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Growing random graphs with a preferential attachment structure

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Abstract. The aim of this paper is to develop a method for proving almost sure convergence in the Gromov–Hausdorff–Prokhorov topology for a class of models of growing random graphs that generalises Rémy's algorithm for binary trees. We describe the obtained limits using some iterative gluing construction that generalises the famous line-breaking construction of Aldous' Brownian tree, and we characterize some of them using the self-similarity property in law that they satisfy.

To do that, we develop a framework in which a metric space is constructed by gluing smaller metric spaces, called *blocks*, along the structure of a (possibly infinite) discrete tree. Our growing random graphs seen as metric spaces can be understood in this framework, that is, as evolving blocks glued along a growing discrete tree structure. Their scaling limit convergence can then be obtained by separately proving the almost sure convergence of every block and verifying some relative compactness property for the whole structure. For the particular models that we study, the discrete tree structure behind the construction has the distribution of an affine preferential attachment tree or a weighted recursive tree. We strongly rely on results concerning those two models and their connection, obtained in the companion paper Sénizergues (2021).

1. Introduction

We start by introducing a particular model of growing random graphs, which we call the *generalised Rémy algorithm*, as an example of a model to which our results apply. Other such models are discussed at the end of the introduction.

1.1. *A generalised version of the Rémy algorithm.* Consider $(G_n)_{n\geq 1}$ a sequence of finite connected rooted graphs and construct the sequence $(H_n)_{n\geq 1}$ recursively as follows. Let $H_1 = G_1$. Then, for any $n \geq 1$, conditionally on the structure H_n already constructed, take an edge in H_n uniformly at random, split it into two edges by adding a vertex "in the middle" of this edge, and glue a copy of G_{n+1} to the structure by identifying the root vertex of G_{n+1} with the newly created vertex. Call the obtained graph H_{n+1}.

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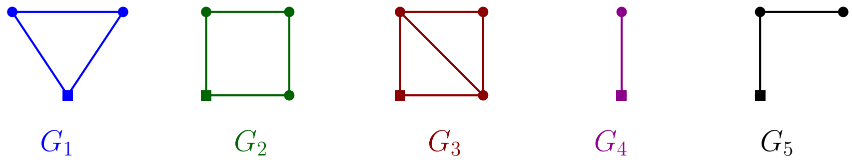


FIGURE 1.1. An example of a sequence of graphs used to run the algorithm, the root of each graph is represented by a square vertex

When all the graphs $(G_n)_{n\geq 1}$ are equal to the single-edge graph, we obtain the so-called *Rémy algorithm*, which produces planted binary trees. In that particular case, if we label the leaves of the obtained tree H_n with their time of creation, then for each n, the graph H_n is a uniform planted binary tree with n labelled leaves. Some generalisations of this algorithm have already been studied for particular sequences $(G_n)_{n\geq 1}$, deterministic Haas and Stephenson (2015); Ross and Wen (2018) or random Haas and Stephenson (2021). Here we try to be as general as possible in the choice of the sequence $(G_n)_{n\geq 1}$.

For any $n \geq 1$, we see the graph H_n as a measured metric space $(V(H_n), \mathrm{d_{gr}}, \mu_{\mathrm{unif}})$, by viewing its set of vertices $V(H_n)$ as endowed with the usual graph distance $\mathrm{d_{gr}}$ and the uniform measure μ_{unif}. In what follows and in the rest of the paper, we will always abuse notation and identify H_n to its set of vertices $V(H_n)$ and write $(H_n, \mathrm{d_{gr}}, \mu_{\mathrm{unif}})$ for the metric space corresponding to H_n.

It is well-known (see Curien and Haas (2013, Theorem 5) for example) that the sequence of trees created through the original Rémy algorithm with distances rescaled by $n^{-1/2}$ almost surely converges in the Gromov–Hausdorff–Prokhorov topology to a constant multiple of Aldous' Brownian tree, see Aldous (1991); Le Gall (2006). Under some mild condition on the sequence $(G_n)_{n\geq 1}$, we give here an analogous result that ensures that the sequence of graphs $(H_n)_{n\geq 1}$ appropriately rescaled almost surely converges in the Gromov–Hausdorff–Prokhorov topology to a random compact metric space.

