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Darling–Erdős theorem for Lévy processes at zero

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Abstract. We establish two equivalent versions of the Darling–Erdős theorem for Lévy processes in the domain of attraction of a stable process at zero with index $\alpha \in (0,2)$. In the course of our proofs we obtain a number of maximal and exponential inequalities for general Lévy processes, which should be of separate interest.

1. Introduction

Let $\{\xi_k\}_{k\geq 1}$ be a sequence of independent mean zero and variance one random variables and for each $n \geq 1$ set $S_n = \xi_1 + \cdots + \xi_n$. Darling and Erdös (1956) proved that if the third absolute moments of the $\{\xi_k\}_{k\geq 1}$ are uniformly bounded then for all x, as $n \to \infty$,

$$\mathbf{P}\left(A(n) \max_{1\leq k\leq n} S_k/\sqrt{k} - B(n) \leq x \right) \to \exp\left(-\exp\left(-x\right)\right), \tag{1.1}$$

where we use the notation for $T > 0$, $A(T) = (2LLT)^{1/2}$ and $B(T) = 2LLT + 2^{-1}LLLT - 2^{-1}L(4\pi)$, with $LT = \log(T \vee e)$. Such a limiting distribution result is now often called a Darling–Erdős theorem. Einmahl (1989) showed in the i.i.d. mean zero and variance one case that for (1.1) to hold it is necessary and sufficient that

$$LLt\,\mathbf{E}\left\{\xi_1^2 1\{|\xi_1| \geq t\}\right\} \to 0, \text{ as } t \to \infty.$$

Einmahl and Mason (1989) have obtained martingale Darling–Erdős theorems, and recently Dierickx and Einmahl (2018) have established multivariate versions. Corresponding results for Brownian motion were established by Khoshnevisan et al. (2005).

In the infinite-variance case Bertoin (1998) proved Darling–Erdős theorems for sums of i.i.d. random variables in the domain of normal attraction of an α-stable

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law. More precisely, if $\mathbf{P}(\xi > x) \sim cx^{-\alpha}$ and $\mathbf{P}(\xi \leq -x) = O(x^{-\alpha})$, as $x \to \infty$, for some $c > 0$ and $\alpha \in (0,1) \cup (1,2)$, and $\mathbf{E}\xi = 0$ for $\alpha > 1$ then for any $x \geq 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\max_{k\leq n} k^{-1/\alpha} S_k \leq x(\log n)^{1/\alpha} \right) = e^{-cx^{-\alpha}}. \tag{1.2}$$

Our work was motivated by this result. In fact, our Theorem 3.7 is a Lévy process version of Theorem 1 in Bertoin (1998).

Let $X_t, t \geq 0$, be a Lévy process in the domain of attraction of a stable process at zero with index $\alpha \in (0,2)$. Introduce the running supremum and the maximum jump processes as

$$\overline{X}_t = \sup_{s\leq t} X_s, \quad m_t = \sup_{0<s\leq t} \Delta X_s = \sup_{0<s\leq t} (X_s - X_{s-}).$$

We consider for an appropriate positive increasing function $a(t)$ of $t > 0$ the maximum of the scaled running supremum, the maximum of the scaled process, and the maximum of the scaled maximum jump process, defined as

$$Y_t = \sup_{t\leq s\leq 1} \frac{\overline{X}_s}{a(s)}, \quad Z_t = \sup_{t\leq s\leq 1} \frac{X_s}{a(s)}, \quad M_t = \sup_{t\leq s\leq 1} \frac{m_s}{a(s)}. \tag{1.3}$$

For $\alpha = 1$ the definitions of Y and Z are slightly different, see Theorems 3.6 and 3.7. Our goal is to derive analogues of (1.1) and (1.2) for the Lévy process $X_t, t > 0$. In particular, we shall prove in our Theorem 3.5 that under suitable regularity conditions for all $x > 0$, in the case $\alpha \neq 1$,

$$\lim_{t\downarrow 0} \mathbf{P}\left(Y_t \, (-\log t)^{-1/\alpha} \leq x \right) = e^{-x^{-\alpha}},$$

and from this result we shall derive its Darling–Erdős version in Theorem 3.7

$$\lim_{t\downarrow 0} \mathbf{P}\left(Z_t \, (-\log t)^{-1/\alpha} \leq x \right) = e^{-x^{-\alpha}}.$$

Along the way, in our Theorem 3.2 we establish a similar result for the scaled maximum jump process M_t. We fix our notation in Section 2, state our results in Section 3 and detail our proofs in Sections 4 and 5, where we derive some maximal and exponential inequalities for general Lévy processes, which should be of separate interest.

2. Notation

In this section we give our basic setup. Let $X_t, t \geq 0$, be a Lévy process with Lévy measure Λ and without a normal component. For $x > 0$ put $\overline{\Lambda}_+(x) = \Lambda((x,\infty))$, $\overline{\Lambda}_-(x) = \Lambda((-\infty,-x))$, and for $u > 0$ let

$$\varphi(u) = \sup\{x : \overline{\Lambda}_+(x) > u\}. \tag{2.1}$$

Note that $\overline{\Lambda}_+(x) > u$ iff $\varphi(u) > x$. Let N be a Poisson random measure on $[0,\infty)\times\mathbb{R}$ with intensity measure $\mu(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t \times \Lambda(\mathrm{d}x)$ and let $\widetilde{N}(\mathrm{d}t, \mathrm{d}y) = N(\mathrm{d}t, \mathrm{d}y) - \mathrm{d}t\Lambda(\mathrm{d}y)$ be the compensated Poisson measure. By the Lévy-Itô representation for suitable shift parameters γ_+ and γ_-, with $\gamma = \gamma_+ + \gamma_-$,

$$\begin{aligned} X_t &= \gamma t + \int_0^t \int_{|y|>1} y N(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{|y|\le 1} y \widetilde{N}(\mathrm{d}s, \mathrm{d}y) \\ &= \gamma_+ t + \int_0^t \int_{(1,\infty)} y N(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{(0,1]} y \widetilde{N}(\mathrm{d}s, \mathrm{d}y) \\ &\quad + \gamma_- t + \int_0^t \int_{(-\infty,-1)} y N(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{[-1,0)} y \widetilde{N}(\mathrm{d}s, \mathrm{d}y) \\ &=: X_t^+ + X_t^-. \end{aligned} \tag{2.2}$$

We assume that X^+ belongs to the domain of attraction at zero of an α-stable law for some $\alpha \in (0,2)$, which means that for some norming and centering functions $a(t), c(t)$

$$\frac{X_t^+ - c(t)}{a(t)} \xrightarrow{\mathcal{D}} X, \quad \text{as } t \downarrow 0, \tag{2.3}$$

where X is an α-stable law. This happens if and only if

$$\overline{\Lambda}_+(x) = x^{-\alpha} \ell(x), \tag{2.4}$$

where ℓ is a slowly varying function at 0; see Bertoin (1996, p.82), Maller and Mason (2008, Theorem 2.3). In what follows we assume that the constants $\gamma_\pm$ are chosen such that

$$\begin{aligned} \gamma_+ &= \begin{cases} \int_{(0,1]} y \Lambda(\mathrm{d}y), & \text{if } \int_{(0,1]} y \Lambda(\mathrm{d}y) < \infty, \\ 0, & \text{otherwise,} \end{cases} \\ \gamma_- &= \begin{cases} \int_{[-1,0)} y \Lambda(\mathrm{d}y), & \text{if } \int_{[-1,0)} |y| \Lambda(\mathrm{d}y) < \infty, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

Note that the integral $\int_{(0,1]} y \Lambda(\mathrm{d}y)$ is always finite for $\alpha \in (0,1)$ and infinite for $\alpha \in (1,2)$, while for $\alpha = 1$ both cases can happen.

Since the function $b(t) = \varphi(1/t)$ can be shown to satisfy with, obvious changes of notation, (2.9) and (2.10) on page 320 of Maller and Mason (2008), we can without loss of generality assume that $a(t)$ in (2.3) is increasing, moreover

$$a(t) = \varphi(1/t). \tag{2.6}$$

Furthermore, using the same remark in Maller and Mason (2008) the function $c(t)$ in (2.3) can be chosen as

$$c(t) = t\nu(a(t)) = t \left(\gamma_+ - \int_{(a(t),1]} y \Lambda(\mathrm{d}y) \right), \tag{2.7}$$

where for $y > 0$

$$\nu(y) = \gamma_+ - \int_{(y,1]} u \Lambda(\mathrm{d}u).$$

For $0 < \alpha < 2$, with $\alpha \neq 1$, it can be shown using standard properties of regularly varying functions that, by the choice of γ_+ given in (2.5),

$$\lim_{t \downarrow 0} \frac{c(t)}{a(t)} = \frac{\alpha}{1-\alpha}.$$

This says that (2.3) holds with $c(t) = 0$ when $0 < \alpha < 2$, with $\alpha \neq 1$; moreover, in this case, we shall from now on assume that $c(t) = 0$.

3. Results

From the monotonicity of a it is simple that

$$M_t = \sup_{t \le s \le 1} \frac{m_s}{a(s)} = \sup \left\{ \frac{\Delta X_s}{a(s)} : s \in (t, 1], \Delta X_s > 0 \right\} \vee \frac{m_t}{a(t)},$$

where $a \vee b = \max\{a, b\}$. This simple observation allows us to calculate the distribution of M_t. Indeed, for $x > 0$ put

$$A_{t,x} = \left\{ (u, y) : \frac{y}{a(u)} > x,\ u \in (t, 1] \right\}$$
$$B_{t,x} = \left\{ (u, y) : \frac{y}{a(t)} > x,\ u \in (0, t] \right\}.$$

Then, recalling the definition of N in (2.2),

$$\mathbf{P}\left(\sup \left\{ \frac{\Delta X_s}{a(s)} : s \in (t, 1], \Delta X_s > 0 \right\} \le x \right) = \mathbf{P}(N(A_{t,x}) = 0) = e^{-\mu(A_{t,x})},$$

and

$$\mathbf{P}(m_t \le a(t)x) = \mathbf{P}(N(B_{t,x}) = 0) = e^{-\mu(B_{t,x})}.$$

As $\mu(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}t \times \Lambda(\mathrm{d}x)$, we have

$$\mu(A_{t,x}) = \int_t^1 \overline{\Lambda}_+(a(u)x)\mathrm{d}u \text{ and } \mu(B_{t,x}) = t\overline{\Lambda}_+(a(t)x).$$

Since $A_{t,x}$ and $B_{t,x}$ are disjoint, we obtain

$$\mathbf{P}(M_t \le x) = \exp\left\{ -\int_t^1 \overline{\Lambda}_+(a(u)x)\mathrm{d}u - t\overline{\Lambda}_+(a(t)x) \right\}. \tag{3.1}$$

Remark 3.1. If X is a spectrally positive α-stable process, $\alpha \in (0, 2)$, with $\overline{\Lambda}_+(x) = x^{-\alpha}$, then $\varphi(u) = u^{-1/\alpha}$, $a(t) = t^{1/\alpha}$. Substituting into (3.1) a short calculation gives for $0 < t \le 1$

$$\mathbf{P}(M_t \le x) = \exp\left\{ -x^{-\alpha}(1 - \log t) \right\}.$$

Therefore, we obtain for any fixed $t > 0$ the scaled maximum has Fréchet distribution, i.e.

$$\mathbf{P}\left(M_t \le x(1 - \log t)^{1/\alpha} \right) = e^{-x^{-\alpha}}. \tag{3.2}$$

In what follows, we show that (3.2) remains true in the limit as $t \downarrow 0$ for Lévy processes in the domain of attraction of a stable law at zero under regularity.

