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A result on power moments of Lévy-type perpetuities and its application to the L_p-convergence of Biggins' martingales in branching Lévy processes

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Abstract. Lévy-type perpetuities being the a.s. limits of particular generalized Ornstein-Uhlenbeck processes are a natural continuous-time generalization of discrete-time perpetuities. These are random variables of the form $S := \int_{[0,\infty)} e^{-X_{s-}} \mathrm{d}Z_s$, where (X, Z) is a two-dimensional Lévy process, and Z is a drift-free Lévy process of bounded variation. We prove an ultimate criterion for the finiteness of power moments of S. This result and the previously known assertion due to Erickson and Maller (2005) concerning the a.s. finiteness of S are then used to derive ultimate necessary and sufficient conditions for the L_p-convergence for $p > 1$ and $p = 1$, respectively, of Biggins' martingales associated to branching Lévy processes. In particular, we provide final versions of results obtained recently in Bertoin and Mallein (2018).

1. Introduction

Let $(M_k, Q_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of an $\mathbb{R}^2$-valued random vector (M, Q) with arbitrary dependence of components. Further, denote by $(\Pi_n)_{n\in\mathbb{N}_0}$ the multiplicative (ordinary) random walk with factors M_n for $n \in \mathbb{N}$

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which starts at 1, that is, $\Pi_0 := 1$ and $\Pi_n := \prod_{i=1}^n M_i$, $n \in \mathbb{N}$. Then define its perturbed variant $(\Theta_n)_{n\in\mathbb{N}}$, that may be called a *perturbed multiplicative random walk*, by

$$\Theta_n := \Pi_{n-1} Q_n, \quad n \in \mathbb{N}. \tag{1.1}$$

When M_k and Q_k are a.s. positive, the random sequence $(\log \Theta_n)_{n\in\mathbb{N}}$ is known in the literature as a *perturbed (additive) random walk*. A major part of the recent book (Iksanov, 2016) is concerned with the so defined perturbed random walks, both multiplicative and additive. We refer to the cited book for numerous applications of these random sequences and to Alsmeyer et al. (2017); Buraczewski et al. (2018); Damek and Kołodziejek (2018); Iksanov et al. (2018, 2017) for more recent contributions.

Recall that, provided that the series $\sum_{k\geq 1} \Theta_k$ converges a.s., its sum

$$\Xi := \sum_{k\geq 1} \Theta_k$$

is called *perpetuity*. The term stems from the fact that such random series may be used in insurance mathematics and financial mathematics to model sums of discounted payment streams. The state of the art concerning various aspects of perpetuities is discussed in Buraczewski et al. (2016) and Iksanov (2016). We think that the most valuable feature of the perturbed multiplicative random walks is their link with perpetuities.

There is also an unexpected connection, unveiled in Lyons (1997) and detailed in Iksanov (2004) and Alsmeyer and Iksanov (2009), between perpetuities and branching random walks. The connection, which is not immediately seen, emerges when studying the weighted random tree associated with the branching random walk under a size-biased measure. In particular, criteria for the uniform integrability and the L_p-convergence for $p > 1$ of the Biggins martingale (also known as the additive martingale or the intrinsic martingale in the branching random walk) are closely linked with criteria for the a.s. finiteness and the existence of the pth moment of perpetuities, respectively. In this way one arrives at a final version of the famous Biggins martingale convergence theorem which was originally proved by Biggins himself in Biggins (1977) with the help of a different argument and under additional moment assumptions. The recent article Bertoin and Mallein (2018) is aimed at obtaining sufficient conditions for the uniform integrability and the L_p-convergence for $p \in (1, 2]$ of the Biggins martingale in a *branching Lévy process*. To this end, a connection similar to that described at the beginning of the paragraph is exploited between certain continuous-time perpetuities and branching Lévy process. The conditions obtained in Bertoin and Mallein (2018) are not optimal.

In this article we first define *perturbed multiplicative Lévy processes* which are natural continuous-time counterparts of the perturbed multiplicative random walks. These are then used to construct *Lévy-type perpetuities* in the same way as the perturbed multiplicative random walks are used to construct the discrete-type perpetuities. The Lévy-type perpetuities are a particular instance of the limit random variables for generalized Ornstein-Uhlenbeck processes. This restriction (that is, that we consider the particular rather than any limit) is motivated by a prospective application, see the end of this section for more details. Necessary and sufficient conditions for the a.s. finiteness of the Lévy-type perpetuities can be derived from Erickson and Maller (2005, Theorem 2).

Our main contribution is two-fold. First, we prove an ultimate criterion for the finiteness of the pth moment of the Lévy-type perpetuity for all $p > 0$. Second, we apply this criterion and the aforementioned result from Erickson and Maller (2005) to derive necessary and sufficient conditions for the a.s. and the L_p-convergence for $p \geq 1$ of the Biggins martingale in the branching Lévy process. Thus, we obtain final versions of Theorem 1.1 and Proposition 1.4 in Bertoin and Mallein (2018) which was our primary motivation.

2. Lévy-type perpetuities

In this section we first define a continuous-time counterpart of the perturbed multiplicative random walks, described in (1.1).

Let Λ be a sigma-finite measure on $\mathbb{R} \times \mathbb{R}$ with $\Lambda(\{0, 0\}) = 0$. Define the projections Λ_1 and Λ_2 of Λ by

$$\Lambda_1(B) := \int_{\mathbb{R}} \Lambda(B, \mathrm{d}y) \quad \text{and} \quad \Lambda_2(B) := \int_{\mathbb{R}} \Lambda(\mathrm{d}x, B)$$

for Borel sets B in $\mathbb{R}\backslash\{0\}$. Throughout the article our standing assumption is that

$$\int_{\mathbb{R}} (x^2 \wedge 1)\Lambda_1(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} (|y| \wedge 1)\Lambda_2(\mathrm{d}y) < \infty. \tag{2.1}$$

Denote by $N := \sum_k \varepsilon_{(\tau_k,(i_k,j_k))}$ a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^2$ with mean measure $\mathrm{LEB} \otimes \Lambda$, where $\mathbb{R}_+ := [0, \infty)$, $\varepsilon_{(t,(x,y))}$ denotes the Dirac mass at $(t, (x, y)) \subset \mathbb{R}_+ \times \mathbb{R}^2$, and LEB is the Lebesgue measure on $\mathbb{R}_+$. Define $N_1 := \sum_k \varepsilon_{(\tau_k,i_k)}$ and $N_2 := \sum_k \varepsilon_{(\tau_k,j_k)}$, the projections of N. These are Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}$ with mean measures $\mathrm{LEB} \otimes \Lambda_j$, $j = 1, 2$.

