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A construction of the Stable Web

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Abstract. We provide a process on the space of collections of coalescing cadlag stable paths and show convergence in an appropriate topology for coalescing stable random walks on the integer lattice.

1. Introduction

A system of coalescing Brownian Motions starting at "every" point in $\mathbb{R}$ and evolving independently before coalescence was first introduced by Arratia (1981a,b). This system has been studied by several authors and motivated the question about the existence of a system of coalescing Brownian Motions starting at "every" point in the space-time plane $\mathbb{R}^2$. Such an object is called the Brownian Web and was introduced by Fontes, Isopi, Newman and Ravishankar in Fontes et al. (2004). In the same paper they prove weak convergence to the Brownian Web under diffusive scaling of the system of simple symmetric one-dimensional coalescing random walks starting on each point in the space-time lattice $\mathbb{Z}^2$. Later Newman, Ravishankar and Sun (Newman et al., 2005) proved an invariance principle related to the Brownian Web; they established the convergence to Brownian web for systems of one-dimensional coalescing random walks under finite absolute fifth moment of the transition probability (allowing for crossing of paths unlike the nearest neighbour walks).

More recently, Evans, Morris and Sen (Evans et al., 2013) studied a system of coalescing α-stable processes, $\alpha > 1$, starting at every point in $\mathbb{R}$. As Arratia

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(1981a) did for Brownian Motion, they proved that the system of α-stable processes are locally finite for every time $t > 0$. And based on this, our main motivation here is to build a stable version of the Brownian Web or simply a "Stable Web" and also prove an invariance principle for it. We point out that an alternative weak topology was introduced in Berestycki et al. (2015) to deal with the convergence of other systems of random coalescing paths that do not have the non-crossing property.

In this note we make a first step at defining the stable web. In subsequent work Hao Xue and the last two authors will generalize the domain of applicability and show that the object defined is equivalent to an object with a more general "smoother" topology. Our objective here is simply to define a reasonable metric on the space of collections of cadlag paths that gives convergence to the "stable web" for suitably normalized coalescing random walks.

The paper is organized as follows. In Section 2 we define the stable aged process or collection of stable aged paths. This is a collection of coalescing stable processes equipped with an associated age process, which will be essential for our approach. Section 3 gives a topology for the space of cadlag aged paths defined in the previous section which we use in order to discuss weak convergence of processes to "our" system of coalescing stable processes. We then give a (somewhat involved) set of criteria for a collection of aged paths to be compact. Section 4 takes this condition for compact sets and shows that the distribution of stable aged paths is tight. The penultimate section introduces finite approximations to the stable web which are used in the last section to establish that in the space presented the suitably renormalized system of coalescing random walks converges in distribution to the stable aged process introduced in Section 2.

2. The Age Process

In the following we have for $n \in \mathbb{Z}_+,\ D_n = \mathbb{Z}/2^n$.

We follow Evans et al. (2013) and consider systems of coalescing identically distributed stable processes $\overline{X}^{D_n} = \{X^{n,x} = (X^{n,x}_t)_{t\geq 0} : x \in D_n\}$ with stable index $\alpha \in (1,2)$ such that $X^{n,x}_0 = x = i2^{-n}$ for integers i and n. It is not essential but since establishing our results in greatest generality is not paramount, we will suppose that the processes are symmetric. The stable processes evolve independently until coalescence. The rules of precedence will be arbitrary for points in D_n/D_{n-1} but lower order points will have coalescence precedence so that $\overline{X}^{D_n} \subset \overline{X}^{D_{n+1}}$ for each $n \geq 1$. For $x \in D = \cup_{n\geq 1} D_n$, take n such that $x \in D_n/D_{n-1}$ and simply write $X^{n,x} = X^x$. Moreover we denote $\overline{X} = \cup_{n\geq 1}\overline{X}^{D_n}$ and for every $t > 0$, $\overline{X}^{D_n}_t = \cup_{x\in D_n} X^x_t$ and $\overline{X}_t = \cup_{x\in D} X^x_t$ which are the time level sets associated to the set valued processes $\overline{X}^{D_n}$ and $\overline{X}$ respectively.

Proposition 2.1. *There exists $K < \infty$ so that $\forall\ n \geq 1$ the density of the process, $\overline{X}^{D_n}$ at time $t < \infty$ $D(n,t)$, given by the a.s. limit of*

$$\lim_{M\to\infty} \frac{1}{2M} \#\big(\overline{X}^{D_n}_t \cap [-M, M]\big),$$

satisfies $D(n,t) \leq \frac{K}{t^{\frac{1}{\alpha}}}$.

Proof: To prove this we simply adapt the ideas of Bramson and Griffeath (1980). Given $t > 0$ and a positive integer n, we divide up time interval $[0, t]$ into intervals

$$[0, 2^{-n\alpha}],\ [2^{-n\alpha}, 2^{-(n-1)\alpha}], \ldots, [2^{-(r+1)\alpha}, 2^{-r\alpha}],\ [2^{-r\alpha}, t]$$

where $r < n$ is an integer such that $2^{-r\alpha} \le t \le 2^{-(r-1)\alpha}$. We may assume that $t \ge 2^{-n\alpha}$ since otherwise the density of $\bar{X}^{D_n}$ is less than $2^n \ \le\ t^{-1/\alpha}$.

We now fix $\varepsilon > 0$ so that for $X^1_{\cdot}$ and $X^2_{\cdot}$ independent stable processes starting at distance less than or equal to ε apart, the probability that the two processes meet before time 1 is at least $2\left((1\ -\ (\frac{2\sqrt{2}-1}{3})\right)$. We fix

$$K \ = \ \frac{82^{\frac{1}{\alpha}}}{\varepsilon(2^\alpha - 1)^{\frac{1}{\alpha}}}.$$

Lemma 2.2. *If the density of $\bar{X}^{D_n}_s$ is less than or equal to $K/s^{1/\alpha}$, then the density of $\bar{X}^{D_n}_{s2^\alpha}$ is less than $K/(2s^{1/\alpha})$*

The proof uses only the scaling property of the system of coalescing stable processes. Write $s_0 < s_1 < s_2$ for $s = s_0,\ s2^\alpha = s_2$ and s_1 the midpoint between the two.

We first suppose for $i\ =\ 0$ or 1 that $\overline{X}^{D_n}_{s_i}$ has density greater than $K/(2s^{1/\alpha})$. Then the density of points, x in $\overline{X}^{D_n}_{s_i}$ so that $(x, x + \frac{8s^{1/\alpha}}{K}]$ contains no points of $\overline{X}^{D_n}_{s_i}$ has density less than $\frac{1}{4}$ of the overall density. We denote these half isolated points by B. The remaining points are all within distance $\frac{8s^{1/\alpha}}{K}$ of another point of the process at time s_i. Accordingly by our choice of K and simple scaling the probability of such a process coalescing in the next $s\frac{2^\alpha-1}{2}$ time units is at least $2\left((1\ -\ (\frac{2\sqrt{2}-1}{3})\right)$. From this we see that the density of $\bar{X}^{D_n}_{s_{i+1}}$ will be less than $\frac{1}{\sqrt{2}}$ that of the density at time s_i.

This implies that either at time s_0 or at time s_1 the density of $\bar{X}^{D_n}_{\cdot}$ will be less than $K/(2s^{1/\alpha})$ (and so by monotonicity, the density at time s_2 will also be) or the density at time s_2 will be $(\frac{1}{\sqrt{2}})^2$ that at time s_0 which again implies that the density at time $s_2 = s2^\alpha$ will be less than $K/(2s^{1/\alpha})$.

Applying Lemma 2.2 successively at time $2^{-j\alpha}$ yields that the density at time $2^{-r\alpha}$ will be less than $K2^{-r}\ \le\ K2^{1/\alpha}/t^{1/\alpha}$, provided K is fixed large (or equivalently ε was chosen sufficiently small).

□

It follows from the fact that 0 is regular for the stable processes that if we choose a $x \notin D$ and start a stable process at x at time 0, it will coalesce with processes starting at D before time t for any $t > 0$ (given precedence to the latter), then $P(X^x_t \in \overline{X}_t\ \forall t > 0) = 1$. For any $x \in \mathbb{R}$ we can indeed unambiguously define a stable process $(X^x_t)_{t \ge 0}$ which has the same distribution as a stable process starting at x and such that $t > 0,\ X^x_t \in \overline{X}_t$. So we can think of the above process as a collection of coalescing stable processes starting "on" $\mathbb{R}$. For any countable set $E \subset \mathbb{R}$ we denote $\overline{X}^E = \{X^x : x \in E\}$. It follows that if for any nested collection of "translation invariant" points V_n with $V\ =\ \cup_{n \ge 1} V_n$ dense we have (with the coalescence rules with $\overline{X}_t$ as above) that $\overline{X}^{V_n}_t \subset \overline{X}_t\ \forall\ t > 0$ with probability 1. But equally we can show a.s that $\overline{X}_t \subset \overline{X}^V_t$. Furthermore for any strictly positive

c we can choose V_0 to be points spaced c apart and containing 0 and V_n obtained from V_{n-1} by adding the midpoints between neighbours. Then we have that

$$\overline{X}_t^{V_n} \xrightarrow{D} \overline{X}_t\,, \quad \overline{X}_t^{D_n} \xrightarrow{D} \overline{X}_t^V \quad \text{and} \quad \frac{1}{c}\ \overline{X}_{c^\alpha t}^{V_n} \overset{D}{=} \overline{X}_t^{D_n}\,,$$

together this yields.