Proposition 1.1. *For any $n \geq 1$, we denote by a_n the number of edges in the graph G_n. Suppose there exists $c > 0$ and $0 \leq c' < \frac{1}{c+1}$ and $\epsilon > 0$ such that, as $n \to \infty$,*

$$\sum_{i=1}^{n} a_i = c \cdot n \cdot \left(1 + O\left(n^{-\epsilon}\right)\right), \qquad \textit{and} \qquad a_n \leq n^{c'+o(1)},$$

then we have the following almost sure convergence in the Gromov–Hausdorff–Prokhorov topology

$$(H_n, n^{\frac{-1}{c+1}} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{H}, \tag{1.1}$$

where $\mathcal{H}$ is some non-trivial random measured metric space.

The limiting random compact measured metric space $\mathcal{H}$, which depends on the whole sequence $(G_n)_{n\geq 1}$, can be described as the result of an *iterative gluing construction*, as defined in Sénizergues (2019). It is a natural extension of the *line-breaking construction* invented by Aldous to describe his Brownian tree Aldous (1991) by gluing together a collection of line segments, but where we allow for more complex shapes than just segments.

An iterative gluing construction. The construction of the limiting random measured metric space $\mathcal{H}$ appearing in Proposition 1.1 is described as follows. We first run a particular increasing time-inhomogeneous Markov chain $(\mathsf{M}^{\mathbf{a}}_n)_{n\geq 1}$ which takes values in $\mathbb{R}_+$, and whose law depends only on the sequence $\mathbf{a} = (a_n)_{n\geq 1}$ (a definition of the law of this increasing Markov chain is given in Section 3.2). Then we cut the semi-infinite line $\mathbb{R}_+$ at the values $\mathsf{M}^{\mathbf{a}}_1, \mathsf{M}^{\mathbf{a}}_2 \dots$ taken by the chain. This creates an ordered sequence of segments with length $\mathsf{M}^{\mathbf{a}}_1, (\mathsf{M}^{\mathbf{a}}_2 - \mathsf{M}^{\mathbf{a}}_1), (\mathsf{M}^{\mathbf{a}}_3 - \mathsf{M}^{\mathbf{a}}_2), \dots$. Now for any $n \geq 1$,

(i) Cut the n-th segment into a_n sub-segments by throwing $a_n - 1$ independent uniform points along its length, and call $(L_{n,1}, L_{n,2}, \dots, L_{n,a_n})$ the respective length of the obtained sub-segments.
(ii) Take the graph G_n and replace every edge $e_k \in \{e_1, \dots, e_{a_n}\}$, where the edges of G_n are labelled in an arbitrary order, with a segment of length $L_{n,k}$. The result, considered as a metric space, is called $\mathcal{G}_n$.

Now, start from $\mathcal{H}_1 := \mathcal{G}_1$ and recursively when $\mathcal{H}_n$ is already constructed, sample a point according to the length measure on $\mathcal{H}_n$ and identify the root of $\mathcal{G}_{n+1}$ to the chosen point. The space $\mathcal{H}$, as a metric space, is obtained as the completion of the increasing union

$$\mathcal{H} = \overline{\bigcup_{n\geq 1} \mathcal{H}_n}.$$

The probability measure carried on $\mathcal{H}$ is the weak limit of the normalised length measure carried by the $\mathcal{H}_n$'s. Additional precision as to how to define a structure by gluing metric spaces onto one another is given in Section 1.2 below.

Remark 1.2. We made the choice of labelling the result discussed above as a *proposition* instead of a *theorem* as to emphasize the fact that this result follows from the more general Theorem 5.2 that appears in Section 5. Other results in the same vein as Proposition 1.1 but that concern other models of randomly growing graphs are stated in Section 5; we discuss them at the end of this introduction.

1.2. *Metric spaces glued along a tree structure.* We introduce a general framework that allows us to handle objects that are defined as the result of gluing together metric spaces along a discrete tree structure. We will use this framework to interpret the construction of the graphs $(H_n)_{n\geq 1}$ in the next paragraph.

