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. It is easy to see that the expression in Eq. (3.1) has a representation of the form $I_4(\varphi_{n,4}) + I_2(\varphi_{n,2}) + \varphi_{n,0}$ for $\varphi_{n,l} \in H^{\odot l}$, $l = 0, 2, 4$, and $\lim_n \varphi_{n,0} = 0$. We find with Slutsky's theorem that $I_4(\varphi_{n,4}) + I_2(\varphi_{n,2}) \overset{\text{Law}}{\to} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{j=1}^{k_2} \left[c_j(P_j^2 - 1) + d_j P_j\right]$, as $n \to \infty$.

Remark 3.2. In the present paper we give necessary and sufficient conditions for weak convergence towards target variables as in Eq. (3.2). We illustrate now why this class of target variables is important.

Consider a random variable X living in the sum of the first two Wiener chaoses:

$$X := I_1(f_1) + I_2(f_2), \tag{3.3}$$

with $f_1 \in H$ and $f_2 \in H^{\odot 2}$. It is known, see Janson (1997, Theorem 6.1) or Nourdin and Peccati (2012, Proposition 2.7.13), that $I_2(f_2) = \sum_{k=1}^{\infty} \alpha_k (I_1(h_k)^2 - 1)$ for an orthonormal system $\{h_i | i \in \mathbb{N}\} \subset H$. Suppose from now on that $\alpha_k \neq 0$ for $1 \leqslant k \leqslant N$ and $\alpha_k = 0$ for every $k \geqslant N$. Consider the projection of f_1 on $\text{span}(h_1, \ldots, h_N)$, then we find $f_1 = \beta_1 h_1 + \ldots + \beta_N h_N + \beta_0 h_0$, where $\|h_0\|_H = 1$ and $h_0 \perp \text{span}\ (h_1, \ldots, h_N)$. Hence:

$$X = \beta_0 I_1(h_0) + \sum_{i=1}^{N} \left[\beta_i I_1(h_1) + \alpha_i (I_1(h_i)^2 - 1)\right]. \tag{3.4}$$

Since $\langle h_i, h_j\rangle_H = \delta_{i,j}$, we have that $\{I_1(h_0), \ldots, I_1(h_N)\}$ is a set of independent standard normal variables and some of the coefficients $\beta_0, \ldots, \beta_N$ may be equal to 0. The representation of X found in Eq. (3.4) is thus equivalent to the representation in Eq. (3.2):

$$X = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2} [c_i(P_i^2 - 1) + d_i P_i], \tag{3.5}$$

where $b_i \neq 0$, $c_j d_j \neq 0$ for $1 \leqslant i \leqslant k_1$, $1 \leqslant j \leqslant k_2$ and N, R_i, P_j are independent standard normal variables living in the first Wiener chaos. A similar argument applied together with the multiplication formula for Wiener integrals shows that $X = I_1(f_1) + I_2(f_2)$ for:

$$f_1 = ah_0 + \sum_{i=1}^{k_2} d_i h'_i \ , \ f_2 = \sum_{i=1}^{k_1} b_i h_i \tilde{\otimes} h_i + \sum_{i=1}^{k_2} c_i h'_i \tilde{\otimes} h'_i, \tag{3.6}$$

for a set of orthonormal functions $h_0, h_1, \ldots, h_{k_1}, h'_1, \ldots, h'_{k_2}$.

If $k_1 = 0$, we set $b_j = 0$ for every j and the empty sum in the representations above is removed. Notice that an empty product equals 1. We proceed similarly if $k_2 = 0$. If on the other hand $k_1 \neq 0$, we suppose that $b_j \neq 0$ for every $j = 1, \ldots, k_1$. We proceed similarly if $k_2 \neq 0$.

The following Lemma shows that random variables with a representation as in Eq. (3.5) extend the class of random variables with a representation as in Eq. (1.2) by adding a (possibly correlated) normal variable.

Lemma 3.3. *Consider the following families of random variables:*

(A) $X \overset{Law}{=} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2} \left[c_i \left(P_i^2 - 1\right) + d_i P_i\right]$, *where* $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ *and* $b_i \neq 0$ *for* $1 \leqslant i \leqslant k_1$ *if* $k_1 > 0$ *and* $c_i d_i \neq 0$ *for* $1 \leqslant i \leqslant k_2$ *if* $k_2 > 0$.

(B) $Y \overset{Law}{=} aU_0 + \sum_{i=1}^{n} \lambda_i (U_i^2 - 1)$ *where* $\lambda_i \neq 0$ *for* $1 \leqslant i \leqslant n$ *if* $n > 0$ *and* $(U_0, U_1, \ldots, U_n)^\top$ *is a centered normal vector such that* $U_1, \ldots, U_n \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$.

Then class (A) coincides with class (B). In other words every random variable in (A) has a representation as in (B) and vice versa.

Proof: (1) Consider $X \overset{\text{Law}}{=} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i]$ as in (A), then, if $a^2 + \sum_{i=1}^{k_2} d_i^2 \neq 0$:

$$\begin{aligned} X &\overset{\text{Law}}{=} \left(aN + \sum_{i=1}^{k_2} d_i P_i\right) + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2} c_i(P_i^2 - 1) \\ &\overset{\text{Law}}{=} \sqrt{a^2 + \sum_{i=1}^{k_2} d_i^2}\, \frac{aN + \sum_{i=1}^{k_2} d_i P_i}{\sqrt{a^2 + \sum_{i=1}^{k_2} d_i^2}} + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2} c_i(P_i^2 - 1). \end{aligned}$$

Define:

$$U_i = \begin{cases} \frac{aN + \sum_{i=1}^{k_2} d_i P_i}{\sqrt{a^2 + \sum_{i=1}^{k_2} d_i^2}} & \textit{for } i = 0, \\ R_i & \textit{for } 1 \leqslant i \leqslant k_1, \\ P_{i-k_1} & \textit{for } k_1 + 1 \leqslant i \leqslant k_1 + k_2, \end{cases}$$

and drop the corresponding terms if $k_1 = 0$ or $k_2 = 0$. After renaming the coefficients, we find the following representation with $n := k_1 + k_2$:

$$X \overset{\text{Law}}{=} \sqrt{a^2 + \sum_{i=1}^{k_2} d_i^2} U_0 + \sum_{i=1}^{n} \lambda_i (U_i^2 - 1),$$

and $(U_0, U_1, \ldots, U_n)^\top$ is clearly centered and normal since every linear combination $\sum_{i=1}^{n} \alpha_i U_i$ is normal. This last property follows directly from $N, R_1, \ldots,$

$R_{k_1}, P_1, \ldots, P_{k_2} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. The definition of $U_1, \ldots, U_n$ yields that $U_1, \ldots, U_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. If $a^2 + \sum_{i=1}^{k_2} d_i^2 = 0$, then $a = d_1 = \ldots = d_{k_2} = k_2 = 0$. The equivalence of both representations is trivial in this case.

(2) Consider $Y \overset{\text{Law}}{=} aU_0 + \sum_{i=1}^n \lambda_i (U_i^2 - 1)$ as in (B). Since the case $a = 0$ is clear, we suppose that $a \neq 0$ and $U_0 \sim \mathcal{N}(0,1)$. Let $U := (U_0, U_1, \ldots, U_n)^\top \sim \mathcal{N}(0_{(n+1)\times 1}, \Sigma)$ for a positive semi-definite matrix Σ. Since $U_1, \ldots, U_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, we have:

$$\Sigma = \left(\begin{array}{c|ccc} 1 & \sigma_1 & \cdots & \sigma_n \\ \hline \sigma_1 & & & \\ \vdots & & \mathbf{I_{n\times n}} & \\ \sigma_n & & & \end{array}\right).$$

We suppose first that $\det \Sigma = 1 - \sum_{i=1}^n \sigma_i^2 > 0$ then $\Sigma = B\,B^\top$ for B defined by:

$$B := \left(\begin{array}{c|ccc} \sqrt{\det \Sigma} & \sigma_1 & \cdots & \sigma_n \\ \hline \mathbf{0_{n\times 1}} & & \mathbf{I_{n\times n}} & \end{array}\right) \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Consider a $n+1$-dimensional standard normal vector $V^\top = (V_0, V_1, \ldots, V_n)^\top$, then $U \overset{\text{Law}}{=} BV$. Hence:

$$\begin{aligned} Y &\overset{\text{Law}}{=} a\left(\sqrt{\det \Sigma} V_0 + \sigma_1 V_1 + \ldots + \sigma_n V_n\right) + \sum_{i=1}^n \lambda_i (V_i^2 - 1) \\ &\overset{\text{Law}}{=} a\sqrt{\det \Sigma} V_0 + \sum_{i=1}^n \left[\lambda_i (V_i^2 - 1) + a\sigma_i V_i\right]. \end{aligned} \tag{3.7}$$

Noticing that some of the covariances σ_i may be zero, Eq. (3.7) yields the representation (A) after renaming the independent standard normal random variables and the coefficients.

Consider now the case $\det \Sigma = 0$, a standard normal vector $(V_1, \ldots, V_n)^\top$ and define $B \in \mathbb{R}^{(n+1)\times n}$:

$$B := \left(\begin{array}{ccc} \sigma_1 & \cdots & \sigma_n \\ \hline & \mathbf{I_{n\times n}} & \end{array}\right) \in \mathbb{R}^{(n+1)\times n}.$$

Then $B\,B^\top = \Sigma$ and $U \overset{\text{Law}}{=} BV$, hence:

$$Y \overset{\text{Law}}{=} a\,(\sigma_1 V_1 + \ldots + \sigma_n V_n) + \sum_{i=1}^n \lambda_i (V_i^2 - 1) \overset{\text{Law}}{=} \sum_{i=1}^n \left[\lambda_i (V_i^2 - 1) + a\sigma_i V_i\right].$$

The statement follows now as above.

□

Remark 3.4. The random variable X in (A) may lead to a degenerate normal vector $(U_0, U_1, \ldots, U_n)^\top$ in (B). In particular $a = k_1 = 0$, $k_2 = 1$ leads to $X \overset{\text{Law}}{=} d_1 P_1 + c_1(P_1^2 - 1)$ and the corresponding normal vector $(U_0, U_1)^\top$ is degenerate. It can be easily deduced from the proof of Lemma 3.3 that $(U_0, U_1, \ldots, U_n)^\top$ is non-degenerate if $a \neq 0$ or $k_2 = 0$. In order to simplify our calculations, we shall consider in this paper target variables of class (A).

We shall need the characteristic function and the cumulants of X, defined in Eq. (3.3) and (3.5).

Lemma 3.5. *Consider $k_1, k_2 \geqslant 0$ and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i],$$

as defined in Eq. (3.3) and (3.5) where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. We have with $\Delta_l := 4c_l^2 + d_l^2$ for the characteristic function φ_X of X:

$$\varphi_X(x) = \exp\left(-\frac{a^2x^2}{2} - ix\sum_{j=1}^{k_1} b_j + \sum_{j=1}^{k_2} \frac{x^2\Delta_j + 2ic_jx}{4ixc_j - 2}\right)$$
$$\times \prod_{j=1}^{k_1}(1 - 2ixb_j)^{-1/2} \prod_{j=1}^{k_2}(1 - 2ixc_j)^{-1/2},$$

where, as usual, an empty product equals 1 and an empty sum equals 0. Moreover φ_X is the unique solution of the initial value problem $y(0) = 1$ and:

$$\begin{aligned}
&y'(x)\prod_{j=1}^{k_1}(1 - 2ixb_j)\prod_{j=1}^{k_2}(1 - 2ixc_j)^2 \\
&= y(x)\left(-xa^2 - i\sum_{j=0}^{k_1} b_j\right)\prod_{j=1}^{k_1}(1 - 2ixb_j)\prod_{j=1}^{k_2}(1 - 2ixc_j)^2 \\
&\quad + y(x)\prod_{j=1}^{k_1}(1 - 2ixb_j)\sum_{l=1}^{k_2}\left(\prod_{j\neq l}(1 - 2ixc_j)^2\right)\left(2xc_l^2 - x\Delta_l(1 - 2ixc_l) - x^2ic_l\Delta_l\right) \\
&\quad + y(x)\prod_{j=1}^{k_2}(1 - 2ixc_j)^2\sum_{l=1}^{k_1} ib_l\prod_{j\neq l}(1 - 2ixb_j). \qquad (3.8)
\end{aligned}$$

Proof: Using the characteristic function of the non-central χ^2 distribution and the representation:

$$c_j(P_j^2 - 1) + d_jP_j = c_j\left(P_j + \frac{d_j}{2c_j}\right)^2 - \frac{d_j^2 + 4c_j^2}{4c_j},$$

we find for the characteristic functions:

$$\varphi_{b_j(R_j^2-1)}(x) = (1 - 2ixb_j)^{-1/2}\exp\left(-ixb_j\right),$$
$$\varphi_{c_j(P_j^2-1)+d_jP_j}(x) = (1 - 2ixc_j)^{-1/2}\exp\left(\frac{x^2\Delta_j + 2ic_jx}{4ixc_j - 2}\right).$$

Hence, with the independence of the standard normal random variables:

$$\begin{aligned}
\varphi_X(x) &= \varphi_{aN}(x)\prod_{j=1}^{k_1}\varphi_{b_j(R_j^2-1)}(x)\prod_{j=1}^{k_2}\varphi_{c_j(P_j^2-1)+d_jP_j}(x) \\
&= \exp\left(-\frac{a^2x^2}{2}\right)\prod_{j=1}^{k_1}(1 - 2ixb_j)^{-1/2}\exp\left(-ix\sum_{j=1}^{k_1} b_j\right)\prod_{j=1}^{k_2}(1 - 2ixc_j)^{-1/2} \\
&\quad \times \exp\left(\sum_{j=1}^{k_2}\frac{x^2\Delta_j + 2ic_jx}{4ixc_j - 2}\right)
\end{aligned}$$

$$= \exp\left(-\frac{a^2x^2}{2} - ix\sum_{j=1}^{k_1} b_j + \sum_{j=1}^{k_2} \frac{x^2\Delta_j + 2ic_jx}{4ixc_j - 2}\right)$$
$$\times \prod_{j=1}^{k_1}(1-2ixb_j)^{-1/2} \prod_{j=1}^{k_2}(1-2ixc_j)^{-1/2}.$$

Notice that $\frac{d}{dx}\frac{1}{\sqrt{1-2ix\alpha}} = \frac{1}{\sqrt{1-2ix\alpha}}\frac{i\alpha}{1-2ix\alpha}$, for every real constant α, thus $\varphi_X'(x)$ equals:

$$\varphi_X(x)\left(-xa^2 - i\sum_{j=1}^{k_1} b_j + \sum_{j=1}^{k_2}\frac{2x\Delta_j + 2ic_j}{4ixc_j - 2} - \sum_{j=1}^{k_2}\frac{(x^2\Delta_j + 2ic_jx)\,4ic_j}{(4ixc_j-2)^2}\right.$$
$$\left. + \sum_{j=1}^{k_1}\frac{ib_j}{1-2ib_j} + \sum_{j=1}^{k_2}\frac{ic_j}{1-2ixc_j}\right)$$
$$= \varphi_X(x)\left(-xa^2 - i\sum_{j=1}^{k_1} b_j - x\sum_{j=1}^{k_2}\frac{\Delta_j}{1-2ixc_j} - x^2\sum_{j=1}^{k_2}\frac{ic_j\Delta_j}{(1-2ixc_j)^2}\right.$$
$$\left. +2x\sum_{j=1}^{k_2}\frac{c_j^2}{(1-2ixc_j)^2} + \sum_{j=1}^{k_1}\frac{ib_j}{1-2ixb_j}\right).$$

Multiplying by $\prod_{j=1}^{k_1}(1-2ixb_j)\prod_{j=1}^{k_2}(1-2ixc_j)^2 \neq 0$ yields:

$$\varphi_X'(x)\prod_{j=1}^{k_1}(1-2ixb_j)\prod_{j=1}^{k_2}(1-2ixc_j)^2$$
$$= \varphi_X(x)\left(-xa^2 - i\sum_{j=0}^{k_1} b_j\right)\prod_{j=1}^{k_1}(1-2ixb_j)\prod_{j=1}^{k_2}(1-2ixc_j)^2$$
$$+ \varphi_X(x)\prod_{j=1}^{k_1}(1-2ixb_j)\sum_{l=1}^{k_2}\left(\prod_{j\neq l}(1-2ixc_j)^2\right)\left(2xc_l^2 - x\Delta_l(1-2ixc_l) - x^2ic_l\Delta_l\right)$$
$$+ \varphi_X(x)\prod_{j=1}^{k_2}(1-2ixc_j)^2\sum_{l=1}^{k_1} ib_l\prod_{j\neq l}(1-2ixb_j).$$

Thus φ_X is a solution of the initial value problem. The uniqueness of the solution follows with the Cauchy-Lipschitz theorem since

$$x \mapsto -xa^2 - i\sum_{j=1}^{k_1} b_j - x\sum_{j=1}^{k_2}\frac{\Delta_j}{1-2ixc_j} - x^2\sum_{j=1}^{k_2}\frac{ic_j\Delta_j}{(1-2ixc_j)^2}$$
$$+ 2x\sum_{j=1}^{k_2}\frac{c_j^2}{(1-2ixc_j)^2} + \sum_{j=1}^{k_1}\frac{ib_j}{1-2ixb_j}$$

is continuous and bounded on every (real) interval. □

Remark 3.6. Notice that it may be possible to simplify the differential equation if not all coefficients are pairwise different. In Theorem 1.1, 3.8 and 3.11, this simplification may yield a polynomial of smaller degree. For a special case, this problem is discussed in Theorem 4.9 and Remark 4.10. For the rest of this section we shall allow that not all coefficients are pairwise different. In the case of pairwise

different coefficients, the differential equation cannot be simplified and the same observation holds for the polynomials in the previously cited theorems.

Lemma 3.7. *Consider $k_1, k_2 \geqslant 0$ and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i],$$

as defined in Eq. (3.3) and (3.5) where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Then, for $r \geqslant 2$:

$$\kappa_r(X) = a^2 1_{[r=2]} + \sum_{j=1}^{k_1} 2^{r-1}(r-1)! b_j^r + \sum_{j=1}^{k_2}[2^{r-1}(r-1)! c_j^r + 2^{r-3} r! c_j^{r-2} d_j^2].$$

Proof: We have $\kappa_2(N) = a^2$, $\kappa_r(N) = 0$ for $r > 2$ and $\kappa_r(R_j^2 - 1) = 2^{r-1}(r-1)!$. We notice:

$$c_j(P_j^2 - 1) + d_j P_j = c_j \left(P_j + \frac{d_j}{2c_j}\right)^2 - \frac{d_j^2}{4c_j} - c_j,$$

hence for $r \geqslant 2$:

$$\kappa_r(c_j(P_j^2 - 1) + d_j P_j) = c_j^r \kappa_r \left[\left(P_j + \frac{d_j}{2c_j}\right)^2\right].$$

Using the formula for the cumulants of the non-central χ^2 distribution, we have for $S \sim \mathcal{N}(\mu, 1)$:

$$\kappa_r(S^2) = 2^{r-1}(r-1)!(1 + r\mu^2),$$

thus:

$$\kappa_r[c_j(P_j^2 - 1) + d_j P_j] = 2^{r-1}(r-1)! c_j^r \left[1 + \left(\frac{d_j}{2c_j}\right)^2 r\right] = 2^{r-1}(r-1)! c_j^r + 2^{r-3} r! c_j^{r-2} d_j^2.$$

The result follows now with the independence of the random variables. □

We can now prove the first part of our main result: a sufficient criterion for the convergence in law to X.