Proposition 2.3. *For the process $\overline{X}_t$ the density at time t is equal to $k/t^{\frac{1}{\alpha}}$ for some k depending on our choice of the stable process.*

To construct the stable web we now consider coalescing stable processes starting at *times* $t \in D$.

The first step is for $D_0 = \mathbb{Z}^1$. We define the stable coalescing processes starting at $\{i\} \times \mathbb{R}$. For $t \in (i, i+1]$ let this process be $\overline{X}_t^i$. At time $t = i+1$, $\overline{X}_{i+1}^i$ will be a countable collection of points on $\{i+1\} \times \mathbb{R}$. As such they can be continued on interval $[i+1, i+2]$ so that $\overline{X}_s^i \subset \overline{X}_s^{i+1}$ for every $s \in [i+1, i+2]$ with probability 1. Continuing we have $\forall\ i < j\ \ \overline{X}_s^i \subset \overline{X}_s^j\ \ \forall\ \ s > j$ a.s.

We now proceed in an analogous manner adding in stable coalescing processes at times D_n/D_{n-1} to obtain a collection of processes $\{\overline{X}_t^d\}_{t>d}$ for $d \in D$ with the property that a.s. for every $d < d'$, $t > d'$ we have that $\overline{X}_t^d \subset \overline{X}_t^{d'}$. We use the notation $X^{d',d} = (X_t^{d',d})_{t \geq d}$ to denote the stable process beginning at (dyadic) time d at spatial (dyadic) point d'.

We now define the *age* of a process (or path) $(\gamma(s))_{s>d}$ of $\overline{X}^d$: the age of $(\gamma(s), s)$ is simply $s - \inf(d' < d : \gamma(s) \in \overline{X}_s^{d'})$. So the age of $(\gamma(s), s)$ increases continuously at rate 1 but then jumps when the path coalesces with an older path. We note that these age processes are compatible in the sense that if for $d' > d$ if $(\gamma(s))_{s>t}$ of $\overline{X}^d$ is equal to $(\gamma'(s))_{s>t}$ of $\overline{X}^{d'}$ on (t, ∞) for some $t > d'$), the two age processes agree on (t, ∞).

While at time $t > d$ the ages of the different processes of $\overline{X}_t^d$ will be unbounded, it should be noted that by Proposition 2.3 above, the density of points in $\overline{X}^{-N}$ at time t is equal to $k/(N-t)^{1/\alpha}$ which tends to zero as N becomes large for fixed t. The event that any process of $\overline{X}^d$ at time t in the spatial interval $[-M, M]$ having age greater than $N - t$ is precisely the event that $\overline{X}^{-N}$ at time t has points in $[-M, M]$. This tends to zero as N becomes large (with t and M fixed) . This ensures that with probability one the ages are all finite for processes of $\overline{X}^d$ in $[-M, M]$ a time t. Since M is arbitrary we have that all ages are finite.

This idea of age is by no means novel, see Fontes et al. (2006).

We can now define our aged path process. This is a collection of pairs of cadlag functions $(\gamma,\ a)$ defined on (σ, ∞) for some finite σ so that for each $d \in D$ strictly greater than σ, γ restricted to (d, ∞) will be an element of $\overline{X}^d$ and a on this interval will be the corresponding age process and σ is the smallest possible value in the sense that $a(t)$ tends to zero as t tends down to σ (or equivalently the functions $(\gamma,\ a)$ are defined on maximal open intervals).

We call this (random) collection of paths the stable web and denote it by $\mathcal{X}$. It is defined as a (H, ρ) valued random variable where H is a set whose elements are closed subsets of the set (G, ρ') of ordered pair of cadlag aged paths (γ, a). ρ' is a compactified Skorohod metric while ρ is the induced Hausdorff metric on the closed subsets of G. The spaces and metrics will be defined in the next section. It may be useful to note that for each dyadic space time point (d, d'), for every $t > d'$, there will exist (infinitely many) elements (γ, a) defined on (σ, ∞) so that $t > \sigma$ and γ agrees with $X^{d,d'}$ on time interval $[t, \infty)$ but there will not typically exist in $\mathcal{X}$ a single (γ, a) so that $\gamma(d')$ is defined and equals d. Again this is not dissimilar to Brownian web behaviour and in no way prevents us talking about a stable process beginning at (d, d').

We could equally have defined the stable web $\mathcal{X}$ as the limit of the aged paths $\{X^{d,d'}\}_{(d,d') \in D_n}$ under the topology to be introduced in the next section, where D_n are finite subsets increasing to D.

We note here one of many similarities with preceding works, in particular Fontes et al. (2004): For a fixed non random (x, t), we can define a stable process $X^{x,t}$ defined on time interval $[t, \infty)$ with $X_t^{x,t} = x$ by taking the limiting process of a fixed sequence of "dyadic" processes $X^{d,d'}$ where d converges to x rapidly and d' converges to t rapidly. Given any countable, dense collection of space time points $\{(x_i, t_i)\}_{i=1}^{\infty}$ we can thus obtain a system of coalescing stable processes $\{X_.^{x_i,t_i}\}_{i=1}^{\infty}$. Arguing as before Proposition 2.3 we have that with probability one for any $(\gamma, a) \in \mathcal{X}$, $a(s) > c$ if and only if γ has coalesced by time s with some $X^{(x_i,t_i)}$ with $t_i < s - c$. Thus we could have defined the system $\mathcal{X}$ via such a system of coalescing stable processes. This shows that the distribution of $\mathcal{X}$ is independent of the dense countable set of space time points used to obtain a system of coalescing stable processes.

A key part of our understanding of $\mathcal{X}$ will be via its image through the operator Φ_ε. Given $(\gamma, a) \in \mathcal{X}$, $\Phi_\varepsilon((\gamma, a))$ is the function pair (γ, a) restricted to $[\sigma_\varepsilon, \infty)$ where $\sigma_\varepsilon = \inf\{s > \sigma : a(s) \geq \varepsilon\}$. $\mathcal{X}_\varepsilon = \Phi_\varepsilon(\mathcal{X})$ will be called the set of ε paths (for $\mathcal{X}$).

We note (it is proved in the Section Four) that for any bounded space time region A and any $\varepsilon > 0$ the subset of ε paths $(\gamma^\varepsilon, a^\varepsilon)$ so that $\gamma^\varepsilon(t) = x$ for some $(x, t) \in A$ is finite. We note that the set of (γ, a) in $\mathcal{X}$ so that $\Phi_\varepsilon((\gamma, a))$ has this property will typically be infinite. A.s. the map Φ_ε is infinitely many to one.

The associated age process $a(.)$ and the function Φ_ε may seem artificial but they are important for our approach as they remove a massive source of irregularities for our collection of coalescing stable processes. Let us consider the space time rectangle $[0, 1]^2$ (though it could be any bounded rectangle with nonempty interior). For each positive integers n and M we can cut it into $2^n \times \lfloor 2^{n\alpha} \rfloor$ subrectangles of spatial side 2^{-n} and temporal side $2^{-n\alpha}$ (plus a small remaining area). If we consider independent coalescing processes beginning at the centre of each rectangle, then the number of processes $X_.$ such that (X_t, t) stays within within their space time rectangle before making a jump of order M will be of order $2^{n\alpha}$ with high probability. This shows that any kind of criterion for tightness such as in Fontes et al. (2004) is not possible. But overwhelmingly these large jumps result from processes of small age. By cutting out the (many) stable processes when their age is small, we remove from consideration the greater part of the overall processes' wildness.

We now wish to define a general class of "aged path" spaces which generalize the above, using the preceding construction as motivation, we define first an *aged path*

Definition 2.4. An *aged path* is a cadlag pair of functions (γ, a) defined on a common open half interval (σ, ∞) to $\mathbb{R} \times \mathbb{R}_+$, so that $a(t+s) - a(t) \geq s \ \forall \ s \geq 0$ and $t > \sigma$.

For $A \subset \mathbb{R} \times \mathbb{R}$, we say that γ *hits* A if there exists $(x,t) \in A$ so that $\gamma(t) = x$. Implicitly t will be such that $\gamma(t)$ is defined. Given an aged path (γ, a), we say it hits A if γ does so. The operator Φ_ε (by abuse of notation we use the same notation for this operator on pairs of paths) from the set of aged paths to itself is defined simply by taking $\Phi_\varepsilon((\gamma, a))$ to be the restriction of (γ, a) to $(\sigma_\varepsilon, \infty)$ where $\sigma_\varepsilon = \inf\{s \geq \sigma : a(s) \geq \varepsilon\}$.

Definition 2.5. A *space* or *collection of aged paths* is a set of aged paths $\{(\gamma_i, a_i)\}_{i \in I}$ having the property that for for every positive integer N the set of pairs $\Phi_{2^{-N}}((\gamma_i, a_i))$ that hit $[-N,N]^2$ is finite and $\gamma(b_N^\gamma)$ is (perhaps as a limit) well defined, where $b_N^\gamma = \inf\{s \geq \sigma_{2^{-N}} \wedge -N : \gamma(s) \in [-N,N]\}$.