A measurable function ℓ is super-slowly varying at 0 with auxiliary function ξ, if for some $\Delta > 0$

$$\lim_{t \downarrow 0} \frac{\ell(t\xi(t)^\delta)}{\ell(t)} = 1 \quad \text{uniformly in } \delta \in [0, \Delta]. \tag{3.3}$$

This is exactly the definition in Bingham et al. (1989, Section 3.12.2), changing x to t^{-1} and $\xi(x)$ to $\xi(t^{-1})^{-1}$. See also Bingham et al. (1989, Section 2.3). We further assume that $\lim_{t \downarrow 0} \xi(t) = 0$ and that ξ is nondecreasing in $(0, c)$ for some $c > 0$. If (3.3) holds for some $\Delta > 0$, and ξ is nondecreasing then (3.3) holds for any $\Delta > 0$; see Bingham et al. (1989, p.186). In what follows we fix the function $\xi(t) = (-\log t)^{-1}$.

Theorem 3.2. *Assume that for $x > 0$, $\overline{\Lambda}_+(x) = x^{-\alpha}\ell(x)$, $\alpha \in (0,2)$, where ℓ is a super-slowly varying function at 0 with auxiliary function $\xi(t) = (-\log t)^{-1}$. Then for all $x > 0$*

$$\lim_{t\downarrow 0} \mathbf{P}\left(M_t\,(-\log t)^{-1/\alpha} \le x\right) = e^{-x^{-\alpha}}. \tag{3.4}$$

Remark 3.3. The super-slowly varying condition is not very restrictive. The slowly varying functions $\ell(t) = (-\log t)^\beta$, $\beta > 0$, $\ell(t) = \exp\{(-\log t)^\beta\}$, $\beta \in (0,1)$ are super-slowly varying with auxiliary function $\xi(t) = (-\log t)^{-1}$. The function $\ell(t) = \exp\{(-\log t)/\log(-\log t)\}$ is slowly varying, but not super-slowly varying with auxiliary function ξ.

Remark 3.4. We also note that Theorem 3.2 is a result on the maximum of a Poisson point process, therefore $\Lambda(\mathrm{d}x)$ does not have to be a Lévy measure. Thus Theorem 3.2 remains true for any $\alpha > 0$.

For our next result assume that the spectrally negative part does not dominate in the sense

$$\limsup_{x\downarrow 0} \frac{\overline{\Lambda}_-(x)}{\overline{\Lambda}_+(x)} < \infty. \tag{3.5}$$

Theorem 3.5. *Assume that X_t is a Lévy process without normal component such that for $x > 0$, $\overline{\Lambda}_+(x) = x^{-\alpha}\ell(x)$, $\alpha \in (0,2)$ with $\alpha \ne 1$, where ℓ is a super-slowly varying function at 0 with auxiliary function $\xi(t) = (-\log t)^{-1}$, and (3.5) holds. Then for all $x > 0$*

$$\lim_{t\downarrow 0} \mathbf{P}\left(Y_t\,(-\log t)^{-1/\alpha} \le x\right) = e^{-x^{-\alpha}}.$$

This result also holds for $\alpha = 1$ but, as usual, a different centering is needed. As in (2.7), for $\alpha = 1$ let

$$c(t) = \begin{cases} t\int_{(0,a(t)]} y\Lambda(\mathrm{d}y), & \text{if } \int_{(0,1]} y\Lambda(\mathrm{d}y) < \infty, \\ -t\int_{(a(t),1]} y\Lambda(\mathrm{d}y), & \text{if } \int_{(0,1]} y\Lambda(\mathrm{d}y) = \infty. \end{cases} \tag{3.6}$$

Theorem 3.6. *Assume that X_t is a Lévy process without normal component such that for $x > 0$, $\overline{\Lambda}_+(x) = x^{-1}\ell(x)$, where ℓ is a super-slowly varying function at 0 with auxiliary function $\xi(t) = (-\log t)^{-1}$, (3.5) holds, and $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$. Then for all $x > 0$*

$$\lim_{t\downarrow 0} \mathbf{P}\left(\sup_{t\le s\le 1} \frac{\sup_{u\le s}(X_u - c(u))}{a(s)}\,(-\log t)^{-1} \le x\right) = e^{-x^{-1}}.$$

As a consequence, we obtain the following Darling–Erdős result.

Theorem 3.7. *Assume that X_t is a Lévy process without normal component such that for $x > 0$, $\overline{\Lambda}_+(x) = x^{-\alpha}\ell(x)$, $\alpha \in (0,2)$, where ℓ is a super-slowly varying function at 0 with auxiliary function $\xi(t) = (-\log t)^{-1}$, and (3.5) holds. For $\alpha = 1$ additionally assume $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$. Then for all $x > 0$*

$$\lim_{t\downarrow 0} \mathbf{P}\left(\sup_{t\le s\le 1} \frac{X_s - c(s)}{a(s)}\,(-\log t)^{-1/\alpha} \le x\right) = e^{-x^{-\alpha}}, \tag{3.7}$$

where $c(s) \equiv 0$ for $\alpha \ne 1$, and is given in (3.6) for $\alpha = 1$.

Remark 3.8. We note that the conditions for the corresponding result for sums of i.i.d. random variables in Bertoin (1998, Theorem 1) are more stringent. The non-dominating negative tail assumption is the same as (3.5), but in Bertoin (1998) it is assumed that the slowly varying function ℓ in (2.4) is constant, and the $\alpha = 1$ case is excluded. It will be apparent from the proofs that the nontrivial slowly varying function significantly complicates the arguments.

We also mention that large time results similar to (3.7) for stable processes are stated in Theorem 5 of Bertoin (1998) based on the correspondence between stable processes and stable Ornstein–Uhlenbeck processes. Theorem 5 in Bertoin (1998) can be deduced from Corollary 5.3 in Rootzén (1978), since Ornstein–Uhlenbeck processes can be represented as stable moving average processes with exponential kernel function (see e.g. Applebaum, 2009, Section 4.3.5).

4. Proof of Theorem 3.2

Since here the spectrally negative part does not play a role, to ease the notation we suppress the lower index, i.e. $\overline{\Lambda} = \overline{\Lambda}_+$. From (3.1) we get for fixed $x > 0$

$$\begin{aligned} &\mathbf{P}\left(M_t \left(-\log t\right)^{-1/\alpha} \le x\right) \\ &= \exp\left\{ - \int_t^1 \overline{\Lambda}(a(u)(-\log t)^{1/\alpha} x) \mathrm{d}u - t \overline{\Lambda}(a(t)(-\log t)^{1/\alpha} x) \right\}. \end{aligned} \tag{4.1}$$

In what follows, we need that

$$\overline{\Lambda}(a(u)) \sim u^{-1}, \text{ as } u \downarrow 0. \tag{4.2}$$

To see this, define for $x > 0$

$$f(x) = \overline{\Lambda}(1/x) = x^{\alpha} \ell(1/x).$$

Clearly f is increasing and regularly varying with index α at ∞. Recall (2.1) and set for $y > 0$

$$\begin{aligned} f^{-1}(y) &= \inf\{x : f(x) > y\} \\ &= \inf\left\{x : \overline{\Lambda}(1/x) > y\right\} \\ &= 1/\sup\left\{x^{-1} : \overline{\Lambda}(1/x) > y\right\} = 1/\varphi(y). \end{aligned}$$

By (2.6) and Theorem 1.5.12 of Bingham et al. (1989) we have that as $y \to \infty$

$$f\left(f^{-1}(y)\right) = \overline{\Lambda}(\varphi(y)) \sim y,$$

which by the change of variable $y = u^{-1}$ gives (4.2).

Let h be an auxiliary function to be chosen later, which is continuous, increasing on $(0,1)$ and $1 > h(t) > t$. We can write the exponent in (4.1) as

$$\begin{aligned} &- \int_t^{h(t)} \overline{\Lambda}(a(u)(-\log t)^{1/\alpha} x) \mathrm{d}u - \int_{h(t)}^1 \overline{\Lambda}(a(u)(-\log t)^{1/\alpha} x) \mathrm{d}u \\ &- t \overline{\Lambda}(a(t)(-\log t)^{1/\alpha} x). \end{aligned} \tag{4.3}$$

By the assumption on $\overline{\Lambda}$

$$\frac{\overline{\Lambda}\left(a(u) x (-\log t)^{1/\alpha}\right)}{\overline{\Lambda}\left(a(u)\right)} = x^{-\alpha} (-\log t)^{-1} \frac{\ell\left(a(u) x (-\log t)^{1/\alpha}\right)}{\ell(a(u))}. \tag{4.4}$$

By the definition of super-slowly varying functions, for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $s \in (0, t_0)$

$$\sup_{\xi(s)^\Delta \le y \le 1} \left| \frac{\ell(sy)}{\ell(s)} - 1 \right| < \varepsilon, \tag{4.5}$$

where $\Delta > 0$. To see this note that for any $\xi(s)^\Delta \le y \le 1$ there exists a $0 \le \rho \le \Delta$ such that $\xi(s)^\rho = y$. We choose for $0 < \beta < 1$ and $t \in (0,1)$

$$h\,(t) = h_\beta(t) = \exp\left\{ - \left(-\log t\right)^\beta \right\}.$$

We claim that

$$\lim_{t \downarrow 0} \sup_{u \in [t, h(t)]} \left| \frac{\ell(a(u))}{\ell\left(a(u)x(-\log t)^{1/\alpha}\right)} - 1 \right| = 0. \tag{4.6}$$

In (4.5) choose

$$s = s(u,t,x) = a(u)x\left(\log t^{-1}\right)^{1/\alpha} \quad \text{and} \quad y = y(t,x) = x^{-1}\left(\log t^{-1}\right)^{-1/\alpha}. \tag{4.7}$$

Clearly, $y \le 1$ for $t > 0$ small enough. Thus, in order to use (4.5) we have to check that

$$\lim_{t \downarrow 0} \sup_{u \in [t, h(t)]} a(u)x\left(\log t^{-1}\right)^{1/\alpha} = 0, \tag{4.8}$$

and, with s, y in (4.7) and $u \in [t, h(t)]$,

$$\xi(s)^\Delta \le y = x^{-1}\left(\log t^{-1}\right)^{-1/\alpha}. \tag{4.9}$$

Since a is regularly varying at 0 with parameter $1/\alpha$

$$\log\left(a(h(t))(-\log t)^{1/\alpha}\right) \sim \frac{-\left(-\log t\right)^\beta}{\alpha}, \quad \text{as } t \downarrow 0. \tag{4.10}$$

Using the monotonicity of a and (4.10), for $u \in [t, h(t)]$, $t > 0$ small enough

$$a(u)(-\log t)^{1/\alpha} \le a(h(t))(-\log t)^{1/\alpha} \le \exp\left\{ \frac{-\left(-\log t\right)^\beta}{2\alpha} \right\}. \tag{4.11}$$

The latter upper bound tends to 0 as $t \downarrow 0$, therefore (4.8) follows.