Theorem 3.8. *Consider $k_1, k_2 \geqslant 0$ and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_i P_i],$$

as defined in Eq. (3.3) and (3.5) where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Suppose that at least one of the parameters a, k_1, k_2 is non-zero. Consider a sequence $\{F_n\}_n$ of non-zero random variables with $F_n = \sum_{i=1}^p I_i(f_{n,i})$ for $p \geqslant 2$ fixed and $\{f_{n,i}\}_n \subset H^{\odot i}$ for $1 \leqslant i \leqslant p$. Define:

$$P(x) = x^{1+1_{[a\neq 0]}} \prod_{j=1}^{k_1}(x - b_j) \prod_{j=1}^{k_2}(x - c_j)^2.$$

If the following conditions hold, as $n \to \infty$:

(1) $\kappa_r(F_n) \to \kappa_r(X)$, *for $r = 1, \ldots, deg(P)$,*

(2) $\mathbb{E}\left[\left|\mathbb{E}\left[\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r! 2^{r-1}} \left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]\right) \middle| F_n\right]\right|\right] \to 0,$

then $F_n \overset{Law}{\to} X$*, as* $n \to \infty$*.*

Proof: Notice that an empty product equals 1 and an empty sum equals 0. We prove the result for $a \neq 0$, the other case can be treated similarly. We shall use and extend an idea of Nourdin and Peccati, see Nourdin and Peccati (2009a, Paragraph 3.5). Since $\sup_n \kappa_2(F_n) < \infty$, we have with Chebychev's inequality that the sequence $\{F_n\}_n$ is tight. We shall use the following corollary of Prokhorov's Theorem: *Consider a tight sequence* $\{F_n\}_n$ *of random variables. If every subsequence* $\{F_{n_k}\}_k$ *which converges in law has the same limit* Y*, then the initial sequence* $\{F_n\}_n$ *converges in law to* Y*.* We consider thus a subsequence $\{F_{n_k}\}_k$ which converges in law to some random variable Y. We notice that $\lim_k \kappa_2(F_{n_k}) = \lim_n \kappa_2(F_n) = \kappa_2(X)$, hence $\sup_k \mathbb{E}[F_{n_k}^2] < \infty$. Since $\{F_n\}_n$ lives in a fixed finite sum of Wiener chaoses, the hypercontractivity property implies $\sup_k \mathbb{E}[|F_{n_k}|^r] < \infty$ for every $r \geqslant 2$. With $F_{n_k} \overset{\text{Law}}{\to} Y$, as $k \to \infty$, we have $\mathbb{E}[Y^2] = \lim_k \mathbb{E}[F_{n_k}^2] = \lim_n \mathbb{E}[F_n^2] = \kappa_2(X) \neq 0$. Hence Y is a non-zero random variable and we have $\lim_k \kappa_r(F_{n_k}) = \kappa_r(Y)$ for every r. On the other hand we have $\lim_n \kappa_r(F_n) = \kappa_r(X)$ for $r = 1, \ldots, \deg(P)$, thus:

$$\kappa_r(X) = \kappa_r(Y), \quad \text{for } r = 1, \ldots, \deg(P).$$

To simplify the notations and to avoid complicated indices we shall write from now on $\{F_n\}_n$ for the subsequence. By the previous corollary, the proof is complete if we can prove that φ_{F_n} converges to φ_X. We shall prove this by showing that $\varphi_Y = \lim_n \varphi_{F_n}$ solves the initial value problem of Lemma 3.5. This implies that $\varphi_Y = \varphi_X$ and hence $Y \overset{\text{Law}}{=} X$. The proof is divided in 5 steps:

- Step 1: We show that $\varphi_Y'(0) = 0$ and find an alternative representation for $\mathbb{E}[\exp(ixF_n)\Gamma_r(F_n)]$.
- Step 2: We calculate:

$$\mathbb{E}\left[\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \exp(ixF_n)\Gamma_{r-1}(F_n)\right], \tag{3.9}$$

  and:

$$\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \mathbb{E}[\exp(ixF_n)]\, \mathbb{E}[\Gamma_{r-1}(F_n)]. \tag{3.10}$$

- Step 3: We calculate $\varphi_Y'(x) \prod_{j=1}^{k_1}(1 - 2ixb_j) \prod_{j=1}^{k_2}(1 - 2ixc_j)^2$.
- Step 4: We find an expression for:

$$\begin{aligned}
&\varphi_Y(x)\left(-xa^2 - i\sum_{j=0}^{k_1} b_j\right)\prod_{j=1}^{k_1}(1-2ixb_j)\prod_{j=1}^{k_2}(1-2ixc_j)^2 \\
&\quad + \varphi_Y(x)\prod_{j=1}^{k_1}(1-2ixb_j)\sum_{l=1}^{k_2}\prod_{j\neq l}(1-2ixc_j)^2\left(2xc_l^2 - x\Delta_l(1-2ixc_l) - x^2 ic_l\Delta_l\right) \\
&\quad + \varphi_Y(x)\prod_{j=1}^{k_2}(1-2ixc_j)^2\sum_{l=1}^{k_1} ib_l \prod_{j\neq l}(1-2ixb_j).
\end{aligned}$$

- Step 5: The proof is completed by showing that the expressions found in the last two steps are equal. Then φ_Y is the unique solution to the initial

value problem in Lemma 3.5, Y and X are thus equal in distribution and $F_n \overset{\text{Law}}{\to} X$, as $n \to \infty$.

For the ease of notation, we define $k := 2k_2 + k_1$, $G_1(x) := \prod_{j=1}^{k_1}(1 - 2ixb_j)$ and $G_2(x) := \prod_{j=1}^{k_2}(1 - 2ixc_j)^2$.

Step 1. The random variable Y is non-zero and has moments of every order. We notice that for $x = 0$, the differential equation of Lemma 3.5 holds for φ_Y if we can prove $\varphi'_Y(0) = 0$. We notice that $\mathbb{E}[|F_n|] < \infty$, hence $\varphi'_{F_n}(0) = i\,\mathbb{E}[F_n] = 0$. On the other hand $\lim_n \varphi'_{F_n}(x) = \varphi'_Y(x)$ for every $x \in \mathbb{R}$ since $\sup_n \mathbb{E}[F_n^2] < \infty$ and $iF_n \exp(ixF_n) \overset{\text{Law}}{\to} iY \exp(ixY)$ by the continuous mapping theorem. We have thus $\varphi'_Y(0) = \lim_n \varphi'_{F_n}(0) = 0$. We suppose now that $x \neq 0$ and calculate $\mathbb{E}[\exp(ixF_n)\Gamma_r F_n]$ for $r \in \mathbb{N}$. For $r \in \{0, 1\}$:

$$\begin{aligned}
&\mathbb{E}[\exp(ixF_n)\Gamma_0(F_n)] = -i\,\mathbb{E}[\exp(ixF_n)iF_n] = -i\varphi'_{F_n}(x),\\
&\mathbb{E}[\exp(ixF_n)\Gamma_1(F_n)] = \mathbb{E}[\exp(ixF_n)\langle DF_n, -DL^{-1}F_n\rangle_H]\\
&= \frac{1}{ix}\,\mathbb{E}[\langle D\exp(ixF_n), -DL^{-1}F_n\rangle_H] = \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)(-\delta DL^{-1}F_n)]\\
&= \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)F_n] = -\frac{i}{ix}\varphi'_{F_n}(x),
\end{aligned}$$

and for $r > 1$:

$$\begin{aligned}
&\mathbb{E}[\exp(ixF_n)\Gamma_r(F_n)] = \mathbb{E}[\exp(ixF_n)\langle DF_n, -DL^{-1}\Gamma_{r-1}(F_n)\rangle_H]\\
&= \frac{1}{ix}\,\mathbb{E}[\langle D\exp(ixF_n), -DL^{-1}\Gamma_{r-1}(F_n)\rangle_H] = \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)[-\delta DL^{-1}\Gamma_{r-1}(F_n)]]\\
&= \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)[\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]]]\\
&= \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)\Gamma_{r-1}(F_n)] - \frac{1}{ix}\,\mathbb{E}[\exp(ixF_n)]\,\mathbb{E}[\Gamma_{r-1}(F_n)].
\end{aligned}$$

Iteration yields:

$$\mathbb{E}[\exp(ixF_n)\Gamma_r(F_n)] = -i(ix)^{-r}\varphi'_{F_n}(x) - \varphi_{F_n}(x)\sum_{j=1}^{r-1}(ix)^{-j}\,\mathbb{E}[\Gamma_{r-j}(F_n)]. \tag{3.11}$$

Step 2. We calculate now the sum in (3.9). We have:

$$\begin{aligned}
&-i\varphi'_{F_n}(x)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}(ix)^{-r+1} = -i\varphi'_{F_n}(x)(2ix)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!}(2ix)^{-r}\\
&= -i\varphi'_{F_n}(x)(2ix)P\left(\frac{1}{2ix}\right)\\
&= -i\varphi'_{F_n}(x)(2ix)\left(\frac{1}{2ix}\right)^2\prod_{j=1}^{k_1}\left(\frac{1}{2ix} - b_j\right)\prod_{j=1}^{k_2}\left(\frac{1}{2ix} - c_j\right)^2\\
&= -i\varphi'_{F_n}(x)(2ix)^{-1-k}G_1(x)G_2(x).
\end{aligned}$$

With Eq. (3.11) we have thus:

$$\mathbb{E}\left[\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}\exp(ixF_n)\Gamma_{r-1}(F_n)\right] = -i\varphi'_{F_n}(x)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}(ix)^{-r+1} + \varphi_{F_n}(x)$$

$$
\begin{aligned}
&\times \mathbb{E}\left[\sum_{r=3}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\frac{-\mathbb{E}[\Gamma_{r-2}(F_n)]}{(ix)^1} - \frac{\mathbb{E}[\Gamma_{r-3}(F_n)]}{(ix)^2} - \ldots - \frac{\mathbb{E}[\Gamma_1(F_n)]}{(ix)^{(r-2)}}\right)\right] \\
&= -i\varphi'_{F_n}(x)\,(2ix)^{-k-1} G_1(x) G_2(x) \\
&\quad - \frac{\varphi_{F_n}(x)}{(ix)^1}\left[\frac{P^{(3)}(0)}{3!2^2}\,\mathbb{E}[\Gamma_1(F_n)] + \frac{P^{(4)}(0)}{4!2^3}\,\mathbb{E}[\Gamma_2(F_n)] + \ldots\right. \\
&\quad \left. + \frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}}\,\mathbb{E}[\Gamma_k(F_n)]\right] \\
&\quad - \frac{\varphi_{F_n}(x)}{(ix)^2}\left[\frac{P^{(4)}(0)}{4!2^3}\,\mathbb{E}[\Gamma_1(F_n)] + \frac{P^{(5)}(0)}{5!2^4}\,\mathbb{E}[\Gamma_2(F_n)] + \ldots\right. \\
&\quad \left. + \frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}}\,\mathbb{E}[\Gamma_{k-1}(F_n)]\right] - \ldots - \frac{\varphi_{F_n}(x)}{(ix)^k}\,\frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}}\,\mathbb{E}[\Gamma_1(F_n)]. \qquad (3.12)
\end{aligned}
$$

We calculate the sum in (3.10) using $\mathbb{E}[\Gamma_r(F_n)] = \kappa_{r+1}(F_n)/r!$ and $\mathbb{E}[\Gamma_0(F_n)] = \mathbb{E}[F_n] = 0$:

$$
\sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}}\,\mathbb{E}[\exp(ixF_n)]\,\mathbb{E}[\Gamma_{r-1}(F_n)] = \sum_{r=2}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}}\,\frac{\kappa_r(F_n)}{(r-1)!}\varphi_{F_n}(x). \qquad (3.13)
$$

Step 3. We have thus for the limit of the expression on the right-hand side of Eq. (3.13), as $n \to \infty$:

$$
\begin{aligned}
&\varphi_Y(x) \sum_{r=2}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}}\,\frac{\kappa_r(X)}{(r-1)!} = \varphi_Y(x)\frac{P''(0)}{2!2}\,a^2 + \varphi_Y(x) \\
&\quad \times \sum_{r=2}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}}\left[\sum_{j=1}^{k_1} \frac{2^{r-1}(r-1)!b_j^r}{(r-1)!} + \sum_{j=1}^{k_2}\left(\frac{2^{r-1}(r-1)!c_j^r + 2^{r-3}r!c_j^{r-2}d_j^2}{(r-1)!}\right)\right] \\
&= \varphi_Y(x)\left[\sum_{j=1}^{k_1}\sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!} b_j^r + \sum_{j=1}^{k_2}\sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!} c_j^r + \sum_{j=1}^{k_2} \frac{d_j^2}{4c_j} \sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!}\, r\, c_j^{r-1}\right. \\
&\quad \left. + \frac{(-1)^{k_1} a^2}{2} \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j\right] \\
&= \varphi_Y(x)\left[\sum_{j=1}^{k_1} P(b_j) + \sum_{j=1}^{k_2} P(c_j) + \sum_{j=1}^{k_2} c_j^{-1} d_j^2 P'(c_j)/4 + \frac{1}{2}(-1)^{k_1} a^2 \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j\right] \\
&= \frac{1}{2}\varphi_Y(x)(-1)^{k_1} a^2 \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j. \qquad (3.14)
\end{aligned}
$$

We have with $M_{r-1} := \Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]$, as $n \to \infty$:

$$
\left|\mathbb{E}\left[\exp(ixF_n) \sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}} M_{r-1}(F_n)\right]\right| \leqslant \mathbb{E}\left[\left|\mathbb{E}\left[\sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}} M_{r-1}(F_n)\middle| F_n\right]\right|\right] \to 0,
$$

hence:

$$
\lim_n \mathbb{E}\left[\exp(ixF_n) \sum_{r=1}^{k+2} \frac{P^{(r)}(0)}{r!2^{r-1}} \Gamma_{r-1}(F_n)\right]
$$

$$= \lim_n \mathbb{E}\left[\exp(ixF_n)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}M_{r-1}(F_n)\right] + \lim_n \varphi_{F_n}(x)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}\,\mathbb{E}[\Gamma_{r-1}(F_n)]$$

$$= 0 + \lim_n \varphi_{F_n}(x)\sum_{r=1}^{k+2}\frac{P^{(r)}(0)}{r!2^{r-1}}\,\mathbb{E}[\Gamma_{r-1}(F_n)],$$

using Eq. (3.12) and Eq. (3.14), we have with $\lim_n \mathbb{E}[\Gamma_r(F_n)] = \kappa_{r+1}(X)/r!$ for $r = 1, \ldots, \deg(P) - 1$:

$$-i\varphi_Y'(x)(2ix)^{-k-1}G_1(x)G_2(x) - \sum_{m=1}^{k}\varphi_Y(x)(ix)^{-m}\sum_{r=m}^{k}\frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}}\frac{\kappa_{r-m+2}(X)}{(r-m+1)!}$$

$$= \frac{1}{2}\varphi_Y(x)(-1)^{k_1}a^2\prod_{j=1}^{k_2}c_j^2\prod_{j=1}^{k_1}b_j. \tag{3.15}$$

We have used that $\lim_n \varphi_{F_n}'(x) = \lim_n \varphi_Y'(x)$ pointwise. This can be seen using the continuous mapping theorem and $\sup_n \mathbb{E}[F_n^2] < \infty$. Multiplying Eq. (3.15) by $i(2ix)^{k+1}$ yields:

$$\varphi_Y'(x)G_1(x)G_2(x) = \sum_{m=1}^{k}(2ix)^{k+1}i\varphi_Y(x)(ix)^{-m}\sum_{r=m}^{k}\frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}}\frac{\kappa_{r-m+2}(X)}{(r-m+1)!}$$
$$+ i\varphi_Y(x)(2ix)^{k+1}\frac{1}{2}(-1)^{k_1}a^2\prod_{j=1}^{k_2}c_j^2\prod_{j=1}^{k_1}b_j. \tag{3.16}$$

Step 4. We have:

$$\lim_n \varphi_{F_n}(x)\Big[\Big(-xa^2 - i\sum_{j=1}^{k_1}b_j\Big)\prod_{j=1}^{k_1}(1-2ixb_j)\prod_{j=1}^{k_2}(1-2ixc_j)^2$$
$$+ \prod_{j=1}^{k_2}(1-2ixc_j)^2\sum_{l=1}^{k_1}ib_l\prod_{j\neq l}(1-2ixb_j)$$
$$+ \prod_{j=1}^{k_1}(1-2ixb_j)\sum_{l=1}^{k_2}\left(\prod_{j\neq l}(1-2ixc_j)^2\right)\left[2xc_l^2 - x\Delta_l(1-2ixc_l) - x^2ic_l\Delta_l\right]\Big]$$

$$= i\varphi_Y(x)(ixa^2 - \sum_{j=1}^{k_1}b_j)G_1(x)G_2(x) + i\varphi_Y(x)G_2(x)\sum_{l=1}^{k_1}b_l\prod_{j\neq l}(1-2ixb_j)$$
$$+ i\varphi_Y(x)G_1(x)\sum_{l=1}^{k_2}\left(\prod_{j\neq l}(1-2ixc_j)^2\right)\left[ix\Delta_l(1-2ixc_l) + (ix)^2c_l\Delta_l - 2ixc_l^2\right]$$

$$= i\varphi_Y(x)ixa^2G_1(x)G_2(x) + i\varphi_Y(x)G_2(x)\sum_{l=1}^{k_1}[-b_l(1-2ixb_l) + b_l]\prod_{j\neq l}(1-2ixb_j)$$
$$+ i\varphi_Y(x)G_1(x)\sum_{l=1}^{k_2}\left(\prod_{j\neq l}(1-2ixc_j)^2\right)\left[ix\Delta_l(1-2ixc_l) + (ix)^2c_l\Delta_l - 2ixc_l^2\right]$$

$$= i\varphi_Y(x)ixa^2G_1(x)G_2(x) + i\varphi_Y(x)G_2(x)2ix\sum_{l=1}^{k}b_l^2\prod_{j\neq l}(1-2ixb_j) + i\varphi_Y(x)G_1(x)$$

$$\times \sum_{l=1}^{k_2} \left(\prod_{j \neq l} (1 - 2ixc_j)^2 \right) \left[ix\Delta_l(1 - 2ixc_l) + (ix)^2 c_l \Delta_l - 2ixc_l^2 \right]. \tag{3.17}$$

Step 5. We have that $Y \overset{\text{Law}}{=} X$ if and only if the right-hand side of Eq. (3.17) equals $\varphi_Y'(x)G_1(x)G_2(x)$ or, using the previous results and Eq. (3.16) in particular, if the following equality holds:

$$\begin{aligned}
&\sum_{l=1}^{k} (2ix)^{k+1} i\varphi_Y(x)(ix)^{-l} \sum_{r=l}^{k} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \frac{\kappa_{r-l+2}(X)}{(r-l+1)!} \\
&\quad + i\varphi_Y(x)(2ix)^{k+1} \frac{(-1)^{k_1} a^2}{2} \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j \\
&= i\varphi_Y(x) ixa^2 G_1(x)G_2(x) + i\varphi_Y(x) 2ixG_2(x) \sum_{l=1}^{k_1} b_l^2 \prod_{j \neq l} (1 - 2ixb_j) \\
&\quad + i\varphi_Y(x)G_1(x) \sum_{l=1}^{k_2} \left(\prod_{j \neq l} (1 - 2ixc_j)^2 \right) \left[ix\Delta_l(1 - 2ixc_l) + (ix)^2 c_l \Delta_l - 2ixc_l^2 \right].
\end{aligned} \tag{3.18}$$

If $\varphi_Y(x) \neq 0$, we can divide by $i\varphi_Y(x)$ and compare the coefficients of $x, x^2, x^3, \ldots$ on the left- and right-hand side of Eq. (3.18). For this final part of the proof, see Appendix. Considering the previous remarks, this concludes the proof. □

Remark 3.9. (1) Notice that the proof of Eq. (3.18) for the general case is rather lengthy and technical. This is basically due to the differential equation derived in Lemma 3.5 from which follows Eq. (3.18). For special cases, such as the case considered in Azmoodeh et al. (2015), the differential equation simplifies considerably and therefore a relatively simple recurrence for the moments of the target variable can be proved. In Azmoodeh et al. (2015), the authors have used this recurrence to prove their main result. In the general case however, this recurrence is hard to handle, therefore we have chosen to use differential equations rather than recurrence relations.

We illustrate now how the proof of the crucial equation can be simplified for the class of target variables considered in Azmoodeh et al. (2015). Notice that for $a = c_i = d_i = 0$, $k = k_1$ and $X \overset{Law}{=} \sum_{i=1}^{k} b_i(N_i^2 - 1)$ with $N_1, \ldots, N_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, we have to check the following equation:

$$\begin{aligned}
&\sum_{l=1}^{k-1} \frac{(2ix)^k}{(ix)^l} \sum_{r=l}^{k-1} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \frac{\kappa_{r-l+2}(X)}{(r-l+1)!} - (-1)^k (2ix)^k \sum_{i=1}^{k} b_i \prod_{j=1}^{k} b_j \\
&= 2ix \sum_{m=1}^{k} b_m^2 \prod_{j \neq m} (1 - 2ixb_j),
\end{aligned} \tag{3.19}$$

where $P(x) := x \prod_{i=1}^{k} (x - b_i)$. We use the following relation which can be proved by induction over l:

$$\sum_{r=l}^{k-1} \frac{P^{(r+2)}(0)}{(r+2)!} \sum_{j=1}^{k} b_j^{r+2-l} = (-1)^{k-l-1} \sum_{j=1}^{k} b_j^2 T_{k-1-l}^{(j)}, \tag{3.20}$$

where $1 \leqslant l \leqslant k-1$ and:

$$T_m^{(j)} := \sum_{\substack{i_1<\ldots<i_m \\ i_1,\ldots,i_m \neq j}} b_{i_1} \times \ldots \times b_{i_m}, \quad 1 \leqslant m \leqslant k-2$$

and $T_0^{(j)} := 1$, see Definition 6.1 for details. We have thus for the left-hand side of Eq. (3.19):

$$\begin{aligned}
&\sum_{l=1}^{k-1} (2ix)^k (ix)^{-l} \sum_{r=l}^{k-1} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \frac{\kappa_{r-l+2}(X)}{(r-l+1)!} - (-1)^k (2ix)^k \sum_{i=1}^{k} b_i \prod_{j=1}^{k} b_j \\
&= \sum_{l=1}^{k-1} (2ix)^k (ix)^{-l} \sum_{r=l}^{k-1} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \sum_{j=1}^{k} \frac{2^{r-l+1}(r-l+1)! b_j^{r-l+2}}{(r-l+1)!} \\
&\quad - (-1)^k (2ix)^k \sum_{i=1}^{k} b_i \prod_{j=1}^{k} b_j \\
&= \sum_{l=1}^{k-1} (2ix)^{k-l} (-1)^{k-l-1} \sum_{j=1}^{k} b_j^2 T_{k-1-l}^{(j)} - (-1)^k (2ix)^k \sum_{i=1}^{k} b_i \prod_{j=1}^{k} b_j,
\end{aligned}$$

and for the right-hand side of Eq. (3.19):

$$\begin{aligned}
&2ix \sum_{m=1}^{k} b_m^2 \sum_{j=1}^{k-1} \sum_{\substack{i_1<\ldots<i_j \\ i_1,\ldots,i_j \neq m}} (-2ixb_{i_1}) \times \ldots \times (-2ixb_{i_j}) + 2ix \sum_{m=1}^{k} b_m^2 \\
&= \sum_{j=0}^{k-1} (2ix)^{1+j} (-1)^j \sum_{m=1}^{k} b_m^2 T_j^{(m)} = \sum_{l=0}^{k-1} (2ix)^{k-l} (-1)^{k-l-1} \sum_{m=1}^{k} b_m^2 T_{k-l-1}^{(m)} \\
&= \sum_{l=1}^{k-1} (2ix)^{k-l} (-1)^{k-l-1} \sum_{m=1}^{k} b_m^2 T_{k-l-1}^{(m)} + (2ix)^k (-1)^{k-1} \sum_{m=1}^{k} b_m \, b_m T_{k-1}^{(m)} \\
&= \sum_{l=1}^{k-1} (2ix)^{k-l} (-1)^{k-l-1} \sum_{m=1}^{k} b_m^2 T_{k-l-1}^{(m)} + (2ix)^k (-1)^{k-1} \sum_{m=1}^{k} b_m \prod_{j=1}^{k} b_j.
\end{aligned}$$

This proves that Eq. (3.19) holds.