We will define a topology on the sets of aged paths in the next section.

We return to our stable web and note that Proposition 2.3 yields the following corollary:

Corollary 2.6. *The density of points of $\mathcal{X}$ with age in the interval $(a, a+\varepsilon)$ at a given time t is equal to $k/a^{1/\alpha} - k/(a+\varepsilon)^{1/\alpha}$.*

We will also need the following result which follows from the fact that the density of processes of age at least M at a particular time tends to zero as M becomes large.

Lemma 2.7. *Given $\varepsilon > 0$ and $N < \infty$, there exists $M = M(\varepsilon, N) < \infty$ so that outside probability ε every path of $\mathcal{X}$ that intersects space time square $[-N,N]^2$ has age less than M at time N.*

Proof: This simply follows from the fact that by Proposition 2.3, the density of coalescing processes, started at time $-(N+M)$ has density $k/(2N+M)^{1/\alpha}$ at time N. So the chance that one such process is in spatial interval $[-N,N]$ at time N is less than $2kN/(2N+M)^{1/\alpha}$. Let $c(N) > 0$ be the infimum of the conditional probability a path be in $[-N,N]$ at time N given that it hits $[-N,N]^2$. The probability that the event of interest occurs is bounded above by $2kN/c(N)(2N+M)^{1/\alpha}$ which will be less than ε for large enough M depending on N and ε. $\square$

3. Topology

First recall (see e.g. Ethier and Kurtz, 1986) the definition of the δ-modulus of continuity for a cadlag path $\gamma : [c,d] \to \mathbb{R}$:

$$\omega(\delta, \gamma, [e,f]) = \inf_{t_i - t_{i-1} \geq \delta} \sup_i \sup_{s,t \in [t_{i-1}, t_i)} |\gamma(t) - \gamma(s)|$$

for $[e,f] \subset [c,d]$. This quantity is important for determining the compactness of sets of cadlag paths.

We use the metric d_1 between two cadlag paths $\gamma_1 : [a,b] \to \mathbb{R}$ and $\gamma_2 : [c,d] \to \mathbb{R}$ (typically but not always we will have $b = d = \infty$) where

$$d_1(\gamma_1, \gamma_2) = |\tanh(a) - \tanh(c)| + \inf_{g:[a,b]\to[c,d]} \left[\sup_{a \leq t \leq b} e^{-|t|} |((\gamma_1(t), t) - (\gamma_2(g(t)), g(t))| \wedge 1) + \sup_{a<s<b} e^{-|s|} |g'(s) - 1| \right].$$

where the infimum is over continuous piecewise differentiable bijections g. This amounts to a compactification of space-time as in Fontes et al. (2004). When dealing with paths defined in a finite time rectangle, we will use the equivalent metric

$$d(\gamma_1, \gamma_2) = |a-c| + \inf_{g:[a,b]\to[c,d]} \left[\sup_{a\le t\le b} (|(\gamma_1(t),t) - (\gamma_2(g(t)), g(t))| \wedge 1) + \sup_{a<s<b} |g'(s)-1| \right].$$

In dealing with aged paths defined over finite intervals

$$(\gamma_i(t), a_i(t)) : [c_i, b_i] \to \mathbb{R}\times(0,\infty)$$

for $i=1,2$, we simply take

$$d((\gamma_1,a_1),(\gamma_2,a_2)) \;=\; d(\gamma_1,\gamma_2) \vee d(a_1,a_2).$$

and similarly for d_1. In the following, when speaking of distance between aged paths $(\gamma_1,a_1),(\gamma_2,a_2)$, we will abuse notation and write $d(\gamma_1,\gamma_2)$.

A similar topology on cadlag functions was introduced in the recent work of Etheridge et al. (2017). It follows immediately,

Lemma 3.1. *For a cadlag function $f:[0,T]\to\mathbb{R}$ let f^η be its restriction to $[\eta,T]$ (for $\eta>0$). $\forall\sigma>0$, there exists η_0 so that $d(f,f^\eta)<\sigma$ for every $0\le\eta\le\eta_0$.*

Proof: Take h to be such that $\sup_{s\le h}|f(s)-f(0)| \le \sigma/10$. Now (for $h>\eta>0$) define path $g:\ [0,T]\to[\eta,T]$ by

$$g(s) = \; s \;\text{ for } s\in[h,T]; \quad g(.) \;\text{ is linear bijection } [0,h] \;\to\; [\eta,h].$$

Then this shows that $d(f,f^\eta) < 2\sigma/10 + \;\eta/h \;+\; \eta$ and so the result follows. □

The above argument in fact yields

Lemma 3.2. *For any $h>\eta$,*

$$d(f^\eta,f) \le \eta \;+\; \frac{\eta}{h} + 2\sup_{0\le s\le h}|f(0)-f(s)|\,.$$

If we are dealing with aged paths

$$d((f^\eta,a^\eta),(f,a)) \le \eta \;+\; \frac{\eta}{h} + 2(\sup_{0\le s\le h}|f(0)-f(s)| \vee a(h)-a(0))\,.$$

Corollary 3.3. *For a stable process $(X(t):t\ge0)$, $\forall\sigma>0$ there exists $\eta_0>0$ so that $\forall\ T>0$,*

$$P(\forall\ \eta\le\eta_0, d(X,X^\eta)<\sigma) > 1-\frac{\sigma^2}{10^6}\,,$$

where we consider the restrictions of X and X^η to $[0,T]$ and $[\eta,T]$ respectively.

Similarly, if for process X and time interval I within its domain of definition, we write X^I as the process with time restricted to I, then we have

Corollary 3.4. *For a stable process $(X(t):t\ge0)$, $\forall\sigma>0$, $\varepsilon>0$ there exists $\eta_0>0$ so that $\forall\ t>\varepsilon$ and $0\le\eta_1,\eta_2\le\eta_0$,*

$$d(X^{[0,t]}, X^{[\eta_1,t+\eta_2]}) < \sigma$$

outside probability $\frac{\sigma^2}{10^6}$.

We will examine the systems of aged paths $\mathcal{X}$ and $\mathcal{X}_\delta$ considering them as random elements of a proper path space which we now define in a natural way based on our previous considerations. Let G be the space of aged paths: $\{(\sigma,\gamma,a)\}$ where γ and a are cadlag functions defined on (σ,∞) and satisfying the conditions given in the previous section.

Denote by Ψ_N the map that associates to an aged path (γ,a) in G its restriction to the interval $[b_N^\gamma, N]$ (where b_N^γ is as in the definition of collections of aged paths). We note that this is not a continuous operator for the given metric between paths.

We denote the composition $\Psi_N \circ \Phi_{2^{-N}}$ by Π_N (recall Φ_δ is defined at the end of Section 2).

We have the semimetric $\rho_{[-N,N]^2}$ on G defined by

$$\rho_{[-N,N]^2}\left((\gamma,a),(\gamma',a')\right) \;=\; d(\Pi_N((\gamma,a)),\Pi_N((\gamma',a'))).$$

We now consider the metric between aged paths (which by abuse of notation we also denote as ρ) by

$$\rho'((b,\gamma,a),(b',\gamma',a')) \;=\; \sum_{N=1}^{\infty} 2^{-N}\min\left\{1,\rho_{[-N,N]^2}\left(\Pi_N(b,\gamma,a),\Pi_N(b',\gamma',a')\right)\right\}.$$

The metric is artificial in that it privileges certain cutoffs and rectangles, whereas the spatial or temporal integer values are not special for stable processes and the ages 2^{-N} are not significant for our coalescing system. However this is a positive in our approach as we will be able to argue that the lack of continuity of our projections Π_N is ultimately not a problem.

We, as usual, take H to be the set of closed subsets of G with the Hausdorff metric, which, is denoted by ρ. We have the usual criterion for tightness (see Fontes et al., 2004).

For every $N \ge 1$ fix $\varepsilon_N > 0$, $M_N > 0$ and $\delta_N \in [0,1)^{\mathbb{N}}$ a sequence tending to zero and put $\vartheta = (\varepsilon_N, M_N, \delta_N)_{N\ge 1}$. We denote by $K(\vartheta)$ the set of collections of aged paths in G such that for each collection $C \in K(\vartheta)$ and for each integer $N \ge 1$:

(i) the number of paths in $\Pi_N(C)$ is less than M_N;
(ii) the age of every path in $\Pi_N(C)$ is less than M_N throughout;
(iii) every path in $\Pi_N(C)$ is contained in $[-M_N, M_N] \times [-N,N]$;
(iv) every path, $(b,\gamma,a) \in \Pi_N(C)$ has $\omega(2^{-r},\gamma,[T_N,N]) \le \delta_N(r)$ for every $r \in \mathbb{N}$, where $T_N = b_N^\gamma$;
(v) every path, $(b,\gamma,a) \in \Pi_{N+1}(C)$ has $\gamma(T_N) \in [-N+\varepsilon_N, N-\varepsilon_N]$ and prior to this it did not enter $[-N-\varepsilon_N, N+\varepsilon_N]$;
(vi) the age process of every path $\gamma \in \Pi_{N+1}(C)$ makes no jump while the age has value in $[2^{-N}-\varepsilon_N, 2^{-N}+\varepsilon_N]$;
(vii) the age process of every path $\gamma \in \Pi_N(C)$ makes no jumps within time ε_N of each other;
(viii) the age process of every path $\gamma \in \Pi_{N+1}(C)$ does not have age in interval $(2^{-N}-\varepsilon_N, 2^{-N}+\varepsilon_N)$ at times in $[-N-\varepsilon_N, -N+\varepsilon_N]$ or $[N-\varepsilon_N, N+\varepsilon_N]$.
(ix) every path, $(b,\gamma,a) \in \Pi_{N+1}(C)$ has $d(\gamma^{[T_N,N]}, \gamma^{[T_N+\eta_1,N+\eta_2]}) \le 2^{-r}$ for every r with $0 \le \delta_N(r) < \varepsilon_N/2$ and $0 \le \eta_1,\eta_2 \le \delta_N(r)$, where again $T_N = b_N^\gamma$.