By (4.11) for any $\Delta > 1$ and $t > 0$ small enough

$$\xi\left(a(h(t))x(-\log t)^{1/\alpha}\right)^\Delta \le \left(\frac{1}{3\alpha}\log\left(1/h(t)\right)\right)^{-\Delta} = (3\alpha)^\Delta(-\log t)^{-\beta\Delta}. \tag{4.12}$$

By the monotonicity of ξ and a, and by (4.12) we have

$$\begin{aligned}\sup_{t \le u \le h(t)} \xi(s)^\Delta &= \sup_{t \le u \le h(t)} \xi\left(a(u)x(\log(1/t))^{1/\alpha}\right)^\Delta = \xi\left(a(h(t))x(\log(1/t))^{1/\alpha}\right)^\Delta \\ &\le (3\alpha)^\Delta(-\log t)^{-\beta\Delta} \le x^{-1}(\log 1/t)^{-1/\alpha},\end{aligned}$$

where the last inequality holds for $t > 0$ small enough if $\beta\Delta > 1/\alpha$. Since, by the remark before Theorem 3.2, Δ can be chosen to be large, (4.9) holds, and (4.6) follows.

Thus, by (4.4) uniformly in $u \in (t, h(t))$

$$\frac{\overline{\Lambda}\left(a(u)x(-\log t)^{1/\alpha}\right)}{\overline{\Lambda}\left(a(u)\right)} \sim x^{-\alpha}(-\log t)^{-1}.$$

Therefore, using also (4.2),

$$\int_t^{h(t)} \overline{\Lambda}\left(a(u)x(-\log t)^{1/\alpha}\right) \mathrm{d}u \sim x^{-\alpha}(-\log t)^{-1} \int_t^{h(t)} u^{-1}\mathrm{d}u$$
$$\sim x^{-\alpha}\left(1 - \frac{\log 1/h(t)}{-\log t}\right) \sim x^{-\alpha}, \text{ as } t \downarrow 0.$$

Next we see that

$$\int_{h(t)}^1 \overline{\Lambda}\left(a(u)x(-\log t)^{1/\alpha}\right) \mathrm{d}u = \int_{h(t)}^1 \overline{\Lambda}\left(a(u)\right) \left[\frac{\overline{\Lambda}\left(a(u)x(-\log t)^{1/\alpha}\right)}{\overline{\Lambda}\left(a(u)\right)}\right] \mathrm{d}u,$$

which by (4.4)

$$= x^{-\alpha}(-\log t)^{-1} \int_{h(t)}^1 \overline{\Lambda}\left(a(u)\right) \left[\frac{\ell\left(a(u)x(-\log t)^{1/\alpha}\right)}{\ell\left(a(u)\right)}\right] \mathrm{d}u.$$

Applying part (ii) of Theorem 1.5.6 in Bingham et al. (1989) we see that this last bound is for any $\delta > 0$, some $A_\delta > 0$ and for all small enough $t > 0$

$$\leq A_\delta x^{-\alpha}(-\log t)^{-1} \int_{h(t)}^1 \overline{\Lambda}\left(a(u)\right) \left(x(-\log t)^{1/\alpha}\right)^\delta \mathrm{d}u.$$

By (4.2) we can infer that there exists a $B > 0$ such that for all $u \in (0, 1]$

$$\overline{\Lambda}(a(u)) \leq Bu^{-1}.$$

Thus with $C = A_\delta B$

$$A_\delta x^{-\alpha}(-\log t)^{-1} \int_{h(t)}^1 \overline{\Lambda}\left(a(u)\right) \left(x(-\log t)^{1/\alpha}\right)^\delta \mathrm{d}u$$
$$\leq Cx^{\delta-\alpha}(-\log t)^{\delta/\alpha-1} \int_{h_\beta(t)}^1 u^{-1}\mathrm{d}u$$
$$= Cx^{\delta-\alpha}(-\log t)^{\delta/\alpha-1+\beta},$$

which for small enough $\delta > 0$ converges to zero as $t \downarrow 0$.

Therefore it follows that

$$\lim_{t\downarrow 0} \int_t^1 \overline{\Lambda}\left(a(u)x(-\log t)^{1/\alpha}\right) \mathrm{d}u = x^{-\alpha}.$$

Finally, for the third term in (4.3) we have, by (4.2) and (4.6),

$$t\overline{\Lambda}\left(a(t)x(-\log t)^{1/\alpha}\right) \sim \frac{\overline{\Lambda}\left(a(t)x(-\log t)^{1/\alpha}\right)}{\overline{\Lambda}\left(a(t)\right)} \sim x^{-\alpha}\left(-\log t\right)^{-1},$$

which converges to zero as $t \downarrow 0$, and statement (3.4) follows.

5. Proofs of Theorems 3.5, 3.6, and 3.7

5.1. *Exponential inequalities for general Lévy processes.* In this subsection for convenience of presentation we state and prove the exponential inequalities that are needed in the proof of Theorem 3.5. All of them are derived from Proposition 5.1 below, which may be of separate interest.

Let X_t, $t \geq 0$, be a Lévy process without a normal component with Lévy measure Λ. As before for $x > 0$, $\overline{\Lambda}_+(x) = \Lambda((x,\infty))$ and $\overline{\Lambda}_-(x) = \Lambda((-\infty,-x))$. For any fixed $a > 0$ introduce the Lévy processes

$$X_s^{(a)} = \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u,\mathrm{d}y) \text{ and } X_s^{(-a)} = \int_0^s \int_{[-a,0)} y\widetilde{N}(\mathrm{d}u,\mathrm{d}y), \quad s \geq 0. \quad (5.1)$$

Set for $a \geq 0$

$$B(a) = \int_0^a y^2\Lambda(\mathrm{d}y) \text{ and } B(-a) = \int_{-a}^0 y^2\Lambda(\mathrm{d}y). \quad (5.2)$$

We note that the following proposition holds for general Lévy process, regular variation of the Lévy measure is not needed here.

Proposition 5.1. *For all $a > 0$, $b > 0$, $p \geq 1$ integer, and $0 < t$*

$$\mathbf{P}\left(\sup_{s\leq t} X_s^{(a)} > b\right) \leq \exp\left\{\frac{b}{(1+1/p)\,a}\left(1+\log\left(\frac{tB(a)\,(p!)^{1/p}}{ab}\right)\right)\right\} \quad (5.3)$$

and for all $a > 0$, $b > 0$ and $0 < t$

$$\mathbf{P}\left(\sup_{s\leq t}\left(-X_s^{(a)}\right) > b\right) \leq \exp\left(-\frac{b^2}{2tB(a)}\right). \quad (5.4)$$

Moreover, inequality (5.3) holds with $\sup_{s\leq t} X_s^{(a)}$ replaced by $\sup_{s\leq t}\left(-X_s^{(-a)}\right)$ and inequality (5.4) remains true with $\sup_{s\leq t}\left(-X_s^{(a)}\right)$ replaced by $\sup_{s\leq t} X_s^{(-a)}$, and where $B(a)$ is replaced by $B(-a)$ in both cases.

Proof. We shall borrow steps from the proof of Lemma 1 of Sato (1973). Clearly, $X_s^{(a)}$ is a martingale, thus by Doob's martingale inequality, for any $\theta > 0$

$$\begin{aligned}\mathbf{P}\left(\sup_{s\leq t} X_s^{(a)} > b\right) &= \mathbf{P}\left(\exp\left\{\theta\sup_{s\leq t} X_s^{(a)}\right\} > e^{\theta b}\right)\\ &\leq e^{-\theta b}\mathbf{E}\exp\{\theta X_t^{(a)}\}.\end{aligned} \quad (5.5)$$

The difficult issue here is to choose the right θ.

Set for $\theta \in \mathbb{R}$,

$$\xi_t(\theta) := \log\mathbf{E}\exp\{\theta X_t^{(a)}\} = t\int_0^a \left(e^{\theta y} - 1 - \theta y\right)\Lambda(\mathrm{d}y).$$

Since $|e^v - 1 - v| \leq v^2\exp(|v|)/2$ for all $v \in \mathbb{R}$, we see that

$$|\xi_t(\theta)| \leq \frac{\theta^2}{2}\int_0^a y^2 t\Lambda(\mathrm{d}y)\exp(a|\theta|) < \infty, \quad \theta \in \mathbb{R}.$$

Thus $\mathbf{E}\exp\{\theta X_t^{(a)}\} < \infty$ for all $\theta \in \mathbb{R}$. Differentiating $\xi_t(\theta)$ with respect to θ we obtain for all $\theta \in \mathbb{R}$

$$\xi_t'(\theta) = \int_0^a y\left(e^{\theta y} - 1\right)t\Lambda(\mathrm{d}y),$$

and differentiating again, for all $\theta \in \mathbb{R}$

$$\xi_t''(\theta) = \int_0^a y^2 e^{\theta y} t\Lambda(\mathrm{d}y) > 0, \quad (5.6)$$

from which we see that

$$\xi_t'(\theta) \downarrow -\mu =: -\int_0^a yt\Lambda(\mathrm{d}y), \text{ as } \ \theta \downarrow -\infty,$$

where $-\infty \leq -\mu < 0$, $\xi_t'(0) = 0$ and $\xi_t'(\theta) \uparrow \infty$, as $\theta \uparrow \infty$.