For $t \geq 0$, set

$$X_t := vB_t + bt + \int_{[0,\,t]\times\mathbb{R}} x\, \mathbb{1}_{[-1,1]}(x) N_1^c(\mathrm{d}s\mathrm{d}x) + \int_{[0,\,t]\times\mathbb{R}} x\, \mathbb{1}_{\mathbb{R}\backslash[-1,1]}(x) N_1(\mathrm{d}s\mathrm{d}x) \tag{2.2}$$

$$Z_t := \int_{[0,\,t]\times\mathbb{R}} y N_2(\mathrm{d}s\mathrm{d}y)$$

where $v^2 \geq 0$, $b \in \mathbb{R}$ and $(B_t)_{t\geq 0}$ is a Brownian motion independent of N. The first integral in (2.2) is a compensated Poisson integral (hence, the notation N_1^c) which can be defined as the following limit in L_2

$$\lim_{\delta\downarrow 0} \int_{[0,\,t]\times\mathbb{R}} x\, \mathbb{1}_{(\delta,1]}(|x|) N_1(\mathrm{d}s\mathrm{d}x) - t \int_{\delta<|x|\leq 1} x\Lambda_1(\mathrm{d}x).$$

In view of the second assumption in (2.1) the process $Z := (Z_t)_{t\geq 0}$ is a drift-free Lévy process of bounded variation. In particular, Z can be represented as the difference of two independent subordinators. The random measure N is the measure of jumps of the two-dimensional Lévy process $(X_t, Z_t)_{t\geq 0}$.

Define the random process $Y := (Y_t)_{t\geq 0}$ by

$$Y_t = \begin{cases} y, & \text{if } N_2(\{t\} \times \{y\}) = 1; \\ 0, & \text{if } N_2(\{t\} \times \mathbb{R}) = 0, \end{cases}$$

that is, $Y = (Z_t - Z_{t-})_{t\geq 0}$ is the process of jumps of Z. The process $(Y_t e^{-X_{t-}})_{t\geq 0}$ which is a natural continuous-time generalization of the process $(\Theta_n)_{n\in\mathbb{N}}$ defined in (1.1) will be called *perturbed multiplicative Lévy process*. For $t \geq 0$, set

$$S_t := \sum_{0 \leq s \leq t} e^{-X_{s-}} Y_s = \sum_{\tau_k \leq t} e^{-X_{\tau_k -}} j_k = \int_{[0,\, t]} e^{-X_{s-}} \mathrm{d}Z_s. \tag{2.3}$$

Whenever the a.s. limit $S := \lim_{t \to \infty} S_t$ exists and is finite, we call the random variable

$$S = \sum_{s \geq 0} e^{-X_{s-}} Y_s = \sum_{k} e^{-X_{\tau_k -}} j_k = \int_{\mathbb{R}_+} e^{-X_{s-}} \mathrm{d}Z_s \tag{2.4}$$

Lévy-type perpetuity.

The following result which gives necessary and sufficient conditions for the a.s. finiteness of Lévy-type perpetuities is a specialization[1] of Theorem 2 in Erickson and Maller (2005). For $x \geq 1$, set

$$A(x) := 1 + \int_1^x \Lambda_1((y, \infty))\mathrm{d}y = 1 + \int_{\mathbb{R}} (x \wedge z - 1)_+ \Lambda_1(\mathrm{d}z),$$

where $z_+ = \max(z, 0)$ and $y \wedge z = \min(y, z)$ for all $y, z \in \mathbb{R}$.

Proposition 2.1. *Assume that*

$$\lim_{t \to \infty} X_t = +\infty \quad \text{a.s.} \quad \textit{and} \quad \int_{\mathbb{R} \setminus [-e, e]} \frac{\log |y|}{A(\log |y|)} \Lambda_2(\mathrm{d}y) < \infty. \tag{2.5}$$

Then

$$\mathbb{P}\{\lim_{t \to \infty} S_t \textit{ exists and is finite}\} = 1. \tag{2.6}$$

Conversely, if (2.5) *fails, then* (2.6) *fails.*

It should not come as a surprise that Proposition 2.1 is very similar to Theorem 2.1 in Goldie and Maller (2000) which provides a criterion for the a.s. finiteness of discrete-time perpetuities Ξ.

3. Power moments of Lévy-type perpetuities

3.1. *Main result.* The purpose of this section is to point out necessary and sufficient conditions for the finiteness of power moments of S. Before formulating the corresponding result we note that the distribution of S is degenerate if, and only if, it is degenerate at 0, and that the latter occurs if, and only if, Λ_2 is trivial which means that $\Lambda_2 \equiv 0$. The non-obvious part of this statement, that is, that the distribution of S cannot be degenerate at a nonzero point follows from the fact that Z does not have a Brownian component and Theorem 2.2 in Bertoin et al. (2008).

Theorem 3.1. *Assume that Λ_2 is nontrivial and let $p > 0$. The following assertions are equivalent:*

$$\mathbb{E}\, e^{-pX_1} < 1 \quad \textit{and} \quad \int_{\mathbb{R} \setminus [-1,1]} |y|^p \Lambda_2(\mathrm{d}y) < \infty; \tag{3.1}$$

$$\mathbb{E}\, |S|^p < \infty. \tag{3.2}$$

[1] In the cited result Z is allowed to be an arbitrary Lévy process. The random process $(S_t)_{t \geq 0}$ in (2.3) is then called a generalized Ornstein-Uhlenbeck process. In view of the second condition in (2.1) which is motivated by a forthcoming application of our results to branching Lévy processes we only consider a subclass of generalized Ornstein-Uhlenbeck processes.

3.2. *Auxiliary results.* Proposition 3.2 and Proposition 3.3 given below are our main technical tools for the proof of Theorem 3.1. We start by recalling a criterion obtained in Theorem 1.4 of Alsmeyer et al. (2009) for the finiteness of power moments of discrete-time perpetuities Ξ.

Proposition 3.2. *Let $p > 0$ and suppose that*

$$\mathbb{P}\{M = 0\} = 0 \quad \textit{and} \quad \mathbb{P}\{Q = 0\} < 1 \tag{3.3}$$

and that

$$\mathbb{P}\{Q + Mr = r\} < 1 \quad \textit{for all } r \in \mathbb{R}. \tag{3.4}$$

The following assertions are equivalent:

$$\mathbb{E}\,|M|^p < 1 \quad \textit{and} \quad \mathbb{E}\,|Q|^p < \infty; \tag{3.5}$$

$$\mathbb{E}\,|\Xi|^p < \infty. \tag{3.6}$$

The next proposition gives sufficient conditions for the finiteness of the pth moment of the integral of an adapted process against the Lévy process Z defined in Section 2.

Proposition 3.3. *Let $(Z_s)_{s\geq 0}$ be a drift-free Lévy process of finite variation (as defined in Section 2) and $(H_s)_{s\geq 0}$ an adapted càdlàg process. Suppose that there exists $p > 0$ such that $\mathbb{E}\,|Z_1|^p < \infty$ and $\mathbb{E}\sup_{s\in(0,1]}|H_s|^p < \infty$. Then*

$$\mathbb{E}\Big|\int_{(0,1]} H_{s-}\mathrm{d}Z_s\Big|^p < \infty.$$

Proof: When $p \geq 1$ the assertion follows from Lemma 6.1 in Behme (2011).