Proposition 3.5. *The sets $K(\vartheta)$ are compact.*

Proof: Given a sequence of collections $C_1, C_2, \dots$ in K, we prove that we can find a convergent subsequence which converges to an element of K.

If we fix N and consider the paths γ that are in $\Pi_N(C_n)$ for some n, then by conditions (iii) and (iv) this set is compact. Similarly (ii) and (vii) ensure that the age processes will be compact. Since the set of compact sets of trajectories on a bounded domain endowed with the Hausdorff metric is compact, that is $\{\Pi_N(C) | C \in K\}$ is compact for each N, we can take a subsequence, $(C_{n_j^N})$ of (C_n) so that $(\Pi_N(C_{n_j^N}))$ converges. By Cantor diagonal method, we arrive at a subsequence $C_{n_i}, i \geq 1$, such that for every N the sequence $(\Pi_N(C_{n_i}))$ converges to some collection D_N. We must show that from this collection $(D_N)_{N\geq 1}$, we can find an aged path collection C so that C_{n_i} converges to C. It will be clear that any such limit is in $K(\vartheta)$ so the principal task is to produce aged path collection$C$ so that for every N, $\Pi_N(C) = D_N$. The essential step is to show that for each N, $\Pi_N(D_{N+1}) = D_N$. Since $\Pi_N(C_{n_i}) \subset \Pi_N \Pi_{N+1}(C_{n_i})$ It is clear that $D_N \subset \Pi_N(D_{N+1})$. Now we show that $\Pi_N(D_{N+1}) \subset D_N$ by contradiction. Suppose that $\Pi_N(D_{N+1})$ contains a path (b, γ, a) not in D_N. We have by hypothesis that there is a $\delta > 0$ so that (b, γ, a) is distance greater than δ from D_N. We consider a sequence of paths $\gamma_{n_i} \in C_{n_i}$ so that $\Pi_{N+1}\gamma_{n_i}$ converge to (b', γ', a') such that $\Pi_N((b', \gamma', a') = (b, \gamma, a)$. By condition (v) and (viii) $T_N(\gamma_{n_i})$ must be in time $[-N+\varepsilon_N, N-\varepsilon_N]$ or the age $a(T_N)$ must be greater than $2^{-N}+\varepsilon_N$and $\gamma_{n_i}(T_N)$ must be in spatial interval $[-N+\varepsilon_N, N-\varepsilon_N]$. From this and (ix), we see that $\Pi_N\gamma_{n_i}$ must converge to (b, γ, a) and the desired contradiction is achieved.

To construct our limit set C (which will clearly be in K), we need to find a collection of aged paths C so that for each N, $\Pi_N C = D_N$. Fix $(\gamma_N, a_N) \in D_N$. By the above paragraph we can find inductively $(\gamma_M, a_M) \in D_M$ $\forall M > N$ so that $\Pi_N((\gamma_M, a_M)) = (\gamma_N, a_N)$. So γ_M and a_M are cadlag functions defined on intervals $[c_M, d_M]$ so that

$$\forall\ M > M' \ \geq N, \quad [c_{M'}, d_{M'}] \ \subset \ [c_M, d_M] \text{ and}$$
$$\gamma_M|_{[c_{M'}, d_{M'}]} = \gamma_{M'},\ a_M|_{[c_{M'}, d_{M'}]} = a_{M'}.$$

We also have that d_M tends to infinity as M tends to infinity but that (by condition (ii)), $\lim_{M->\infty} c_M > -\infty$. We define γ on $(\lim_{M->\infty} c_M, \infty)$ by $\gamma(s) = \gamma_M(s)$ for any (and by the consistency all) M with $s \in [c_M, d_M]$. Similarly for a.

We have that (γ, a) has the desired property. We take C to be the totality of paths that can be obtained in this way (i.e. starting from some N and taking a convergent sequence of aged paths. It is clear C is our limit. □

4. Tightness of the Stable web

In this section our main purpose is to show

Proposition 4.1. *For each σ, $\exists\vartheta = (\varepsilon_N)_{N\geq 1}, (M_N)_{N\geq 1}, (\delta_N)_{N\geq 1}$ so that*

$$P[\mathcal{X} \notin K(\vartheta))] \ < \ \sigma^2/10^6$$

Remark 4.2. This shows that our measure on aged paths is tight. Given Proposition 3.5 this section will consist of simply working through (not necessarily in order) the hypotheses of Proposition 3.5.

For $N = 1, 2, \ldots$, we consider Γ_N as the random collection of aged paths $\Pi_N(\mathcal{X})$. Let us fix a space time square $S_N = [-N, N]^2$.

Definition 4.3. For a cadlag path $F : I \to \mathbb{R}$, the *variation* of f on $[a, b] \subset I$ is $\sup_{a \le s,t \le b} |f(t) - f(s)|$.

Proposition 4.4. *Given a path (b, γ, a) that intersects S_N while of age at least ε, let*

$$\beta = \beta(\gamma, \varepsilon) = \inf\{t : (\gamma(t), t) \in S_N \text{ and } (\gamma(t), t) \text{ has age } \ge \varepsilon\}\,.$$

Given $N \ge 1$ and $1 > \sigma > 0$ then $\exists\ \eta_1 > 0$ and $N' < \infty$ so that

$$P\big[\exists\ \gamma \text{ that intersects } S_N \text{ while of age at least } \varepsilon \text{ and has variation greater than } \sigma/100 \text{ on } [\beta, \beta + 3\eta_1]\big] \le 10^{-6}\sigma^2\,.$$

and

$$P\big[\exists\ \gamma \text{ that intersects } S_N \text{ while of age at least } \varepsilon \text{ and after time } \beta \text{ leaves } [-N', N']\big] \le 10^{-6}\sigma^2\,.$$

Remark 4.5. Of course we are particularly interested in ε of the form 2^{-N}.

Proof: We only explicitly show the first probability bound, the second following in similar fashion.

We consider paths γ for which the β as defined above lies in $\big(i\frac{\varepsilon}{3}, (i+1)\frac{\varepsilon}{3}\big]$ for fixed i. There are $\le \frac{7N}{\varepsilon}$ such i's for ε small). We fix such an i.

We are interested in the paths' behaviour after the time $\beta(\varepsilon)$. As such it is only necessary to treat a "good" representative. While it may be true that the evolution of the age of path γ immediately before $\beta(\varepsilon, \gamma)$ was very rapid due to several coalescences, for every γ, the behaviour of the path γ on interval $[\beta(\gamma), \infty)$ will equal that of a path γ' on interval $[\beta(\gamma), \infty) = [\beta(\gamma'), \infty)$ for some path whose age at time $i\varepsilon/3$ is at least $\varepsilon/2$. As such to establish the proposition it is enough to treat γ having this property. Henceforth we drop the dependence of β on γ from the notation.

So we are interested in the behaviour on interval $[\beta, \beta + 3\eta_1]$ of paths γ having the property that at time $i\varepsilon/3$ the path has age at least $\varepsilon/2$ and such that in time interval $(i\varepsilon/3, (i+1)\varepsilon/3]$ the path γ meets spatial interval $[-N, N]$. We first note that by Proposition 2.3, the density of the translation invariant collection of processes of age at least $\varepsilon/2$ at time $i\varepsilon/3$ (or indeed any time) is equal to $K/(\varepsilon/2)^{1/\alpha}$. We now consider (for comparison purposes) the system of stable processes beginning with these walkers evolving independently on time interval $[i\varepsilon/3, (i+1)\varepsilon/3]$ without coalescence. This system at time $(i+1)\varepsilon/3$ will have the same density $(K/(\varepsilon/2)^{1/\alpha})$ and will again be translation invariant. Thus the expected number of points for our comparison system in $[-N, N]$ at time $(i+1)\varepsilon/3$ is exactly $2Nk/(\varepsilon/2)^{1/\alpha}$. By the Markov property (applied when a process first enters $[-N, N]$) each process that touches $[-N, N]$ has a probability $\ge C_{N,\varepsilon} > 0$ of being within interval $[-N, N]$ at time $\frac{(i+1)\varepsilon}{3}$ where by symmetry e.g. $C_{N,\varepsilon} > 1/3$ for ε small enough and N large enough. Thus provided ε was fixed sufficiently small, we have that the expectation of the number of comparison processes that touch spatial interval$[-N, N]$ in time interval $[i\varepsilon/3, (i+1)\varepsilon/3]$ is bounded above by

$$6Nk/(\varepsilon/2)^{1/\alpha}\,.$$

This bound must then also apply to our original system of coalescing paths by simple stochastic domination.