For any $-\mu < x < \infty$ introduce the inverse to $\xi_t'(\theta)$:

$$\xi_t'(\eta_t(x)) = x. \tag{5.7}$$

The function η_t is well defined on $(-\mu, \infty)$, since by (5.6), $\xi_t'(\theta)$ is strictly increasing and continuous as a function of θ. Furthermore by the inverse function theorem we have

$$\xi_t''(\eta_t(x))\eta_t'(x) = 1, \text{ for } \ -\mu < x < \infty, \tag{5.8}$$

and we know from the above that $\eta_t(x) > 0$ if and only if $x > 0$. Now by (5.5) with $\theta = \eta_t(b)$ and (5.7) for any $b > 0$,

$$\begin{aligned}\mathbf{P}\left(\sup_{s\leq t} X_s^{(a)} > b\right) &\leq \mathbf{P}\left(\eta_t(b)\sup_{s\leq t} X_s^{(a)} > \eta_t(b)b\right) \\ &\leq \exp\left\{\xi_t(\eta_t(b)) - \eta_t(b)\xi_t'(\eta_t(b))\right\}.\end{aligned}$$

Observe that

$$\begin{aligned}\xi_t(\eta_t(b)) - \eta_t(b)\,\xi_t'(\eta_t(b)) &= \int_0^{\eta_t(b)} \xi_t'(s)\,\mathrm{d}s - \eta_t(b)\,\xi_t'(\eta_t(b)) \\ = -\int_0^{\eta_t(b)} s\xi_t''(s)\,\mathrm{d}s &= -\int_0^b \eta_t(x)\,\xi_t''(\eta_t(x))\,\eta_t'(x)\,\mathrm{d}x,\end{aligned}$$

which by (5.8) is equal to

$$= -\int_0^b \eta_t(x)\,\mathrm{d}x.$$

Thus for all $b > 0$

$$\mathbf{P}\left(\sup_{s\leq t} X_s^{(a)} > b\right) \leq \exp\left(-\int_0^b \eta_t(x)\,\mathrm{d}x\right). \tag{5.9}$$

Since $\exp(v) - 1 \leq v\exp(v)$ for all $v \geq 0$, for $\tau \geq 0$

$$\begin{aligned}\xi_t'(\tau) &= \int_0^a y\left(e^{\tau y} - 1\right)t\Lambda(\mathrm{d}y) \leq t\int_0^a y^2\Lambda(\mathrm{d}y)\tau\exp(a\tau) \\ &= tB(a)\tau\exp(a\tau),\end{aligned}$$

from which it follows by (5.7) that

$$x \leq tB(a)\eta_t(x)\exp(a\eta_t(x)), \text{ for } \ x \geq 0.$$

The inequality $v \leq (p!)^{1/p}\exp\left(\frac{v}{p}\right)$ for $p \geq 1$ and $v \geq 0$, gives

$$x \leq \frac{tB(a)}{a}(p!)^{1/p}\exp\left((1+1/p)\,a\eta_t(x)\right), \text{ for } x > 0,$$

and thus

$$\frac{\log x}{(1+1/p)\,a} - \frac{1}{(1+1/p)\,a}\log\left(\frac{tB(a)\,(p!)^{1/p}}{a}\right) \leq \eta_t(x)\,, \text{ for } x > 0.$$

Hence after a little algebra we get

$$\exp\left(-\int_0^b \eta_t(x)\mathrm{d}x\right) \le \exp\left(-\int_0^b \frac{\log x\,\mathrm{d}x}{(1+1/p)\,a} + \frac{b}{(1+1/p)\,a}\log\left(\frac{tB(a)\,(p!)^{1/p}}{a}\right)\right)$$
$$= \exp\left\{\frac{b}{(1+1/p)\,a}\left(1+\log\left(\frac{tB\,(a)\,(p!)^{1/p}}{ab}\right)\right)\right\},$$

which on account of (5.9) gives (5.3).

Next consider inequality (5.4). The process $-X_s^{(a)}$, $s \ge 0$, is also a martingale. Therefore exactly as above for all $\theta > 0$

$$\mathbf{P}\left(\sup_{s\le t}\left(-X_s^{(a)}\right) > b\right) \le e^{-\theta b}\mathbf{E}\exp\{-\theta X_t^{(a)}\} = \exp\left(-\theta b + \gamma_t\,(\theta)\right),$$

where $\gamma_t\,(\theta) = \xi_t\,(-\theta)$. We get

$$\gamma_t'\,(\theta) = -\xi_t'\,(-\theta) = t\int_0^a y\left(1-e^{-\theta y}\right)\Lambda(\mathrm{d}y),$$

and $\gamma_t''\,(\theta) = \xi_t''\,(-\theta) > 0$, from which we see that

$$\gamma_t'\,(\theta) \uparrow \mu = t\int_0^a y\Lambda(\mathrm{d}y), \text{ as } \theta\uparrow\infty,$$

where $0 < \mu \le \infty$, $\gamma_t'\,(0) = 0$ and $\gamma_t'\,(\theta) \downarrow -\infty$, as $\theta \downarrow -\infty$. For any $-\infty < x < \mu$ introduce the inverse to $\gamma_t'\,(\theta)$:

$$\gamma_t'\,(\kappa_t\,(x)) = x.$$

The function κ_t is well defined on $(-\infty, \mu)$, since by $\gamma_t''\,(\theta) = \xi_t''\,(-\theta) > 0$, $\gamma_t'\,(\theta)$ is strictly increasing and continuous as a function of θ. Furthermore by the inverse function theorem we have

$$\gamma_t''\,(\kappa_t\,(x))\,\kappa_t'\,(x) = 1, \text{ for } -\infty < x < \mu,$$

and we know from the above that $\kappa_t\,(x) > 0$ if and only if $x > 0$.

Now just as in the proof (5.3), for all $b > 0$

$$\mathbf{P}\left(\sup_{0\le s\le t}\left(-X_s^{(a)}\right) > b\right) \le \exp\left(-\int_0^b \kappa_t\,(x)\,\mathrm{d}x\right). \tag{5.10}$$

Since $1-\exp(-v) \le v$ for $v > 0$, we have for all $\theta \ge 0$,

$$\begin{aligned}\gamma_t'\,(\theta) &= -\xi_t'\,(-\theta) = t\int_0^a y\left(1-e^{-\theta y}\right)\Lambda(\mathrm{d}y)\\ &\le t\theta\int_0^a y^2\Lambda(\mathrm{d}y) = \theta tB\,(a)\,,\end{aligned} \tag{5.11}$$

from which it follows by setting $\theta = \kappa_t\,(x)$ into (5.11) that $x \le tB\,(a)\,\kappa_t\,(x)$. This gives (5.4) by (5.10).

The validity of the moreover part of the statement of Proposition 5.1 is obvious. □

5.2. *Applications of Proposition 5.1.* In what follows we assume that (2.4) holds. Then

$$
\begin{aligned}
B(x) &= \int_{(0,x]} y^2 \Lambda(\mathrm{d}y) \\
&= 2\int_0^x y\overline{\Lambda}_+(y)\mathrm{d}y - x^2\,\overline{\Lambda}_+(x) \\
&= 2\int_0^x y^{1-\alpha}\,\ell(y)\mathrm{d}y - x^{2-\alpha}\,\ell(x),
\end{aligned}
$$

which, by an application of the obvious at zero version of Proposition 1.5.10 in Bingham et al. (1989) to the integral in the last equation gives

$$
B(x) \sim \frac{\alpha}{2-\alpha} x^2 \overline{\Lambda}_+(x) \quad \text{as } x \downarrow 0. \tag{5.12}
$$

For any $0 < \beta < \alpha$, select $0 < \rho$ small and $\kappa > 1$ depending on α and β so that for all $0 < xu \le \rho$ with $x \ge 1$ such that by the Potter bounds (Bingham et al., 1989, p. 25, Section 2.3, Theorem 1.5.6),

$$
\frac{\ell(ux)}{\ell(u)} \le \kappa x^{\beta}. \tag{5.13}
$$

Corollary 5.2. *Assume (2.4). For any $\varepsilon \in (0,1)$ there exist $\alpha' > \alpha$, $t_0 > 0$, and $A > 0$, such that whenever $\max\{a(t)x, a(t), t\} < t_0$ and $x(1-\varepsilon) > 1$*

$$
\mathbf{P}\left(\sup_{s\le t} X_s^{(a(t)x(1-\varepsilon))} > a(t)x(1-\varepsilon/2)\right) \le A x^{-\alpha'}. \tag{5.14}
$$

Proof. Let $a = a(t)x(1-\varepsilon)$ with $x(1-\varepsilon) \ge 1$, $b = a(t)\,x(1-\varepsilon/2)$, and set $q = (1-\varepsilon/2)/(1-\varepsilon)$. Choose the integer p so large that $1 + 1/p < q$. By (5.3)

$$
\mathbf{P}\left(\sup_{s\le t} X_s^{(a)} > b\right) \le \exp\left(\frac{q}{1+1/p}\right)\left(\frac{tB(a)\,(p!)^{1/p}}{ab}\right)^{q/(1+1/p)}. \tag{5.15}
$$

Let $\beta < \alpha$ be defined later. If both $a(t)x$ and $t > 0$ are small enough, we get from (5.12), (4.2), and (5.13) that for some $K > 0$

$$
\frac{tB(a)\,(p!)^{1/p}}{ab} \le K x^{\beta-\alpha}.
$$

Substituting back into (5.15) we obtain

$$
\mathbf{P}\left(\sup_{s\le t} X_s^{(a)} > b\right) \le \exp\left(\frac{q}{1+1/p}\right)\left(K x^{-\alpha+\beta}\right)^{q/(1+1/p)},
$$

which, by choosing $\beta > 0$ small enough, implies (5.14). □

Corollary 5.3. *Assume (2.4). For any $\beta \in (0,\alpha)$ there exists $t_0 > 0$ and $D > 0$ such that if $\max\{a(t)x, a(t), t\} < t_0$ and $x \ge 1$, then for any $\tau > 0$*

$$
\mathbf{P}\left(\sup_{0\le s\le t}\left(-X_t^{(a(t)x)}\right) > \tau a(t)x\right) \le \exp\left(-D\tau^2 x^{\alpha-\beta}\right). \tag{5.16}
$$

In particular, for any $0 \le s \le t < t_0$ and $0 < a(t)x < t_0$ with $x \ge 1$

$$
\mathbf{P}\left(-X_s^{(a(t)x)} > \tau a(t)x\right) \le \exp\left(-D\tau^2 x^{\alpha-\beta}\right). \tag{5.17}
$$

Proof. Set $a = a(t)x$, and $b = \tau a$, with $x \geq 1$, and $\tau > 0$. If both $a(t)x$ and $t > 0$ are small enough, we get from (5.12), (4.2), and (5.13) that for some $D > 0$

$$\frac{b^2}{2tB(a)} \geq D\tau^2 x^{\alpha-\beta}.$$

Thus, an application of inequality (5.4) implies (5.16). □

Recall from (2.2) that X^- is the spectrally negative part of the Lévy process X.