Assume that $p \in (0,1)$. Subadditivity of $x \mapsto x^p$ on $\mathbb{R}_+$ and the triangle inequality entail

$$\mathbb{E}\Big|\int_{(0,1]} H_{s-}\mathrm{d}Z_s\Big|^p \leq \mathbb{E}\left(\int_{(0,1]} |H_{s-}|\,\mathrm{d}Z_s^{(1)}\right)^p + \mathbb{E}\left(\int_{(0,1]} |H_{s-}|\,\mathrm{d}Z_s^{(2)}\right)^p, \tag{3.7}$$

where, for $t \geq 0$,

$$Z_t^{(1)} := \int_{[0,\,t]\times\mathbb{R}} |y|\,\mathbb{1}_{[-1,1]}(y)N_2(\mathrm{d}s\mathrm{d}y) = \sum_{\tau_k\leq t} |j_k|\,\mathbb{1}_{\{|j_k|\leq 1\}}$$

and

$$Z_t^{(2)} := \int_{[0,\,t]\times\mathbb{R}} |y|\,\mathbb{1}_{\mathbb{R}\backslash[-1,1]}(y)N_2(\mathrm{d}s\mathrm{d}y) = \sum_{\tau_k\leq t} |j_k|\,\mathbb{1}_{\{|j_k|>1\}}\,.$$

Note that $Z^{(i)} := (Z_t^{(i)})_{t\geq 0}$, $i = 1,2$ are drift-free subordinators. We shall prove finiteness of the two summands on the right-hand side of (3.7) separately.

We start by observing that $Z^{(2)}$ is a compound Poisson process with jumps sizes larger than one. Denote by $T_1, T_2, \ldots$ the times at which $Z^{(2)}$ jumps, ranked in the increasing order, and set $R_i := Z^{(2)}_{T_i} - Z^{(2)}_{T_i-}$ for $i \in \mathbb{N}$. The sequence $(T_k)_{k\in\mathbb{N}}$ forms the arrival times of a Poisson process with intensity $c := \Lambda_2(\mathbb{R}\backslash[-1,1])$, and $(R_k)_{k\in\mathbb{N}}$ are i.i.d. random variables with distribution $\mathbb{P}\{R_1 > x\} = c^{-1}\Lambda_2(\mathbb{R}\backslash[-x,x])$ for $x > 1$ and $\mathbb{P}\{R_1 > x\} = 1$ for $x \leq 1$. Moreover, for each fixed $i \in \mathbb{N}$, (H_{T_i-}, T_i)

is independent of R_i. Using these facts in combination with the aforementioned subadditivity we obtain

$$\begin{aligned}\mathbb{E}\Big|\int_{(0,\,1]} H_{s-}\mathrm{d}Z_s^{(2)}\Big|^p \leq & \,\mathbb{E}\Big(\sum_{i\geq 1}|H_{T_i-}|^p R_i^p\,\mathbb{1}_{\{T_i\leq 1\}}\Big)\\ = & \,\mathbb{E}\Big(\sum_{i\geq 1}|H_{T_i-}|^p\,\mathbb{1}_{\{T_i\leq 1\}}\Big)\,\mathbb{E}\,R_1^p\\ = & \,c\,\mathbb{E}\Big(\int_0^1 |H_s|^p\mathrm{d}s\Big)c^{-1}\int_{\mathbb{R}\backslash[-1,\,1]}|y|^p\Lambda_2(\mathrm{d}y),\end{aligned}$$

where, recalling that $(H_s)_{s\geq 0}$ is an adapted process, the second equality is justified by the compensation formula for Poisson random measures. As a result,

$$\mathbb{E}\Big|\int_{(0,\,1]} H_{s-}\mathrm{d}Z_s^{(2)}\Big|^p \leq \mathbb{E}\Big(\sup_{s\in[0,1]}|H_s|^p\Big)\int_{\mathbb{R}\backslash[-1,\,1]}|y|^p\Lambda_2(\mathrm{d}y)<\infty.$$

Here, the inequality $\int_{\mathbb{R}\backslash[-1,\,1]}|y|^p\Lambda_2(\mathrm{d}y)<\infty$ is guaranteed by the assumption $\mathbb{E}\,|Z_1|^p<\infty$ (see Theorem 25.3 in Sato (2013)).

It remains to show that

$$\mathbb{E}\left(\int_{(0,1]}|H_{s-}|\,\mathrm{d}Z_s^{(1)}\right)^p<\infty. \tag{3.8}$$

For each $A>0$ and each $t\in[0,1]$, set $K_t^A=|H_t|\wedge A$. Also, for each $n\in\mathbb{N}$ and integer $1\leq k\leq n$, set $I_{k,n}:=((k-1)/n,k/n]$ and let $\mathcal{F}_{k,n}$ denote the σ-algebra generated by $(H_{s-},Z_s^{(1)})_{0\leq s\leq k/n}$ (we also denote by $\mathcal{F}_{0,n}$ the trivial σ-algebra). Recalling that $Z^{(1)}$ is a drift-free subordinator we write

$$\begin{aligned}\mathbb{E}\Big(\int_{(0,\,1]}K_{s-}^A\mathrm{d}Z_s^{(1)}\Big)^p &= \mathbb{E}\Big(\sum_{k=1}^n\int_{I_{k,n}}K_{s-}^A\mathrm{d}Z_s^{(1)}\Big)^p\\ &\leq 2\,\mathbb{E}\Big(\sum_{k=1}^n\mathbb{E}\Big(\int_{I_{k,n}}K_{s-}^A\mathrm{d}Z_s^{(1)}\Big|\mathcal{F}_{k-1,n}\Big)\Big)^p\\ &\leq 2\,\mathbb{E}\Big(\sum_{k=1}^n\int_{I_{k,n}}\mathbb{E}(K_s^A|\mathcal{F}_{k-1,n})\mathrm{d}s\Big)^p\Big(\int_{[-1,\,1]}|y|\Lambda_2(\mathrm{d}y)\Big)^p,\end{aligned}$$

where the first inequality follows by an application of Lemma 6 on p. 411 in Chow and Teicher (1988), and the second inequality is a consequence of subadditivity of $x\mapsto x^p$ on $\mathbb{R}_+$ and the equality

$$\mathbb{E}\Big(\int_{I_{k,n}}K_{s-}^A\mathrm{d}Z_s^{(1)}\Big|\mathcal{F}_{k-1,n}\Big)=\int_{I_{k,n}}\mathbb{E}(K_s^A|\mathcal{F}_{k-1,n})\mathrm{d}s\int_{[-1,\,1]}|y|\Lambda_2(\mathrm{d}y)$$

which is implied by the compensation formula for Poisson random measures. Further, letting $n\to\infty$ and using the fact that $(K_s^A)_{s\geq 0}$ is an adapted bounded process, an appeal to Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty}\mathbb{E}\Big(\sum_{k=1}^n\int_{I_{k,n}}\mathbb{E}(K_s^A|\mathcal{F}_{k-1,n})\mathrm{d}s\Big)^p=\mathbb{E}\Big(\int_0^1 K_s^A\mathrm{d}s\Big)^p\leq\mathbb{E}(\sup_{s\in[0,\,1]}(K_s^A)^p).$$

Thus, we have proved that, for each $A > 0$,

$$\mathbb{E}\Big(\int_{(0,1]}(|H_{s-}|\wedge A)\mathrm{d}Z_s^{(1)}\Big)^p \le 2\,\mathbb{E}(\sup_{s\in[0,1]}|H_s|\wedge A)^p\Big(\int_{[-1,\,1]}|y|\Lambda_2(\mathrm{d}y)\Big)^p$$
$$\le \mathbb{E}(\sup_{s\in[0,1]}|H_s|)^p\Big(\int_{[-1,\,1]}|y|\Lambda_2(\mathrm{d}y)\Big)^p < \infty.$$

Letting $A\to\infty$ in the latter formula, we infer (3.8) with the help of Lévy's monotone convergence theorem. □

The result given next is a consequence of Theorem 25.18 in Sato (2013). A direct proof can be found in Lemma 2.1 (a) of Aurzada et al. (2015).