In the general case, the calculation of $\sum_{r=l}^{\deg(P)-2} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \frac{\kappa_{r-l+2}(X)}{(r-l+1)!}$ is lengthy. The following relations are needed and can be proved by induction over l, for the ease of notation define $k' := 2k_2 + k_1 + 1_{[a \neq 0]}$:

$$\begin{aligned}
&\sum_{r=l}^{k'-1} \frac{P^{(r+2)}(0)}{(r+2)!} b_j^{r+2-l} = (-1)^{k'-l-1} \sum_{j=1}^{k_1} b_j^2 \sum_{\substack{i_1+i_2+i_3 \\ =k'-1-l}} T_{i_1}^{(j)} S_{i_2} S_{i_3}, \\
&\sum_{r=l}^{k'-1} \frac{P^{(r+2)}(0)}{(r+2)!} \sum_{j=1}^{k_2} c_j^{r+2-l} = (-1)^{k'-l-1} \\
&\qquad \times \sum_{j=1}^{k_2} \left(c_j^2 \sum_{\substack{i_1+i_2+i_3 \\ =k'-1-l}} T_{i_1} S_{i_2}^{(j)} S_{i_3}^{(j)} + c_j^3 \sum_{\substack{i_1+i_2+i_3 \\ =k'-2-l}} T_{i_1} S_{i_2}^{(j)} S_{i_3}^{(j)} \right),
\end{aligned} \tag{3.21}$$

and:

$$\sum_{r=l}^{k'-1} \frac{P^{(r+2)}(0)}{(r+2)!} \sum_{j=1}^{k_2} (r-l+2) c_j^{r-l} d_j^2$$
$$= (-1)^{k'-l-1} \sum_{j=1}^{k_2} \left(2d_j^2 \sum_{\substack{i_1+i_2+i_3 \\ =k'-l-1}} T_{i_1} S_{i_2}^{(j)} S_{i_3}^{(j)} + cjd_j^2 \sum_{\substack{i_1+i_2+i_3 \\ =k'-l-2}} T_{i_1} S_{i_2}^{(j)} S_{i_3}^{(j)} \right), \quad (3.22)$$

where S_i and $S_i^{(j)}$ are defined similarly to $T_i^{(j)}$ with the coefficients $b_1, \ldots, b_{k_1}$ replaced by $c_1, \ldots, c_{k_2}$, see Definition 6.1 for details.
In particular, equations (3.21) and (3.22) imply that the proof of Eq. (3.18) is technical. In Proposition 6.2 we present a proof of the latter equation which does not require Eq. (3.21) and (3.22). The coefficients of the polynomials on both sides of Eq. (3.18) are compared directly. This proof still remains complicated and lengthy. As mentioned above, this is due to the form of the differential equation for the characteristic function of the target variable in the general case.

(2) Theorem 3.8 extends Azmoodeh et al. (2015, Theorem 3.2,(ii) → (i)) since it holds for a more general set of target random variables X and we have L^1-convergence instead of L^2-convergence.

(3) We notice that for the proof it is essential that, as $n \to \infty$:

$$\mathbb{E}\left[\exp(ixF_n) \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} M_{r-1}(F_n) \right]$$
$$= \mathbb{E}\left[\exp(ixF_n)\, \mathbb{E}\left[\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} M_{r-1}(F_n) \Big| F_n \right] \right] \to 0.$$

Since $|\exp(ixF_n)| = 1$, the triangle inequality shows that it is sufficient to have:

$$\mathbb{E}\left[\left| \mathbb{E}\left[\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)] \right) \Big| F_n \right] \right| \right] \to 0,$$

as $n \to \infty$. This L^1-convergence is weaker than L^2-convergence which results typically from the Cauchy-Schwarz inequality, used to control unbounded factors.

(4) Different random variables X may lead to the same polynomial P. Let N be a standard normal variable, $X_1 \overset{\text{Law}}{=} N^2 - 1 + d_1 N$ and $X_2 \overset{\text{Law}}{=} N^2 - 1 + d_2 N$ with $d_1 \neq d_2$ non-zero, then Theorem 3.8 yields the polynomial $P(x) = x(x-1)^2$. A similar observation holds for Nourdin and Poly (2012, Theorem 3.4) where the constant μ_0 only appears in the limit of the second cumulant. In our case, the condition (1) of Theorem 3.8 discerns both cases. For special cases it may be useful to consider a polynomial which is different from the 'standard polynomial' defined in Theorem 3.8. This observation is founded in the initial value problem established in Lemma 3.5. It can be shown that the differential equation in Lemma 3.5 can be simplified if the b_i or the (c_j, d_j) are not pairwise different. For instance, if $X_1 \overset{\text{Law}}{=} (N_1^2 - 1) + (N_2^2 - 1)$ and

$N_1, N_2 \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1)$, Lemma 3.5 yields a differential equation which can be simplified to yield Nourdin and Peccati (2009a, Eq. (1.9)).

We proceed now to the proof of the converse of Theorem 3.8. We shall need the following Lemma which generalizes Azmoodeh et al. (2015, Eq. (3.6)).

Lemma 3.10. *Let* $P(x) := x^{1+1_{[a\neq 0]}} \prod_{i=1}^{k_1}(x-b_i)\prod_{i=1}^{k_2}(x-c_i)^2$ *and* $X = I_1(f_1) + I_2(f_2)$*, see Eq.* (3.3)*, then:*

$$\mathbb{P}\left(\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)]\right) = 0\right) = 1.$$

Proof: In this proof all the equalities between random variables hold with probability 1. We consider iterated contractions recursively defined as follows for $f_2 \in H^{\odot 2}$:

$$f_2\tilde{\otimes}_1^{(1)} f_2 := f_2 \ ; \ f_2\tilde{\otimes}_1^{(p)} f_2 := \left(f_2\tilde{\otimes}_1^{(p-1)} f_2\right)\tilde{\otimes}_1 f_2,$$

for $p \geqslant 2$. We use the representation for X given in Eq. (3.3) and Eq. (3.6).

(1) We prove by induction over $r \geqslant 1$:

$$\begin{aligned}\Gamma_r(X) - \mathbb{E}[\Gamma_r(X)] &= 2^r I_2(f_2\tilde{\otimes}_1^{(r+1)} f_2) + 2^{r-1}\, 3I_1((\ldots(f_2\tilde{\otimes}_1 f_1)\tilde{\otimes}_1 f_2)\ldots)\tilde{\otimes}_1 f_2) \\ &\quad + 2^{r-1}\sum_{j=3}^{r+1}\sum_{\substack{g_j=f_1\\ g_i=f_2,\ \text{for } i\neq j}} I_1((\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_{r+1}).\end{aligned} \tag{3.23}$$

Notice that all iterated contractions on the right hand side of Eq. (3.23) run over $r+1$ functions.

For $r = 1$, we have with the stochastic Fubini theorem and the multiplication formula for multiple Wiener integrals:

$$\begin{aligned}\Gamma_1(X) &= \langle DX, -DL^{-1}X\rangle_H = \langle 2I_1(f_2(t,\cdot)) + f_1(t), I_1(f_2(t,\cdot)) + f_1(t)\rangle_H \\ &= 2\int_0^T I_2(f_2(t,\cdot)\tilde{\otimes} f_2(t,\cdot)) + f_2(t,\cdot)\tilde{\otimes}_1 f_2(t,\cdot)dt + 3\int_0^T I_1(f_2(t,\cdot)\tilde{\otimes} f_1(t))dt \\ &\quad + \int_0^T f_1(t)^2 dt \\ &= 2I_2(f_2\tilde{\otimes}_1 f_2) + 3I_1(f_2\tilde{\otimes}_1 f_1) + 2\|f_2\|^2_{H\otimes H} + \|f_1\|^2_H.\end{aligned}$$

The claim follows since the expectation of every multiple Wiener integral is 0. Suppose now that the claim holds for some $r \geqslant 1$, then:

$$\begin{aligned}\Gamma_{r+1}(X) &= \langle DX, -DL^{-1}\Gamma_r(X)\rangle_H \\ &= \langle 2I_1(f_2(t,\cdot)) + f_1(t), 2^r I_1((f_2\tilde{\otimes}_1^{(r+1)} f_2)(t,\cdot)) \\ &\quad + 2^{r-1}3((\ldots(f_2\tilde{\otimes}_1 f_1)\tilde{\otimes}_1 f_2)\ldots\tilde{\otimes}_1 f_2)(t)) \\ &\quad + 2^{r-1}\sum_{j=3}^{r+1}\sum_{\substack{g_j=f_1\\ g_i=f_2,\ \text{for } i\neq j}} ((\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots\tilde{\otimes}_1 g_{r+1})(t)\rangle_H \\ &= 2^{r+1} I_2(f_2\tilde{\otimes}_1^{(r+2)} f_2) + 2^{r+1}\langle f_2\tilde{\otimes}_1^{(r+1)} f_2, f_2\rangle_H \\ &\quad + 2^r 3I_1((\ldots(f_2\tilde{\otimes}_1 f_1)\tilde{\otimes}_1 f_2)\ldots)\tilde{\otimes}_1 f_2)\tilde{\otimes}_1 f_2)\end{aligned}$$

$$
\begin{aligned}
&+ 2^r \sum_{j=3}^{r+1} \sum_{\substack{g_j=f_1 \\ g_i=f_2,\ \text{for } i\neq j}} I_1((\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_{r+2}) \\
&+ 2^r I_1((f_2\tilde{\otimes}_1^{(r+1)} f_2)\tilde{\otimes}_1 f_1) + 2^{r-1} \\
&\times \Bigg(3\langle(\ldots(f_2\tilde{\otimes}_1 f_1)\tilde{\otimes}_1 f_2)\ldots)\tilde{\otimes}_1 f_2, f_1\rangle_H \\
&+ \sum_{j=3}^{r+1} \sum_{\substack{g_j=f_1 \\ g_i=f_2,\ \text{for } i\neq j}} \langle(\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_{r+1}, f_1\rangle_H\Bigg).
\end{aligned}
$$

Hence Eq. (3.23) follows for $r+1$.

(2) For $r \geqslant 2$, we have the following equalities:

$$
f_2\tilde{\otimes}_1^{(r)} f_2 = \sum_{i=1}^{k_1} b_i^r h_i\tilde{\otimes}h_i + \sum_{r}^{k_2} c_i^r h_i'\tilde{\otimes}h_i', \tag{3.24}
$$

$$
(\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_r = \sum_{i=1}^{k_2} c_i^{r-1} d_i h_i', \tag{3.25}
$$

where the last equality holds if exactly one of the functions $g_1, \ldots, g_r$ equals f_1, all remaining functions g_i being equal to f_2. We notice that $(h_i\tilde{\otimes}h_i)\tilde{\otimes}_1(h_j'\tilde{\otimes}h_j') = 0$ and $(h_i'\tilde{\otimes}h_i')\tilde{\otimes}_1(h_j'\tilde{\otimes}h_j') = 1_{[i=j]}h_i'\tilde{\otimes}h_i'$ and $(h_i\tilde{\otimes}h_i)\tilde{\otimes}_1(h_j\tilde{\otimes}h_j) = 1_{[i=j]}h_i\tilde{\otimes}h_i$. Hence $f_2\tilde{\otimes}_1 f_2 = \sum_{i=1}^{k_1} b_i^2 h_i\tilde{\otimes}h_i + \sum_{i=1}^{k_2} c_i^2 h_i'\tilde{\otimes}h_i'$ and, generally:

$$
f_2\tilde{\otimes}_1^{(r)} f_2 = \sum_{i=1}^{k_1} b_i^r h_i\tilde{\otimes}h_i + \sum_{i=1}^{k_2} c_i^r h_i'\tilde{\otimes}h_i'.
$$

To prove Eq. (3.25), we suppose first that $g_2 = f_1$ and $g_1 = g_3 = \ldots = g_r = f_2$. Then:

$$
g_1\tilde{\otimes}_1 g_2 = f_2\tilde{\otimes}_1 f_1 = \left(\sum_{i=1}^{k_1} b_i h_i\tilde{\otimes}h_i + \sum_{i=1}^{k_2} c_i h_i'\tilde{\otimes}h_i'\right)\tilde{\otimes}_1\left(ah_0 + \sum_{i=1}^{k_2} d_i h_i'\right) = \sum_{i=1}^{k_2} c_i d_i h_i'.
$$

Since $g_1 = g_3 = \ldots = g_r = f_2$, it is now easy to see that:

$$
(\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_r = \sum_{i=1}^{k_2} c_i^{r-1} d_i h_i'.
$$

If, on the other hand $g_1 = \ldots = g_l = f_2$ for $l > 1$, we can use Eq. (3.24) to see that

$$
(\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_l = f_2\tilde{\otimes}_1^{(l)} f_2 = \sum_{i=1}^{k_1} b_i^l h_i\tilde{\otimes}h_i + \sum_{i=1}^{k_2} c_i^l h_i'\tilde{\otimes}h_i',
$$

and we can proceed as above to see that Eq. (3.25) holds.

(3) Considering Eq. (3.23), it is easy to see that

$$
\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)]\right)
$$

lives in the sum of the first two Wiener chaoses. We consider the projection of the random variable above on the first respectively on the second Wiener chaos. We have with Eq. (3.23):

$$
\begin{aligned}
J_1 &\left(\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)]\right) \right) = \frac{P'(0)}{1!2^0} J_1(X) \\
&+ \sum_{r=2}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} 2^{r-2} \left(3I_1((\ldots(f_2\tilde{\otimes}_1 f_1)\tilde{\otimes}_1 f_2)\ldots)\tilde{\otimes}_1 f_2\right) \\
&+ \sum_{j=3}^{r} \sum_{\substack{g_j=f_1 \\ g_i=f_2,\ \text{for } i\neq j}} I_1((\ldots(g_1\tilde{\otimes}_1 g_2)\tilde{\otimes}_1 g_3)\ldots)\tilde{\otimes}_1 g_r) \Bigg) \\
&= \frac{P'(0)}{1!2^0} I_1(f_1) + \sum_{i=1}^{k_2} \sum_{r=2}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} I_1\left(2^{r-2}\left(3h_i' + (r-2)h_i'\right) d_i c_i^{r-1}\right) \\
&= \frac{P'(0)}{1!2^0} I_1(f_1) + \frac{1}{2} \sum_{i=1}^{k_2} \sum_{r=2}^{\deg(P)} \frac{P^{(r)}(0)}{r!} I_1\left(h_i'\right)(r+1) d_i c_i^{r-1}.
\end{aligned}
$$

If $a = 0$, we have:

$$
\frac{P'(0)}{1!2^0} I_1(f_1) = \frac{1}{2} \sum_{i=1}^{k_2} \frac{P'(0)}{1!2^0} I_1(h_i')\, 2d_i. \tag{3.26}
$$

If $a \neq 0$, we have $P'(0) = 0$, hence Eq. (3.26) holds in both cases and:

$$
\begin{aligned}
&J_1 \left(\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)]\right) \right) \\
&= \frac{1}{2} \sum_{i=1}^{k_2} \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} I_1\left(h_i'\right)(r+1) d_i c_i^{r-1} \\
&= \frac{1}{2} \sum_{i=1}^{k_2} \left[d_i \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} r c_i^{r-1} \right] I_1(h_i') + \frac{1}{2} \sum_{i=1}^{k_2} \left[\frac{d_i}{c_i} \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} c_i^r \right] I_1(h_i') \\
&= \frac{1}{2} \sum_{i=1}^{k_2} d_i P'(c_i) I_1(h_i') + \frac{1}{2} \sum_{i=1}^{k_2} \frac{d_i}{c_i} P(c_i) I_1(h_i') = 0.
\end{aligned}
$$

We have for the projection on the second Wiener chaos with Eq. (3.24):

$$
\begin{aligned}
&J_2 \left(\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)]\right) \right) \\
&= \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} 2^{r-1} I_2\left(f_2 \tilde{\otimes}_1^{(r)} f_2\right) \\
&= \sum_{i=1}^{k_1} \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} b_i^r I_2(h_i \tilde{\otimes} h_i) + \sum_{i=1}^{k_2} \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} c_i^r I_2(h_i' \tilde{\otimes} h_i')
\end{aligned}
$$

$$= \sum_{i=1}^{k_1} P(b_i) I_2(h_i \tilde{\otimes} h_i) + \sum_{i=1}^{k_2} P(c_i) I_2(h_i' \tilde{\otimes} h_i') = 0.$$

(4) Since $Z := \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r! 2^{r-1}} (\Gamma_{r-1}(X) - \mathbb{E}[\Gamma_{r-1}(X)])$ equals $J_1(Z) + J_2(Z)$, the claim of the Lemma follows now directly.

□

Theorem 3.11. *Consider $k_1, k_2 \geqslant 0$ and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i (R_i^2 - 1) + \sum_{i=1}^{k_2} [c_i (P_i^2 - 1) + d_i P_i],$$

as defined in Eq.(3.3) and Eq. (3.5) where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Suppose that at least one of the parameters a, k_1, k_2 is non-zero. Consider a sequence $\{F_n\}_n$ of non-zero random variables with $F_n = \sum_{i=1}^{p} I_i(f_{n,i})$ for $p \geqslant 2$ fixed and $\{f_{n,i}\}_n \subset H^{\odot i}$ for $1 \leqslant i \leqslant p$. Define:

$$P(x) = x^{1+1_{[a \neq 0]}} \prod_{j=1}^{k_1} (x - b_j) \prod_{j=1}^{k_2} (x - c_j)^2.$$

If $F_n \overset{Law}{\to} X$, as $n \to \infty$, then the following limits hold, as $n \to \infty$:

(1) $\kappa_r(F_n) \to \kappa_r(X), \quad$ *for* $r = 1, \ldots, deg(P)$,

(2) $\mathbb{E}\left[\left|\mathbb{E}\left[\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r! 2^{r-1}} (\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]) \middle| F_n\right]\right|\right] \to 0.$

Proof: The proof of this theorem is identical to Azmoodeh et al. (2015, Theorem 3.2, Proof of (i)→ (iii)). It is enough to replace Azmoodeh et al. (2015, Lemma 3.1) by Lemma 3.10. □

Proof of Theorem 1.1. The main result of this paper is now a direct consequence of Theorem 3.8 and Theorem 3.11. □

Remark 3.12. (1) We notice that, for any sequences of integrable random variables $\{F_n\}_n$ and $\{G_n\}_n$, we have that $\mathbb{E}[|F_n|] \to 0$ implies $\mathbb{E}[|\,\mathbb{E}[F_n|G_n]|] \to 0$, as $n \to \infty$:

$$\mathbb{E}[|\,\mathbb{E}[F_n|G_n]|] \leqslant \mathbb{E}[\mathbb{E}[|F_n|\ |G_n]] = \mathbb{E}[|F_n|] \to 0, \quad \text{as } n \to \infty.$$

Hence, with the notations of Theorem 1.1, a set of sufficient conditions for $F_n \overset{\text{Law}}{\to} X$, as $n \to \infty$, is:

(1') $\kappa_r(F_n) \to \kappa_r(X), \quad$ for $r = 1, \ldots, \deg(P)$,

(2') $\mathbb{E}\left[\left|\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r! 2^{r-1}} (\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)])\right|^{\alpha}\right] \to 0\ ,$

where $\alpha \geqslant 1$. If $\alpha = 2$,we can use Eq. (2.1) to calculate the expectation in (2').

(2) For $\alpha = 2$, conditions (1') and (2') can be expressed in terms of conditions for contractions. However the resulting conditions are usually complicated as indicates the following example:

Consider $p \geqslant 2$ even, $a \neq 0$, $b \neq 0$ and a sequence of functions $\{f_{n,p}\} \subset H^{\odot p}$ such that, as $n \to \infty$:

$$p!\|f_{n,p}\|^2_{H^{\otimes p}} \to a^2 + 2b^2, \tag{3.27}$$

$$\langle f_{n,p} \tilde{\otimes}_{p/2} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} \to 8b^3 \left[p!(p/2)! \binom{p}{p/2}^2 \right]^{-1}. \tag{3.28}$$

Define sets A_l and B_l for every even integer l with $2 \leqslant l \leqslant 3p-4$:

$$A_l := \left\{ (s,t) \in \mathbb{N}^2 : 1 \leqslant s \leqslant p-1 \ , \ 1 \leqslant t \leqslant p \wedge (2p-2s) \ , \ 3p - 2(s+t) = l \right\},$$
$$B_l := \left\{ s \in \mathbb{N} : 1 \leqslant s \leqslant p-1 \ , \ 2p - 2s = l \right\},$$

and:

$$c(p,s,t) := p(s-1)!(t-1)! \binom{p-1}{s-1}^2 \binom{p-1}{t-1} \binom{2p-2s-1}{t-1},$$
$$k(p,s) := (s-1)! \binom{p-1}{s-1}^2.$$

Define for every even integer l a condition C_l by:

$$\left\| \sum_{(s,t) \in A_l} c(p,s,t)(f_{n,p} \tilde{\otimes}_s f_{n,p}) \tilde{\otimes}_t f_{n,p} - 2b \sum_{s \in B_l} k(p,s) f_{n,p} \tilde{\otimes}_s f_{n,p} \right\|_{H^{\otimes l}} \to 0.$$

If conditions C_l hold for every even integer l with $2 \leqslant l \leqslant 3p-4$, we have, as $n \to \infty$:

$$I_p(f_{n,p}) \overset{Law}{\to} aN + b(\xi^2 - 1), \quad N, \xi \overset{i.i.d.}{\sim} \mathcal{N}(0,1).$$

The proof of this result is omitted, another example for such conditions can be found in Azmoodeh et al. (2015, Theorem 4.1).