Corollary 5.4. *Assume (2.4) and (3.5). Further assume that* $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$ *if* $\alpha = 1$. *For every* $0 < \varepsilon < 1$ *there exist* $t_0 > 0$ *and* $x_0 \geq 1$, *such that for all* $0 < t < t_0$ *and* $x > x_0$

$$\mathbf{P}\left(\sup_{s\leq t} X_s^- > \frac{\varepsilon}{4}a(t)x\right) \leq e^{-x\varepsilon/8}. \tag{5.18}$$

Proof. First note that if $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$, in particular if $\alpha \leq 1$, then $-X_t^-$ is a subordinator, therefore the probability in question is 0.

Assume that $\alpha > 1$. Since X_s^- is a spectrally negative Lévy process for any $0 < a \leq 1$

$$\mathbf{P}\left(\sup_{s\leq t} X_s^- > \frac{\varepsilon}{4}a(t)x\right) \leq \mathbf{P}\left(\sup_{s\leq t} X_s^{(-a)} - t\mu_-(a) > \frac{\varepsilon}{4}a(t)x\right), \tag{5.19}$$

where

$$\mu_-(a) = \int_{[-1,-a)} y\Lambda(\mathrm{d}y).$$

Now

$$-t\mu_-(a) = -t\int_{[-1,-a)} y\Lambda(\mathrm{d}y) = t\left(a\overline{\Lambda}_-(a) - \overline{\Lambda}_-(1) + \int_a^1 \overline{\Lambda}_-(y)\mathrm{d}y\right),$$

which by (3.5) and (2.4) for all small enough $a > 0$ is for some $C_\alpha > 0$

$$\leq C_\alpha a t\overline{\Lambda}_+(a).$$

Similarly, we can verify that for some $D_\alpha > 0$

$$tB(-a) = t\int_{[-a,0)} y^2\Lambda(\mathrm{d}y) \leq D_\alpha a^2 t\overline{\Lambda}_+(a).$$

Setting $a = a(t)$ we see by (4.2) that for all $t > 0$ small enough for some $c_\alpha > 0$ and $d_\alpha > 0$ both

$$-\int_{[-1,-a)} ty\Lambda(\mathrm{d}y) \leq c_\alpha a(t) \ \text{ and } \ tB(-a) \leq d_\alpha a^2(t).$$

Choose x so large so that $\frac{\varepsilon}{8}x > \max\{c_\alpha, 2d_\alpha\}$. Thus by (5.19)

$$\mathbf{P}\left(\sup_{s\leq t} X_s^- > \frac{\varepsilon}{4}a(t)x\right) \leq \mathbf{P}\left(\sup_{s\leq t} X_s^{(-a(t))} > \frac{\varepsilon}{8}a(t)x\right),$$

which by inequality (5.4) in the $\sup_{s\leq t} X_s^{(-a)}$ case with $a = a(t)$ and $b = \varepsilon a(t)x/8$ is

$$\leq \exp\left(-\frac{b^2}{2tB(-a)}\right) \leq \exp\left(-\frac{\left(\frac{\varepsilon}{8}x\right)^2}{2d_\alpha}\right) < \exp\left(-\frac{\varepsilon}{8}x\right).$$

This gives (5.18), with $x_0 = 8\varepsilon^{-1}\max\{c_\alpha, 2d_\alpha\}$ and for $t_0 > 0$ sufficiently small. □

5.3. *Four auxiliary lemmas.* The i.i.d. counterpart of the next result is due to Bertoin (1998, Lemma 1).

Lemma 5.5. *Let $\alpha \in (0,2)$, $\alpha \neq 1$. Assume (2.4) and (3.5). For any $0 < \varepsilon < 1$ there exist $A > 0$, $t_0 > 0$, $x_0 \geq 1$, $\alpha' > \alpha$, such that if $t < t_0$, $x > x_0$, $a(t)x < 1$, and for $\alpha < 1$ additionally assume $a(t)x < t_0$, then*

$$\mathbf{P}\left(m_t \leq a(t)x(1-\varepsilon), \overline{X}_t > a(t)x\right) \leq Ax^{-\alpha'}. \tag{5.20}$$

Proof. **Step 1.** Assume that X_t is spectrally positive. Note that in this case $\gamma_- = 0$. For $a \in (0,1)$, $c > 0$ we have

$$\begin{aligned}&\left\{m_t \leq a, \overline{X}_t > c\right\} = \left\{N((0,t] \times (a,\infty)) = 0,\ \overline{X}_t > c\right\}\\ &= \{N((0,t] \times (a,\infty)) = 0\} \cap \\ &\qquad \cap \left\{\sup_{s \leq t}\left(\gamma_+ s + \int_0^s \int_{y \leq a} y\widetilde{N}(\mathrm{d}u, \mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > c\right\},\end{aligned}$$

where the latter two events are independent. Therefore

$$\begin{aligned}&\mathbf{P}\left(m_t \leq a, \overline{X}_t > c\right)\\ &= \mathbf{P}(m_t \leq a)\,\mathbf{P}\left(\sup_{s \leq t}\left(\gamma_+ s + \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u, \mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > c\right)\\ &\leq \mathbf{P}\left(\sup_{s \leq t}\left(\gamma_+ s + \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u, \mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > c\right).\end{aligned} \tag{5.21}$$

Recall the definition from (5.1). We see by (5.21) that when $1 < \alpha < 2$

$$\mathbf{P}\left(m_t \leq a, \overline{X}_t > c\right) \leq \mathbf{P}\left(\sup_{0 \leq s \leq t} X_s^{(a)} > c\right),$$

as $\gamma_+ = 0$ by (2.5), and when $0 < \alpha < 1$, again by (2.5)

$$\mathbf{P}\left(m_t \leq a, \overline{X}_t > c\right) \leq \mathbf{P}\left(\sup_{0 \leq s \leq t} X_s^{(a)} > c - t\int_{(0,a]} y\Lambda\,(\mathrm{d}y)\right).$$

Fix $0 < \varepsilon < 1$. Put

$$a = a(t)x(1-\varepsilon) \quad \text{and} \quad c = a(t)x. \tag{5.22}$$

In the case $0 < \alpha < 1$, Proposition 1.5.10 in Bingham et al. (1989) can be applied as in (5.12) to show that for $a > 0$ small enough

$$t\int_{(0,a]} y\Lambda(\mathrm{d}y) \leq t\int_0^a \overline{\Lambda}_+(y)\mathrm{d}y \sim \frac{1}{1-\alpha} at\overline{\Lambda}_+(a).$$

Thus, by using (4.2) and the Potter bounds there exists a $c_1 > 0$, such that for all $a = a(t)x(1-\varepsilon) > 0$ small enough

$$t\int_{(0,a]} y\Lambda(\mathrm{d}y) \leq a(t)x(1-\varepsilon)t\overline{\Lambda}_+(a(t)x(1-\varepsilon)) \leq c_1 a(t)x^{1-\alpha+\alpha/2}.$$

Hence for all $t > 0$ small enough and x large enough

$$a(t)x - t\int_0^a y\Lambda(\mathrm{d}y) > a(t)x\left(1 - c_1 x^{-\alpha+\alpha/2}\right) > a(t)x\left(1 - \frac{\varepsilon}{2}\right).$$

Therefore, for any $0 < \alpha < 2$, $\alpha \neq 1$, there exist $t_0 > 0$, $x_0 \geq 1$, $\alpha' > \alpha$, such that for $0 < t < t_0$, $x > x_0$, and for $0 < \alpha < 1$ additionally assume $0 < a(t)x < t_0$, we have with a as in (5.22)

$$\mathbf{P}\left(m_t \leq a(t)x(1-\varepsilon), \overline{X}_t > a(t)x\right) \leq \mathbf{P}\left(\sup_{0\leq s\leq t} X_s^{(a)} > a(t)x\left(1-\frac{\varepsilon}{2}\right)\right),$$

which by inequality (5.14) is less than or equal to $Ax^{-\alpha'}$ for some $\alpha' > \alpha$ and constant $A > 0$. This proves (5.20).

Step 2. Finally, we extend the statement from spectrally positive processes. Recall from (2.2) that X_t^- is the spectrally negative part of X_t. Notice that by arguing as in Step 1, for $a \in (0,1)$, $c > 0$ we have

$$\mathbf{P}\left(m_t \leq a, \overline{X}_t > c\right) \leq \mathbf{P}\left(\sup_{s\leq t}\left(X_s^- + \gamma_+ s + X_s^{(a)} - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > c\right)$$

In the case $0 < \alpha < 1$

$$0 < -\int_{[-1,0)} y\Lambda(\mathrm{d}y) < \infty,$$

so $-X_t^-$ is a subordinator and thus $X_t^- < 0$ for any $t > 0$. Therefore, the result follows immediately from the $0 < \alpha < 1$ case of Step 1, since

$$\sup_{s\leq t}\left(X_s^- + \gamma_+ s + X_s^{(a)} - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) \leq \sup_{s\leq t} X_s^{(a)} + t\int_{(0,a]} y\Lambda(\mathrm{d}y).$$

On the other hand $\gamma_+ = 0$ in the case $1 < \alpha < 2$, thus

$$\begin{aligned}&\left\{\sup_{s\leq t}\left(X_s^- + \gamma_+ s + X_s^{(a)} - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > a(t)x\right\}\\ &\subset \left\{\sup_{s\leq t}\left(X_s^{(a)} - s\int_{(a,1]} y\Lambda(\mathrm{d}y)\right) > a(t)x\left(1-\frac{\varepsilon}{2}\right)\right\} \cup \left\{\sup_{s\leq t} X_s^- > \frac{\varepsilon}{2}a(t)x\right\}.\end{aligned}$$

By a slight modification of the first part of the proof given in Step 1, the probability of the first event on the right-hand side of the last inclusion is bounded by $Ax^{-\alpha'}$ for some $\alpha' > \alpha$ and $A > 0$, while the probability of the second event is exponentially small by (5.18). □

Lemma 5.6. *Let $\alpha \in (0,2)$, $\alpha \neq 1$. Assume (2.4) and (3.5). For any $0 < \varepsilon < 1$ there exist a constant $A > 0$, $t_0 > 0$, $x_0 \geq 1$, $\alpha' > \alpha$, such that if $t < t_0$, $x > x_0$, and $a(t)x < t_0$, then*

$$\mathbf{P}\left(\overline{X}_t \leq a(t)x(1-\varepsilon), m_t > a(t)x\right) \leq Ax^{-\alpha'}. \tag{5.23}$$

Proof. **Step 1.** Assume first that X is spectrally positive. For $a \in (0,1)$ let $\tau = \tau_a = \inf\{s : \Delta X_s > a\}$. Then $\{m_t > a\} = \{\tau \leq t\}$, and τ is exponentially distributed with parameter $\overline{\Lambda}_+(a)$. Conditioning on τ, and using part (iii) of Proposition 0.5.2 in Bertoin (1996), which says that the Poisson random measure N on