Lemma 3.4. *Let $p > 0$. If $\mathbb{E}\,e^{-pX_1} < \infty$, then*

$$\mathbb{E}\sup_{s\in[0,1]} e^{-pX_s} = \mathbb{E}\exp(-p\inf_{s\in[0,1]}X_s) < \infty.$$

3.3. *Proof of Theorem 3.1.*

Proof of (3.1)⇒(3.2). We first show that conditions (3.1) ensure $|S| < \infty$ a.s. Indeed, by Jensen's inequality $\mathbb{E}\,e^{-pX_1} < 1$ entails $\mathbb{E}\,X_1 \in (0,\infty]$, whence $\lim_{t\to\infty}X_t = +\infty$ a.s. Further, $\int_{|y|>1}|y|^p\Lambda_2(\mathrm{d}y) < \infty$ ensures $\int_{|y|>1}\log|y|\Lambda_2(\mathrm{d}y) < \infty$ and, a fortiori, the second condition in (2.5). Now $|S| < \infty$ a.s. follows from Proposition 2.1.

Now observe that the random variable S can be obtained as a discrete-time perpetuity generated by the pair of random variables

$$(M_*, Q_*) := (e^{-X_1}, \int_{[0,\,1]} e^{-X_{s-}}\mathrm{d}Z_s).$$

In view of the discussion at the beginning of Section 3.1 and our assumption that Λ_2 is nontrivial, the distribution of S is nondegenerate. Therefore, $\mathbb{P}\{Q_*+M_*r = r\} < 1$ for all $r\in\mathbb{R}$. This enables us to invoke Proposition 3.2 which states that $\mathbb{E}\,|S|^p < \infty$ if, and only if, $\mathbb{E}\,M_*^p = \mathbb{E}\,e^{-pX_1} < 1$ and $\mathbb{E}\,|Q_*|^p = \mathbb{E}\,|\int_{[0,\,1]}e^{-X_{s-}}\mathrm{d}Z_s|^p < \infty$.

It is well-known that the second assumption in (3.1) is equivalent to $\mathbb{E}\,|Z_1|^p < \infty$ (see, for instance, Theorem 25.3 on p. 159 in Sato (2013)). By Lemma 3.4, the first condition in (3.1) guarantees $\mathbb{E}\sup_{s\in[0,1]}e^{-pX_s} < \infty$. With these at hand we infer $\mathbb{E}\,|Q_*|^p < \infty$ by Proposition 3.3. □

Proof of (3.2)⇒(3.1). We assume that Λ_2 charges all the punctured line $\mathbb{R}\backslash\{0\}$. Otherwise, the proof becomes simpler. We have $\mathbb{E}\,M_*^p = \mathbb{E}\,e^{-pX_1} < 1$ by another appeal to Proposition 3.2. Using the inequality

$$|x+y|^p \ge (2^{1-p}\wedge 1)|x|^p - |y|^p,\quad x,y\in\mathbb{R}$$

which is implied by convexity (respectively subadditivity) of $s\mapsto s^p$ for $s\ge 0$ when $p\ge 1$ (resp. when $p\in(0,1)$) we obtain

$$\infty > \mathbb{E}\,|S|^p = \mathbb{E}\Big|\int_{[0,\,1]}e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(1)} + \int_{[0,\,1]}e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(2)}\Big|^p$$
$$\ge (2^{1-p}\wedge 1)\,\mathbb{E}\Big|\int_{[0,\,1]}e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(2)}\Big|^p - \mathbb{E}\Big|\int_{[0,\,1]}e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(1)}\Big|^p,$$

where, for $t \geq 0$,

$$\tilde{Z}_t^{(1)} := \int_{[0,\,t]\times\mathbb{R}} y\,\mathbb{1}_{[-1,1]}(y)N_2(\mathrm{d}s\mathrm{d}y) = \sum_{\tau_k \leq t} j_k\,\mathbb{1}_{\{|j_k|\leq 1\}}$$

$$\tilde{Z}_t^{(2)} := Z_t - \tilde{Z}_t^{(1)} = \int_{[0,\,t]\times\mathbb{R}} y\,\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(y)N_2(\mathrm{d}s\mathrm{d}y) = \sum_{\tau_k \leq t} j_k\,\mathbb{1}_{\{|j_k|>1\}}\,.$$

By Theorem 25.3 on p. 159 in Sato (2013), the random variable $|\tilde{Z}_1^{(1)}|$ has finite power moments of all positive orders. In particular, $\mathbb{E}\,|\tilde{Z}_1^{(1)}|^p < \infty$. Hence, according to Proposition 3.3, $\mathbb{E}\left|\int_{[0,\,1]} e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(1)}\right|^p < \infty$. Recall the notation $(T_i, R_i)_{i\in\mathbb{N}}$ introduced in the proof of Proposition 3.3 for the jump times and jump sizes of $Z^{(2)}$, respectively. Noting that $T_1, T_2, \ldots$ are also the jump times of $\tilde{Z}^{(2)}$, set $V_i := \tilde{Z}_{T_i}^{(2)} - \tilde{Z}_{T_i-}^{(2)}$ for $i \in \mathbb{N}$ and observe that $|V_i| = R_i$. We infer

$$\begin{aligned}\infty > \mathbb{E}\left|\int_{[0,\,1]} e^{-X_{s-}}\mathrm{d}\tilde{Z}_s^{(2)}\right|^p &\geq \mathbb{E}\left|\sum_{T_k\leq 1} e^{-X_{T_k-}}V_k\right|^p \mathbb{1}_{\{T_1\leq 1<T_2\}} = \mathbb{E}\,|e^{-X_{T_1-}}V_1|^p e^{-c}c\\ &= \mathbb{E}\,e^{-pX_{T_1-}}\,\mathbb{E}\,|V_1|^p e^{-c}c,\end{aligned}$$

where $c = \Lambda_2(\mathbb{R}\setminus[-1,1])$, thereby proving that $\mathbb{E}\,|V_1|^p < \infty$ or, equivalently, that the second inequality in (3.1) holds. The proof of Theorem 3.1 is complete. $\square$

4. Applications to branching Lévy processes

4.1. *Definitions and main result.* Branching Lévy processes are a continuous-time generalization of branching random walks. Similarly to Lévy processes (see (2.2)), branching Lévy processes are characterized by a triplet (σ^2, a, Π), where $\sigma^2 \geq 0$, $a \in \mathbb{R}$ and Π is a sigma-finite measure on

$$\mathcal{P} := \left\{\mathbf{x} = (x_n) \in [-\infty,\infty)^{\mathbb{N}} : x_1 \geq x_2 \geq \cdots \quad \text{and} \quad \lim_{n\to\infty} x_n = -\infty\right\}.$$

Also, it is assumed that Π satisfies

$$\int_{\mathcal{P}} (x_1^2 \wedge 1)\Pi(\mathrm{d}\mathbf{x}) < \infty, \tag{4.1}$$

and that there exists $\theta > 0$ such that

$$\int_{\mathcal{P}} \left(e^{\theta x_1}\,\mathbb{1}_{(1,\infty)}(x_1) + \sum_{j\geq 2} e^{\theta x_j}\right)\Pi(\mathrm{d}\mathbf{x}) < \infty. \tag{4.2}$$

In the sequel we reserve the letter θ to denote a fixed (possibly unique) positive number for which (4.2) holds.