4. Applications

We notice that our results about convergence in law of sequences living in a finite sum of Wiener chaoses can be extended to match convergence in total variation, see for instance Nourdin and Peccati (2012, Appendix C). Indeed a direct application of our results together with Azmoodeh et al. (2015, Lemma 3.3) or Nourdin and Poly (2013, Theorem 3.1) proves that we can replace the convergence in law of sequences living in a finite sum of Wiener chaoses by convergence in total variation.

4.1. *Recovering classical criteria.* As a direct consequence of Theorem 1.1, we get the following Corollary 4.1 which extends Azmoodeh et al. (2015, Theorem 3.2). We have proved that (2) is necessary and sufficient for convergence in law whereas the authors in the cited reference need L^2-convergence of the conditional expectation to prove $F_n \overset{\text{Law}}{\to} X$, as $n \to \infty$. As pointed out in Remark 3.9 this results from the Cauchy-Schwarz inequality. Moreover, Corollary 4.1 extends the first part of Döbler and Peccati (2016, Proposition 1.7) to more general linear combinations of independent central χ^2 distributed random variables. As anticipated, if $b_1 = \ldots = b_k = 1$, the polynomial P can be simplified.

Corollary 4.1. *Consider* $X = I_2(f_2) = \sum_{i=1}^k b_i(R_i^2 - 1)$ *for* $R_1, \dots, R_{k_1} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ *and* $b_i \neq 0$ *for* $i = 1, \dots, k$, *see Eq.* (3.3) *and* (3.5). *Define* $P(x) = x \prod_{j=1}^k (x - b_i)$. *Let* $\{F_n\}_n$ *be a sequence of non-zero random variables such that each* F_n *lives in a finite sum of Wiener chaoses, i.e.* $F_n = \sum_{i=1}^m I_i(f_{n,i})$ *for* $n \geqslant 1$ *and* $m \geqslant 2$ *fixed,* $\{f_{n,i}\}_n \subset H^{\odot i}$ *for* $1 \leqslant i \leqslant m$. *The following two asymptotic relations (1) and (2) are equivalent, as* $n \to \infty$*:*

(1) $F_n \overset{Law}{\to} X$*;*
(2) (a) $\kappa_r(F_n) \to \kappa_r(X)$, *for* $r = 2, \dots, k+1$,
(b) $\mathbb{E}\left[\left|\mathbb{E}\left[\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]\right) \middle| F_n\right]\right|\right] \to 0.$

Proof: Use Theorem 1.1 with $a = k_2 = 0$. □

We come now to the seminal paper Nualart and Peccati (2005) of Nualart and Peccati in which the authors have characterised the convergence in law to a standard normal random variable. Since this first paper, the conditions given in Nualart and Peccati (2005) have been extended, see for instance Nourdin and Peccati (2009a). Taking $k_1 = k_2 = 0$ and $a = 1$, we have with Theorem 1.1 the following characterisation which corresponds to condition (iii') cited in Section 1.2:

Corollary 4.2. *Consider* $X = aI_1(h_0)$ *for* $I_1(h_0) \sim \mathcal{N}(0,1)$ *and* $a \neq 0$. *Let* $\{F_n\}_n$ *be a sequence of non-zero random variables such that each* F_n *lives in a finite sum of Wiener chaoses, i.e.* $F_n = \sum_{i=1}^m I_i(f_{n,i})$ *for* $n \geqslant 1$ *and* $m \geqslant 2$ *fixed,* $\{f_{n,i}\}_n \subset H^{\odot i}$ *for* $1 \leqslant i \leqslant m$. *The following two asymptotic relations (1) and (2) are equivalent, as* $n \to \infty$*:*

(1) $F_n \overset{Law}{\to} X$,
(2) (a) $\kappa_2(F_n) \to a^2$,
(b) $\mathbb{E}\left[\left|\mathbb{E}\left[\langle DF_n, -DL^{-1}F_n\rangle_H - a^2 \middle| F_n\right]\right|\right] \to 0.$

Proof: The equivalence of (1) and (2) follows directly from Theorem 1.1 with $P(x) = x^2$, since:

$$\sum_{r=1}^{2} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]\right) = \frac{1}{2}\left(\Gamma_1(F_n) - \kappa_2(F_n)\right).$$

If (2) holds, we have clearly $\lim_n \kappa_2(F_n) = a^2$. If (1) holds, the latter limit follows from Azmoodeh et al. (2015, Lemma 3.3). □

Remark 4.3. The equivalence of (1) and (2) in the last corollary can be extended to sequences $\{F_n\}_n$ living in $\mathbb{D}^{1,2}$ if some assumptions on the boundedness of the Γ_1-operator is added. Indeed we can prove that (1) implies (2) if $\sup_n \mathbb{E}[\Gamma_1(F_n)^2] < \infty$. It is unsure whether the equivalence of (1) and (2) holds without any additional assumptions if $\{F_n\}_n \subset \mathbb{D}^{1,2}$.

We use now Remark 3.12 to recover more criteria for the convergence in law, see Corollary 4.4. Notice that (1) is Nourdin and Peccati (2012, Theorem 5.3.1) for the case of a sequence $\{F_n\}_n$ living in a fixed sum of Wiener chaoses, whereas (2) is the sufficient part of the criterion given in Nourdin and Peccati (2009a, Eq. (1.3)) with L^2-convergence replaced by L^1-convergence. Since all L^p-norms inside a fixed sum of Wiener chaoses are equivalent, the criterion given in (2) below is thus necessary and sufficient.

Corollary 4.4. *Consider $X = aI_1(h_0)$ for $I_1(h_0) \sim \mathcal{N}(0,1)$ and $a \neq 0$. Let $\{F_n\}_n$ be a sequence of non-zero random variables with $\kappa_2(F_n) \to a^2$, as $n \to \infty$.*

(1) If $F_n = \sum_{i=1}^m I_i(f_{n,i})$ for $n \geqslant 1$, $m \geqslant 2$ fixed with $\{f_{n,i}\}_n \subset H^{\odot i}$ for $1 \leqslant i \leqslant m$ and, as $n \to \infty$:

$$\mathbb{E}\left[\left|\langle DF_n, -DL^{-1}F_n\rangle_H - a^2\right|\right] \to 0,$$

then $F_n \overset{Law}{\to} X$, as $n \to \infty$.

(2) If $F_n = I_p(f_{n,p})$ with $\{f_{n,p}\}_n \subset H^{\odot p}$ and, as $n \to \infty$:

$$\mathbb{E}\left[\left|1/p\|DF_n\|_H^2 - a^2\right|\right] \to 0,$$

then $F_n \overset{Law}{\to} X$, as $n \to \infty$.

4.2. *New applications.* For the rest of the section we suppose that p is even. Notice that if p is odd, we cannot have:

$$I_p(f_{n,p}) \overset{\text{Law}}{\to} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1),$$

as $n \to \infty$ for $k_1 \geqslant 1$, $\sup_n \|f_{n,p}\|_{H^{\otimes p}} < \infty$ and $\sum_{i=1}^k b_i^3 \neq 0$. This can be checked using the third cumulant and Eq. (2.1), see Nourdin and Peccati (2009a, Remark 1.3). More generally, if $p \geqslant 3$ is odd, we can exclude a large set of possible target random variables by using the fact that all odd-order cumulants are zero, see Remark 4.10 for details.

The following Lemma shall be needed to prove a sufficient criterion based on the convergence of some contractions and cumulants.

Lemma 4.5. *Consider p_1, $p_2 \geqslant 1$, $f \in H^{\odot p_1}$ and $g \in H^{\odot p_2}$. We have for $0 \leqslant r \leqslant p_1 \wedge p_2$:*

$$\begin{aligned}\|f\tilde{\otimes}_r g\|^2_{p_1+p_2-2r} &\leqslant \|f\otimes_r g\|^2_{p_1+p_2-2r} = |\langle f \otimes_{p_1-r} f, g \otimes_{p_2-r} g\rangle_{H^{\otimes(2r)}}| \\ &\leqslant \|f \otimes_{p_1-r} f\|_{H^{\otimes(2r)}} \|g \otimes_{p_2-r} g\|_{H^{\otimes(2r)}} \leqslant \|f \otimes_{p_1-r} f\|_{H^{\otimes(2r)}} \|g\|^2_{H^{\otimes p_2}} \\ &\leqslant \|f\|^2_{H^{\otimes p_1}} \|g\|^2_{H^{\otimes p_2}}.\end{aligned}$$

Proof: The first inequality is a standard result. The first equality follows from Nourdin and Rosiński (2014, Eq. (2.5)). The next inequality follows from the Cauchy-Schwarz inequality. The last inequalities follow from the following general result:

$$\|f \otimes_r g\|_{H^{\otimes(p_1+p_2-2r)}} \leqslant \|f\|_{H^{\otimes p_1}} \|g\|_{H^{\otimes p_2}},$$

which is, again, a consequence of the Cauchy-Schwarz inequality. □

Theorem 4.6. *Consider $k_1, k_2 \geqslant 0$ and*

$$X = I_1(f_1) + I_2(f_2) = aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2 - 1) + d_iP_i],$$

as defined in Eq. (3.3) and Eq. (3.5) where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Suppose that at least one of the parameters a, k_1, k_2 is non-zero. Consider a sequence $\{F_n\}_n$ of non-zero random variables with $F_n = I_p(f_{n,p})$ for $p \geqslant 4$ fixed,

even, and $\{f_{n,p}\}_n \subset H^{\odot p}$. *Define:*

$$P(x) = x^{1+1_{[a\neq 0]}} \prod_{j=1}^{k_1}(x-b_j)\prod_{j=1}^{k_2}(x-c_j)^2 \ ; c_p := 2(p/2)!\binom{p-1}{p/2-1}^2 \ ;$$

$$f_{n,p}\tilde{\otimes}^{(1)}_{p/2}f_{n,p} := f_{n,p} \ ; \ f_{n,p}\tilde{\otimes}^{(l+1)}_{p/2}f_{n,p} := \left(f_{n,p}\tilde{\otimes}^{(l)}_{p/2}f_{n,p}\right)\tilde{\otimes}_{p/2}f_{n,p}, \quad \textit{for } l\in\mathbb{N}_{>0}.$$

As $n\to\infty$, *we have* $I_p(f_{n,p}) \overset{Law}{\to} X$ *if the following hold:*

$$\kappa_r(I_p(f_{n,p})) \to \kappa_r(X), \quad \textit{for } r=1,\dots,\textit{deg }(P), \tag{4.1}$$

$$\|f_{n,p}\tilde{\otimes}_l f_{n,p}\|_{H^{\otimes(2p-2l)}} \to 0, \quad \textit{for every } 1\leqslant l\leqslant p-1 \textit{ with } l\neq p/2, \tag{4.2}$$

$$\left\|\sum_{r=1}^{degP}\frac{P^{(r)}(0)}{r!2^{r-1}}f_{n,p}\tilde{\otimes}^{(r)}_{p/2}f_{n,p}c_p^r\right\|_{H^{\otimes p}} \to 0. \tag{4.3}$$

Proof: (1) We notice that, for $p\geqslant 4$ even, condition (4.2) is equivalent to:

$$\|f_{n,p}\otimes_l f_{n,p}\|_{H^{\otimes(2p-2l)}} \to 0, \quad \text{for every } 1\leqslant l\leqslant p-1 \text{ with } l\neq p/2,$$

see Nourdin and Peccati (2009a, Proposition 3.1). If $p=2$, we have necessarily $k_2=0$ and the convergence in law of sequences $\{I_2(f_{n,2})\}_n$ is completely characterised by necessary and sufficient criteria in Nourdin and Poly (2012), see Remark 4.7.

(2) We first prove that, except for the contractions $f_{n,p}\tilde{\otimes}^{(r)}_{p/2}f_{n,p}$ for $r=2,\dots,\deg(P)$, all the contractions, appearing in the representation of $M_{r-1} = \Gamma_{r-1}(I_p(f_{n,p})) - \mathbb{E}[\Gamma_{r-1}(I_p(f_{n,p}))]$ for $r=2,\dots,\deg(P)$, converge to zero (in the corresponding Hilbert-space norm). With Azmoodeh et al. (2015, Proposition 2.1), we have for $i\geqslant 1$ that $\Gamma_i(I_p(f_{n,p})) - \mathbb{E}[\Gamma_i(I_p(f_{n,p}))]$ equals:

$$\sum_{(r_1,\dots,r_i)\in S_i} c_p(r_1,\dots,r_i) \times I_{(i+1)p-2r_1-\dots-2r_i}((\dots(f_{n,p}\tilde{\otimes}_{r_1}f_{n,p})\tilde{\otimes}_{r_2}f_{n,p})\dots f_{n,p})\tilde{\otimes}_{r_i}f_{n,p}),$$

where S_i is the set of elements $(r_1,\dots,r_i)$ such that:

$$1\leqslant r_1\leqslant p \ , \ \dots \ , \ 1\leqslant r_i\leqslant (ip-2r_1-\dots-2r_{i-1})\wedge p;$$
$$r_1<p \ , \ \dots \ , \ r_1+\dots+r_{i-1}<ip/2 \ , \ (i+1)p-2r_1-\dots-2r_i\neq 0,$$

and $c_p(r_1,\dots,r_i)$ is defined recursively, see Azmoodeh et al. (2015, Proposition 2.1). For $(r_1,\dots,r_i)\in S_i\setminus\{(p/2,\dots,p/2)\}$, we prove that, as $n\to\infty$:

$$\|(\dots(f_{n,p}\tilde{\otimes}_{r_1}f_{n,p})\tilde{\otimes}_{r_2}f_{n,p})\dots f_{n,p})\tilde{\otimes}_{r_i}f_{n,p}\|_{H^{\otimes[(i+1)p-2r_1-\dots-2r_i]}} \to 0. \tag{4.4}$$

If $r_1\neq p/2$, we have $\|f_{n,p}\tilde{\otimes}_{r_1}f_{n,p}\|_{H^{\otimes(2p-2r_1)}}\to 0$, as $n\to\infty$, since $r_1\neq p$. If $i\geqslant 2$ and $r_1=p/2$, let $r_1=\dots=r_j=p/2$ and $r_{j+1}\neq p/2$ with $1\leqslant j\leqslant i-1$. We have $r_{j+1}\neq p$, since otherwise the definition of S_i yields the following contradictions:

$$\frac{jp}{2}+p = \sum_{k=1}^{j+1} r_k < \frac{(j+2)p}{2}, \quad \text{if } j+1\leqslant i-1,$$

and:

$$0\neq (i+1)p - 2\sum_{k=1}^{i} r_k = (i+1)p - 2(i-1)p/2 - 2p = 0, \quad \text{if } j+1=i.$$

Thus, with Lemma 4.5:

$$\begin{aligned}
&\|(\dots(f_{n,p}\tilde{\otimes}_{r_1}f_{n,p})\tilde{\otimes}_{r_2}f_{n,p})\dots f_{n,p})\tilde{\otimes}_{r_{j+1}}f_{n,p}\|^2_{H^{\otimes[(j+2)p-2r_1-\dots-2r_{j+1}]}}\\
&=\left\|\left(f_{n,p}\tilde{\otimes}^{(j+1)}_{p/2}f_{n,p}\right)\tilde{\otimes}_{r_{j+1}}f_{n,p}\right\|^2_{H^{\otimes[(j+2)p-2r_1-\dots-2r_{j+1}]}}\\
&\leqslant \|f_{n,p}\|^{2j+2}_{H^{\otimes p}}\ \|f_{n,p}\otimes_{p-r_{j+1}}f_{n,p}\|_{H^{\otimes(2r_{j+1})}}.
\end{aligned}$$

Since $1\leqslant r_{j+1}\leqslant p-1$ and $r_{j+1}\neq p/2$, we have that $1\leqslant p-r_{j+1}\leqslant p-1$ and $p-r_{j+1}\neq p/2$. Thus $\|f_{n,p}\otimes_{p-r_{j+1}}f_{n,p}\|_{H^{\otimes(2r_{j+1})}}\to 0$, as $n\to\infty$, since $\sup_n\|f_{n,p}\|_{H^{\otimes p}}<\infty$, Eq. (4.4) is proved.

(3) With Eq. (4.4), we have for $i\geqslant 1$, as $n\to\infty$:

$$\begin{aligned}
&\mathbb{E}\left[\left(\Gamma_i(I_p(f_{n,p}))-\mathbb{E}[\Gamma_i(I_p(f_{n,p}))]-c_p(p/2,\dots,p/2)I_p\left(f_{n,p}\tilde{\otimes}^{(i+1)}_{p/2}f_{n,p}\right)\right)^2\right]\\
&=\mathbb{E}\left[\left(\Gamma_i(I_p(f_{n,p}))-\mathbb{E}[\Gamma_i(I_p(f_{n,p}))]-c_p^iI_p\left(f_{n,p}\tilde{\otimes}^{(i+1)}_{p/2}f_{n,p}\right)\right)^2\right]\to 0,
\end{aligned}$$

where we have used the definition of c_p and Azmoodeh et al. (2015, Proposition 2.1). For $i=0$, we have trivially $\mathbb{E}[(\Gamma_0(I_p(f_{n,p}))-\mathbb{E}[\Gamma_0(I_p(f_{n,p}))]-I_p(f_{n,p}))^2]=0$. Thus with $M_{r-1}:=\Gamma_{r-1}(I_p(f_{n,p}))-\mathbb{E}[\Gamma_{r-1}(I_p(f_{n,p}))]$, as $n\to\infty$:

$$\mathbb{E}\left[\left(\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\left(M_{r-1}-c_p^{r-1}I_p\left(f_{n,p}\tilde{\otimes}^{(r)}_{p/2}f_{n,p}\right)\right)\right)^2\right]\to 0. \tag{4.5}$$

On the other hand, as $n\to\infty$:

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}c_p^{r-1}I_p\left(f_{n,p}\tilde{\otimes}^{(r)}_{p/2}f_{n,p}\right)\right)^2\right]\\
&=c_p^{-2}p!\left\|\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}c_p^rf_{n,p}\tilde{\otimes}^{(r)}_{p/2}f_{n,p}\right\|^2_{H^{\otimes p}}\to 0.
\end{aligned}\tag{4.6}$$

The result follows with Eq. (4.5) and (4.6), the reverse triangle inequality, Theorem 1.1 and Remark 3.12.

□

Remark 4.7. If $p=2$ and $\{f_{n,2}\}_n\subset H^{\odot 2}$, it is known that $I_2(f_{n,2})\overset{\text{Law}}{\to}X'$ as $n\to\infty$ implies that X' has a representation as in Eq. (1.2) with $k_1\leqslant\infty$ and $k_2=0$. In particular, if $k_1<\infty$, $k_2=a=0$ and $p=2$, Theorem 4.6 holds and is the sufficient part of Azmoodeh et al. (2015, Proposition 3.1). It is shown in Azmoodeh et al. (2015) that the conditions of Theorem 4.6 are also necessary for the case $p=2$. The case $p=2$, $k_1<\infty$, $a\neq 0$ and $k_2=0$ is not covered by Azmoodeh et al. (2015) but it can be shown that the conditions given in Theorem 4.6 are also necessary conditions for this case. This can be proved using Nourdin and Poly (2012, Theorem 3.4) and Nourdin and Peccati (2012, Eq. (2.7.17)).

We give now an example for the use of Theorem 4.6 in the case $k_2=0$ where the approximating sequence lives in a fixed Wiener chaos. We notice that it is so

far unknown whether a sequence living in a fixed chaos can converge to a target variable as in Theorem 4.6 with $k_2 \neq 0$, see Remark 4.10.