Consider the Ulam tree $\mathbb{U}$ with its usual representation as

$$\mathbb{U} = \bigcup_{n\geq 0} \mathbb{N}^n, \tag{1.2}$$

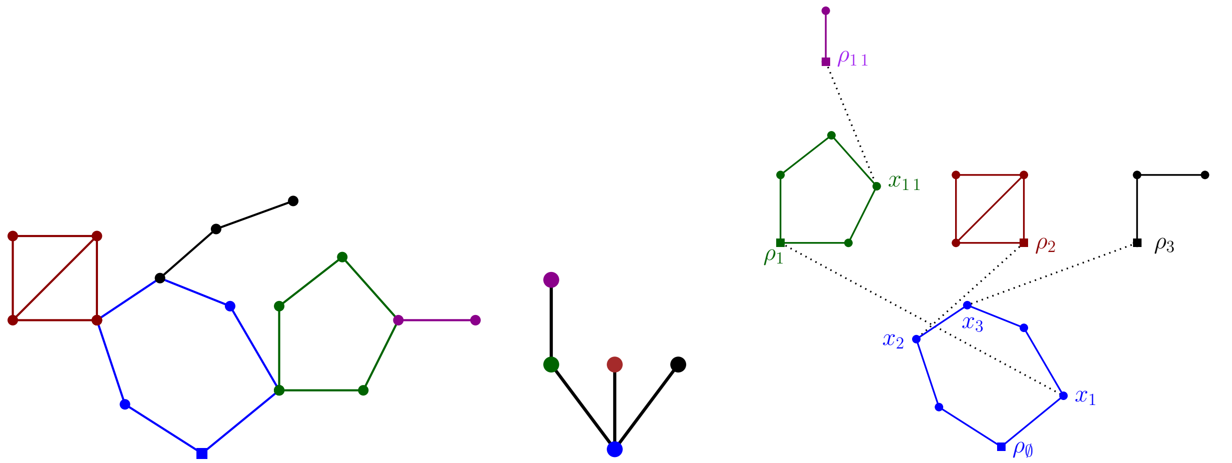
the set of words on the alphabet $\mathbb{N} := \{1, 2, 3 \dots\}$. We say that $\mathcal{D} = (\mathcal{D}(u))_{u\in\mathbb{U}}$ is a *decoration* on the Ulam tree if for any $u \in \mathbb{U}$,

$$\mathcal{D}(u) = (D_u, d_u, \rho_u, (x_{ui})_{i\geq 1}),$$

is a compact rooted metric space, with underlying set D_u, distance d_u, rooted at a point ρ_u and endowed with a sequence $(x_{ui})_{i\geq 1} \in D_u$. We say that $\mathcal{D}(u)$ is a *block* of the decoration $\mathcal{D}$. Note that here, for any $u \in \mathbb{U}$ and $i \in \mathbb{N}$, we see ui as the concatenation of the word u and the letter i so that ui is also an element of $\mathbb{U}$.

Then for any such decoration $\mathcal{D}$, we make sense of the following metric space $\mathscr{G}(\mathcal{D})$, which is informally what we get if we take the disjoint union $\bigsqcup_{u\in\mathbb{U}} D_u$ and identify every root $\rho_{ui} \in D_{ui}$ to the distinguished point $x_{ui} \in D_u$ for every $u \in \mathbb{U}$ and every $i \in \mathbb{N}$, and take the metric completion of the obtained metric space.

This setting encompasses in particular the case where we only glue a finite number of blocks along a plane tree. If τ is a plane tree, it can be natural to consider a decoration $\mathcal{D} = (\mathcal{D}(u))_{u\in\tau}$ which is only defined on the vertices of τ and which is such that for all $u \in \tau$, the block $\mathcal{D}(u) = (D_u, d_u, \rho_u, (x_{ui})_{1\leq i\leq \deg^+_\tau(u)})$ is only endowed with a finite number of distinguished points that corresponds to the number $\deg^+_\tau(u)$ of children of u in τ. In this case we automatically extend $\mathcal{D}$ to the rest of $\mathbb{U}$ by letting the block $\mathcal{D}(u)$ be the one-point space $(\star, 0, \star, (\star)_{i\geq 1})$ for all $u \notin \tau$ and by letting $x_{ui} = \rho_i$ for all $u \in \tau$ and $i > \deg^+_\tau(u)$. Thanks to this identification, we can always assume that the decoration that we are working with are defined on the whole Ulam tree $\mathbb{U}$ and that their blocks are endowed with infinitely many distinguished points.