So the expectation of the total number (i.e. for every relevant i) is thus $\leq \frac{7N}{\varepsilon}\frac{6kN}{(\frac{\varepsilon}{2})^{\frac{1}{\alpha}}}$. By the Markov property (for each stable process) if we choose η_1 so that

$$P(\sup_{s\leq 3\eta_1}|X(s)-X(0)|\geq \sigma/100)\leq \sigma^2/\left(\frac{7N}{\varepsilon}\frac{6kN}{(\frac{\varepsilon}{2})^{\frac{1}{\alpha}}}.10^7\right),$$

we have that outside probability $\frac{\sigma^2}{10^7}$, the variation on $[\beta,\beta+3\eta_1]]$ is less than $\sigma/100$ for all these paths. □

Proposition 4.6. *Given a path (b,γ,a) that intersects S_N, while of age at least ε, γ, let $\beta=\beta(\gamma,\varepsilon)$ be as in Proposition 4.4. Given N,σ $\exists$ $(\delta(r))_{r\geq 1}$ a positive sequence tending to zero as r tends to infinity so that $P\big[\exists\ \gamma$ that intersects $[-N,N]^2$ while of age at least ε and so that for some $r\geq 1, \omega(2^{-r},\gamma,[\beta,N])>\delta(r)\big]\ \leq \frac{\sigma^2}{10^6}$.*

Proof: The claim follows from the observations $\omega(2^{-r},\gamma,[\beta,N])\to 0$ as $r\to\infty$ and the total number of paths is bounded in probability (from the previous proposition). □

Remark 4.7. The results above speak to properties (i), (iii) and (iv) in the definition of compact set $K((\varepsilon_N)_{N\geq 1},(M_N)_{N\geq 1},(\delta_N)_{N\geq 1})$ while the next proposition addresses (iii). Property (ii) follows from Lemma 2.7.

The following result is important in establishing properties (vi) and (vii)

Proposition 4.8. *Given $\eta_2,\sigma>0$, there is N' so that the probability that a path (b,γ,a) satisfies*

(i) γ hits S_N while of age at least 2^{-N},
(ii) there exists $t\in[-N,N]$ so that $a_\gamma(t)\geq\eta_2$ and $\gamma(t)\in[-N',N']^c$,

is bounded by σ.

Remark 4.9. This result is useful in that it argues that the age is "locally" determined (to within a certain precision). We do not rule out that the true age is determined out at infinity, our claim is that the age within η_2 is decided locally.

Proof of Proposition 4.8: Fix a positive integer k and let A_k be the event that for some i, there is a path γ that intersects interval I_k (defined below) in time in the time interval $\left[\frac{i\eta_2}{3},(i+1)\frac{\eta_2}{3}\right]$ so that

(i) $\left[\frac{i\eta_2}{3},(i+1)\frac{\eta_2}{3}\right]\cap[-N,N]\neq\emptyset$;
(ii) the path has age $\geq\frac{\eta_2}{2}$ at time $\frac{i\eta_2}{3}$;
(iii) $I_k=[2^kN^\beta,2^{k+1}N^\beta]\cup[-2^{k+1}N^\beta,-2^kN^\beta]$ for $\beta\geq\frac{2}{\alpha}$.

We have, since the number of such i is bounded above by $\frac{7N}{\eta_2}$ and the expected number of such paths is dominated by $2\frac{K2^kN^\beta}{\eta_2^{\frac{1}{\alpha}}}$, then the expectation of this number is bounded above by

$$\frac{7N}{\eta_2}\frac{2K2^kN^\beta}{\eta_2^{\frac{1}{\alpha}}}q_N\,.$$

where q_N is the upper bound over possible initial points of the probability that a stable process starting at some point in interval I_k visits$[-N,N]$ before time $2N$ which is of order $N/N^{\beta(\alpha+1)}2^{k(\alpha+1)}$. Thus for some $K'>0$

$$P(A_k)\leq \frac{7KK'}{\eta_2^{1+\frac{1}{\alpha}}}N^{2-\beta\alpha}2^{-k\alpha}\,.$$

Since $\beta\geq\frac{2}{\alpha}$, summing this over $k\geq k_o$ gives $P(\cup_{k\geq k_0}A_k)<\sigma$ for k_0 sufficiently large. Since $\cup_{k\geq k_0}A_k$ contains the event in the statement we are done. □

We use Proposition 4.8 above to show:

Lemma 4.10. *For every $N\in\mathbb{N}$ and $\varepsilon>0$ there exists $\eta>0$ so that $P\big(\exists\,(b,\gamma,a)$ so that $\Pi_N\gamma\neq\emptyset$ and so that $a(\cdot)$ jumps twice in an interval of length η while of age at least $2^{-N}\big)$ is less than ε.*

Remark 4.11. Lemma 4.10 above addresses (vii) of the definition of the compact sets $K(\vartheta)$.

Proof of Lemma 4.10: We fix N and ε. First pick N' according to Propositions 4.8 and 4.4 applied to $N+1$ with $\eta_2=2^{-N}/10$ and $\sigma\;=\;\varepsilon/100$. Then pick N'' in this way with N' substituted for N. Pick M so that the probability that the number of paths in $\Phi_{N+2}(C)$ to touch $(-N'',N'')\times(-N,N)$ is greater than M is less than $\varepsilon/100$. Let the complements of these “expected events” be denoted by B_1,B_2,B_3:

B_1 is the event that there exists a path which touches spatial interval $[-N,N]$ in time interval $[-N-1,N+1]$ while having been outside spatial interval $[-N',N']$ in this time interval while of age greater than $\eta_2=2^{-N}/10$, or there exists such a path which subsequently leaves interval $[-N',N']$.

B_2 is the event that there exists a path which touches spatial interval $[-N',N']$ in time interval $[-N-1,N+1]$ while having been outside spatial interval $[-N'',N'']$ while of age greater than $\eta_2=2^{-N}/10$.

B_3 is the event that the number of paths to touch spatial interval $[-N'',N'']$ during temporal interval $[-N-1,N+1]$ while having age greater than 2^{-N-2} exceeds M.

We divide up the event in question into the union of events $A(i,N)$ where $A(i,N)$ is the event that a path of age $\geq 2^{-N}$ having touched spatial interval $[-N-1,N+1]$, meets two paths also of age $\geq 2^{-N}$ in time interval $[i\eta,(i+2)\eta]$ (which intersects $[-N,N]$) and all three paths were in $(-N'',N'')$ at time $i\eta-2^{-N-1}$.

It is easy to see that the probability of $\cup_{i\eta\in(-N-1,N)}A(i,N)$ occurring but not one of the B_i is less than $Const(N/\eta)2^{2N/\alpha}M^3\eta^{2/\alpha}$: we simply note that for a fixed such interval, for the event to occur (and none of the B_i) at time $i\eta-2^{-N}/4$ all three of the processes must be among the at most M processes of age at least $2^{-N}/2$ in spatial interval $[-N'',N'']$. We have at most M^3 choices for the three processes. Secondly we see that uniformly over the spatial initial points, the probability for two independent stable processes to meet in time interval $[1,1+\delta]$ (one beginning at time 0) is less than $K\delta^{1/\alpha}$ for some universal K. So, by scaling and the Markov property uniformly over the positions of the three processes a time $i\eta-2^{-N}/4$, the probability that the first two meet and then the third in the given time interval is bounded by $K\eta^{2/\alpha}2^{2N/\alpha}$ for some universal K.

Given that our value M has been fixed (and that $\alpha<2$) this upper bound will be less than $\varepsilon/100$ if η is chosen small enough. Thus with these choices of N',N'',M

and η the probability of the event occurring is less than $\varepsilon/100 + \varepsilon/100 + \varepsilon/100 + \varepsilon/100 < \varepsilon$ and the result follows.

□

In a similar way we can show the following which is relevant to (vi) of the definition of $K(\vartheta)$.

Proposition 4.12. *For each $\sigma, \varepsilon > 0$, $N < \infty$, $\exists \eta > 0$ so that the probability that there exists (γ, a) so that*

(i) $\exists s$ so that $\gamma(s) \in [-N, N]$ and $a(s) \in (\varepsilon - \eta, \varepsilon + \eta)$ and so that for some other path $(\gamma' a')$

(ii) $|\gamma(s) - \gamma'(s)| < \eta$ and $a'(s) > a(s)$

is bounded by $\sigma^2/10^6$.

Again by Proposition 4.8 we can restrict attention to paths within N' of the origin. Again we bound the number of such paths that touch this spatial interval during time interval $(-N, N)$ while of age at least 2^{-N-1}. The argument is now as with Lemma 4.10.

Proposition 4.13. *For each $\sigma, \varepsilon > 0$, $N < \infty$, $\exists \varepsilon_N > 0$ so that for event $A(N, \varepsilon_N) \equiv \{\exists \gamma$ having age in interval $(2^{-N} - \varepsilon_N, 2^{-N}, +\varepsilon_N)$ at times in $(-N - \varepsilon_N, -N + \varepsilon_N)$ or $(N - \varepsilon_N, N + \varepsilon_N)$ while in interval spatial $[-N, N]\}$ satisfies*

$$P[A(N, \varepsilon_N)] \ < \ \sigma^2/10^6.$$

Again we sketch. To see this for the time interval $(N - \varepsilon_N, N + \varepsilon_N)$, we first choose N' so that (using Proposition 4.8), outside a set of probability $\sigma^2/10^7$. any path that meets $(-N-1, N+1)$ in time interval $(-N-1, N+1)$ must be within N' of the origin while having age at least $2^{-N}/3$. Outside this small probability event, the claimed event lies in the existence of a path such that at time $N - 2^{-N-1}$ lies in $(-N', N')$ and has age in interval of length $4\varepsilon_N$ around 2^{-N-1}. Given Corollary 2.6, we obtain the result.