Example 4.8. Consider $p \geqslant 2$ and sequences of non-zero functions $\{f_{n,p}\}_n$, $\{g_{n,p}\}_n \subset H^{\odot p}$ and $\{h_{n,2p}\}_n \subset H^{\odot(2p)}$ such that the sequences $\{I_p(f_{n,p})\}_n$, $\{I_p(g_{n,p})\}_n$ and $\{I_{2p}(h_{n,2p})\}_n$ converge in law to a standard normal variable. We also suppose that $\langle f_{n,p}, g_{n,p}\rangle_{H^{\otimes p}} \to 0$, as $n \to \infty$. We have then, see Peccati and Tudor (2005, Theorem 1), as $n \to \infty$:

$$(I_p(f_{n,p}), I_p(g_{n,p}), I_{2p}(h_{n,2p}))^\top \overset{\text{Law}}{\to} (R_1, R_2, N)^\top,$$

where $(R_1, R_2, N)^\top$ has a 3-dimensional standard normal distribution. Using the continuous mapping theorem, Nualart and Peccati (2005, Theorem 1) and proceeding as in Nourdin and Peccati (2009a, Proposition 4.1), it is easy to see that, as $n \to \infty$:

$$I_{2p}(\varphi_{n,2p}) \overset{\text{Law}}{\to} X = N + (R_1^2 - 1) - (R_2^2 - 1),$$

where $\varphi_{n,2p} := f_{n,p}\tilde{\otimes} f_{n,p} - g_{n,p}\tilde{\otimes} g_{n,p} + h_{n,2p}$. Alternatively the latter convergence in law can be proved using Theorem 4.6. Define:

$$k_p := \frac{1}{(p/2)!\binom{p-1}{p/2-1}^2} = \frac{4}{(p/2)!\binom{p}{p/2}^2}\ ,\ c_p := 2(p/2)!\binom{p-1}{p/2-1}^2. \tag{4.7}$$

We shall need the following limits, as $n \to \infty$:

(a) $\langle f_{n,p}\tilde{\otimes} f_{n,p}, g_{n,p}\tilde{\otimes} g_{n,p}\rangle_{H^{\otimes(2p)}} \to 0$,
(b) $\langle f_{n,p}\tilde{\otimes} f_{n,p}, h_{n,2p}\rangle_{H^{\otimes(2p)}} \to 0$ and $\langle g_{n,p}\tilde{\otimes} g_{n,p}, h_{n,2p}\rangle_{H^{\otimes(2p)}} \to 0$,
(c) $\|(f_{n,p}\tilde{\otimes} f_{n,p})\tilde{\otimes}_p(g_{n,p}\tilde{\otimes} g_{n,p})\|_{H^{\otimes(2p)}} \to 0$,
(d) $\|(f_{n,p}\tilde{\otimes} f_{n,p})\tilde{\otimes}_p h_{n,2p}\|_{H^{\otimes(p)}} \to 0$ and $\|(g_{n,p}\tilde{\otimes} g_{n,p})\tilde{\otimes}_p h_{n,2p}\|_{H^{\otimes(p)}} \to 0$,
(e) $\|\varphi_{n,2p}\tilde{\otimes}_p^{(r)}\varphi_{n,2p} - k_{2p}^{r-1} f_{n,p}\tilde{\otimes} f_{n,p} - k_{2p}^{r-1} g_{n,p}\tilde{\otimes} g_{n,p}\|_{H^{\otimes(2p)}} \to 0$ for $r = 2, 3, 4$,
(f) $\|\varphi_{n,2p}\tilde{\otimes}_l\varphi_{n,2p}\|_{H^{\otimes(4p-2l)}} \to 0$ for $1 \leqslant l \leqslant 2p-1$ and $l \neq p$.

These statements (a)-(f). can be proved using Nualart and Peccati (2005, Theorem 1) and Nourdin and Peccati (2009a, Theorem 1.2) We check the conditions of Theorem 4.6.

(1) We prove that $\lim_n \kappa_4(I_{2p}(\varphi_{n,2p})) = \kappa_4(X)$. The convergence of the cumulants of order 2 and 3 can be seen similarly using (a)-(f) and Nourdin and Peccati (2009a, Eq. (3.4)). We use $\kappa_4(I_{2p}(\varphi_{n,2p})) = \mathbb{E}[I_{2p}(\varphi_{n,2p})^4] - 3\,\mathbb{E}[I_{2p}(\varphi_{n,2p})^2]^2$ and Nourdin and Peccati (2009a, Eq. (3.6)):

$$\begin{aligned}
&\lim_n \kappa_4(I_{2p}(\varphi_{n,2p})) \\
&= \lim_n \frac{3}{2p}\sum_{r=1}^{2p-1}(2p)^2(r-1)!\binom{2p-1}{r-1}^2 r!\binom{2p}{r}^2 (4p-2r)! \\
&\quad \times \|\varphi_{n,2p}\tilde{\otimes}_r\varphi_{n,2p}\|^2_{H^{\otimes(4p-2r)}} \\
&= \lim_n \frac{3}{2p}(2p)^2(p-1)!\binom{2p-1}{p-1}^2 p!\binom{2p}{p}^2 (2p)!\|\varphi_{n,2p}\tilde{\otimes}_p\varphi_{n,2p}\|^2_{H^{\otimes(2p)}} \\
&= \lim_n \frac{3}{2p}(2p)^2(p-1)!\binom{2p-1}{p-1}^2 p!\binom{2p}{p}^2 (2p)! \\
&\quad \times \|k_{2p} f_{n,p}\tilde{\otimes} f_{n,p} + k_{2p} g_{n,p}\tilde{\otimes} g_{n,p}\|^2_{H^{\otimes(2p)}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_n \frac{3}{2p}(2p)^2(p-1)! \binom{2p-1}{p-1}^2 p! \binom{2p}{p}^2 (2p)! \\
&\quad \times \left(k_{2p}^2 \|f_{n,p}\tilde{\otimes} f_{n,p}\|^2_{H^{\otimes(2p)}} + k_{2p}^2 \|g_{n,p}\tilde{\otimes} g_{n,p}\|^2_{H^{\otimes(2p)}}\right) \\
&= \lim_n \frac{3}{2p}(2p)^2(p-1)! \binom{2p-1}{p-1}^2 p! \binom{2p}{p}^2 (2p)! \\
&\quad \times \left(\|(f_{n,p}\tilde{\otimes} f_{n,p})\tilde{\otimes}_p(f_{n,p}\tilde{\otimes} f_{n,p})\|^2_{H^{\otimes(2p)}}\right. \\
&\quad \left. + \|(g_{n,p}\tilde{\otimes} g_{n,p})\tilde{\otimes}_p(g_{n,p}\tilde{\otimes} g_{n,p})\|^2_{H^{\otimes(2p)}}\right) \\
&= \kappa_4(R_1^2-1) + \kappa_4(R_2^2-1) = \kappa_4(X).
\end{aligned}
$$

The fifth equality follows from Nourdin and Peccati (2009a, Theorem 1.2.(iv)). For the last equality we have used that, for independent random variables, the cumulant of the sum equals the sum of the cumulants and $\kappa_r(N) = 0$ for every $r \geqslant 3$. We have thus proved Eq. (4.1). Notice that Eq. (4.2) holds because of (f). We have $P(x) = x^2(x-1)(x+1)$ and:

$$P'(0) = P'''(0) = 0\ ,\ P''(0)/(2!2) = -1/2\ ,\ P^{(4)}(0)/(4!2^3) = 1/8.$$

We check now Eq. (4.3):

$$
\begin{aligned}
&\left\| \sum_{r=1}^{4} \frac{P^{(r)}(0)}{r!2^{r-1}} \varphi_{n,2p}\tilde{\otimes}_p^{(r)} \varphi_{n,2p} c_{2p}^r \right\|_{H^{\otimes(2p)}} \\
&\leqslant \left\| -\frac{c_{2p}^2}{2}\left(\varphi_{n,2p}\tilde{\otimes}_p\varphi_{n,2p} - k_{2p} f_{n,p}\tilde{\otimes} f_{n,p} - k_{2p} g_{n,p}\tilde{\otimes} g_{n,p}\right)\right\|_{H^{\otimes(2p)}} \\
&\quad + \left\| \frac{c_{2p}^4}{8}\left(\varphi_{n,2p}\tilde{\otimes}_p^{(4)}\varphi_{n,2p} - k_{2p}^3 f_{n,p}\tilde{\otimes} f_{n,p} - k_{2p}^3 g_{n,p}\tilde{\otimes} g_{n,p}\right)\right\|_{H^{\otimes(2p)}} \\
&\quad + c_{2p}^2 k_{2p}\left(\left\| -\frac{1}{2} f_{n,p}\tilde{\otimes} f_{n,p} + \frac{c_{2p}^2 k_{2p}^2}{8} f_{n,p}\tilde{\otimes} f_{n,p}\right\|_{H^{\otimes(2p)}}\right. \\
&\quad \left. + \left\| -\frac{1}{2} g_{n,p}\tilde{\otimes} g_{n,p} + \frac{c_{2p}^2 k_{2p}^2}{8} g_{n,p}\tilde{\otimes} g_{n,p}\right\|_{H^{\otimes(2p)}}\right) \to 0,
\end{aligned}
$$

as $n \to \infty$. Theorem (4.6) yields now $I_{2p}(\varphi_{n,2p}) \overset{\text{Law}}{\to} X$, as $n \to \infty$.

We consider now the convergence in law to random variable with a centered χ^2 law and compare the new criterion of Theorem 4.6 with the main result of Nourdin and Peccati (2009a). We shall see that in this case, both criteria are equivalent.

Theorem 4.9. *Consider $k_1 > 0$, $a = k_2 = 0$ and $X = \sum_{i=1}^{k_1}(R_i^2 - 1)$ as defined in Eq. (3.3) and Eq. (3.5), where $R_1, \ldots, R_{k_1} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Consider a sequence $\{F_n\}_n$ of non-zero random variables with $F_n = I_p(f_{n,p})$ for $p \geqslant 2$ even fixed and $\{f_{n,p}\}_n \subset H^{\odot p}$. Define $P(x) = x(x-1)^{k_1}$ and suppose that $\kappa_2(F_n) \to 2k_1$, as $n \to \infty$. The following conditions a., b. and c. are equivalent, as $n \to \infty$:*

a. $F_n \overset{Law}{\to} X$,

b. 1) $\|f_{n,p}\tilde{\otimes}_l f_{n,p}\|_{H^{\otimes(2p-2l)}} \to 0$, *for every* $1 \leqslant l \leqslant p-1$ *with* $l \neq p/2$,,

2) $\|f_{n,p}\tilde{\otimes}_{p/2} f_{n,p} - 2/c_p f_{n,p}\|_{H^{\otimes p}} \to 0$

c. 1) $\|f_{n,p}\tilde{\otimes}_l f_{n,p}\|_{H^{\otimes(2p-2l)}} \to 0$, *for every* $1 \leqslant l \leqslant p-1$ *with* $l \neq p/2$,,

2) $\left\| \sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} f_{n,p} \tilde{\otimes}_{p/2}^{(r)} f_{n,p} c_p^r \right\|_{H^{\otimes p}} \to 0$

3) $\kappa_r(I_p(f_{n,p})) \to \kappa_r(X), \quad for\ r = 1, \dots, deg\ (P).$

Proof: The equivalence of a. and b. is proved in Nourdin and Peccati (2009a). We prove the equivalence of b. and c. for $k_1 \geqslant 2$ since b. and c. are clearly equivalent if $k_1 = 1$.

Suppose that b. holds, than a. holds as well and $\sup_n \mathbb{E}[F_n^2] < \infty$ together with the hypercontractivity property yields that $\sup_n \mathbb{E}[|F_n|^r] < \infty$ for every $r \geqslant 1$, thus $\mathbb{E}[F_n^r] \to \mathbb{E}[X^r]$ for every $r \in \mathbb{N}$. Hence c.3) holds. We have with $k = k_1$:

$$\begin{aligned}
&\sum_{l=1}^{k+1} \frac{P^{(l)}(0)}{l!2^{l-1}} c_p^l f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} = (-1)^k \sum_{l=1}^{k+1} \binom{k}{l-1} (-2)^{1-l} c_p^l f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \\
&= (-1)^k \left[\sum_{l=1}^{k} c_p^{l+1} (-2)^{-l} \sum_{r=l}^{k} \binom{k}{r} (-1)^{r-l} \left(f_{n,p} \tilde{\otimes}_{p/2}^{(l+1)} f_{n,p} - \frac{2}{c_p} f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \right) \right. \\
&\quad \left. - c_p (-2)^{-1+1} \sum_{r=1}^{k} \binom{k}{r} (-1)^{r-1} f_{n,p} + \binom{k}{0} (-2)^0 c_p f_{n,p} \right] \\
&= (-1)^k \left[\sum_{l=1}^{k} c_p^{l+1} (-2)^{-l} \sum_{r=l}^{k} \binom{k}{r} (-1)^{r-l} \left(f_{n,p} \tilde{\otimes}_{p/2}^{(l+1)} f_{n,p} - \frac{2}{c_p} f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \right) \right. \\
&\quad \left. - (-1) c_p f_{n,p} \sum_{r=0}^{k} \binom{k}{r} (-1)^r \right] \\
&= (-1)^k \left[\sum_{l=1}^{k} c_p^{l+1} (-2)^{-l} \sum_{r=l}^{k} \binom{k}{r} (-1)^{r-l} \left(f_{n,p} \tilde{\otimes}_{p/2}^{(l+1)} f_{n,p} - \frac{2}{c_p} f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \right) \right],
\end{aligned}$$

where we have used that $\sum_{r=0}^{k} \binom{k}{r} (-1)^r = (-1+1)^k = 0$. We have for $l \geqslant 2$, as $n \to \infty$:

$$\begin{aligned}
&\left\| f_{n,p} \tilde{\otimes}_{p/2}^{(l+1)} f_{n,p} - 2/c_p f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \right\|_{H^{\otimes p}} \\
&= \left\| (\dots (f_{n,p} \tilde{\otimes}_{p/2} f_{n,p} - 2/c_p f_{n,p}) \tilde{\otimes}_{p/2} f_{n,p}) \dots) \tilde{\otimes}_{p/2} f_{n,p} \right\|_{H^{\otimes p}} \\
&\leqslant \left\| f_{n,p} \tilde{\otimes}_{p/2} f_{n,p} - 2/c_p f_{n,p} \right\|_{H^{\otimes p}} \left\| f_{n,p} \right\|_{H^{\otimes p}}^{l-1} \to 0,
\end{aligned}$$

hence c.2) follows with the triangle inequality. Suppose now that c. holds. We have to prove that $\|c_p^2 f_{n,p} \tilde{\otimes}_{p/2}^{(2)} f_{n,p} - 2c_p f_{n,p}\|_{H^{\otimes p}}^2 \to 0$, as $n \to \infty$, or equivalently:

$$\begin{aligned}
&c_p^4 \|f_{n,p} \tilde{\otimes}_{p/2}^{(2)} f_{n,p}\|_{H^{\otimes p}}^2 - 4c_p^3 \langle f_{n,p} \tilde{\otimes}_{p/2}^{(2)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} + 4c_p^2 \|f_{n,p}\|_{H^{\otimes p}}^2 \\
&= c_p^4 \langle f_{n,p} \tilde{\otimes}_{p/2}^{(3)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} + 4c_p^2 \|f_{n,p}\|_{H^{\otimes 2}}^2 \\
&\quad - 4c_p^3 \langle f_{n,p} \tilde{\otimes}_{p/2} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} \to 0.
\end{aligned} \tag{4.8}$$

We have used in the last step that:

$$\langle f_{n,p} \tilde{\otimes}_{p/2}^{(k)} f_{n,p}, f_{n,p} \tilde{\otimes}_{p/2}^{(l)} f_{n,p} \rangle_{H^{\otimes p}} = \langle f_{n,p} \tilde{\otimes}_{p/2}^{(k+1)} f_{n,p}, f_{n,p} \tilde{\otimes}_{p/2}^{(l-1)} f_{n,p} \rangle_{H^{\otimes p}},$$

for all integers k, l with $k \geqslant 1$ and $l \geqslant 2$. This can be checked directly using the integral representation of the contractions.

- If $k_1 = 2$, c.2) yields that, as $n \to \infty$:
$$\left\| c_p f_{n,p} - c_p^2 f_{n,p} \tilde{\otimes}_{p/2} f_{n,p} + c_p^3/4 f_{n,p} \tilde{\otimes}_{p/2}^{(3)} f_{n,p} \right\|_{H^{\otimes p}} \to 0,$$
hence, since $\sup_n \|f_{n,p}\|_{H^{\otimes p}} < \infty$, as $n \to \infty$:
$$\langle 4c_p f_{n,p} \ , \ c_p f_{n,p} - c_p^2 f_{n,p} \tilde{\otimes}_{p/2} f_{n,p} + c_p^3/4 f_{n,p} \tilde{\otimes}_{p/2}^{(3)} f_{n,p} \rangle_{H^{\otimes p}} \to 0.$$
The linearity of the scalar product yields that relation (4.8) holds.
- If $k_1 \geqslant 3$, the calculations made in the proof of Theorem 4.6, together with Nourdin and Peccati (2012, Theorem 8.4.4) and c.3) show that:
$$0 = \lim_n \left[\kappa_i(F_n) - p!(i-1)! c_p^{i-2} \langle f_{n,p} \tilde{\otimes}_{p/2}^{(i-1)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} \right]$$
$$0 = \lim_n \left[\langle f_{n,p} \tilde{\otimes}_{p/2}^{(i-1)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} - \frac{\kappa_i(F_n)}{p!(i-1)! c_p^{i-2}} \right]$$
$$= \lim_n \left[\langle f_{n,p} \tilde{\otimes}_{p/2}^{(i-1)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} \right] - \frac{2^{i-1} k_1}{p! c_p^{i-2}},$$
for $i = 3, \ldots, k_1 + 1$. Thus:
$$\lim_n \Big[c_p^4 \langle f_{n,p} \tilde{\otimes}_{p/2}^{(3)} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} + 4c_p^2 \|f_{n,p}\|_{H^{\otimes 2}}^2$$
$$-4c_p^3 \langle f_{n,p} \tilde{\otimes}_{p/2} f_{n,p}, f_{n,p} \rangle_{H^{\otimes p}} \Big] = \frac{k_1 c_p^2 (8 + 8 - 16)}{p!} = 0.$$

This completes the proof. □

Remark 4.10. (1) The equivalence of a. and b. above is the main result of Nourdin and Peccati (2009a, Theorem 1.2). Theorem 4.9 shows that, although the polynomials defined throughout this paper may not always be of minimal degree if some of the coefficients are equal, the criterion of Theorem 4.6 is *necessary* and sufficient in some situations. It is yet unknown if the conditions of Theorem 4.6 are always *necessary* and sufficient.

(2) Notice that the problem of characterising possible target variables amongst all random variables with a representation as in Eq. (3.5) is far from being solved if the approximating sequence lives in a fixed Wiener chaos. Indeed further research on this topic is needed in order to make a comprehensive statement, but it seems doubtful that $c_1(P_1^2 - 1) + d_1 P_1$ with $c_1, d_1 \neq 0$ is a possible target for a sequence living in a chaos of fixed order. For every sequence living in a Wiener chaos of odd order, all moments of odd order vanish which reduces the class of possible target variables $X = \sum_{i=1}^k [c_i(P_i^2 - 1) + d_i P_i]$ with $P_1, \ldots, P_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ considerably. The following criterion allows to exclude such target variables by just considering the coefficients c_i and d_j. Suppose without loss of generality that the coefficients satisfy the following conditions:
- $c_1, \ldots, c_m \in \mathbb{R}^*$,
- $c_{m+1} = \ldots = c_k = 0$,
- the coefficients $c_1, \ldots, c_m$ are sorted in increasing order of their absolute values and:
$$|c_1| = |c_2| = \ldots = |c_{k_1}| < |c_{k_1+1}| = \ldots = |c_{k_2}|$$
$$< \ldots < |c_{k_{l-1}+1}| = \ldots = |c_{k_l}|,$$
where $k_0 := 0 < k_1 \leqslant k_2 \leqslant \ldots \leqslant k_l = m$.

Then $I_p(f_{n,p}) \overset{\text{Law}}{\to} X$, as $n \to \infty$, for $p \geqslant 3$ odd, implies that:

(i) m is even,

(ii) k_i is even for every $i \in \{1, \dots, l\}$ and, more precisely:

$$|\{k_{i-1}+1 \leqslant j \leqslant k_i : c_j > 0\}| = |\{k_{i-1}+1 \leqslant j \leqslant k_i : c_j < 0\}|.$$

(iii) we have for every $i \in \{1, \dots, l\}$:

$$\sum_{i=k_{i-1}+1}^{k_i} d_i^2 \cdot c_i = 0.$$

An analogue version holds for target variables in class (B) of Lemma 3.3.

5. Stable convergence

In this section we consider sequences of non-zero random variables living in a finite sum of Wiener chaoses. The sequences are supposed to converge in law to a non-zero target variable X and we ask whether the sequence also converges stably. Our first result follows from Nourdin et al. (2016b, Theorem 1.3). As before, we shall write φ_Y for the characteristic function of a random variable Y.

Theorem 5.1. *Consider $p \geqslant 3$ and the random variable $X \overset{Law}{=} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2}[c_i(P_i^2-1)+d_iP_i]$ where $N, R_1, \dots, R_{k_1}, P_1, \dots, P_{k_2} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$. Assume that $\{g_{n,p}\}_n \subset H^{\odot p}$ and, as $n \to \infty$:*

$$\|g_{n,p} \otimes_{p-1} g_{n,p}\|_{H\otimes H} \to 0. \tag{5.1}$$

If $I_p(g_{n,p}) \overset{Law}{\to} X$, as $n \to \infty$, then:

$$I_p(g_{n,p}) \overset{st.}{\to} X, \tag{5.2}$$

or, equivalently, for every $\mathcal{F}$-measurable random variable Z:

$$(I_p(g_{n,p}), Z)^\top \overset{Law}{\to} (X, Z)^\top,$$

where $\mathcal{F}$ is introduced in Section 2.1, and X is independent of the underlying Brownian motion.