$(0,a]\times\mathbb{R}$ is independent of τ, we get for $b>0$

$$\begin{aligned}
&\mathbf{P}\left(\overline{X}_t \le b, m_t > a\right) = \mathbf{P}\left(\overline{X}_t \le b, \tau \le t\right)\\
&\le \mathbf{P}\left(X_\tau \le b, \tau \le t\right)\\
&\le \int_0^t \mathbf{P}\left(\gamma_+ s + \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u,\mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y) \le b-a\right)\overline{\Lambda}_+(a)e^{-\overline{\Lambda}_+(a)s}\mathrm{d}s\\
&\le \left(1-e^{-\overline{\Lambda}_+(a)t}\right)\sup_{s\le t}\mathbf{P}\left(X_s^{(a)} + \gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) \le b-a\right).
\end{aligned} \tag{5.24}$$

Put

$$b = a(t)x(1-\varepsilon) \quad \text{and} \quad a = a(t)x. \tag{5.25}$$

For $\alpha\in(1,2)$ integration by parts and the at zero version of Proposition 1.5.8 in Bingham et al. (1989) give

$$\begin{aligned}
\int_{(z,1]} y\Lambda(\mathrm{d}y) &= \int_z^1 \overline{\Lambda}_+(y)\mathrm{d}y + z\overline{\Lambda}_+(z) - \overline{\Lambda}_+(1)\\
&\sim \frac{\alpha}{\alpha-1} z\overline{\Lambda}_+(z), \qquad \text{as } z\downarrow 0.
\end{aligned}$$

Moreover, by (4.2) and by Potter's bounds, there exist $t_0>0$ and $x_0\ge 1$ such that for $t<t_0$, $x>x_0$, $a(t)x<t_0$

$$t\overline{\Lambda}_+(a)a = t\overline{\Lambda}_+(a(t))\frac{\overline{\Lambda}_+(a)}{\overline{\Lambda}_+(a(t))}a \le 2x^{-\alpha/2}a(t)x.$$

Therefore for any $0<\varepsilon<1$ fixed, there exist $t_0>0$ and $x_0\ge 1$ such that for $0\le s\le t<t_0$, $x>x_0$, $a(t)x<t_0$,

$$b-a+s\int_{(a,1]} y\Lambda(\mathrm{d}y) \le -\frac{\varepsilon}{2}a(t)x.$$

For $\alpha\in(0,1)$ by the definition of γ_+ in (2.5), simply $X_s^{(a)}+\gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) \ge X_s^{(a)}$.

Summarizing, for any $\alpha\in(0,1)\cup(1,2)$, $0\le s\le t<t_0$ and $x>x_0$, we get the bound

$$\mathbf{P}\left(X_s^{(a)} + \gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) \le b-a\right) \le \mathbf{P}\left(X_s^{(a)} \le -\frac{\varepsilon}{2}a(t)x\right).$$

Inequality (5.17) gives for any choice of $0<\beta<\alpha$ there exist $t_0>0$, such that for $0<t<t_0$ and $0<a(t)x<t_0$ with $x\ge 1$

$$\mathbf{P}\left(-X_s^{(a(t)x)} > \frac{\varepsilon}{2}a(t)x\right) \le \exp\left(-D\left(\frac{\varepsilon}{2}\right)^2 x^{\alpha-\beta}\right) =: \exp\left(-D\left(\frac{\varepsilon}{2}\right)^2 x^{\delta}\right),$$

which is clearly stronger than (5.23).

Step 2. We extend the proof to the general case. As in (5.24)

$$\begin{aligned}
&\mathbf{P}\left(\overline{X}_t \le b, m_t > a\right) \\
&\le \left(1 - e^{-t\overline{\Lambda}_+(a)}\right) \sup_{s \le t} \mathbf{P}\left(X_s^- + X_s^{(a)} + \gamma_+ s - s\int_a^1 y\Lambda(\mathrm{d}y) \le b - a\right) \\
&\le \left(1 - e^{-t\overline{\Lambda}_+(a)}\right) \sup_{s \le t} \left[\mathbf{P}\left(X_s^{(a)} + \gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) \le \frac{b-a}{2}\right)\right. \\
&\qquad\qquad \left. + \mathbf{P}\left(X_s^- \le \frac{b-a}{2}\right)\right].
\end{aligned} \tag{5.26}$$

The first term in the square bracket is exponentially small by the first part of the proof, where a, b are as in (5.25).

Note that $-X_s^-$ in the second term is a spectrally positive Lévy process, therefore we can use the methods of the first part of the proof of Lemma 5.5. Let $m_t^- = \sup_{s \le t} |X_s^- - X_{s-}^-|$, we have

$$\begin{aligned}
&\mathbf{P}\left(-X_s^- > \frac{a-b}{2}\right) = \mathbf{P}\left(-X_s^- > \frac{\varepsilon}{2} a\,(t)\,x\right) \\
&\le \mathbf{P}\left(m_s^- > \frac{\varepsilon}{4} a\,(t)\,x\right) + \mathbf{P}\left(-X_s^- > \frac{\varepsilon}{2} a\,(t)\,x,\ m_s^- \le \frac{\varepsilon}{4} a\,(t)\,x\right).
\end{aligned} \tag{5.27}$$

For the first term in (5.27) we have by (3.5), (4.2), and (5.10)

$$\begin{aligned}
\mathbf{P}\left(m_s^- > \frac{\varepsilon}{4} a(t)x\right) &= 1 - \exp\left\{-s\Lambda_-(a(t)x\varepsilon/4)\right\} \\
&\le c_1 t \Lambda_+(a(t)x\varepsilon/8) \le c_2 x^{-\alpha 7/8},
\end{aligned}$$

whenever $a(t)x$ and $t > 0$ are small enough. For the second term in (5.27), by assumption (3.5), Lemma 5.5 is applicable, therefore it is of order $x^{-\alpha'}$ for some $\alpha' > \alpha$. Finally, note that the first factor in the right-hand side of (5.26)

$$1 - e^{-t\overline{\Lambda}_+(a)} \le 2t\overline{\Lambda}_+(a) \le c_3 x^{-7\alpha/8},$$

and the result follows. □

The next result is the continuous analogue of Lemma 2, Bertoin (1998). Recall the notation in (1.3).

Lemma 5.7. *Let $\alpha \in (0,2)$, $\alpha \ne 1$. Assume (2.4) and (3.5). For any $y > 0$ and $\delta \in (0,1)$*

$$\lim_{u \downarrow 0} \mathbf{P}\left(M_u \le (-\log u)^{1/\alpha} y(1-\delta), Y_u > (-\log u)^{1/\alpha} y\right) = 0.$$

Proof. For $0 < q < 1$ consider the sequence q^n. We have for $s \in [q^{n+1}, q^n]$

$$\frac{m_s}{a(s)} \ge \frac{m_{q^{n+1}}}{a(q^{n+1})} \frac{a(q^{n+1})}{a(q^n)}.$$

As $a(s)$ is regularly varying, the second factor in the lower bound converges to $q^{1/\alpha}$. Therefore for any $q^{1/\alpha} > \varepsilon_1 > 0$ there is a $t_0 > 0$ such that for all $0 < u \le u' \le t_0$

$$\sup_{u \le s \le u'} \frac{m_s}{a(s)} \ge \max_{n' \le n \le n_u} (q^{1/\alpha} - \varepsilon_1) \frac{m_{q^n}}{a(q^n)} =: (q^{1/\alpha} - \varepsilon_1)\widetilde{M}_u, \tag{5.28}$$

where

$$n_u = \lceil \log u / \log q \rceil \quad \text{and} \quad n' = n_{u'} = \lceil \log u' / \log q \rceil. \tag{5.29}$$

Since $\overline{X}_u$ is monotone increasing, for $s \in [q^{n+1}, q^n]$

$$\frac{\overline{X}_s}{a(s)} \leq \frac{\overline{X}_{q^n}}{a(q^n)} \frac{a(q^n)}{a(q^{n+1})}.$$

Similarly, the second factor in the upper bound converges to $q^{-1/\alpha}$. Thus for any $q^{1/\alpha} > \varepsilon_1 > 0$ there is a $t_0 > 0$ such that for all $0 < u \leq u' \leq t_0$

$$\sup_{u \leq s \leq u'} \frac{\overline{X}_s}{a(s)} \leq (q^{-1/\alpha} + \varepsilon_1) \max_{n'-1 \leq n \leq n_u} \frac{\overline{X}_{q^n}}{a(q^n)} =: (q^{-1/\alpha} + \varepsilon_1) \widetilde{Y}_u, \tag{5.30}$$

with n' and n_u as above. Note that for any $0 < u' < 1$ fixed the random variables

$$\sup_{u' \leq s \leq 1} \frac{m_s}{a(s)} \quad \text{and} \quad \sup_{u' \leq s \leq 1} \frac{\overline{X}_s}{a(s)} \tag{5.31}$$

are almost surely finite. Keeping in mind that $0 < \delta < 1$ is fixed, we can choose $q < 1$ close to 1 and $\varepsilon_1 < q^{1/\alpha}$ small so that

$$(1 - \delta)(q^{1/\alpha} - \varepsilon_1)^{-1} < (q^{-1/\alpha} + \varepsilon_1)^{-1}. \tag{5.32}$$

Then choose t_0 such that both (5.28) and (5.30) hold true. This choice will permit us to use Lemma 5.5. We see for $0 < u \leq u' \leq t_0$

$$\begin{aligned}
&\mathbf{P}\left(M_u \leq (-\log u)^{1/\alpha} y(1-\delta), Y_u > (-\log u)^{1/\alpha} y \right) \\
&\leq \mathbf{P}\left(M_u \leq (-\log u)^{1/\alpha} y(1-\delta), \sup_{u \leq s \leq u'} \frac{\overline{X}_s}{a(s)} > (-\log u)^{1/\alpha} y \right) \\
&\quad + \mathbf{P}\left(\sup_{u' \leq s \leq 1} \frac{\overline{X}_s}{a(s)} > (-\log u)^{1/\alpha} y \right)
\end{aligned}$$

which by (5.31), as $u \downarrow 0$,

$$\begin{aligned}
&= \mathbf{P}\left(M_u \leq (-\log u)^{1/\alpha} y(1-\delta), \sup_{u \leq s \leq u'} \frac{\overline{X}_s}{a(s)} > (-\log u)^{1/\alpha} y \right) + o(1) \\
&\leq \mathbf{P}\left(\sup_{u \leq s \leq u'} \frac{m_s}{a(s)} \leq (-\log u)^{1/\alpha} y(1-\delta), \sup_{u \leq s \leq u'} \frac{\overline{X}_s}{a(s)} > (-\log u)^{1/\alpha} y \right) + o(1) \\
&\leq \mathbf{P}\left(\widetilde{M}_u \leq \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha} - \varepsilon_1}, \widetilde{Y}_u > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha} + \varepsilon_1} \right) + o(1),
\end{aligned}$$

where at the last inequality we used (5.28) and (5.30).