We can equally address property (v) in our definition of compact K:

Lemma 4.14. *For each N and each $\delta > 0$, $\exists\ \varepsilon_N > 0$ so that the probability that there exists a path γ which first hits $[N - \varepsilon_N, N + \varepsilon_N]$ at a time before or equal to its first time of hitting $[-N, N]^2$ while of age at least 2^{-N} is less than δ.*

Proof: We denote the "bad" event whose probability we wish to bound by B_N. It follows from the self similarity properties of the stable process that for a stable process $\{X(s)\}_{s \geq 0}$ starting at 1 with $\tau = \inf\{s : X(s) \leq 0\}$, we have $X(\tau) < 0$. By quasi left continuity we get that

$$c(\varepsilon) \equiv \ P^1(\{X(s)\ 0 \leq s \leq \tau\} \cap [-\varepsilon, \varepsilon] \neq \emptyset) \ \to \ 0$$

as ε tends to zero. So by scaling we have for τ now equal to $\inf\{s : X(s)X(0) \leq 0\}$

$$\sup_{|x| \geq a} P^x(\{X(s)\ 0 \leq s \leq \tau\} \cap [-\varepsilon, \varepsilon] \neq \emptyset) \ = \ c(\varepsilon/a).$$

We divide up B_N into four parts:

- $B_N(1)$: a path enters $[-N, N]^2$ while of age at least 2^{-N} which had been outside spatial interval $[-N', N']$ while of age in $[2^{-N}/3, 2^{-N}]$.
- $B_N(2)$: the number of paths inside $[-N', N'] \times [-N, N]$ of age at least $2^{-N}/3$ is greater than N''

- $B_N(3)$: there exists a path that achieves age 2^{-N} while spatially in $[N - \varepsilon'_N, N + \varepsilon'_N]$.
- $B_N(4)$: B_N occurs through a path that hits $[-N, N]^2$ with age at least 2^{-N} which achieved age 2^{-N} while outside $[N - \varepsilon'_N, N + \varepsilon'_N]$.

In the above definition N', N'' and ε'_N will be specified as the proof progresses.

By Proposition 4.8 if N' is fixed high enough, then $P(B_N(1)) < \delta/4$. Similarly we have that for N'' sufficiently large $P(B_N(2)\backslash B_N(1)) < \delta/4$. By applying the Markov property at age time $2^{-N}/3$ we easily see that $P(B_N(3)\backslash(B_N(1) \cup B_N(2))) < CN''\varepsilon'_N 2^{N/\alpha} < \delta/4$ if ε'_N is fixed small enough. Finally

$$P(B_N(4)\backslash(B_N(1) \cup B_N(2) \cup B_N(3))) < CN''c(\frac{\varepsilon_N}{\varepsilon'_N})$$

which is less than $\delta/4$ if ε_N is chosen small enough. □

Propositions 4.6, 4.12, 4.13, Lemma 4.10 and Corollary 3.4 as well as the proof of Proposition 4.4 yield.

Proposition 4.15. *For each σ, there exists $\vartheta = ((\varepsilon_N)_{N\geq 1}, (M_N)_{N\geq 1}, (\delta_N)_{N\geq 1})$ so that*

$$P[\mathcal{X} \notin K(\vartheta))] \ < \ \sigma^2/10^6$$

Remark 4.16. This shows that our measure on aged paths is tight given Proposition 3.5.

5. A Discrete Approximation.

The object in this section is to introduce a coalescing process based on a large but finite number of coalescing stable processes which well approximates the Π_N projection of a given stable web. This approximation result is Proposition 5.3 below. After which some related technical results are prepared. The purpose of the approximation is to facilitate the proof of convergence in distribution of systems of coalescing random walks (which are in the domain of attraction of our stable process) which will be performed in the next section. It is at least plausible that for θ small the web on bounded space-time rectangles will be well approximated by a large but finite system of coalescing stable processes beginning on a fine mesh of space-time points separated by distance θ.

A θ process is simply a stable process beginning at a some space-time point in $\theta\mathbb{Z}^2$. The $(\theta i, \theta j)$-process is simply the stable process beginning at position θi at time θj The θ stable web is the collection of coalescing θ processes . We will also be interested in finite subsets of these processes.

More generally given a space time subset A (either a countable set or a rectangle $[x, y] \times [s, t]$) for a path $(\gamma, a))$ that is a restriction of a path in $\mathcal{X}$ we say its $A-$age, $(a^A(s))$ (on $s \in (\sigma, \infty)$) is the supremum of $s - t$ over paths $X^{x,t}$ so that

(i) $X^{x,t}$ has coalesced with γ by time s and

(ii) (x, t) is in A (if A is countable) or $A \cap D$ (for A a rectangle).

If there has been no such coalescence the A-age will be undefined. Obviously the A-age at time s is less than or equal to $a(s)$.

We say that (given A countable or a rectangle) and $(x, t) \in A$ or in $A \cap D$) that $X^{x,t}$ δ-approximates (γ, a) by time s on time interval I if for each $u \in (s, \infty) \cap I$, $\gamma(u) = X^{x,t}(u)$ and $0 \leq a(u) - a^A(X)(u) < \delta$ where $a^A(X)$ is the $A - age$ for the process $X^{x,t}$.

Proposition 5.1. *$\forall\ \eta,\ \sigma\ >\ 0$ and positive integer N, there exists $\theta > 0$ and finite N'' so that outside probability σ for very path $\gamma\ \in\ \mathcal{X}$ so that $\Pi_N((\gamma,a))$ is nontrivial, there exists $(y,s)\ \in\ A\ =\ \theta\mathbb{Z}^2\cap[-N'',N'']\times.[-N'',N]$ so that $X^{y,s}$ η-approximates (γ,a) by time $\sigma_{2^{-N}}(X^{y,s},a^A(X,.))$ on $[-N,N]$.*

Proof: We use V_N to denote the set of γ which intersect $[-N,N]^2$ after age 2^{-N} tacitly identifying paths which agree after age 2^{-N} so we will consider the number to be finite. For space time rectangle R, we write $U(R,\delta)$ to denote the set of paths that hit R after age δ(again identifying paths which agree after age δ).

Given N we apply Proposition 4.8 twice. First with $(N,\eta/8,\sigma/100)$ to obtain value N' so that with probability at least $1-\sigma/100$, every path (γ,a) that after age 2^{-N} is in V_N has the property that it for $a(s)\geq\eta/8$ $(\gamma,(s),s)\ \in\ [-N',N']\times[-N,N]$.

Next we take M so that every path that hits $[-N',N']\times[-N',N]$ has age less than M outside probability $\sigma/100$ (we can do this by Lemma 2.7). We now apply Proposition 4.8 for $(N^{'}+M,\eta/8,\sigma/100)$ to obtain value N''.

For a path γ in V_N we cannot rule out that the age of the path is defined via paths that "diverge to infinity". But we have that outside probability $3\sigma/100$ every such path has an age determined within $\eta/4$ by paths starting on $[-N'',N'']\times[-N'',N]$ in the sense that outside this probability, for every $(\gamma,a)\in\ V_N)$ there exists a path γ' in $U([-N'',N'']\times[-N'',N],\eta/4)$ so that $(\gamma',a(\gamma')^A)$ (with $A=[-N'',N'']\times[-N'',N]$) $\eta/2$-approximates (γ,a) by time $\sigma_{2^{-N}}(\gamma)$ on $[-N,N]$.

It remains to show that for any relevant γ', we can $\eta/4$-approximate $(\gamma',a(\gamma')^{[-N'',N'']\times[-N'',N]})$ by time $\sigma_{2^{-N}}(\gamma',a(\gamma')^{[-N'',N'']\times[-N'',N]})$ on $[-N,N]$ by a pair $(X^v,a(X)^A)$ for $A=[-N'',N'']\times[-N'',N]\cap\theta\mathbb{Z}^2$ and $v\in\theta\mathbb{Z}^2$ (outside appropriately small probability).

We note that by the argument for Proposition 4.8, there exists $K\ =\ K(N'',\eta)$ so that the number of distinct processes in $U_N([-N'',N'']\times[-N'',N],\eta/4)$ is bounded by K outside of probability $\sigma/100$.

Now (outside the previously " eliminated" event of probability bounded by $3\sigma/100$) every γ' in U_N remains in rectangle $[-N'',N'']\times[-N'',N]$ after age $\eta/4$.