Proof: Consider $f \in H$ with $\|f\|_H = 1$. We have to prove that, as $n \to \infty$:

$$(I_1(f), I_p(g_{n,p}))^\top \overset{\text{Law}}{\to} (I_1(f), X)^\top, \tag{5.3}$$

see Section 2.4, where $I_1(f)$ is independent of X. Since $I_p(g_{n,p})$ is a non-zero random variable, we have $\|g_{n,p}\|_{H\otimes p} \neq 0$ for every $n \in \mathbb{N}$. Define $g'_{n,p} := \frac{g_{n,p}}{\sqrt{p!\|g_{n,p}\|^2_{H\otimes p}}}$ for every n, then $I_p(g'_{n,p}) \overset{\text{Law}}{\to} X/\sqrt{A}$, as $n \to \infty$, where $A := a^2 + 2\sum_{i=1}^{k_1} b_i^2 + \sum_{i=1}^{k_2}(2c_i^2 + d_i^2) \neq 0$. We have $\mathbb{E}[I_p(g'_{n,p})^2] = 1$ and with Eq. (5.1), as $n \to \infty$:

$$\|g'_{n,p} \otimes_1 f\|^2_{H\otimes(p-1)} \leqslant \|g_{n,p} \otimes_{p-1} g_{n,p}\|_{H^{\otimes 2}} \|f\|^2_H \frac{1}{p!\|g_{n,p}\|^2_{H\otimes p}} \to 0.$$

Nourdin et al. (2016b, Theorem 1.3) yields with $a_n := 1/\sqrt{p!\|g_{n,p}\|^2_{H\otimes p}} \to a := 1/\sqrt{A}$, as $n \to \infty$:

$$\varphi_{I_1(f)}(t_1)\,\varphi_X(t_2/\sqrt{A}) = \varphi_{I_1(f)}(t_1)\,\varphi_{X/\sqrt{A}}(t_2)$$

$$
\begin{aligned}
&= \lim_n \mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 I_p(g'_{n,p})\right)\right] \\
&= \lim_n \mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 a_n I_p(g_{n,p})\right)\right] \\
&= \lim_n \left(\mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 a I_p(g_{n,p})\right)\right]\right. \\
&\quad \left. + \mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 a I_p(g_{n,p})\right)\left(\exp(it_2(a_n - a) I_p(g_{n,p})) - 1\right)\right]\right).
\end{aligned}
$$

We have with $|e^{ix} - 1| \leqslant |x|$ for every $x \in \mathbb{R}$:

$$
\begin{aligned}
&\left|\mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 a I_p(g_{n,p})\right)\left(\exp(it_2(a_n - a) I_p(g_{n,p})) - 1\right)\right]\right| \\
&\leqslant \mathbb{E}\left[\left|\exp(it_2(a_n - a) I_p(g_{n,p})) - 1\right|\right] \leqslant \mathbb{E}\left[\left|t_2(a_n - a) I_p(g_{n,p})\right|\right] \\
&\leqslant |t_2(a_n - a)|\, \mathbb{E}[I_p(g_{n,p})^2]^{1/2} \leqslant |t_2(a_n - a)| \sqrt{p! \sup_n \|g_{n,p}\|^2_{H^{\otimes p}}} \to 0,
\end{aligned}
$$

as $n \to \infty$. Hence:

$$
\varphi_{I_1(f)}(t_1)\, \varphi_X(t_2/\sqrt{A}) = \lim_n \mathbb{E}\left[\exp\left(it_1 I_1(f) + it_2 a I_p(g_{n,p})\right)\right],
$$

and with $t'_2 := t_2/\sqrt{A}$:

$$
\varphi_{I_1(f)}(t_1)\, \varphi_X(t'_2) = \lim_n \mathbb{E}\left[\exp\left(it_1 I_1(f) + it'_2 I_p(g_{n,p})\right)\right].
$$

Since $(t_1, t'_2) \mapsto \varphi_{I_1(f)}(t_1)\varphi_X(t'_2)$ is the characteristic function of $(I_1(f), X)^\top$ where $I_1(f)$ is independent of X, we have that Eq. (5.3), or equivalently Eq. (5.2) holds. Since $f \in H$ with $\|f\|_H = 1$ is arbitrary, the statement follows with the remarks of Section 2.4. □

Remark 5.2. The proof of the previous theorem was straightforward since the converging sequence of random variables lives in a fixed Wiener chaos. Under these assumptions, Nourdin et al. (2016b, Theorem 1.3) yields the desired stable convergence. If the sequence of random variables is allowed to live in a finite sum of Wiener chaoses or if assumption (5.1) does not hold, the conditions ensuring stable convergence involve Γ-operators.

For the next results, the target variables are (again) supposed to have the form:

$$
X \overset{\text{Law}}{=} aN + \sum_{i=1}^{k_1} b_i(R_i^2 - 1) + \sum_{i=1}^{k_2} \left[c_i(P_i^2 - 1) + d_i P_i\right], \tag{5.4}
$$

where $N, R_1, \ldots, R_{k_1}, P_1, \ldots, P_{k_2}$ are independent standard normal variables and $b_i \neq 0$ for $1 \leqslant i \leqslant k_1$, as well as $c_j d_j \neq 0$ for $1 \leqslant j \leqslant k_2$. We suppose that at least one of the parameters k_1, k_2 is positive. We shall use the convention $0^0 := 1$.

Theorem 5.3. *Consider $p \geqslant 3$, $k_1, k_2 \geqslant 0$ and X as defined in Eq. (5.4). Suppose that $\{g_{n,l}\}_n \subset H^{\odot l}$ for $1 \leqslant l \leqslant p$ and:*

$$
F_n = \sum_{l=1}^{p} I_l(g_{n,l}),
$$

Define $P(x) = x^{1+1_{[a\neq 0]}} \prod_{i=1}^{k_1}(x - b_i) \prod_{i=j}^{k_2}(x - c_j)^2$ and suppose that, as $n \to \infty$:

$$
\kappa_r(F_n) \to k(r) := 1_{[r=2]} a^2 + \sum_{i=1}^{k_1} 2^{r-1}(r-1)! b_i^r
$$

$$+\sum_{j=1}^{k_2} 2^{r-1}(r-1)!\left(c_j^r+\frac{rc_j^{r-2}d_j^2}{4}\right), \qquad \textit{for } 2\leqslant r\leqslant deg(P), \tag{5.5}$$

$$\sum_{r=1}^{deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n)-\mathbb{E}[\Gamma_{r-1}(F_n)]\right)\overset{L^1}{\to}0. \tag{5.6}$$

If the following two conditions hold, as $n\to\infty$, for every $f\in H$ with $\|f\|_H=1$ and $k:=2k_2+k_1$:

$$\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle DI_1(f),DF_n\rangle_H\right]\to 0, \qquad \textit{for every } (t_1,t_2)\in\mathbb{R}^2, \tag{5.7}$$

$$it_1\,\mathbb{E}\Big[\exp\left(i(it_1I_1(f)+t_2F_n)\right)\langle DI_1(f),\sum_{r=1}^{deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\sum_{\substack{\alpha+\beta=r-1\\ \alpha\geqslant 1,\beta\geqslant 0}}(it_2)^{k+1_{[\alpha\neq 0]}-\alpha}$$
$$\times\left(-DL^{-1}\Gamma_\beta(F_n)\right)\rangle_H\Big]\to 0, \qquad \textit{for every } (t_1,t_2)\in\mathbb{R}^2, \tag{5.8}$$

then $F_n\overset{st.}{\to}X$, as $n\to\infty$, and X is independent of the underlying Brownian motion.

Proof: Theorem 3.12, Eq. (5.5) and Eq. (5.6) imply that $F_n\overset{\text{Law}}{\to}X$, as $n\to\infty$. Define:

$$\begin{aligned}B_0&:=\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle DI_1(f),DF_n\rangle_H\right],\\ A_m&:=\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle DI_1(f),-DL^{-1}\Gamma_m(F_n)\rangle_H\right]1_{[m\geqslant 0]},\\ \varphi_n(t_1,t_2)&:=\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\right].\end{aligned}$$

The obvious dependence of (t_1,t_2) is dropped in the first equalities. The proof is divided in three steps:

- we derive two equations involving derivatives of φ_n,
- we prove that the sequence $\left\{(I_1(f),F_n)^\top\right\}_n$ is tight and that we have $(I_1(f),F_n)^\top\overset{\text{Law}}{\to}(I_1(f),X)^\top$, as $n\to\infty$, where X is independent of $I_1(f)$,
- we conclude that $F_n\overset{\text{st.}}{\to}X$, as $n\to\infty$, and X is independent of the underlying Brownian motion.

(1) Consider $(t_1,t_2)\in\mathbb{R}\times\mathbb{R}^*$. Then for $r\geqslant 2$:

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\Gamma_{r-1}(F_n)\right]\\ &=\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle DF_n,-DL^{-1}\Gamma_{r-2}(F_n)\rangle_H\right]\\ &=\frac{1}{it_2}\,\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle it_2DF_n+it_1DI_1(f),-DL^{-1}\Gamma_{r-2}(F_n)\rangle_H\right]\\ &\quad-\frac{it_1}{it_2}\,\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\langle DI_1(f),-DL^{-1}\Gamma_{r-2}(F_n)\rangle_H\right]\\ &=\frac{1}{it_2}\,\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\Gamma_{r-2}(F_n)\right]-\frac{1}{it_2}\varphi_n(t_1,t_2)\,\mathbb{E}[\Gamma_{r-2}(F_n)]\\ &\quad-\frac{it_1}{it_2}A_{r-2}.\end{aligned}$$

We have used the integration by parts rule and $-\delta DL^{-1}F=F-\mathbb{E}[F]$ for every $F\in L^2(\Omega)$. Repeating this first calculations once again if $r-2\geqslant 1$, we find:

$$\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_n)\right)\Gamma_{r-1}(F_n)\right]$$

$$= \frac{1}{it_2}\left(\frac{1}{it_2}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\Gamma_{r-3}(F_n)\right] - \frac{1}{it_2}\varphi_n(t_1,t_2)\,\mathbb{E}\left[\Gamma_{r-3}(F_n)\right]\right.$$
$$\left. - \frac{it_1}{it_2}A_{r-3}\right) - \frac{1}{it_2}\varphi_n(t_1,t_2)\,\mathbb{E}[\Gamma_{r-2}(F_n)] - \frac{it_1}{it_2}A_{r-2},$$

and iteration yields finally for $r \geqslant 1$:

$$\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\Gamma_{r-1}(F_n)\right]$$
$$= \frac{1}{(it_2)^{r-1}}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\Gamma_0(F_n)\right]$$
$$- \varphi_n(t_1,t_2)\sum_{\substack{\alpha+\beta=r-1\\ \alpha,\beta\geqslant 1}}\frac{1}{(it_2)^{\alpha}}\,\mathbb{E}[\Gamma_\beta(F_n)] - it_1\sum_{\substack{\alpha+\beta=r-1\\ \alpha\geqslant 1,\beta\geqslant 0}}\frac{1}{(it_2)^{\alpha}}A_\beta.$$

With $\frac{\partial\varphi_n}{\partial t_2}(t_1,t_2) = i\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)F_n\right]$, we find:

$$\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\Gamma_{r-1}(F_n)\right]$$
$$= \frac{-i}{(it_2)^{r-1}}\frac{\partial\varphi_n}{\partial t_2}(t_1,t_2) - \varphi_n(t_1,t_2)\sum_{\substack{\alpha+\beta=r-1\\ \alpha,\beta\geqslant 1}}\frac{1}{(it_2)^{\alpha}}\,\mathbb{E}\left[\Gamma_\beta(F_n)\right]$$
$$- it_1\sum_{\substack{\alpha+\beta=r-1\\ \alpha\geqslant 1,\beta\geqslant 0}}\frac{1}{(it_2)^{\alpha}}A_\beta.$$

Hence, for all $(t_1,t_2)\in\mathbb{R}\times\mathbb{R}^*$:

$$\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n)-\mathbb{E}\left[\Gamma_{r-1}(F_n)\right]\right)\right]$$
$$= -\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\frac{i}{(it_2)^{r-1}}\frac{\partial\varphi_n}{\partial t_2}(t_1,t_2)$$
$$- \varphi_n(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\sum_{\substack{\alpha+\beta=r-1\\ \alpha,\beta\geqslant 1}}\frac{1}{(it_2)^{\alpha}}\frac{\kappa_{\beta+1}(F_n)}{\beta!}$$
$$- \varphi_n(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\frac{\kappa_r(F_n)}{(r-1)!} - it_1\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\sum_{\substack{\alpha+\beta=r-1\\ \alpha\geqslant 1,\beta\geqslant 0}}\frac{1}{(it_2)^{\alpha}}A_\beta. \quad (5.9)$$

With $\Gamma_1(I_1(f))-1=0$, we find for $(t_1,t_2)\in\mathbb{R}^*\times\mathbb{R}$:

$$0 = \mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\,(\Gamma_1(I_1(f))-1)\right]$$
$$= \mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\langle DI_1(f), -DL^{-1}I_1(f)\rangle_H\right] - \varphi_n(t_1,t_2)$$
$$= \frac{1}{it_1}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\langle it_1 DI_1(f)+it_2DF_n, -DL^{-1}I_1(f)\rangle_H\right]$$
$$- \frac{it_2}{it_1}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\langle DF_n, -DL^{-1}I_1(f)\rangle_H\right] - \varphi_n(t_1,t_2)$$
$$= \frac{1}{i^2t_1}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)iI_1(f)\right] - \frac{it_2}{it_1}\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f)+t_2F_n)\right)\right.$$
$$\left.\times\langle DF_n, DI_1(f)\rangle_H\right] - \varphi_n(t_1,t_2)$$

$$= -\frac{1}{t_1}\left(\frac{\partial\varphi_n}{\partial t_1}(t_1,t_2) + t_1\varphi_n(t_1,t_2)\right) - \frac{it_2}{it_1}B_0. \tag{5.10}$$

(2) We prove that the sequence $\left\{(I_1(f),F_n)^\top\right\}_n$ is tight. We have with the Markov inequality, for $K\to\infty$:

$$\begin{aligned}\mathbb{P}\left(I_1(f)^2+F_n^2\geqslant K\right) &\leqslant K^{-1}\,\mathbb{E}[I_1(f)^2+F_n^2] = K^{-1}\left(1+\mathbb{E}[F_n^2]\right)\\ &\leqslant K^{-1}\left(1+\sup_n \mathbb{E}[F_n^2]\right)\to 0.\end{aligned}$$

Consider a subsequence $\left\{(I_1(f),F_{n_l})^\top\right\}_l$ which converges in law to a random vector $V=(V_1,V_2)^\top$. If we have that V is the same for every converging subsequence and $V\overset{\text{Law}}{=}(I_1(f),X)^\top$, where $I_1(f)$ and X are independent, then we have that $\left\{(I_1(f),F_n)^\top\right\}_n$ converges in law to the same limit. We have that $\varphi_{n_l}(t_1,t_2)=\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_{n_l})\right)\right]\to\varphi_{(V_1,V_2)^\top}(t_1,t_2)=:\varphi^*(t_1,t_2)$, as $l\to\infty$. Since we have Eq. (5.6) and Eq. (5.8), we have for $(t_1,t_2)\in\mathbb{R}\times\mathbb{R}^*$ and $l\to\infty$ with Eq. (5.9):

$$\begin{aligned}&-\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\frac{i}{(it_2)^{r-1}}\frac{\partial\varphi^*}{\partial t_2}(t_1,t_2)\\ &-\varphi^*(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\sum_{\substack{\alpha+\beta=r-1\\ \alpha,\beta\geqslant 1}}\frac{1}{(it_2)^\alpha}\frac{k(\beta+1)}{\beta!}\\ &-\varphi^*(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\frac{k(r)}{(r-1)!}=0.\end{aligned}\tag{5.11}$$

Notice that, as $l\to\infty$, we have $\frac{\partial\varphi_{n_l}}{\partial t_2}\to\frac{\partial\varphi^*}{\partial t_2}$ by the continuous mapping theorem and $\kappa_{\beta+1}(F_n)/\beta!\to k(\beta+1)/\beta!$, as $n\to\infty$, since Eq. (5.5) holds. We have:

$$\frac{\partial\varphi^*}{\partial t_2}(t_1,t_2)=\lim_l\frac{\partial\varphi_{n_l}}{\partial t_2}(t_1,t_2)=\lim_l\mathbb{E}\left[\exp\left(i(t_1I_1(f)+t_2F_{n_l})\right)iF_{n_l}\right],$$

hence:

$$\begin{aligned}\frac{\partial\varphi^*}{\partial t_2}(t_1,0)&=\lim_l\mathbb{E}\left[\exp\left(it_1I_1(f)\right)iF_{n_l}\right]\\ &=-\lim_l\mathbb{E}\left[\exp\left(it_1I_1(f)\right)\langle t_1DI_1(f),-DL^{-1}F_{n_l}\rangle_p\right]=0.\end{aligned}$$

To see this, apply Eq. (5.8) with $t_2=0$. We find similarly $\frac{\partial\varphi^*}{\partial t_1}(0,t_2)=0$. Multiplying Eq. (5.11) by $(it_2)^{k+1_{[a\neq 0]}}$, we find the following equation for $(t_1,t_2)\in\mathbb{R}^2$:

$$\begin{aligned}&-\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}i(it_2)^{k+1_{[a\neq 0]}-r+1}\frac{\partial\varphi^*}{\partial t_2}(t_1,t_2)\\ &-\varphi^*(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\sum_{\substack{\alpha+\beta=r-1\\ \alpha,\beta\geqslant 1}}(it_2)^{k+1_{[a\neq 0]}-\alpha}\frac{k(\beta+1)}{\beta!}\\ &-(it_2)^{k+1_{[a\neq 0]}}\varphi^*(t_1,t_2)\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!2^{r-1}}\frac{k(r)}{(r-1)!}=0,\end{aligned}\tag{5.12}$$

with $\varphi^*(0,0) = 1$ and $\frac{\partial \varphi^*}{\partial t_1}(0,t_2) = \frac{\partial \varphi^*}{\partial t_2}(t_1,0) = 0$. From Eq. (5.10), we find with n replaced by n_l and $l \to \infty$, for $(t_1,t_2) \in \mathbb{R}^* \times \mathbb{R}$:

$$\frac{\partial \varphi^*}{\partial t_1}(t_1,t_2) + t_1 \varphi^*(t_1,t_2) = 0. \tag{5.13}$$

We have used that condition (5.7) implies that B_0 converges to 0 in Eq. (5.10). Since $\frac{\partial \varphi^*}{\partial t_1}(0,t_2) = 0$, the differential equation (5.13) holds for every $(t_1,t_2) \in \mathbb{R}^2$. We have thus a system of partial differential equations, given by Eq. (5.12) and Eq. (5.13), with the conditions $\varphi^*(0,0) = 1$ and $\frac{\partial \varphi^*}{\partial t_1}(0,t_2) = \frac{\partial \varphi^*}{\partial t_2}(t_1,0) = 0$. We notice that the calculations in the proof of Theorem 3.8, in particular Step 1, Eq. (3.12) and Eq. (3.13) show that the function $(t_1,t_2) \mapsto \varphi_X(t_2)$ satisfies Eq. (5.12) and we have $\varphi_X(0) = 1$ as well as $\varphi'_X(0) = 0$. For a standard normal random variable M, the function $(t_1,t_2) \mapsto \varphi_M(t_1)$ satisfies Eq. (5.13) and we have $\varphi_M(0) = 1$ as well as $\varphi'_M(0) = 0$. A solution of the system is given by the function $(t_1,t_2) \mapsto \varphi_M(t_1)\varphi_X(t_2)$. Suppose that $\tilde{\varphi}$ is another solution of the system. Define $\Psi(t_1,t_2) := \tilde{\varphi}(t_1,t_2)/(\varphi_M(t_1)\varphi_X(t_2))$. Notice that $\varphi_M(t_1)\varphi_X(t_2) \neq 0$ for $t_1,t_2 \in \mathbb{R}$. For $t_2 \neq 0$, Eq. (5.12) yields an expression of the form $\frac{\partial \tilde{\varphi}}{\partial t_2}(t_1,t_2) = \tilde{\varphi}(t_1,t_2)\, Q(t_2)$, where Q is a function which depends only on t_2. Since Eq. (3.12) and Eq. (3.13) hold, we have with the same function Q the representation $\frac{\partial \varphi_X}{\partial t_2}(t_2) = \varphi_X(t_2)\, Q(t_2)$. Hence:

$$\frac{\partial \Psi}{\partial t_2}(t_1,t_2) = \frac{\tilde{\varphi}(t_1,t_2)Q(t_2)\varphi_M(t_1)\varphi_X(t_2) - \tilde{\varphi}(t_1,t_2)\varphi_X(t_2)Q(t_2)\varphi_M(t_1)}{[\varphi_M(t_1)\varphi_X(t_2)]^2} = 0.$$

We find $\frac{\partial \Psi}{\partial t_2}(t_1,0) = 0$ since $\frac{\partial \tilde{\varphi}}{\partial t_2}(t_1,0) = \varphi'_X(0) = 0$. A similar calculation shows that $\frac{\partial \Psi}{\partial t_1}(t_1,t_2) = 0$, hence $\nabla \Psi = 0$ on $\mathbb{R}^2$. We conclude that Ψ is constant and since $\Psi(0,0) = 1$, we have that:

$$\tilde{\varphi}(t_1,t_2) = \varphi_N(t_1)\varphi_X(t_2).$$

(3) We have finally that every subsequence which converges in law, has the same limit $(I_1(f), X)^\top$ with X independent of $I_1(f)$. Since the $\mathbb{P}$-completion of the σ-field generated by $\{I_1(f) : f \in H, \|f\|_H = 1\}$ is the σ-field $\mathcal{F}$ of the Brownian motion, an application of Nourdin et al. (2016a, Lemma 2.3.) concludes the proof, see Section 2.4.

□

The following Corollary 5.4 is a special case of the implication (2) ⇒ (3) in Nourdin et al. (2016b, Theorem 1.3).

Corollary 5.4. *Consider $p \geqslant 3$, $k_1, k_2 \geqslant 0$ and X as defined in Eq. (5.4). Suppose that $\{g_{n,p}\}_n \subset H^{\odot p}$, define $P(x) = x^{1+1_{[a\neq 0]}} \prod_{i=1}^{k_1}(x-b_i) \prod_{j=1}^{k_2}(x-c_j)^2$ and suppose that, as $n \to \infty$:*

$$\kappa_r(F_n) \to k(r) := 1_{[r=2]}a^2 + \sum_{i=1}^{k_1} 2^{r-1}(r-1)!b_i^r + \sum_{j=1}^{k_2} 2^{r-1}(r-1)!\left(c_j^r + \frac{rc_j^{r-2}d_j^2}{4}\right), \quad \text{for } 2 \leqslant r \leqslant deg(P),$$

$$\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - \mathbb{E}[\Gamma_{r-1}(F_n)]\right) \xrightarrow{L^1} 0.$$

If the following condition holds, as $n \to \infty$, for every $f \in H$ with $\|f\|_H = 1$:

$$\|g_{n,p} \otimes_1 f\|_{H^{\otimes(p-1)}} \to 0, \tag{5.14}$$

then $I_p(g_{n,p}) \xrightarrow{st.} X$, as $n \to \infty$, and X is independent of the underlying Brownian motion.