We apply Lemma 5.5 with

$$t = q^n, \; x = x(u) = \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha} + \varepsilon_1}, \; \varepsilon = 1 - \frac{q^{-1/\alpha} + \varepsilon_1}{q^{1/\alpha} - \varepsilon_1}(1 - \delta), \tag{5.33}$$

and note that $\varepsilon \in (0, 1)$ by (5.32). By Lemma 5.5 there exist $0 < t_1 \leq t_0$, $x_1 > 0$, and $\alpha' > \alpha$ such that if $t < t_1$, $x > x_1$, $a(t)x < t_1$ then

$$\mathbf{P}\left(\frac{m_{q^n}}{a(q^n)} \leq \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha} - \varepsilon_1}, \frac{\overline{X}_{q^n}}{a(q^n)} > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha} + \varepsilon_1} \right) \leq A x^{-\alpha'}.$$

With $x(u)$ in (5.33), for $u > 0$ define

$$\eta = \eta(u) = \min\{n : \, a(q^n) x(u) < t_1\}. \tag{5.34}$$

Using Potter's bounds for z small enough, $a(z) \le z^{1/(2\alpha)}$, thus

$$a(q^n)x = a(q^n)\,\frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1} \le \frac{y}{q^{-1/\alpha}+\varepsilon_1} q^{n/(2\alpha)}(-\log u)^{1/\alpha}.$$

For $n \ge 4(\log q^{-1})^{-1} \log\log u^{-1}$ we have

$$q^{n/(2\alpha)}(-\log u)^{1/\alpha} \le (-\log u)^{-1/\alpha}$$

which tends to 0 as $u \downarrow 0$. Therefore, $\eta(u) \le 4(\log q^{-1})^{-1} \log\log u^{-1}$ for $u > 0$ small enough. Recall n_u and n' from (5.29). Simply,

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{M}_u \le \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha}-\varepsilon_1}, \widetilde{Y}_u > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1}\right)\\
&\le \mathbf{P}\left(\overline{X}_{q^{n'}} > t_1\right)\\
&\quad + \sum_{n=\eta(u)}^{n_u} \mathbf{P}\left(\frac{m_{q^n}}{a(q^n)} \le \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha}-\varepsilon_1}, \frac{\overline{X}_{q^n}}{a(q^n)} > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1}\right)\\
&\le \mathbf{P}\left(\overline{X}_{q^{n'}} > t_1\right) + A n_u x^{-\alpha'},
\end{aligned}$$

where the second term goes to 0 as $u \downarrow 0$ for any u'. To finish the proof note that for $t_1 > 0$ fixed as $u' \downarrow 0$ (thus $n' \to \infty$) we have $\mathbf{P}(\overline{X}_{q^{n'}} > t_1) \to 0$. □

Lemma 5.8. *Let $\alpha \in (0,2)$, $\alpha \ne 1$. Assume (2.4) and (3.5). For any $y > 0$ and $0 < \delta < 1$*

$$\lim_{u\downarrow 0} \mathbf{P}(Y_u \le (-\log u)^{1/\alpha} y(1-\delta), M_u > (-\log u)^{1/\alpha} y) = 0.$$

Proof. The proof follows the steps of the previous proof, so we only sketch it.

For $0 < q < 1$ consider the sequence q^n. For any $q^{1/\alpha} > \varepsilon_1 > 0$ there is a $t_0 > 0$ such that for all $0 < u \le u' \le t_0$

$$\sup_{u\le s\le u'} \frac{m_s}{a(s)} \le \max_{n'-1\le n\le n_u} (q^{-1/\alpha}+\varepsilon_1)\frac{m_{q^n}}{a(q^n)} =: (q^{-1/\alpha}+\varepsilon_1)\widetilde{M}_u, \tag{5.35}$$

and

$$\sup_{t\le s\le t'} \frac{\overline{X}_s}{a(s)} \ge \max_{n'\le n\le n_t} (q^{1/\alpha}-\varepsilon_1)\frac{\overline{X}_{q^n}}{a(q^n)} =: (q^{1/\alpha}-\varepsilon_1)\widetilde{Y}_u, \tag{5.36}$$

where n' and n_u are defined as in (5.29).

Choose $q < 1$ close to 1 and $\varepsilon_1 < q^{1/\alpha}$ so small that (5.32) holds. Then choose t_0 such that both (5.35) and (5.36) hold true. This choice will permit us to use Lemma 5.6. We see for $0 < u \le u' \le t_0$

$$\begin{aligned}
&\mathbf{P}\left(Y_u \le (-\log u)^{1/\alpha} y(1-\delta), M_u > (-\log u)^{1/\alpha} y\right)\\
&\le \mathbf{P}\left(Y_u \le (-\log u)^{1/\alpha} y(1-\delta), \sup_{u\le s\le u'} \frac{m_s}{a(s)} > (-\log u)^{1/\alpha} y\right)\\
&\quad + \mathbf{P}\left(\sup_{u'\le s\le 1} \frac{m_s}{a(s)} > (-\log u)^{1/\alpha} y\right),
\end{aligned}$$

where the second term goes to 0 by (5.31). For the first term by (5.35) and (5.36) we have

$$\mathbf{P}\left(Y_u \le (-\log u)^{1/\alpha} y(1-\delta), \sup_{u\le s\le u'} \frac{m_s}{a(s)} > (-\log u)^{1/\alpha} y\right) \le \mathbf{P}\left(\widetilde{Y}_u \le \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha}-\varepsilon_1}, \widetilde{M}_u > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1}\right).$$

Choose t, x, ε as in (5.33). Using Lemma 5.6 we can show there exist $A > 0$, $\alpha' > \alpha$, $0 < t_1 \le t_0$, $x_1 \ge 1$, and $A > 0$ such that if $t < t_1$, $x > x_1$, $a(t)x < t_1$ then

$$\mathbf{P}\left(\frac{\overline{X}_{q^n}}{a(q^n)} \le \frac{(-\log u)^{1/\alpha} y(1-\varepsilon)}{q^{1/\alpha}-\varepsilon_1}, \frac{m_{q^n}}{a(q^n)} > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1}\right) \le A x^{-\alpha'}.$$

For $\eta(u)$ in (5.34), as in the previous proof $\eta(u) \le 4(\log q^{-1})^{-1} \log\log u^{-1}$ for $u > 0$ small enough. We obtain

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{Y}_u \le \frac{(-\log u)^{1/\alpha} y(1-\delta)}{q^{1/\alpha}-\varepsilon_1}, \widetilde{M}_u > \frac{(-\log u)^{1/\alpha} y}{q^{-1/\alpha}+\varepsilon_1}\right)\\
&\le \mathbf{P}\Big(m_{q^{n'}} > t_1\Big) + \sum_{n=\eta(u)}^{n_u} \mathbf{P}\left(\frac{\overline{X}_{q^n}}{a(q^n)} \le \frac{(-\log u)^{\frac{1}{\alpha}} y(1-\varepsilon)}{q^{\frac{1}{\alpha}}-\varepsilon_1}, \frac{m_{q^n}}{a(q^n)} > \frac{(-\log u)^{\frac{1}{\alpha}} y}{q^{-\frac{1}{\alpha}}+\varepsilon_1}\right)\\
&\le \mathbf{P}\left(m_{q^{n'}} > t_1\right) + A n_u x^{-\alpha'},
\end{aligned}$$

where the second term goes to 0 as $u \downarrow 0$ for any u'. To finish the proof note that for $t_1 > 0$ fixed as $t' \downarrow 0$ (thus $n' \to \infty$) we have $\mathbf{P}(m_{q^{n'}} > t_1) \to 0$. □

5.4. *Proof of Theorem 3.5.* Now we are ready to prove Theorem 3.5. Let $0 < \varepsilon < 1$ be arbitrary. Simply,

$$\begin{aligned}
&\mathbf{P}(M_t \le (-\log t)^{1/\alpha} x(1-\varepsilon))\\
&= \mathbf{P}(M_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), Y_t \le (-\log t)^{1/\alpha} x)\\
&\quad + \mathbf{P}(M_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), Y_t > (-\log t)^{1/\alpha} x).
\end{aligned}$$

By Theorem 3.2 the left-hand side converges to $\exp\{-[(1-\varepsilon)x]^{-\alpha}\}$, and by Lemma 5.7 the second term in the right-hand side tends to 0. Therefore

$$\lim_{t\downarrow 0} \mathbf{P}(M_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), Y_t \le (-\log t)^{1/\alpha} x) = \exp\left\{-[(1-\varepsilon)x]^{-\alpha}\right\},$$

thus

$$\liminf_{t\downarrow 0} \mathbf{P}(Y_t \le (-\log t)^{1/\alpha} x) \ge e^{-x^{-\alpha}}. \tag{5.37}$$

On the other hand, for $0 < \varepsilon < 1$

$$\begin{aligned}
&\mathbf{P}\left(Y_t \le (-\log t)^{1/\alpha} x(1-\varepsilon)\right)\\
&= \mathbf{P}\left(Y_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), M_t > (-\log t)^{1/\alpha} x\right)\\
&\quad + \mathbf{P}\left(Y_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), M_t \le (-\log t)^{1/\alpha} x\right).
\end{aligned}$$

Here the first term on the right-hand side goes to 0 by Lemma 5.8, and by Theorem 3.2

$$\limsup_{t\downarrow 0} \mathbf{P}\left(Y_t \le (-\log t)^{1/\alpha} x(1-\varepsilon), M_t \le (-\log t)^{1/\alpha} x\right) \le e^{-x^{-\alpha}}.$$

Combining this with (5.37) the result follows.

5.5. *Proof of Theorem 3.6*. In the $\alpha = 1$ case the result follows similarly, only a minor change is needed in the proof, because one cannot choose the centering to be zero. Note that Theorem 3.2, Proposition 5.1, and Corollaries 5.2, 5.3, and 5.4 hold for any $\alpha \in (0,2)$. Recalling the definition of the centering in (3.6), introduce the notation

$$\widehat{X}_t = \sup_{s\le t}(X_s - c(s)), \quad t \ge 0. \tag{5.38}$$

Lemma 5.5 remains true in the following form.