The set of individuals alive at time t which we denote by $\mathcal{N}_t$ can be encoded using an adaptation of Ulam-Harris notation (see Shi and Watson, 2017 for the proposed encoding in the context of compensated fragmentations). For all $s \leq t$ and all individual u alive at time t, we write $X_s(u)$ for the position at time s of u if $u \in \mathcal{N}_s$, and for the position of its ancestor at time s if $u \notin \mathcal{N}_s$.

We outline the evolution of a branching Lévy process with characteristic triplet (σ^2, a, Π) and refer to Sections 4 and 5 in Bertoin and Mallein (2019+) for more details. Denote by $\mathcal{N} = \sum \varepsilon_{(t_k,\mathbf{x}^{(k)})}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{P}$ with mean measure LEB $\otimes\,\Pi$. The position of the initial particle in the branching Lévy process follows the path of the process $(X_t(\varnothing))_{t\geq 0}$ defined by

$$X_t(\varnothing) := \sigma B_t^* + at + \int_{[0,\,t]\times\mathcal{P}} x_1\,\mathbb{1}_{[-1,1]}(x_1)\mathcal{N}^c(\mathrm{d}s\mathrm{d}\mathbf{x}) + \int_{[0,\,t]\times\mathcal{P}} x_1\,\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x_1)\mathcal{N}(\mathrm{d}s\mathrm{d}\mathbf{x}),\quad t\geq 0, \tag{4.3}$$

where $(B_t^*)_{t\geq 0}$ is a Brownian motion independent of $\mathcal{N}$, and the first Poisson integral is taken in the compensated sense (see Section 2 for more details concerning a similar integral). For each atom $(t_k,\mathbf{x}^{(k)})$ of $\mathcal{N}$, the initial particle gives birth at time t_k to new individuals which are started at position $X_{t_k-}(\varnothing)+x_2^{(k)}, X_{t_k-}(\varnothing)+x_3^{(k)},\ldots$. Each of the newborn particles then starts an independent copy of the branching Lévy process from their birth time and position. Note that $(X_t(\varnothing))_{t\geq 0}$ is a Lévy process with characteristic triplet (σ^2,a,Π_1), where Π_1 is the image measure of Π under the mapping $\mathbf{x}\to x_1$, and (4.3) is its Lévy-Itô decomposition (compare with (2.2)). Condition (4.1) guarantees that this Lévy process is well-defined.

For $z\in\mathbb{C}$ with $\mathrm{Re}(z)=\theta$, set

$$\kappa(z) = \frac{1}{2}\sigma^2 z^2 + az + \int_{\mathcal{P}}\left(\sum_{k\geq 1}(e^{zx_k}-1-zx_1\,\mathbb{1}_{(-1,1)}(x_1))\right)\Pi(\mathrm{d}\mathbf{x}).$$

Condition (4.2) ensures that $\kappa(z)$ is finite on its domain. By Bertoin and Mallein (2019+, Theorem 1.1(ii)), we have, for $t\geq 0$,

$$\mathbb{E}\left(\sum_{u\in\mathcal{N}_t} e^{zX_t(u)}\right) = \exp(t\kappa(z)). \tag{4.4}$$

Therefore, it is natural to say that $\kappa(z)$ is the value at z of the *cumulant generating function* of the branching Lévy process. For later needs we also note that according to the many-to-one formula for branching Lévy processes (Bertoin and Mallein, 2019+, Lemma 2.2), the function $\Psi:\mathbb{R}\to\mathbb{C}$ defined by

$$\Psi(s) := \kappa(\theta+\mathrm{i}s)-\kappa(\theta) \tag{4.5}$$

is the Lévy-Khinchine exponent of a Lévy process that we denote by $\xi=(\xi_t)_{t\geq 0}$.

The branching property of the branching Lévy process tells us that conditionally on the positions of the particles at time t the processes initiated by these particles are i.i.d. branching Lévy processes, shifted by the position of their ancestor, see Bertoin and Mallein (2019+, Fact (B)). The branching property in combination with (4.4) imply that the process $W:=(W_t)_{t\geq 0}$ defined by

$$W_t := \sum_{u\in\mathcal{N}_t} e^{\theta X_t(u)-t\kappa(\theta)},\quad t\geq 0 \tag{4.6}$$

is a non-negative continuous-time martingale with respect to the natural filtration. This martingale, often called Biggins' or McKean's martingale, and its a.s. limit W_∞ are of primary importance for the study of branching Lévy processes. According to a classical result in the field of branching processes

$$\mathbb{P}\{W_\infty=0\}\in\{\mathbb{P}\{\exists t>0:\mathcal{N}_t=\varnothing\},1\},$$

i.e., either W_∞ is strictly positive a.s. on the survival set of the branching Lévy process or $W_\infty = 0$ a.s. While the first case is equivalent to the uniform integrability of the martingale W, the second one is called the degenerate case.

We are ready to state the second main result of the present article.

Theorem 4.1. *Let X be a branching Lévy process satisfying (4.1) and (4.2), W the corresponding Biggins martingale, and ξ the Lévy process with the Lévy-Khinchine exponent given in (4.5).*

(i) *The martingale W is uniformly integrable if, and only if,*

$$\lim_{t\to\infty}(\theta\xi_t - t\kappa(\theta)) = -\infty \text{ a.s.}$$

$$\text{and} \quad \int_{\mathcal{P}} \sum_{k\geq 1} e^{\theta x_k} \frac{\log\left(\sum_{j\neq k} e^{\theta x_j}\right)}{A\left(\log\left(\sum_{j\neq k} e^{\theta x_j}\right)\right)} \mathbb{1}_{(e,\infty)}\left(\sum_{j\neq k} e^{\theta x_j}\right)\Pi(\mathrm{d}\mathbf{x}) < \infty, \tag{4.7}$$

where $A(y) = 1 + \int_{\mathcal{P}} \sum_{k\geq 1} e^{\theta x_k}\left((-x_k)\wedge y - 1\right)_+ \Pi(\mathrm{d}\mathbf{x})$ for $y \geq 1$.

(ii) *Let $p \in (1, 2]$. The martingale W converges in L_p if, and only if,*

$$\kappa(p\theta) < p\kappa(\theta) \quad \text{and} \quad \int_{\mathcal{P}} \sum_{k\geq 1} e^{\theta x_k}\left(\sum_{j\neq k} e^{\theta x_j}\right)^{p-1} \mathbb{1}_{(e,\infty)}\left(\sum_{j\neq k} e^{\theta x_j}\right)\Pi(\mathrm{d}\mathbf{x}) < \infty \tag{4.8}$$

In Bertoin and Mallein (2018, Theorem 1.1) similar necessary and sufficient conditions for the uniform integrability of W were obtained under the additional assumption that $\mathbb{E}\,\xi_1 \in (-\infty, \infty)$. A new aspect of part (i) of Theorem 4.1 is that $\mathbb{E}\,\xi_1$ may be infinite or not exist. In Bertoin and Mallein (2018, Proposition 1.4) it was proved that conditions (4.8) entail the L_p-convergence of W under the additional integrability condition $\kappa(q\theta) < \infty$ for some $q > p$.