Proof: We prove that Eq. (5.7) and Eq. (5.8) hold, Theorem 5.3 yields then the statement.

(1) We have, as $n \to \infty$:

$$\begin{aligned}\mathbb{E}\left[|\langle DI_1(f), DI_p(g_{n,p})\rangle_H|\right]^2 &\leqslant \mathbb{E}\left[|pI_{p-1}(g_{n,p} \otimes_1 f)|^2\right] \\ &= p^2(p-1)!\|g_{n,p} \otimes_1 f\|^2_{H^{\otimes(p-1)}} \to 0.\end{aligned}$$

We have used the stochastic Fubini theorem for multiple Wiener integrals, and Eq. (5.7) holds since $|\exp\left(i(t_1 I_1(f) + t_2 I_p(g_{n,p}))\right)| = 1$.

(2) To prove that Eq. (5.8) holds, we notice that $\Gamma_b(I_p(g_{n,p}))$ has a representation as a finite linear combination of random variables of the following form with $R := \sum_{i=1}^b r_i$:

$$I_{(b+1)p-2R}\left(\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\tilde{\otimes}_{r_2} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p}\right),$$

see Azmoodeh et al. (2015, Proposition 2.1) for details and the set containing $(r_1, \ldots, r_b)$ in particular. We can represent $\langle DI_1(f), -DL^{-1}\Gamma_b(I_p(g_{n,p}))\rangle_H$ as finite linear combination of terms having the following form:

$$\begin{aligned}&\langle f(\cdot), I_{(b+1)p-2R-1}\left(\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p})(\,,\cdot)\right)\rangle_H \\ &= I_{(b+1)p-2R-1}\left(\int_0^T (\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p})(\,,t)f(t)dt\right).\end{aligned} \tag{5.15}$$

- Let $A := (b+1)p - 2(r_1 + \ldots + r_b) > 1$, then we can represent

$$(\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p}(x_1, \ldots, x_A)$$

as finite linear combination of integrals of the following form:

$$\int_{[0,T]^R} g_{n,p}(Y_{1,l}, X_{1,l}) \ldots g_{n,p}(Y_{b+1,l}, X_{b+1,l}) dY_{1,l} \ldots dY_{b+1,l}, \tag{5.16}$$

where l is a summation index, $Y_{i,l}$ and $X_{j,l}$ represent collections of variables such that:

  - $\cup_{i=1}^{b+1} Y_{i,l}$ contains R elements and every element is in exactly two of the (non-empty) sets $Y_{1,l}, \ldots, Y_{b+1,l}$,
  - $\left(\cup_{i=1}^{b+1} Y_{i,l}\right) \cap \{x_1, \ldots, x_A\} = \varnothing$,
  - $\dot{\cup}_{i=1}^{b+1} X_{i,l} = \{x_1, \ldots, x_A\}$.

We have then with $x_m \in \{x_1, \ldots, x_A\}$ and Nourdin and Rosiński (2014, Lemma 2.3), as $n \to \infty$:

$$\int_{[0,T]^{A-1}} \left(\int_{[0,T]^{R+1}} \prod_{i=1}^{b+1} g_{n,p}(Y_{i,l}, X_{i,l}) f(x_m) dY_{1,l} \ldots dY_{b+1,l} dx_m\right)^2$$

$$\times \frac{dx_1 \dots dx_A}{dx_m} \leqslant \|g_{n,p} \otimes_1 f\|^2_{H^{\otimes(p-1)}} \|g_{n,p}\|^{2b}_{H^{\otimes p}} \to 0.$$

- Let $A := (b+1)p - 2(r_1 + \ldots + r_b) = 1$, then we can represent
$$(\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\tilde{\otimes}_{r_2} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p}(x_0)$$
as linear combination of integrals as in Eq. (5.16) and all but one set $X_{i,l}$ is empty. Assume without loss of generality that $X_{1,l}$ is the non-empty set, containing the integration variable, say x_0. Hence, with Nourdin and Rosiński (2014, Lemma 2.3), as $n \to \infty$:

$$\left| \int_{[0,T]^{R+1}} f(x_0) g_{n,p}(Y_{1,l}, x_0) g_{n,p}(Y_{2,l}) \ldots g_{n,p}(Y_{b+1,l}) dY_{1,l} \ldots dY_{b+1,l} dx_0 \right|^2$$
$$= \left| \int_{[0,T]^R} (g_{n,p} \otimes_1 f)(Y_{1,l}) \prod_{i=2}^{b+1} g_{n,p}(Y_{i,l}) dY_{1,l} \ldots dY_{b+1,l} \right|^2$$
$$\leqslant \|g_{n,p} \otimes_1 f\|^2_{H^{\otimes(p-1)}} \|g_{n,p}\|^{2b}_{H^{\otimes p}} \to 0.$$

Combining these results, we find with $(y_1 + \ldots + y_q)^2 \leqslant q\left(y_1^2 + \ldots + y_q^2\right)$ that:

$$\mathbb{E}\left[I_{(b+1)p-2R-1}\left(\int_0^T (\ldots(g_{n,p}\tilde{\otimes}_{r_1} g_{n,p})\tilde{\otimes}_{r_2} g_{n,p})\ldots)\tilde{\otimes}_{r_b} g_{n,p}(\,,t) f(t) dt \right)^2 \right]$$

converges to 0, as $n \to \infty$, hence:

$$\mathbb{E}\left[\langle DI_1(f), -DL^{-1}\Gamma_b(I_p(g_{n,p}))\rangle_H^2\right] \to 0,$$

since $\langle DI_1(f), -DL^{-1}\Gamma_b(I_p(g_{n,p}))\rangle_H$ can be represented as linear combination of integrals as in Eq. (5.15). Eq. (5.8) follows since $|\exp(i(t_1 I_1(f) + t_2 I_p(g_{n,p})))| = 1$.

□

Remark 5.5. Eq. (5.14) holds if $\|g_{n,p}\tilde{\otimes}_{p-1} g_{n,p}\|_{H\otimes H} \to 0$, as $n \to \infty$. This follows directly from Lemma 4.5.

We have the following converse of Theorem 5.3.

Theorem 5.6. *With the notations of Theorem 5.3, consider $p \geqslant 3$, $k_1, k_2 \geqslant 0$ and X as defined in Eq. (5.4). Suppose that $\{g_{n,l}\}_n \subset H^{\odot l}$ for $1 \leqslant l \leqslant p$, and:*

$$F_n = \sum_{l=1}^{p} I_l(g_{n,l}),$$

If condition (5.6) holds and $F_n \xrightarrow{st.} X$, as $n \to \infty$, where X is independent of the underlying Brownian motion, then for every $f \in H$ with $\|f\|_H = 1$:

(1) Eq. (5.7) and Eq. (5.8) hold,

(2) $\lim_n \left(\sum_{l=2}^{p-1} (l+1)! \langle g_{n,l-1}\tilde{\otimes}_{l-1} g_{n,l+1}, f\tilde{\otimes} f\rangle_{H\otimes H} + \mathbb{E}\left[\langle DI_1(f), DF_n\rangle_H^2\right] \right) = 0.$

Proof: (1) Consider an arbitrary element $f \in H$ with $\|f\|_H = 1$. The stable convergence and the independence property imply that $(I_1(f), F_n)^\top \xrightarrow{\text{Law}} (I_1(f), X)^\top$, as $n \to \infty$, where X is independent of $I_1(f)$. Hence:

$$\lim_n \varphi_n(t_1, t_2) = \lim_n \mathbb{E}[\exp(i(t_1 I_1(f) + t_2 F_n))] = \varphi_{I_1(f)}(t_1)\varphi_X(t_2),$$

and Eq. (5.9) together with Eq. (3.11) implies that Eq. (5.8) holds for every $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}^*$. With Eq. (5.10), we have that Eq (5.7) holds for every $t_1, t_2 \neq 0$. To see that Eq. (5.7) holds for $t_1 = 0, t_2 \neq 0$, calculate the derivative with respect to t_1:

$$\frac{\partial \varphi_n}{\partial t_1}(t_1, t_2) = i\,\mathbb{E}\left[\exp\left(i(t_1 I_1(f) + t_2 F_n)\right) I_1(f)\right]$$

hence:

$$\begin{aligned}\frac{\partial \varphi_n}{\partial t_1}(0, t_2) &= i\,\mathbb{E}\left[\exp(it_2 F_n) I_1(f)\right] = i\,\mathbb{E}\left[\exp(it_2 F_n)(-\delta D L^{-1} I_1(f))\right] \\ &= i\,\mathbb{E}\left[\exp\left(it_2 F_n\right)\langle it_2 DF_n, DI_1(f)\rangle_H\right].\end{aligned}$$

By the continuous mapping theorem, we have:

$$\left(\exp(it_2 F_n), I_1(f)\right)^\top \overset{\text{Law}}{\to} \left(\exp(it_2 X), I_1(f)\right)^\top,$$

as $n \to \infty$, and X is independent of $I_1(f)$. Since $\sup_n \mathbb{E}\left[\left|\exp(it_2 F_n) I_1(f)\right|^2\right] = \sup_n \mathbb{E}[1\, I_1(f)^2] = 1 < \infty$, we have:

$$\lim_n \mathbb{E}\left[\exp(it_2 F_n) I_1(f)\right] = \varphi_X(t_2)\,\mathbb{E}[I_1(f)] = 0,$$

thus, as $n \to \infty$:

$$\begin{aligned}0 = i\varphi_X(t_2)\,\mathbb{E}[I_1(f)] &= \lim_n \frac{\partial \varphi_n}{\partial t_1}(0, t_2) \\ &= -\lim_n t_2\,\mathbb{E}\left[\exp\left(it_2 F_n\right)\langle DF_n, DI_1(f)\rangle_H\right].\end{aligned}$$

We prove that Eq. (5.7) holds for $t_2 = 0, t_1 \neq 0$:

$$\begin{aligned}&\mathbb{E}[\exp(it_1 I_1(f))\langle DI_1(f), DF_n\rangle_H] \\ &= (it_1)^{-1}\,\mathbb{E}[\langle it_1 f \exp(it_1 I_1(f)), DF_n\rangle_H] \\ &= (it_1)^{-1}\,\mathbb{E}[\delta(it_1 f \exp(it_1 I_1(f)))\, F_n] \\ &= (it_1)^{-1}\,\mathbb{E}[(it_1 I_1(f) + t_1^2)\,\exp(it_1 I_1(f))\, F_n] \to 0,\end{aligned}$$

as $n \to \infty$.We have used Eq. (2.4) in the last step, the convergence follows as above and Eq. (5.7) holds for $t_2 = 0, t_1 \neq 0$. In the case $t_1 = t_2 = 0$, we find with the independence property and Eq. (2.3) that Eq. (5.7) holds. We see similarly that:

$$0 = \lim_n \frac{\partial \varphi_n}{\partial t_2}(t_1, 0) = -\lim_n t_1\,\mathbb{E}\left[\exp\left(it_1 I_1(f)\right)\langle DI_1(f), -DL^{-1} F_n\rangle_H\right]. \tag{5.17}$$

On the other hand, Eq. (5.8) yields for $t_2 = 0$ the following condition:

$$\frac{it_1 P^{k+1+1_{[a\neq 0]}}(0)}{(k+1+1_{[a\neq 0]})!\,2^{k+1_{[a\neq 0]}}}\,\mathbb{E}\left[\exp\left(it_1 I_1(f)\right)\langle DI_1(f), -DL^{-1} F_n\rangle_H\right] \to 0, \tag{5.18}$$

as $n \to \infty$. Since $P^{k+1+1_{[\alpha_0\neq 0]}}(0) \neq 0$, the convergence in Eq. (5.18) follows clearly from Eq. (5.17). We conclude that Eq. (5.7) and (5.8) hold for every $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$.

(2) Consider an arbitrary element $f \in H$ with $\|f\|_H^2 = 1$. Since $(I_1(f), F_n)^\top \overset{\text{Law}}{\to} (I_1(f), X)^\top$, as $n \to \infty$, where $I_1(f) \sim \mathcal{N}(0,1)$ is independent of X, we have

that $(I_1(f)^2, F_n^2)^\top \overset{\text{Law}}{\to} (I_1(f)^2, X^2)$, as $n \to \infty$, by the continuous mapping theorem, hence $I_1(f)^2 F_n^2 \overset{\text{Law}}{\to} I_1(f)^2 X^2$, as $n \to \infty$. We have that:

$$\sup_n \mathbb{E}\left[\left(F_n^2 I_1(f)^2\right)^2\right] < \infty. \tag{5.19}$$

this follows from the hypercontractivity property and the Cauchy-Schwarz inequality. Hence:

$$0 = \lim_n \left(\mathbb{E}[F_n^2 I_1(f)^2] - \mathbb{E}[F_n^2]\,\mathbb{E}[I_1(f)^2]\right), \tag{5.20}$$

since $I_1(f)$ is independent of X. With Eq. (5.20):

$$\begin{aligned}
0 &= \lim_n \left(\mathbb{E}[F_n^2 I_1(f)^2] - \mathbb{E}[F_n^2]\,\mathbb{E}[I_1(f)^2]\right)\\
&= \lim_n \left(\mathbb{E}\left[F_n^2 I_1(f)(-\delta D L^{-1} I_1(f))\right] - \mathbb{E}[F_n^2]\right)\\
&= \lim_n \left(\mathbb{E}\left[\langle D(F_n^2 I_1(f)), -DL^{-1} I_1(f)\rangle_H\right] - \mathbb{E}[F_n^2]\right)\\
&= \lim_n \left(2\,\mathbb{E}\left[F_n I_1(f)\langle DF_n, DI_1(f)\rangle_H\right] + \mathbb{E}\left[F_n^2 \|DI_1(f)\|_H^2\right] - \mathbb{E}\left[F_n^2\right]\right)\\
&= 2\lim_n \mathbb{E}\left[F_n I_1(f)\langle DF_n, DI_1(f)\rangle_H\right],
\end{aligned} \tag{5.21}$$

and with the stochastic Fubini theorem for multiple Wiener integrals:

$$\begin{aligned}
\langle DI_1(f), DF_n\rangle_H &= \sum_{l=1}^p l I_{l-1}(g_{n,l}\tilde{\otimes}_1 f),\\
F_n I_1(f) &= \sum_{l=1}^p I_{l+1}(g_{n,l}\tilde{\otimes} f) + \sum_{l=1}^p l I_{l-1}(g_{n,l}\tilde{\otimes}_1 f)\\
&= \sum_{l=1}^p I_{l+1}(g_{n,l}\tilde{\otimes} f) + \langle DI_1(f), DF_n\rangle_H.
\end{aligned}$$

Hence with Eq. (5.21):

$$\begin{aligned}
0 &= \lim_n \left(\mathbb{E}\left[\sum_{l=1}^p I_{l+1}(g_{n,l}\tilde{\otimes} f)\ \sum_{l=1}^p l I_{l-1}(g_{n,l}\tilde{\otimes}_1 f)\right] + \mathbb{E}[\langle DI_1(f), DF_n\rangle_H^2]\right)\\
&= \lim_n \left(\mathbb{E}\left[\sum_{l=2}^{p+1} I_l(g_{n,l-1}\tilde{\otimes} f)\ \sum_{l=0}^{p-1}(l+1) I_l(g_{n,l+1}\tilde{\otimes}_1 f)\right]\right.\\
&\quad \left. + \mathbb{E}[\langle DI_1(f), DF_n\rangle_H^2]\right)\\
&= \lim_n \left(\sum_{l=2}^{p-1}(l+1)!\langle g_{n,l-1}\tilde{\otimes} f, g_{n,l+1}\tilde{\otimes}_1 f\rangle_{H^{\otimes l}} + \mathbb{E}[\langle DI_1(f), DF_n\rangle_H^2]\right)\\
&= \lim_n \left(\sum_{l=2}^{p-1}(l+1)!\langle g_{n,l-1}\tilde{\otimes}_{l-1} g_{n,l+1}, f\tilde{\otimes} f\rangle_{H\otimes H} + \mathbb{E}[\langle DI_1(f), DF_n\rangle_H^2]\right).
\end{aligned}$$

The last equality can be checked directly using the definition of contractions. □

Corollary 5.7. *Let the notations and assumptions of Theorem 5.6 prevail and suppose that $F_n = \sum_{l=1}^p I_l(g_{n,l}) \overset{st.}{\to} X$, as $n \to \infty$, where X is independent of the underlying Brownian motion. Consider $f \in H$ arbitrary.*

(1) If $F_n = I_p(g_{n,p})$, then we have $\lim_n \|g_{n,p}\tilde{\otimes}_1 f\|_{H^{\otimes(p-1)}} = 0$.

(2) If $\lim_n \|g_{n,l+1}\tilde{\otimes}_{l-1}g_{n,l-1}\|_{H\otimes H} = 0$*, for every* $2 \leqslant l \leqslant p-1$*, then we have* $\lim_n \|g_{n,l}\tilde{\otimes}_1 f\|_{H^{\otimes(l-1)}} = 0$*, for every* $1 \leqslant l \leqslant p$*.*

(3) If $F_n = I_{p-1}(g_{n,p-1}) + I_p(g_{n,p})$*, then we have* $\lim_n \|g_{n,p-1}\tilde{\otimes}_1 f\|_{H^{\otimes(p-2)}} = \lim_n \|g_{n,p}\tilde{\otimes}_1 f\|_{H^{\otimes(p-1)}} = 0$*.*

We can recover Nourdin and Peccati (2009a, Proposition 4.2. and Remark 4.3.).

Proposition 5.8. *Consider* $p \geqslant 4$ *even and a sequence* $\{g_{n,p}\}_n \subset H^{\otimes p}$ *such that* $I_p(g_{n,p}) \overset{Law}{\to} N^2 - 1$*, as* $n \to \infty$*, where* N *is a standard normal variable. Then* $I_p(g_{n,p}) \overset{st.}{\to} N^2 - 1$*, as* $n \to \infty$*, and* $N^2 - 1$ *is independent of the underlying Brownian motion.*

Proof: Nourdin and Peccati (2009a, Theorem 1.2.) implies with $P(x) = x(x-1)$, as $n \to \infty$:

$$\begin{aligned}
&\sum_{r=1}^{2} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(I_{g_{n,p}}) - \mathbb{E}[\Gamma_{r-1}(I_{g_{n,p}})]\right) \\
&= I_p(g_{n,p}) - \frac{1}{2}\Gamma_1(I_p(g_{n,p})) + \frac{1}{2}\,\mathbb{E}[\Gamma_1(I_p(g_{n,p}))] \\
&= I_p(g_{n,p}) - \frac{1}{2}\Gamma_1(I_p(g_{n,p})) + \frac{1}{2}\,\mathbb{E}[I_p(g_{n,p})^2] \overset{L^2}{\to} 0.
\end{aligned}$$

Moreover the convergence in law implies that $\|g_{n,p}\otimes_{p-1}g_{n,p}\|_{H\otimes H} \to 0$, as $n \to \infty$. Notice that in condition (5.8) in Theorem 5.3 we must have $(\alpha, \beta, r) = (1, 0, 2)$ and condition (5.8) is then satisfied if condition (5.7) holds. We check now condition (5.7).

$$\begin{aligned}
&|\mathbb{E}\left[\exp\left(i(t_1 I_1(f) + t_2 I_p(g_{n,p}))\right)\langle DI_1(f), DI_p(g_{n,p})\rangle_H\right]| \\
&\leqslant \mathbb{E}\left[\langle DI_1(f), DI_p(g_{n,p})\rangle_H^2\right]^{1/2} \\
&= \mathbb{E}[p^2 I_{p-1}(g_{n,p}\otimes_1 f)^2] = p^2(p-1)!\|g_{n,p}\otimes_1 f\|^2_{H^{\otimes(p-1)}} \\
&\leqslant p^2(p-1)!\|g_{n,p}\otimes_{p-1}g_{n,p}\|_{H^{\otimes 2}}\|f\|^2_H \to 0,
\end{aligned}$$

as $n \to \infty$. □

6. Appendix

Definition 6.1. (1) Consider $k_1 \geqslant 0$. If $k_1 > 0$, we suppose that $b_j \neq 0$ for every $j = 1, \ldots, k_1$. We set:

$$T_j = \begin{cases} 1 & \text{for } j = 0, \\ \displaystyle\sum_{1\leqslant i_1<\ldots<i_j\leqslant k_1} b_{i_1} \times \ldots \times b_{i_j} & \text{for } 1 \leqslant j \leqslant k_1, \\ 0 & \text{for } j \notin \{0, 1, \ldots, k_1\}, \end{cases}$$

and for $l = 1, \ldots, k_1$:

$$T_j^{(l)} = \begin{cases} 1 & \text{for } j = 0, \\ \displaystyle\sum_{\substack{1\leqslant i_1<\ldots<i_j\leqslant k_1 \\ i_1,\ldots,i_j\neq l}} b_{i_1} \times \ldots \times b_{i_j} & \text{for } 1 \leqslant j \leqslant k_1 - 1, \\ 0 & \text{for } j \notin \{0, 1, \ldots, k_1 - 1\}. \end{cases}$$

If $k_1 = 0$, we set $b_j = 0$ for every j, $T_0 = 1$ and $T_j = T_j^{(l)} = 0$ for all other values of j and l.