We fix one such path γ' and consider $J\ =\ [t,t+\eta/4]\cap\theta\mathbb{Z}$ where t is the first time the age of γ' exceeds $\eta/4$. For each $s_i\in J$, let x_i be one of the spatial points in $\theta\mathbb{Z}$ closest to $\gamma'(s_i)$. By scaling (and the fact that $\alpha>1$) there is a chance greater than $1/2$ (for θ small enough) chance that γ' and $X^{(x_i,s_i)}$ will coalesce in the next θ time units. Since J has cardinality at least $\eta/(5\theta)$ for θ small we see that outside probability $\sigma/100+K(1/2)^{\eta/(5\theta)}$ every path γ' will be coalesced with a process X^s for $s\in\ [-N'',N'']\times[-N'',N]\cap\theta\mathbb{Z}^2$. The result now follows by taking θ sufficiently small that $K(N'',\eta).(1/2)^{\eta/(5\theta)}$ is less than $\sigma/100$.

$\square$

Notation: Given $A\subset\mathbb{Z}^2$ either countable or a rectangle and $(x,t)\ \in\ A$ (or $A\cap D$), the age process for $X^{x,t}$ (given A, $a^A(X^{x,t})$ has been defined. We can accordingly define $\Pi^A_N(X^{x,t})$ to be the joint process $(X^{x,t},a^A(X^{x,t})$ on the interval $[b^A,\infty)$ where

$$b^A\ =\ \inf\{s:a^A(s)\geq 2^{-N}\text{ and }(X^{s,t}(s),s)\ \in\ [-N,N]^2\,.$$

Proposition 5.1 begets

Corollary 5.2. *Given N and $\sigma > 0$, there exists N' and $\theta > 0$ so that if Ł is the collection of coalescing stable processes in $\theta\mathbb{Z}^2 \cap [-N', N'] \times [-N', N]$, then outside probability σ for every path γ with $(\gamma(s), s) \in [-N, N]^2$ for some $s \in [-N, N]$ with $a(s) \geq 2^{-N}$, we have $\gamma' \in$ Ł with distance*

$$\rho'(\Pi_N(\gamma), \Pi_N^A(\gamma') < \sigma/10.$$

Proof: Given N and σ let us apply Proposition 4.4 with $\varepsilon = 2^{-N}$ to obtain η_1 satisfying the desired condition. We can also, arguing as in Lemma 4.10 suppose that η_1 is sufficiently small that no path hitting $[-N, N]^2$ while of age greater than 2^{-N} has a jump in $[a(2^{-N}) - \eta_1, a(2^{-N}) + \eta_1]$ outside this probability. Now given this η_1 (which we can take to be small compared to 2^{-N}, let η_2 be less than $\sigma\eta_1/100$. We apply Proposition 4.8 with $\eta = \eta_2$ and σ equal to our fixed $\sigma^2/10^6$. This yields our desired N' (We here also suppose that outside this probability no path hitting $[-N, N]^2$ has age greater than N'.). Applying Proposition 5.1, with N, N', η and σ replaced by $\sigma^2/10^6$ we have our θ and outside of probability $2\sigma^2/10^6$, every path γ as above has coalesced with a path in Ł before it has age 2η. The result now follows from Proposition 4.4. □

Propositions 4.4, 4.8 and 5.1 and Corollary 5.2 yield

Proposition 5.3. *$\forall \sigma > 0$ there exists N, N', θ so that outside probability σ the ρ distance between the paths resulting from the stable web and the paths resulting from the $\theta\mathbb{Z}^2 \cap [-N', N'] \times [-N', N+1]$ beginning after age (defined via the processes in $A = \theta\mathbb{Z}^2 \cap [-N', N'] \times [-N', N]$) when the processes touch $[-N, N]$ while of age greater than 2^{-N} is less than σ.*

We denote the system of θ-processes by $\mathcal{X}^\theta$ and the system of θ-processes of age at least δ by $\mathcal{X}^\theta_\delta$. We claim that $\mathcal{X}^\theta$ plays the role of an skeleton for the stable web in an analogous way to the definition of an skeleton for the Brownian Web, see Fontes et al. (2004).

6. Convergence in Distribution

In this Section we only consider discrete time random walks, although analogous results hold for continuous time random walks. Consider a random walk $(W_n)_{n\geq 1}$ such that $W_n = \sum_{i=1}^n Z_n$ with $(Z_n)_{n=1}^\infty$ iid random variables whose distribution is in the domain of attraction of a stable symmetric $\alpha \in (1, 2)$ random variable $X = X_1^{0,0}$ ($X^{0,0}$ defined as in Section 2). Let $p(x) = P(Z_1 = x)$, $x \in \mathbb{Z}$ be its transition probability function. Since the best convergence result is not our focus we assume that $p(\cdot)$ is symmetric and satisfies

$$x^{1+\alpha} p(x) \to C \in (0, \infty)\,, \quad \text{as } |x| \to \infty\,,$$

where the C is chosen to be compatible with X. Thus we have that

$$\left(\frac{W_{\lfloor nt \rfloor}}{n^{\frac{1}{\alpha}}} \right)_{t \geq 0}$$

converges in distribution under the Skorohod topology to $(X_t^{0,0})_{t\geq 0}$. We have the Gnedenko local CLT, see Gnedenko and Kolmogorov (1954),

$$\sup_{|x_0| \leq K n^{\frac{1}{\alpha}}} \left| n^{\frac{1}{\alpha}} P\left(W_n = x_0\right) - f_X\left(\frac{x_0}{n^{\frac{1}{\alpha}}}\right) \right| \to 0\,, \quad \text{as } n \to \infty\,,$$

where f_X is the density of X. This is enough to show convergence of the Green functions: For every $\beta > 0$

$$G_\beta(u) = \lim_{n\to\infty} n^{(\frac{1}{\alpha}-1)} \sum_{j=1}^{\infty} P(W_j = [un^{\frac{1}{\alpha}}])e^{-\beta\frac{j}{n}},$$

uniformly on compact intervals for u.

From this and standard optional stopping, we have

Lemma 6.1. *For every $\beta > 0$, if W^1, W^2 are two iid random walks with increments distributed as $p(\cdot)$ and starting at $W_0^1 = 0$ and $W_0^2 = [n^{\frac{1}{\alpha}}u]$ and $T_n = \frac{1}{n}\inf\{j : W_j^1 = W_j^2\}$ then $\lim_{n\to\infty} Ee^{-\beta T_n} = Ee^{-\beta T}$ where $T = \inf\{t : X_t^1 = X_t^2\}$ for X^1, X^2 iid stable processes distributed as $X^{0,0}$ starting at $X_0^1 = 0$ and $X_u^2 = u$. Therefore T_n converges to T in distribution.*

From this we obtain

Lemma 6.2. *For N, N', ε and θ fixed positive and finite, the system of coalescing stable processes starting from points in $\theta\mathbb{Z}^2 \cap [-N', N'] \times [N', N]$ is the limit as $n \to \infty$ of the system of coalescing random walks starting on $[-N'n^{\frac{1}{\alpha}}, N'n^{\frac{1}{\alpha}}] \times [-Nn, Nn]$ from points in $\lfloor\theta n^{\frac{1}{\alpha}}\rfloor\mathbb{Z} \times \theta n\mathbb{Z}$ and appropriately rescaled.*

Recall the definitions of Section 2 and Proposition 2.3. The system $\overline{X}$ of α-stable processes starting from full occupancy at time 0 is scale invariant and in particular the density scales as $k/t^{\frac{1}{\alpha}}$ for some constant k not depending on t. We now note that the density for coalescing random walks scales (when suitably renormalized) in the same way.

Proposition 6.3. *For coalescing random walks on $\mathbb{Z} \times \mathbb{Z}_+$ beginning with full occupancy at time 0, the density at time n is approximately $k/n^{\frac{1}{\alpha}}$, meaning that the ratio goes to one as n goes to infinity, where k is the constant for the continuous time coalescing processes obtained in Proposition 2.3.*

Proof: By Proposition 2.3 for the system of α-stable coalescing processes the density at time t is the (increasing) limit as $\theta \downarrow 0$ of the processes beginning at $\theta\mathbb{Z}$. Thus for every $\varepsilon > 0$, there exists $\theta > 0$ so that the density of coalescing θ-processes at given times t_1, t_2 are greater than $(k-\varepsilon)/t_1^{\frac{1}{\alpha}}$ and $(k-\varepsilon)/t_2^{\frac{1}{\alpha}}$.

We first take $t_1 = 1$. Now for θ as above, we consider the coalescing random walks beginning at $\lfloor\theta n^{\frac{1}{\alpha}}\rfloor\mathbb{Z}$. By Lemma 6.2, we get that for n large the density of these coalescing random walks at time n is at least $(k-2\varepsilon)/n^{\frac{1}{\alpha}}$. Hence by monotonicity it is at least $(k-2\varepsilon)/n^{\frac{1}{\alpha}}$ for the full process of coalescing random walks (i.e. starting from full occupancy on $\mathbb{Z}$).