Using Doney (2007, Theorem 4.15) one can give an integral test expressed in terms of the characteristics of the branching Lévy process which is equivalent to the first condition in (4.7), that is, $\lim_{t\to\infty}(\theta\xi_t - t\kappa(\theta)) = -\infty$ a.s.

Theorem 4.1 will be proved along the lines of the proof of the corresponding result for branching random walks, see the introduction for more details. To this end, in the next section we define a size-biased measure and the corresponding spinal decomposition. The latter as well as Proposition 2.1 and Theorem 3.1 are essential ingredients for the proof of Theorem 4.1.

4.2. *Spinal decomposition.* The spinal decomposition is a useful tool to construct the branching Lévy process under the size-biased law

$$\bar{\mathbb{P}}\big|_{\mathcal{F}_t} := W_t \mathbb{P}|_{\mathcal{F}_t}, \quad t \geq 0,$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration for W. The resulting process is a branching process with the set of distinguished individuals, called the spine. While the individuals belonging to the spine produce offspring and displace them according to a special law, the rest of the population behaves as in a standard branching Lévy process. This justifies the term 'spinal decomposition'.

To explain the evolution of a branching Lévy process with spine we need more notation. Let $\hat{\Pi}$ be a measure on $\mathcal{P} \times \mathbb{N}$ defined by

$$\hat{\Pi}(\mathrm{d}\mathbf{x}\mathrm{d}k) = e^{\theta x_k}\left(\Pi(\mathrm{d}x)\mathrm{Count}(\mathrm{d}k)\right), \tag{4.9}$$

where Count is the counting measure on $\mathbb{N}$. Set

$$\hat{a} = a + \theta\sigma^2 + \int_{\mathcal{P}} \Big(\sum_{k\geq 1} x_k e^{\theta x_k} \, \mathbb{1}_{[-1,1]}(x_k) - x_1 \, \mathbb{1}_{[-1,1]}(x_1) \Big) \Pi(\mathrm{d}\mathbf{x})$$

and note that $\hat{a}$ is well-defined and finite by (4.1) and (4.2). Also, we denote by $\hat{\mathcal{N}}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ with mean measure $\mathrm{LEB} \otimes \hat{\Pi}$ and by $(\hat{B}_t)_{t\geq 0}$ a Brownian motion which is independent of $\hat{\mathcal{N}}$.

Now we define the spine process $\hat{\xi} = (\hat{\xi}_t)_{t\geq 0}$ by the following Lévy-Itô decomposition: for $t \geq 0$

$$\begin{aligned}\hat{\xi}_t :=& \sigma \hat{B}_t + \hat{a}t + \int_{[0,\,t]\times\mathcal{P}\times\mathbb{N}} x_k \, \mathbb{1}_{[-1,1]}(x_k) \hat{\mathcal{N}}^{(c)}(\mathrm{d}s\mathrm{d}\mathbf{x}\mathrm{d}k) \\ &+ \int_{[0,\,t]\times\mathcal{P}\times\mathbb{N}} x_k \, \mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x_k) \hat{\mathcal{N}}(\mathrm{d}s\mathrm{d}\mathbf{x}\mathrm{d}k).\end{aligned}$$

Plainly, $\hat{\xi}$ is a Lévy process with characteristic triplet $(\sigma^2, \hat{a}, \Lambda_1)$, where the Lévy measure is given by

$$\int_{\mathbb{R}} f(-x)\Lambda_1(\mathrm{d}x) = \int_{\mathcal{P}} \Big(\sum_{k\geq 1} e^{\theta x_k} f(x_k) \Big) \Pi(\mathrm{d}\mathbf{x}). \tag{4.10}$$

Further, it can be checked that the Lévy-Khinchine exponent of $\hat{\xi}$ is Ψ defined in (4.5).

We are now ready to discuss briefly the evolution of a branching Lévy process with spine. The spine particle displaces according to the Lévy process $\hat{\xi}$, and for each atom $(t, \mathbf{x}, k)$ of $\hat{\mathcal{N}}$, the spine particle produces offspring at positions $\hat{\xi}_{t-} + x_j$ for all $j \neq k$. Each of these newborn particles then immediately starts an independent branching Lévy process from their birth place and time. Retaining the notation $\mathcal{N}_t$ and $X_s(u)$ (see Section 4.1) for the branching Lévy process with spine we shall also write w_t for the label at time t of the spine particle. With these at hand we denote by $\hat{\mathbb{P}}$ the law of $((X_t(u))_{u\in\mathcal{N}_t, t\geq 0}, (\mathcal{N}_t)_{t\geq 0}, (w_t)_{t\geq 0})$.

Denote by $(\mathcal{H}_t)_{t\geq 0}$ the filtration associated to $(X_t(u))_{u\in\mathcal{N}_t, t\geq 0}$ for the branching Lévy process with spine which excludes the information concerning the labels of the spine individuals.

Lemma 4.2. *We have $\bar{\mathbb{P}}_{|\mathcal{H}_t} = \hat{\mathbb{P}}_{|\mathcal{H}_t}$ for $t \geq 0$ and*

$$\hat{\mathbb{P}}\{w_t = u | \mathcal{H}_t\} = \frac{e^{\theta X_t(u) - t\kappa(\theta)}}{W_t}, \quad t \geq 0.$$

Furthermore, under $\hat{\mathbb{P}}$, $(X_t(w_t))_{t\geq 0}$ is a Lévy process with Lévy-Khinchine exponent Ψ.

The spinal decomposition was introduced in Lyons et al. (1995) in the context of Galton-Watson processes. Lyons (1997) then proved a spinal decomposition result for branching random walks. This result was further generalized to branching Markov chains and general associated harmonic functions in Biggins and Kyprianou (2004), to general Markov processes and multiple spines in Harris and Roberts (2017), etc. In the context of growth-fragmentation processes a proof of the spinal decomposition appeared in Bertoin et al. (2018) for binary compensated fragmentations, i.e., under the assumption $\Pi(\{x_1 > 0\}) + \Pi(\{x_3 > -\infty\}) = 0$. The first general spinal decomposition result for branching Lévy processes was obtained in

Shi and Watson (2017, Theorem 5.2) under the assumption $\Pi(\{x_1 > 0\}) = 0$. A simple argument was given in Bertoin and Mallein (2018, Lemma 2.3) which enabled one to deduce the spinal decomposition for branching Lévy processes from that for branching random walks.