(2) Consider $k_2 \geqslant 0$. If $k_2 > 0$, we suppose that $c_j d_j \neq 0$ for every $j = 1, \ldots, k_2$. We set:

$$S_j = \begin{cases} 1 & \textit{for } j = 0, \\ \displaystyle\sum_{1 \leqslant i_1 < \ldots < i_j \leqslant k_2} c_{i_1} \times \ldots \times c_{i_j} & \textit{for } 1 \leqslant j \leqslant k_2, \\ 0 & \textit{for } j \notin \{0, 1, \ldots, k_2\}\,; \end{cases}$$

and for $l = 1, \ldots, k_2$:

$$S_j^{(l)} = \begin{cases} 1 & \textit{for } j = 0, \\ \displaystyle\sum_{\substack{1 \leqslant i_1 < \ldots < i_j \leqslant k_2 \\ i_1, \ldots, i_j \neq l}} c_{i_1} \times \ldots \times c_{i_j} & \textit{for } 1 \leqslant j \leqslant k_2 - 1, \\ 0 & \textit{for } j \notin \{0, 1, \ldots, k_2 - 1\}\,. \end{cases}$$

If $k_2 = 0$, we set $c_j = d_j = 0$ for every j, $S_0 = 1$ and $S_j = S_j^{(l)} = 0$ for all other values of j and l.

Proposition 6.2. ***Proof of Eq.*** (3.18)**.**

Proof: We prove that Eq. (3.18) holds for $x \neq 0$. Suppose that $x\varphi_Y(x) \neq 0$ and divide Eq. (3.18) by $i\varphi_Y(x)$:

$$\begin{aligned} &\sum_{l=1}^{k} (2ix)^{k+1} (ix)^{-l} \sum_{r=l}^{k} \frac{P^{(r+2)}(0)}{(r+2)!2^{r+1}} \frac{\kappa_{r-l+2}(X)}{(r-l+1)!} + (2ix)^{k+1} \frac{(-1)^{k_1} a^2}{2} \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j \\ &= ixa^2 G_1(x) G_2(x) + 2ixG_2(x) \sum_{l=1}^{k_1} b_l^2 \prod_{j \neq l} (1 - 2ixb_j) \\ &\quad + G_1(x) \sum_{l=1}^{k_2} \left(\prod_{j \neq l} (1 - 2ixc_j)^2 \right) \left[ix\Delta_l (1 - 2ixc_l) + (ix)^2 c_l \Delta_l - 2ixc_l^2 \right]. \end{aligned} \tag{6.1}$$

We determine an alternative representation for $P(x)$ and calculate $P^{(l)}(0)/l!$. We have with $k = 2k_2 + k_1$

$$\begin{aligned} P(x) &= x^2 \prod_{j=1}^{k_1} (x - b_j) \prod_{j=1}^{k_2} (x - c_j)^2 = x^2 x^k \prod_{j=1}^{k_1} (1 - b_j/x) \prod_{j=1}^{k_2} (1 - c_j/x)^2 \\ &= x^{k+2} \sum_{j_1=0}^{k_1} \sum_{j_2, j_3 = 0}^{k_2} (-1)^{j_1} T_{j_1} (-1)^{j_2} S_{j_2} (-1)^{j_3} S_{j_3} x^{-j_1 - j_2 - j_3} \\ &= x^{k+2} \sum_{p=0}^{k} x^{-p} (-1)^p \sum_{j_1 + j_2 + j_3 = p} T_{j_1} S_{j_2} S_{j_3} \\ &= \sum_{l=2}^{k+2} x^l (-1)^{k+2-l} \left(\sum_{j_1 + j_2 + j_3 = k+2-l} T_{j_1} S_{j_2} S_{j_3} \right). \end{aligned}$$

If for instance $k_1 = 0$, we have $T_j = 1_{[j=0]}$ and the formula above holds also for the cases $k_1 = 0$, $k_2 = 0$ and $k_1 = k_2 = 0$. Hence:

$$\frac{P^{(l)}(0)}{l!} = (-1)^{k+2-l} \sum_{j_1+j_2+j_3=k+2-l} T_{j_1} S_{j_2} S_{j_3}.$$

We compare now the terms in x, on the left-hand side of Eq. (6.1) we have:

$$2^{k+1}(ix)^{k+1-k} \frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}} \kappa_2(X) = ix \left(a^2 + 2\sum_{j=1}^{k_1} b_j^2 + \sum_{j=1}^{k_2} (2c_j^2 + d_j^2) \right),$$

and for the right-hand side of Eq. (6.1):

$$ixa^2 + 2ix \sum_{l=1}^{k_1} b_l^2 + \sum_{l=1}^{k_2} [ix\Delta_l - 2ixc_l^2],$$

the desired equality for the terms in x follows. The term in x^{k+1} on the left-hand side of Eq. (6.1) is $2^k(ix)^{k+1}(-1)^{k_1} a^2 \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j$, and on the right-hand side of Eq. (6.1) we have:

$$\begin{aligned} ixa^2 \prod_{j=1}^{k_2} (-2xic_j)^2 \prod_{j=1}^{k_1} (-2ixb_j) &= (ix)^{k+1}(-2)^k a^2 \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j \\ &= (ix)^{k+1}(-1)^{k_1} 2^k a^2 \prod_{j=1}^{k_2} c_j^2 \prod_{j=1}^{k_1} b_j. \end{aligned}$$

We have for the term in x^m and $1 < m < k+1$ on the left-hand side of Eq. (6.1):

$$\begin{aligned} &2^{k+1}(ix)^{k+1-(k+1-m)} \left(\frac{P^{(k+1-m+2)}(0)\kappa_2(X)}{(k+1-m+2)!2^{k-m+2}} + \ldots + \frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}} \frac{\kappa_{m+1}(X)}{m!} \right) \\ &= 2^{k+1}(ix)^m \Bigg[(-1)^{m-1} \sum_{\substack{j_1+j_2+j_3 \\ =m-1}} T_{j_1} S_{j_2} S_{j_3} \frac{2^{m-1}}{2^{k+1}} \kappa_2(X) \\ &\quad + \ldots + (-1)^0 \sum_{\substack{j_1+j_2+j_3 \\ =0}} T_{j_1} S_{j_2} S_{j_3} \frac{2^0}{2^{k+1}} \frac{\kappa_{m+1}(X)}{m!} \Bigg] \\ &= (ix)^m \Bigg[(-2)^{m-1} \sum_{\substack{j_1+j_2+j_3 \\ =m-1}} T_{j_1} S_{j_2} S_{j_3} \left(a^2 + 2\sum_{j=1}^{k_2} c_j^2 + 2\sum_{j=1}^{k_1} b_j^2 + \sum_{j=1}^{k_2} d_j^2 \right) \\ &\quad + \ldots + (-2)^0 \sum_{\substack{j_1+j_2+j_3 \\ =0}} T_{j_1} S_{j_2} S_{j_3} \Bigg(2^m \sum_{j=1}^{k_2} c_j^{m+1} + 2^m \sum_{j=1}^{k_1} b_j^{m+1} \\ &\quad + 2^{m-2}(m+1) \sum_{j=1}^{k_2} c_j^{m-1} d_j^2 \Bigg] \Bigg). \end{aligned} \tag{6.2}$$

We use the relations $S_j = S_j^{(l)} + c_l S_{j-1}^{(l)}$ and $T_j = T_j^{(l)} + b_l T_{j-1}^{(l)}$ if $k_1, k_2 > 0$, and set $Z_j^{(l)} := \sum_{j_2+j_3=j} S_{j_2}^{(l)} S_{j_3}^{(l)}$. We consider the powers of b_j in Eq. (6.2) and suppose

that $k_1 \neq 0$ for the next calculation.

$$\sum_{l=1}^{k_1}\left[(-2)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\right)2b_l^2+\ldots+(-2)^0\left(\sum_{\substack{j_1+j_2+j_3\\=0}} T_{j_1}S_{j_2}S_{j_3}\right)2^m b_l^{m+1}\right]$$

$$= 2^m\sum_{l=1}^{k_1}\left[(-1)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} (T_{j_1}^{(l)}+b_lT_{j_1-1}^{(l)})S_{j_2}S_{j_3}\right)b_l^2\right.$$

$$\left.+\ldots+\left(\sum_{\substack{j_1+j_2+j_3\\=0}} (T_{j_1}^{(l)}+b_lT_{j_1-1}^{(l)})S_{j_2}S_{j_3}\right)b_l^{m+1}\right]$$

$$= 2^m\sum_{l=1}^{k_1}\left[(-1)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^2+(-1)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^3\right.$$

$$+(-1)^{m-2}\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^3+(-1)^{m-2}\left(\sum_{\substack{j_1+j_2+j_3\\=m-3}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^4$$

$$\left.+\ldots+(-1)^0\left(\sum_{\substack{j_1+j_2+j_3\\=0}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^{m+1}+(-1)^0\left(\sum_{\substack{j_1+j_2+j_3\\=-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^{m+2}\right]$$

$$= 2^m(-1)^{m-1}\sum_{l=1}^{k_1}\left[\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^2\right],$$

hence:

$$\sum_{l=1}^{k_1}\left[(-2)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\right)2b_l^2+\ldots+\left(\sum_{\substack{j_1+j_2+j_3\\=0}} T_{j_1}S_{j_2}S_{j_3}\right)2^m b_l^{m+1}\right]$$

$$= (-2)^{m-1}\sum_{l=1}^{k_1}\left[\left(\sum_{j_1+j_2+j_3=m-1} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)2b_l^2\right].$$

Notice that, with our settings, the previous equality holds also if $k_1 = 0$ or $k_2 = 0$ We consider now the powers of c_j in Eq. (6.2) and suppose that $k_2 > 0$ for the next calculation.

$$\sum_{j=1}^{k_2}\left[(-2)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\right)2c_j^2\right.$$

$$\left.+\ldots+(-2)^0\left(\sum_{\substack{j_1+j_2+j_3\\=0}} T_{j_1}S_{j_2}S_{j_3}\right)2^m c_j^{m+1}\right] \tag{6.3}$$

$$
\begin{aligned}
&= 2^m \sum_{j=1}^{k_2} \left[(-1)^{m-1} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-1}} T_{j_1} S_{j_2} S_{j_3} \right) c_j^2 + \ldots + \left(\sum_{\substack{j_1+j_2+j_3 \\ =0}} T_{j_1} S_{j_2} S_{j_3} \right) c_j^{m+1} \right] \\
&= 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} \sum_{r=0}^{m-1} (-1)^{m-1-r} \left(\sum_{\substack{j_2+j_3 \\ =m-1-r-j_1}} S_{j_2} S_{j_3} \right) c_j^{r+2} \\
&= 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} \sum_{r=0}^{m-1} (-1)^{m-1-r} \left[\sum_{\substack{j_2+j_3 \\ =m-1-r-j_1}} (S_{j_2}^{(j)} + c_j S_{j_2-1}^{(j)}) (S_{j_3}^{(j)} + c_j S_{j_3-1}^{(j)}) \right] c_j^{r+2} \\
&= 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} \sum_{r=0}^{m-1} (-1)^{m-1-r} \Big[Z_{m-1-r-j_1}^{(j)} c_j^{r+2} + 2 Z_{m-2-r-j_1}^{(j)} c_j^{r+3} \\
&\quad + Z_{m-3-r-j_1}^{(j)} c_j^{r+4} \Big] \\
&= 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} (-1)^{m-1} \left[Z_{m-1-j_1}^{(j)} c_j^2 + 2 Z_{m-2-j_1}^{(j)} c_j^3 + Z_{m-3-j_1}^{(j)} c_j^4 \right] \\
&\quad + 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} (-1)^{m-2} \left[Z_{m-2-j_1}^{(j)} c_j^3 + 2 Z_{m-3-j_1}^{(j)} c_j^4 + Z_{m-4-j_1}^{(j)} c_j^5 \right] \\
&\quad + \ldots + 2^m \sum_{j=1}^{k_2} \sum_{j_1=0}^{k_1} T_{j_1} (-1)^1 \left[Z_{1-j_1}^{(j)} c_j^m + 2 Z_{-j_1}^{(j)} c_j^{m+1} + Z_{-1-j_1}^{(j)} c_j^{m+2} \right] \\
&\quad + 2^m \sum_{j=1}^{k_2} c_j^{m+1}.
\end{aligned}
$$

Hence the expression in Eq. (6.3) equals:

$$
\begin{aligned}
&2^m (-1)^{m-1} \sum_{j=1}^{k_2} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-1}} T_{j_1} S_{j_2}^{(j)} S_{j_3}^{(j)} \right) c_j^2 + 2^m \sum_{j=1}^{k_2} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-2}} T_{j_1} S_{j_2}^{(j)} S_{j_3}^{(j)} \right) \\
&\quad \times c_j^3 \left[2(-1)^{m-1} + (-1)^{m-2} \right] + 2^m \sum_{r=2}^{m-1} \sum_{j=1}^{k_2} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-1-r}} T_{j_1} S_{j_2}^{(j)} S_{j_3}^{(j)} \right) \\
&\quad \times \left[(-1)^{m-1-r} + 2(-1)^{m-1-(r-1)} + (-1)^{m+1-r} \right] c_j^{r+2} \\
&= (-2)^{m-1} \sum_{j=1}^{k_2} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-1}} T_{j_1} S_{j_2}^{(j)} S_{j_3}^{(j)} \right) 2c_j^2 - (-2)^{m-2} \sum_{j=1}^{k_2} \left(\sum_{\substack{j_1+j_2+j_3 \\ =m-2}} T_{j_1} S_{j_2}^{(j)} S_{j_3}^{(j)} \right) 4c_j^3.
\end{aligned}
$$

The resulting equality holds also if $k_1 = 0$ or $k_2 = 0$. In the latter case, we have $S_{j_2}^{(l)} = c_j = d_j = 0$ and an empty sum in (6.3) equals 0. For the terms of the form $c_l^r d_l^2$ in Eq. (6.2), we can make a similar calculation. For the calculation below, we

suppose that $k_2 \neq 0$.

$$\sum_{j=1}^{k_2}\Bigg[(-2)^{m-1}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\Bigg)d_j^2+(-2)^{m-2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}S_{j_3}\Bigg)3d_j^2c_j$$

$$+\ldots+(-2)^0\Bigg(\sum_{\substack{j_1+j_2+j_3\\=0}} T_{j_1}S_{j_2}S_{j_3}\Bigg)2^{m-2}(m+1)d_j^2c_j^{m-1}\Bigg]$$

$$=\sum_{j=1}^{k_2}2^{m-1}(-1)^{m-1}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)d_j^2$$

$$-\sum_{j=1}^{k_2}2^{m-2}(-1)^{m-2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)c_jd_j^2$$

$$+2^{m-2}\sum_{j=1}^{k_2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=-1}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)(m+2)c_j^md_j^2+\sum_{j=1}^{k_2}\sum_{l=2}^{m-1}\big[(-1)^{m-l+1}lc_j^ld_j^2$$

$$+2(l+1)(-1)^{m-l}c_j^ld_j^2+(l+2)(-1)^{m-l-1}c_j^ld_j^2\big]\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-l-1}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)2^{m-2}$$

$$=(-2)^{m-1}\sum_{j=1}^{k_2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)d_j^2-(-2)^{m-2}\sum_{j=1}^{k_2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)c_jd_j^2.$$

The resulting equality holds also if $k_1=0$ or $k_2=0$. We have finally for the term in x^m and $1<m<k+1$ on the left-hand side of Eq. (6.2):

$$2^{k+1}(ix)^m\left(\frac{P^{(k+1-m+2)}(0)}{(k+1-m+2)!2^{k-m+2}}\,\mathbb{E}[\Gamma_1(X)]+\ldots+\frac{P^{(k+2)}(0)}{(k+2)!2^{k+1}}\,\mathbb{E}[\Gamma_m(X)]\right)$$

$$=(ix)^m(-2)^{m-1}\sum_{j=1}^{k_2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)(2c_j^2+d_j^2)$$

$$+(ix)^m(-2)^{m-2}\sum_{j=1}^{k_2}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(j)}S_{j_3}^{(j)}\Bigg)(-4c_j^3-c_jd_j^2)$$

$$+(ix)^m(-2)^{m-1}\sum_{j=1}^{k_1}\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(j)}S_{j_2}S_{j_3}\Bigg)2b_j^2$$

$$+(ix)^m(-2)^{m-1}a^2\Bigg(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\Bigg). \tag{6.4}$$

Consider now the term in x^m for $1 < m < k+1$ on the right-hand side of Eq. (6.1). Define:

$$Q_m\left(\sum_{i=0}^{\infty} c_i x^i\right) = c_m x^m.$$

Q_m is thus the projection of a series (or a polynomial) on its term of degree m. The terms in x^m for $1 < m < k+1$ on the right-hand side of Eq. (6.1) are given by:

$$\begin{aligned}
&ixa^2 Q_{m-1}(G_1(x)G_2(x)) + 2ixQ_{m-1}\left(G_2(x)\sum_{l=1}^{k_1} b_l^2 \prod_{j\neq l}(1-2ixb_j)\right)\\
&\quad + ixQ_{m-1}\left(G_1(x)\sum_{l=1}^{k_2}\Delta_l(1-2ixc_l)\prod_{j\neq l}(1-2ixc_j)^2\right)\\
&\quad + (ix)^2 Q_{m-2}\left(G_1(x)\sum_{l=1}^{k_2} c_l\Delta_l\prod_{j\neq l}(1-2ixc_j)^2\right)\\
&\quad - 2ixQ_{m-1}\left(G_1(x)\sum_{l=1}^{k_2} c_l^2\prod_{j\neq l}(1-2ixc_j)^2\right). \qquad (6.5)
\end{aligned}$$

We calculate now the projections appearing in the expression above.

$$G_1(x)G_2(x) = \sum_{j_1=0}^{k_1}(-2ix)^{j_1}T_{j_1}\sum_{j_2=0}^{k_2}(-2ix)^{j_2}S_{j_2}\sum_{j_3=0}^{k_2}(-2ix)^{j_3}S_{j_3},$$

$$Q_{m-1}(G_1(x)G_2(x)) = (-2ix)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\right),$$

$$G_2(x)\sum_{l=1}^{k_1} b_l^2\prod_{j\neq l}(1-2ixb_l) = \sum_{l=1}^{k_1} b_l^2\sum_{j_1=0}^{k_1-1}(-2ix)^{j_1}T_{j_1}^{(l)}\sum_{j_2=0}^{k_2}(-2ix)^{j_2}S_{j_2}\sum_{j_3=0}^{k_2}(-2ix)^{j_3}S_{j_3},$$

$$Q_{m-1}\left(G_2(x)\sum_{l=1}^{k_1} b_l^2\prod_{j\neq l}(1-2ixb_l)\right) = (-2ix)^{m-1}\sum_{l=1}^{k_1} b_l^2\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right).$$

Notice that these equalities hold if $k_1 \geqslant 0$ and $k_2 \geqslant 0$. The following equalities can be derived as above:

$$\begin{aligned}
&Q_{m-1}\left(G_1(x)\sum_{l=1}^{k_2}\Delta_l(1-2ixc_l)\prod_{j\neq l}(1-2ixc_j)^2\right)\\
&= (-2ix)^{m-1}\left[\sum_{l=1}^{k_2}\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right) + \sum_{l=1}^{k_2} c_l\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\right],\\
&Q_{m-2}\left(G_1(x)\sum_{l=1}^{k_2} c_l\Delta_l\prod_{j\neq l}(1-2ixc_j)^2\right) = (-2ix)^{m-2}\sum_{l=1}^{k_2} c_l\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right),
\end{aligned}$$

$$Q_{m-1}\left(G_1(x)\sum_{l=1}^{k_2} c_l^2 \prod_{j\neq l}(1-2ixc_j)^2\right) = (-2ix)^{m-1}\sum_{l=1}^{k_2} c_l^2 \left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1} S_{j_2}^{(l)} S_{j_3}^{(l)}\right).$$

We have thus for expression (6.5):

$$\begin{aligned}
&ixa^2(-2ix)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_2}\right) + 2ix(-2ix)^{m-1}\sum_{l=1}^{k_1} b_l^2\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)\\
&\quad + ix(-2ix)^{m-1}\sum_{l=1}^{k_2}\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\\
&\quad + ix(-2ix)^{m-1}\sum_{l=1}^{k_2} c_l\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\\
&\quad + (ix)^2(-2ix)^{m-2}\sum_{l=1}^{k_2} c_l\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\\
&\quad - 2ix(-2ix)^{m-1}\sum_{l=1}^{k_2} c_l^2\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\\
&= (ix)^m\left\{(-2)^{m-1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}S_{j_3}\right)a^2 + 2(-2)^{m-1}\sum_{l=1}^{k_1}\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}^{(l)}S_{j_2}S_{j_3}\right)b_l^2\right.\\
&\quad + (-2)^{m-1}\sum_{l=1}^{k_2}(2c_l^2+d_l^2)\left(\sum_{\substack{j_1+j_2+j_3\\=m-1}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\\
&\quad \left. - (-2)^{m-2}\sum_{l=1}^{k_2} c_l\Delta_l\left(\sum_{\substack{j_1+j_2+j_3\\=m-2}} T_{j_1}S_{j_2}^{(l)}S_{j_3}^{(l)}\right)\right\}. \qquad (6.6)
\end{aligned}$$

This equality holds (again) for $k_1 \geqslant 0$ and $k_2 \geqslant 0$. Comparing Eq. (6.4) and Eq. (6.6), we find the desired equality of the terms in x^m on the left-hand and the right-hand side of Eq. (6.1). □

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