On the other hand we can via Bramson and Griffeath arguments (Bramson and Griffeath, 1980), see Lemma 2.2, show that there exists $m < \infty$ so that for every choice of the scale parameter n, the density of the full process of coalescing random walks at time T is bounded above by $m/T^{\frac{1}{\alpha}}$. In particular at $T = rn$, r small, the density is bounded above by $m/(r^{\frac{1}{\alpha}}n^{\frac{1}{\alpha}})$. We now couple this to a coalescing system of random walks starting with occupancy at $\lfloor\theta n^{\frac{1}{\alpha}}\rfloor\mathbb{Z}$ at time $T = rn$ and we take $t_2 = (1-r)$ then the density of the full process of coalescing random walks at time $n = rn + (1-r)n$ is equal to the density of the $\lfloor\theta n^{\frac{1}{\alpha}}\rfloor\mathbb{Z}$ random walks at time $(1-r)n$ plus the density of walks of the full process that have not coupled with a θ random walk by time n. But (if $\theta << 1/m$ is sufficiently small) this latter density will be smaller than $\varepsilon/n^{1/\alpha}$. while the former density (by invariance) will be less than $(k+\varepsilon)/((1-r)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}})$. Thus the density of full random walks at time n will be bounded above by

$$\frac{k+\varepsilon}{(1-r)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}}} + \varepsilon/n^{1/\alpha}\,.$$

We now let $r \downarrow 0$ and then $\varepsilon \downarrow 0$ to obtain our result. $\square$

From now on consider that we work with systems of discrete time coalescing random walks starting at points on $[n^{\frac{1}{\alpha}}]\mathbb{Z} \times n\mathbb{Z}$, making jumps from i to j with probability $p(j-i)$ after intervals of time with length one. The age of such a random walk is defined to be continuously increasing at rate 1 on time intervals of no coalescence and when a coalescence occurs the age jumps to the age of the older path. We call it an aged random walk. We denote this collection of aged random walks (γ, a) with space scaled by $n^{-1/\alpha}$ and time by n^{-1} by $\mathcal{W}_n \in (H, \rho)$ for $n \geq 1$ (so n is the scaling parameter).

Proposition 6.4.

$$\mathcal{W}_n \Rightarrow \mathcal{X} \quad \textit{as } n \to \infty$$

Given Proposition 6.3 we can easily prove the following analogue of Proposition 4.4:

Proposition 6.5. *Given an aged random walk path $(\gamma, a) \in \mathcal{W}_n$ that intersects $S_N = [-N, N]^2$, while of age at least ε, let*

$$\beta = \beta(\gamma, \varepsilon) \;=\; \inf\{t : (\gamma(t), t) \in S_N \textit{ and } a(t) \geq \varepsilon\}.$$

Given N and σ there exists η_1 so that for every n

(i)

$$P\big[\exists\ (\gamma, a) \in \mathcal{W}_n \textit{ that intersects } S_N \textit{ while of age at least } \varepsilon \textit{ and has variation}$$
$$\textit{greater than } \sigma \textit{ on } [\beta, \beta + 3\eta_1]\big] \leq \frac{\sigma^2}{10^6}\,,$$

and (ii)

$$P\big[\exists\ \gamma \textit{ that intersects } S_N \textit{ while of age at least } \varepsilon$$
$$\textit{and after time } \beta \textit{ leaves } [-N', N']\big] \leq 10^{-6}\sigma^2\,.$$

We similarly have analogues of Propositions 4.6, 4.12, 4.13, Corollary 3.4 and Lemmas 4.10 and 4.14. This yields:

Proposition 6.6. *For each $\sigma > 0$, there exists $\vartheta = ((\varepsilon_N)_{N\geq 1}, (M_N)_{N\geq 1}, (\delta_N)_{N\geq 1})$ so that for each $\delta > 0$, there exists $n_0 \;=\; n_0(\delta) \;< \infty$ so that for $n \geq n_0$*

$$P[\mathcal{W}_n \notin K^\delta(\vartheta)] < \sigma^2/10^6\,,$$

where $K^\delta = \{\psi : \rho(\psi, K) \leq \delta\}$. In particular $(\mathcal{W}_n)_{n\geq 1}$ is a tight family of random elements of H.

Remark 6.7. We need to consider K^δ rather than simply K since we need the convergence of renormalized random walks to continuous time stable processes that ensures the desired approximation for paths $\Pi_N\gamma$, $N \geq 1$.

The aim from this point is to prove weak convergence of $\mathcal{W}_n$ to $\mathcal{X}$. Our argument uses the approximation of $\mathcal{X}$ by θ-processes and we need an analogous approximation for the system of aged random walks. So we consider the system of θ-random walks associated to the scaling parameter n as the collection of discrete time rescaled coalescing random walks obtained from coalescing random walks starting at $\lfloor \theta n^{\frac{1}{\alpha}} \rfloor \mathbb{Z} \times \theta n \mathbb{Z}$ that evolves as before (jumps from i to j with probability $p(j-i)$ after one unit of time), with space scaled by $n^{-\frac{1}{\alpha}}$ and time by n^{-1}. To simplify notation we write $X^{\theta i, \theta j}$ for the θ-random walk starting at $(\lfloor \theta n^{\frac{1}{\alpha}} \rfloor n^{-\frac{1}{\alpha}} i, \theta j)$.

Given a collection of θ-random walks, we can, just as in the original process, speak of ages of paths: the age of a path $X^{\theta i, \theta j}$ at time $s > j\theta$ is simply

$$s - \inf\{\theta l : \exists k : X_s^{\theta i, \theta j} = X_s^{\theta k, \theta l}\}.$$

We then denote by $\mathcal{W}_n^\theta$ the system of aged coalescing θ-random walks with scaling parameter n and given N and N' we write $\mathcal{W}_n^{\theta, N'}$ for the system of coalescing θ-random walks with scaling parameter n beginning at points inside the space-time box $[-N-1, N] \times [-N', N']$. (Usually N is given and so is dropped from the notation.)

As in Proposition 6.6, we can show that $(\mathcal{W}_n^\theta)_{n \geq 1}$ is a tight family of random elements of H. Moreover, we can prove as in Corollary 5.2 the following result:

Proposition 6.8. *Given N, N' and θ, the coalescing renormalized systems $\mathcal{W}_n^{\theta, N'}$ converge in distribution to $\mathcal{X}^{\theta, N'}$ as n tends to infinity. Furthermore $\Pi_N(\mathcal{W}^{\theta, N'_n})$ converges in distribution to $\Pi_N(\mathcal{X}^{\theta, N'})$.*

We are now ready to establish weak convergence of $\mathcal{W}_n$ to $\mathcal{X}$. (Proof of Proposition 6.4)

Proof: To establish weak convergence it is sufficient to show that for a bounded and continuous F on our space

$$E[F(\mathcal{W}_n)] \ \to \ E[F(\mathcal{X})]$$

as n tends to infinity.

We fix $\varepsilon > 0$ and a bounded continuous function F on the set of aged path collections. Given $\varepsilon > 0$, we fix a compact set K of collections of paths as in Proposition 3.5 so that the probability that $\mathcal{X} \in K$ is at least $1 - \varepsilon$. By the compactness of K we have that there exists $\eta > 0$ so that

$$\forall \psi \in K, \quad \sup_{\psi' : \rho(\psi, \psi') < 2\eta} |F(\psi) - F(\psi')| \quad < \ \varepsilon/2$$

which immediately implies that

$$\forall \psi \in K^\eta, \quad \sup_{\psi' : \rho(\psi, \psi') < \eta} |F(\psi) - F(\psi')| \quad < \ \varepsilon$$

where again $K^\eta = \{\psi : \rho(\psi, K) \leq \eta\}$.

By Proposition 6.8 for any $\theta > 0$ (and N, N'),

$$\lim_{n \to \infty} \left| E[F(\mathcal{W}_n^\theta)] - E[F(\mathcal{X}^\theta)] \right| = 0.$$

We choose θ, N and N' according to Proposition 5.3 so that the distance between $\mathcal{X}$ and $\mathcal{X}^\theta$ is less than η outside probability ε. We can take this θ, N and N' so that we have equally for each n, the distance between $\mathcal{W}_n$ and $\mathcal{W}_n^\theta$ is less than η outside this probability. We now have

$$\begin{aligned}\left|E[F(\mathcal{X})] - E[F(\mathcal{X}^\theta)]\right| &\leq E\left[\left|F(\mathcal{X})] - F(\mathcal{X}^\theta)\right|\right]\\ &\leq E\left[\left|F(\mathcal{X}) - F(\mathcal{X}^\theta)\right| I_{\mathcal{X}\notin K}\right]\\ &\qquad + E\left[\left|F(\mathcal{X}) - F(\mathcal{X}^\theta)\right| I_{\rho(\mathcal{X},\mathcal{X}^\theta)<\eta} I_{\mathcal{X}\in K}\right]\\ &\qquad\quad + E\left[\left|F(\mathcal{X}) - F(\mathcal{X}^\theta)\right| I_{\rho(\mathcal{X},\mathcal{X}^\theta)\geq\eta}\right].\end{aligned}$$

This latter sum is bounded by $2\varepsilon||F||_\infty \ + \ \varepsilon \ + 2\varepsilon||F||_\infty \leq 5\varepsilon\,(1 \vee ||F||_\infty)$.

We also have that for any $\eta > 0$, $P(\mathcal{W}_n \in K^\eta) > 1 - \varepsilon/2$ for n large and so we can argue as above that for universal C

$$\left|E[F(\mathcal{W}_n)] - E[F(\mathcal{W}_n^\theta)]\right| \leq C\varepsilon\,(1 \vee ||F||_\infty)$$

for n large which gives that $\left|E[F(\mathcal{W}_n)] - E[F(\mathcal{X})]\right| \leq C'\varepsilon\,(1 \vee ||F||_\infty)$ for n large. $\square$

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