4.3. *Proof of Theorem 4.1.* We start with some preliminary work. Denote by Ω_s the multiset[2] of children's positions at time s relative to the positions of their parents belonging to the spine, i.e.,

$$\Omega_s = \begin{cases} \oslash, & \text{if } \hat{\mathcal{N}}(\{s\} \times \mathcal{P} \times \mathbb{N}) = 0 \\ \{(x_j)_{j \neq k}\}, & \text{if } \hat{\mathcal{N}}(\{(s, \mathbf{x}, k\}) = 1. \end{cases}$$

Setting

$$S_t := \sum_{0 \le s \le t} e^{\theta X_{s-}(w_{s-}) - s\kappa(\theta)} \sum_{z \in \Omega_s} e^{\theta z}, \quad t \ge 0$$

we note that the $\hat{\mathbb{P}}$-a.s. limit $\lim_{t\to\infty} S_t$, provided it is finite, is a Lévy-type perpetuity (see (2.4)) in which the role of X is played by $(-\theta X_t(w_t) + t\kappa(\theta))_{t \ge 0}$ under $\hat{\mathbb{P}}$, and the associated Lévy measures Λ_1 and Λ_2 are given, respectively, by (4.10) and

$$\int_{\mathbb{R}_+} f(x)\Lambda_2(\mathrm{d}x) = \int_{\mathcal{P}} \sum_{k \ge 1} e^{\theta x_k} f\left(\sum_{j \neq k} e^{\theta x_j}\right) \Pi(\mathrm{d}\mathbf{x}).$$

It can be checked that assumptions (4.1) and (4.2) guarantee that the so defined Λ_1 and Λ_2 satisfy (2.1).

To facilitate a forthcoming application of Proposition 2.1 let us note that the second condition in (4.7) is equivalent to

$$\int_{(e,\infty)} \frac{\log y}{A(\log y)} \Lambda_2(\mathrm{d}y) < \infty, \tag{4.11}$$

where $A(x) = 1 + \int_1^x \Lambda_1((y, \infty))\mathrm{d}y$ for $x \ge 1$ as in Section 2 but with Λ_1 as defined above. As far as an application of Theorem 3.1 is concerned observe that $\kappa(p\theta) < p\kappa(\theta)$ which is the first condition in (4.8) is equivalent to

$$\hat{\mathbb{E}} \exp((p-1)(\theta X_t(w_t) - t\kappa(\theta))) = \exp(\kappa(p\theta) - p\kappa(\theta)) < 1. \tag{4.12}$$

The latter is the first condition in (3.1) with X as defined in the previous paragraph. Further, the second condition in (4.8) is equivalent to

$$\int_{(1,\infty)} y^{p-1} \Lambda_2(\mathrm{d}y) < \infty. \tag{4.13}$$

Now we write a basic representation for what follows:

$$W_t^* := \hat{\mathbb{E}}\left(W_t | \mathcal{G}\right) = e^{\theta X_t(w_t) - t\kappa(\theta)} + S_t, \quad t \ge 0, \tag{4.14}$$

where $\mathcal{G}$ is the σ-algebra which contains the information concerning the trajectory of the spine as well as the birth place and the birth times of its offspring.

Passing to the proof of Theorem 4.1 we first deal with the uniform integrability of W.

Lemma 4.3. *Under the assumptions of Theorem 4.1 the martingale W is uniformly integrable if, and only if, conditions (4.7) hold.*

[2] I.e., the set of elements counted with their multiplicity.

Proof: We use the classical observation (see, for instance, p. 220 in Lyons, 1997) that

$$W \text{ is uniformly integrable under } \mathbb{P} \iff \bar{W}_\infty := \limsup_{t\to\infty} W_t < \infty \quad \hat{\mathbb{P}}-\text{a.s.} \tag{4.15}$$

Therefore, it is enough to prove that conditions (4.7) are equivalent to the $\hat{\mathbb{P}}$-a.s. finiteness of $\bar{W}_\infty$.

Assume that conditions (4.7) hold. Since the law of the Lévy process $(\xi_t)_{t\geq 0}$ is the same as the $\hat{\mathbb{P}}$-law of $(X_t(w_t))_{t\geq 0}$, the first condition in (4.7) ensures that

$$\lim_{t\to\infty}(\theta X_t(w_t) - t\kappa(\theta)) = -\infty \quad \hat{\mathbb{P}} - \text{a.s.} \tag{4.16}$$

This entails $\lim_{t\to\infty} W_t^* = \lim_{t\to\infty} S_t$ $\hat{\mathbb{P}}$-a.s. With (4.11) and (4.16) at hand, an application of Proposition 2.1 (recall our specific choice of X) yields $\lim_{t\to\infty} S_t < \infty$ $\hat{\mathbb{P}}$-a.s. and thereupon $\lim_{t\to\infty} W_t^* < \infty$ $\hat{\mathbb{P}}$-a.s. Invoking the conditional Fatou Lemma we further infer

$$\liminf_{t\to\infty} W_t < \infty \quad \hat{\mathbb{P}} - \text{a.s.} \tag{4.17}$$

According to Proposition 2 in Harris and Roberts (2009), $1/W$ is a positive supermartingale under $\hat{\mathbb{P}}$. Thus, $1/W_t$ converges $\hat{\mathbb{P}}$-a.s. as $t\to\infty$. In view of (4.17) the limit cannot be zero. Therefore, $\bar{W}_\infty < \infty$ $\hat{\mathbb{P}}$-a.s. which is equivalent to the uniform integrability of W.

Conversely, assume that W is uniformly integrable or equivalently $\bar{W}_\infty < \infty$ $\hat{\mathbb{P}}$-a.s. Then

$$W_t \geq \sum_{u\in\mathcal{N}_t} e^{\theta X_t(u) - t\kappa(\theta)} \geq e^{\theta X_t(w_t) - t\kappa(\theta)}, \quad t\geq 0$$

entails $\limsup_{t\to\infty}(\theta X_t(w_t) - t\kappa(\theta)) < \infty$ $\hat{\mathbb{P}}$-a.s, whence $\lim_{t\to\infty}(\theta X_t(w_t) - t\kappa(\theta)) = -\infty$ $\hat{\mathbb{P}}$-a.s. This proves that the first condition in (4.7) holds.

Passing to the proof of the second condition in (4.7) we first observe that, for all $0 \leq s \leq t$,

$$W_t \geq \sum_{0\leq r\leq s} e^{\theta X_{r-}(w_r) - r\kappa(\theta)} \sum_{z\in\Omega_r} e^{\theta z} W_t^{(r,z)} \quad \hat{\mathbb{P}} - \text{a.s.},$$

where the random variables

$$W_t^{(r,z)} := \sum_{u\in\mathcal{N}_t} e^{\theta(X_t(u) - X_r(u)) - (t-r)\kappa(\theta)} \mathbb{1}_{\{u \text{ descendant of } z\}}$$

are independent of $\mathcal{G}$ and have the same $\hat{\mathbb{P}}$-distribution as the $\mathbb{P}$-distribution of W_{t-r}. Letting now $t\to\infty$ we infer, for all $s \geq 0$,

$$\bar{W}_\infty \geq \sum_{r\leq s} e^{\theta X_{r-}(w_r) - r\kappa(\theta)} \sum_{z\in\Omega_r} e^{\theta z} W_\infty^{(r,z)} \quad \hat{\mathbb{P}} - \text{a.s.}, \tag{4.18}$$

where $W_\infty^{(r,z)}$ is the limit of the Biggins martingale associated to the descendant of the spine born at time r at position z.