Lemma 5.9. *Assume (2.4) with $\alpha = 1$, (3.5), and $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$. For any $0 < \varepsilon < 1$ there exist $A > 0$, $t_0 > 0$, $x_0 \ge 1$, $\alpha' > 1$, such that if $t < t_0$, $x > x_0$, $a(t)x < t_0$, then*

$$\mathbf{P}\left(m_t \le a(t)x(1-\varepsilon), \widehat{X}_t > a(t)x\right) \le Ax^{-\alpha'}.$$

Proof: **Step 1.** First let X_t be spectrally positive. Note that in this case $\gamma_- = 0$. For $a = a(t)x(1-\varepsilon) \in (0,1)$, $c = a(t)x$ we have

$$\begin{aligned}
&\left\{m_t \le a, \widehat{X}_t > c\right\} = \left\{N((0,t]\times(a,\infty)) = 0,\ \widehat{X}_t > c\right\} \\
&= \{N((0,t]\times(a,\infty)) = 0\}\cap \\
&\qquad \cap \left\{\sup_{s\le t}\left(\gamma_+ s + \int_0^s \int_{y\le a} y\widetilde{N}(\mathrm{d}u,\mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s)\right) > c\right\},
\end{aligned}$$

where the latter two events are independent. Therefore

$$\begin{aligned}
&\mathbf{P}\left(m_t \le a, \widehat{X}_t > c\right) \\
&\le \mathbf{P}\left(\sup_{s\le t}\left(\gamma_+ s + \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u,\mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s)\right) > c\right).
\end{aligned} \tag{5.39}$$

Recall the definition of γ_+ and the centering in (2.5) and in (3.6). Since, for $s \le t$ and x large enough, $a(t)x(1-\varepsilon) > a(s)$, if $\int_{(0,1]} y\Lambda(\mathrm{d}y) = \infty$ we obtain

$$\begin{aligned}
\gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s) &= -s\int_{(a,1]} y\Lambda(\mathrm{d}y) + s\int_{(a(s),1]} y\Lambda(\mathrm{d}y) \\
&= s\int_{(a(s),a(t)x(1-\varepsilon)]} y\Lambda(\mathrm{d}y) > 0.
\end{aligned} \tag{5.40}$$

While, if $\int_{(0,1]} y\Lambda(\mathrm{d}y) < \infty$,

$$\begin{aligned}
\gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s) &= s\int_{(0,1]} y\Lambda(\mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - s\int_{(0,a(s)]} y\Lambda(\mathrm{d}y) \\
&= s\int_{(a(s),a(t)x(1-\varepsilon)]} y\Lambda(\mathrm{d}y) > 0.
\end{aligned} \tag{5.41}$$

Therefore in both cases we get the same term. Next, we claim that for all x large enough and $t > 0$ small enough

$$\sup_{s \le t} s \int_{(a(s), a(t)x]} y \Lambda(\mathrm{d}y) \le \frac{\varepsilon}{2} a(t) x. \tag{5.42}$$

We have for $x > 1$ and $t \ge s > 0$ small

$$\begin{aligned} s \int_{(a(s), a(t)x]} y \Lambda(\mathrm{d}y) &= s \left(\int_{a(s)}^{a(t)x} \overline{\Lambda}_+(y) \mathrm{d}y - \ell(a(t)x) + \ell(a(s)) \right) \\ &\le s \int_1^{\frac{xa(t)}{a(s)}} \frac{\ell(a(s)u)}{u} \mathrm{d}u + s\ell(a(s)). \end{aligned} \tag{5.43}$$

By Potter's bounds, whenever $a(t)x$ is small enough

$$\int_1^{\frac{xa(t)}{a(s)}} \frac{\ell(a(s)u)}{\ell(a(s))u} \mathrm{d}u \le \int_1^{\frac{xa(t)}{a(s)}} 2u^{-1/2} \mathrm{d}u < 4\sqrt{x} \sqrt{\frac{a(t)}{a(s)}}.$$

Substituting back into (5.43) and using that $\overline{\Lambda}_+(a(t)) = \ell(a(t))/a(t) \sim t^{-1}$ by (4.2), we obtain uniformly in $s \le t$

$$\begin{aligned} & s \int_{(a(s), a(t)x]} y \Lambda(\mathrm{d}y) \le s\ell(a(s)) \left(4\sqrt{x} \sqrt{\frac{a(t)}{a(s)}} + 1 \right) \\ & \le 5x^{-1/2} x a(t) \le \frac{\varepsilon}{2} x a(t) \end{aligned}$$

for x large enough and $t > 0$ small enough. This proves (5.42).

Using the bound (5.42) in inequality (5.39) we obtain

$$\mathbf{P} \left(m_t \le a, \widehat{X}_t > c \right) \le \mathbf{P} \left(\sup_{s \le t} X_s^{(a)} > a(t) x \left(1 - \frac{\varepsilon}{2} \right) \right),$$

and the result follows from (5.14).

Step 2. The extension to the general case is immediate now, because $-X_t^-$ is a subordinator by our assumption $\int_{[-1,0)} -y\Lambda(\mathrm{d}y) < \infty$. □

The corresponding version of Lemma 5.6 also holds. Recall the definition in (5.38).

Lemma 5.10. *Assume (2.4) with* $\alpha = 1$ *and (3.5). For any* $0 < \varepsilon < 1$ *there exist a constant* $A > 0$, $t_0 > 0$, $x_0 \ge 1$, $\alpha' > 1$, *such that if* $t < t_0$, $x > x_0$, *and* $a(t)x < t_0$, *then*

$$\mathbf{P} \left(\widehat{X}_t \le a(t)x(1-\varepsilon), m_t > a(t)x \right) \le A x^{-\alpha'}.$$

Proof. Assume first that X_t is spectrally positive. Let $\tau = \tau_a = \inf\{s : \Delta X_s > a\}$, for $a \in (0,1)$. As in the proof of Lemma 5.6 for $b > 0$

$$\begin{aligned}
&\mathbf{P}\left(\widehat{X}_t \le b, m_t > a\right) = \mathbf{P}\left(\widehat{X}_t \le b, \tau \le t\right) \\
&\le \mathbf{P}\left(X_\tau - c(\tau) \le b, \tau \le t\right) \\
&\le \int_0^t \mathbf{P}\left(\gamma_+ s + \int_0^s \int_{(0,a]} y\widetilde{N}(\mathrm{d}u, \mathrm{d}y) - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s) \le b - a\right) \times \\
&\qquad\qquad\qquad\qquad \times \overline{\Lambda}_+(a) e^{-\overline{\Lambda}_+(a)s}\mathrm{d}s \\
&\le \left(1 - e^{-\overline{\Lambda}_+(a)t}\right) \sup_{s\le t} \mathbf{P}\left(X_s^{(a)} + \gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s) \le b - a\right).
\end{aligned}$$

Put $b = a(t)x(1-\varepsilon)$ and $a = a(t)x$. From (5.40) and (5.41) we obtain

$$\mathbf{P}\left(X_s^{(a)} + \gamma_+ s - s\int_{(a,1]} y\Lambda(\mathrm{d}y) - c(s) \le b - a\right) \le \mathbf{P}\left(X_s^{(a)} \le b - a\right).$$

Therefore, the result follows as in the proof of Lemma 5.6.

The general case follows exactly as in the proof of Lemma 5.6. □

After having the appropriate versions of Lemma 5.5 and 5.6 the proof of the theorem is identical to the proof in the $\alpha \ne 1$ case.

5.6. *Proof of Theorem 3.7.* We shall prove that

$$\lim_{t\downarrow 0} \mathbf{P}\left(Y_t = \sup_{t\le s\le 1} \frac{X_s - c(s)}{a(s)}\right) = 1, \tag{5.44}$$

which clearly implies the theorem.

First assume that $\alpha \ne 1$, in which case $c(s) = 0$. Note that

$$Y_t = \sup_{t\le s\le 1} \frac{\sup_{u\le s} X_u}{a(s)} \ge \sup_{t\le s\le 1} \frac{X_s}{a(s)} =: Z_t.$$

Assume that $Y_t > Z_t$. Then $Y_t = \frac{X_{u_0}}{a(s_0)}$, for some $s_0 \in [t,1]$ and $u_0 \le s_0$. Since $Y_t > Z_t$, we have $u_0 < t$, thus the monotonicity of a implies $Y_t = \overline{X}_t/a(t)$. Therefore

$$\mathbf{P}(Y_t > Z_t) \le \mathbf{P}\left(Y_t = \frac{\overline{X}_t}{a(t)}\right).$$

Now for all $t > 0$ and $x > 0$

$$\begin{aligned}
\mathbf{P}\Big(Y_t = \frac{\overline{X}_t}{a(t)}\Big) &\le \mathbf{P}\big(Y_t \le x(-\log t)^{\frac{1}{\alpha}}\big) + \mathbf{P}\Big(Y_t = \frac{\overline{X}_t}{a(t)}, \frac{\overline{X}_t}{a(t)} \ge x(-\log t)^{\frac{1}{\alpha}}\Big) \\
&=: p_t(x).
\end{aligned} \tag{5.45}$$

By Theorem 3.5 for all $x > 0$ the first term on the right-hand side tends to $\exp(-x^{-\alpha})$, which converges to 0 as $x \downarrow 0$. Next we show that $\overline{X}_t/a(t)$ is stochastically bounded. By (2.2)

$$\frac{1}{a(t)} \sup_{s\le t} X_s \le \frac{1}{a(t)} \sup_{s\le t} X_s^+ + \frac{1}{a(t)} \sup_{s\le t} X_s^-. \tag{5.46}$$

The second term on the right side of inequality (5.46) is stochastically bounded by (5.18), while the first term on the right side of inequality (5.46) is stochastically bounded since the process

$$\frac{X_{ts}^{+}}{a(t)},\ 0 \le s \le 1,$$

converges weakly in $D_0[0,1]$. (See Remark *(iv)* on page 322 of Maller and Mason (2008) and the methods of the proofs of Proposition 4.1 and Corollary 4.2 of Maller and Mason (2010).) Thus the second term in the right side of (5.45) converges to 0 for all $x > 0$. We see now that

$$\limsup_{t\downarrow 0} \mathbf{P}\left(Y_t = \frac{\overline{X}_t}{a(t)}\right) \le \lim_{x\downarrow 0}\limsup_{t\downarrow 0} p_t(x) = \lim_{x\downarrow 0} \exp\left(-x^{-\alpha}\right) = 0,$$

which implies that

$$\lim_{t\downarrow 0} \mathbf{P}\left(Y_t = Z_t\right) = 1,$$

and this is (5.44).

For $\alpha = 1$ the proof is almost identical, except that here, instead of Theorem 3.5, we apply Theorem 3.6. There is a small difference in the verification of the stochastic boundedness of $\widehat{X}_t/a(t)$. Note that

$$\frac{1}{a(t)}\sup_{s\le t}(X_s - c(s)) \le \frac{1}{a(t)}\sup_{s\le t}(X_s^{+} - c(s)) + \frac{1}{a(t)}\sup_{s\le t} X_s^{-}. \qquad (5.47)$$

The second term on the right side of inequality (5.47) is again stochastically bounded by (5.18), while for the first term on the right side of inequality (5.47) it follows from the convergence of $(X_t - c(t))/a(t)$ as above. □

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