The random variables $(W_\infty^{(r,z)})_{r\geq 0, z\in\Omega_r}$ are i.i.d. In view of the assumption $\bar{W}_\infty < \infty$ $\hat{\mathbb{P}}$-a.s. equivalence (4.15) ensures $\mathbb{E}\, W_\infty^{(r,z)} = 1$. As a consequence, there exists $\delta > 0$ such that $\mathbb{P}\{W_\infty^{(r,z)} \geq 1\} = \delta$. Setting $e^{(r,z)} = \mathbb{1}_{[1,\infty)}(W_\infty^{(r,z)})$ we conclude that the random variables $(e^{(r,z)})_{r\geq 0, z\in\Omega_r}$ are independent Bernoulli random variables with parameter δ. Now (4.18) implies that, for all $s \geq 0$,

$$\bar{W}_\infty \geq \sum_{r \leq s} e^{\theta X_{r-}(w_r) - r\kappa(\theta)} \sum_{z \in \Omega_r} e^{\theta z} e^{(r,z)} =: \Gamma_s \quad \hat{\mathbb{P}} - \text{a.s.}$$

In particular, there exists a sequence (s_j) such that $\lim_{j\to\infty} \Gamma_{s_j} < \infty$ $\hat{\mathbb{P}}$-a.s.

Assume now that $\lim_{t\to\infty} S_t = \infty$ $\hat{\mathbb{P}}$-a.s., so that $\lim_{j\to\infty}(\Gamma_{s_j}/S_{s_j}) = 0$ $\hat{\mathbb{P}}$-a.s. Since $\Gamma_{s_j}/S_{s_j} \leq 1$ $\hat{\mathbb{P}}$-a.s. Γ_{s_j}/S_{s_j} must converge to 0 in $\hat{\mathbb{P}}$- mean. However, this is not the case, for $\hat{\mathbb{E}}(\Gamma_{s_j}/S_{s_j}) = \delta$, a contradiction. Thus, we have shown that $\lim_{t\to\infty} S_t < \infty$ $\hat{\mathbb{P}}$-a.s. By Proposition 2.1 this implies that the second condition in (4.7) holds. The proof of Lemma 4.3 is complete. □

The proof of the second part of Theorem 4.1 follows by a similar reasoning. We first use the fact that $(W_n)_{n\in\mathbb{N}_0}$ is the Biggins martingale of a branching random walk with the underlying point process $\sum_{u\in\mathcal{N}_1} \varepsilon_{X_1(u)}$. The following result is well-known and can be found in Theorem 3.1 of Alsmeyer and Kuhlbusch (2010), Corollary 5 of Iksanov (2004), Theorem 2.1 of Liu (2000) and perhaps some other articles: the L_p-convergence of $(W_n)_{n\in\mathbb{N}_0}$ for $p > 1$ is equivalent to the following two conditions

$$\kappa(p\theta) < p\kappa(\theta) \quad \text{and} \quad \mathbb{E}\, W_1^p < \infty. \tag{4.19}$$

Another form of the left-hand inequality is given by the first inequality in

$$1 > \mathbb{E} \sum_{u\in\mathcal{N}_1} e^{p(\theta X_1(u) - \kappa(\theta))} = e^{\kappa(p\theta) - p\kappa(\theta)}.$$

As the L_p-convergence of W is obviously equivalent to that of $(W_n)_{n\in\mathbb{N}_0}$, it only remains to check that conditions (4.8) and (4.19) are equivalent.

Lemma 4.4. *Let $p \in (1, 2]$. Assume (4.1) and (4.2) hold and that $\kappa(p\theta) < p\kappa(\theta)$. Then*

$$\mathbb{E}\, W_1^p < \infty \iff \int_{\mathcal{P}} \sum_{k\geq 1} e^{\theta x_k} \Big(\sum_{j\neq k} e^{\theta x_j}\Big)^{p-1} \mathbb{1}_{(e,\infty)} \Big(\sum_{j\neq k} e^{\theta x_j}\Big) \Pi(\mathrm{d}\mathbf{x}) < \infty.$$

Proof: $\Leftarrow$: We intend to prove that $\mathbb{E}\, W_1^p < \infty$ or equivalently $\hat{\mathbb{E}} W_1^{p-1} < \infty$. By Theorem 3.1, conditions (4.12) and (4.13) ensure that $S := \lim_{t\to\infty} S_t < \infty$ $\hat{\mathbb{P}}$-a.s. and that $\hat{\mathbb{E}} S^{p-1} < \infty$. Recalling (4.14) we obtain

$$\hat{\mathbb{E}} W_1^{p-1} \leq \hat{\mathbb{E}}[\hat{\mathbb{E}}(W_1|\mathcal{G})^{p-1}] \leq \hat{\mathbb{E}}\big(e^{(p-1)(\theta X_1(w_1) - \kappa(\theta))} + S_1^{p-1}\big) < \infty$$

having used the conditional Jensen inequality for the first inequality, subadditivity of $x \mapsto x^{p-1}$ on $\mathbb{R}_+$ for the second, and (4.12) together with $\hat{\mathbb{E}} S_1^{p-1} \leq \hat{\mathbb{E}} S^{p-1}$ for the third.

$\Rightarrow$: For $s > 0$ and $z \in \Omega_s$, denote by $(W_u^{(s,z)})_{u\geq 0}$ the Biggins martingale associated to the descendant of the spine born at time s at position z. Setting $\underline{W}_1^{(s,z)} := \inf_{u\in[0,1]} W_u^{(s,z)}$ we obtain

$$\begin{aligned} W_1 &\geq \sum_{0\leq s\leq 1} e^{\theta X_{s-}(w_{s-}) - s\kappa(\theta)} \sum_{z\in\Omega_s} e^{\theta z} W_{1-s}^{(s,z)} \\ &\geq \sum_{0\leq s\leq 1} e^{\theta X_{s-}(w_{s-}) - s\kappa(\theta)} \sum_{z\in\Omega_s} e^{\theta z} \underline{W}_1^{(s,z)} \quad \hat{\mathbb{P}} - \text{a.s.} \end{aligned}$$

The random variables $\underline{W}_1^{(s,z)}$ are $\hat{\mathbb{P}}$-i.i.d., positive with positive probability and independent of all the other random variables occurring under the sum. Using concavity of $x \mapsto x^{p-1}$ on $\mathbb{R}_+$ yields

$$W_1^{p-1} \geq S_1^{p-1} \times \frac{\sum_{0\leq s\leq 1, z\in\Omega_s} e^{\theta X_{s-}(w_{s-})-s\kappa(\theta)} e^{\theta z} (\underline{W}_1^{(s,z)})^{p-1}}{S_1} \quad \hat{\mathbb{P}} - \text{a.s.}$$

Denoting by $\underline{W}_1$ a generic copy of $\underline{W}_1^{(s,z)}$, we deduce $\hat{\mathbb{E}} W_1^{p-1} \geq \hat{\mathbb{E}} S_1^{p-1} \hat{\mathbb{E}} \underline{W}_1^{p-1}$, thereby showing that $\hat{\mathbb{E}} S_1^{p-1} < \infty$.

Using Proposition 3.2 in the same way as in the proof of Theorem 3.1, implication (3.1)⇒ (3.2) we conclude that $\hat{\mathbb{E}} S_1^{p-1} < \infty$ together with (4.12) ensure that $\hat{\mathbb{E}} S^{p-1} < \infty$. Formula (4.13) then follows by Theorem 3.1. The proof of Lemma 4.4 is complete. □

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