(A) A realisation of H_5 (B) The tree $\mathtt{P}_5$ (C) The graph H_5 seen as the gluing of a decoration

FIGURE 1.2. Decomposition of H_5 as the gluing of some decoration

Representing the generalised Rémy algorithm using decorations. As an example, we now interpret $(H_n)_{n\geq 1}$ in the framework described above by constructing a sequence of decorations $(\mathcal{D}_n)_{n\geq 1}$, in such a way that for all $n \geq 1$ the graph H_n seen as a metric space coincides with $\mathscr{G}(\mathcal{D}_n)$. That way, the problem of understanding the whole structure of H_n is decomposed into the easier problem of understanding separately all the blocks $\mathcal{D}_n(u)$ for $u \in \mathbb{U}$.

For this particular construction, this is done in the following way: As pictured in Figure 1.1, we colour the vertices and edges of each graph in the sequence $(G_n)_{n\geq 1}$ with distinct colours for different graphs. Then we keep those colours in the construction of the graphs $(H_n)_{n\geq 1}$, and use the rule that every time that an edge is split by the algorithm, the two resulting edges have the same colour as the original edge. See Figure 1.2a for a realisation of H_5 using the coloured graphs of Figure 1.1. We can naturally couple this construction with that of an increasing sequence of plane trees $(\mathtt{P}_n)_{n\geq 1}$, in which every one of the n vertices of $\mathtt{P}_n$ corresponds to one of the n colours present in H_n: two vertices of $\mathtt{P}_n$ are linked by an edge if and only if their corresponding colours are adjacent in the graph H_n (the left-to-right order of children in the plane tree is given by the order of creation of the vertices), see Figure 1.2b. It can be checked that the sequence $(\mathtt{P}_n)_{n\geq 1}$ has the distribution of a *preferential attachment tree with sequence of fitnesses* $\mathbf{a} = (a_n)_{n\geq 1}$, whose definition is recalled in Section 3. The construction of the decoration $\mathcal{D}_n$ is pictured in Figure 1.2c and simply corresponds to decomposing the graph H_n into n pieces that correspond to the n different colours and that are glued along the tree $\mathtt{P}_n$.

With this particular construction, each process $(\mathcal{D}_n(u))_{n\geq 1}$ for a fixed $u \in \mathbb{U}$ only evolves at times when the degree of u evolves in the tree $\mathtt{P}_n$ and stays constant otherwise. Also, at times where the block $\mathcal{D}_n(u)$ evolves, it does so independently of all the other blocks and follows some simple dynamics. This allows us to study the evolution of the processes $(\mathcal{D}_n(u))_{n\geq 1}$, including their scaling limit, separately for every $u \in \mathbb{U}$.

From there, the last step in order to get to the convergence stated in Proposition 1.1 is a sort of continuity argument that ensures that understanding the scaling limit of each block $(\mathcal{D}_n(u))_{n\geq 1}$ for every $u \in \mathbb{U}$ is enough to understand the scaling limit of the full structure $(\mathscr{G}(\mathcal{D}_n))_{n\geq 1}$. This is discussed in a general setting in the paragraph below.

Convergence of metric spaces glued along a tree structure. For a sequence $(\mathcal{D}_n)_{n\geq 1}$ of decorations, we prove a sufficient condition for the convergence of the sequence $(\mathscr{G}(\mathcal{D}_n))_{n\geq 1}$ in the Gromov–Hausdorff topology: it is the content of Theorem 2.2 that if

- for every $u \in \mathbb{U}$, we have the convergence $\mathcal{D}_n(u) \to \mathcal{D}_\infty(u)$ for some decoration $\mathcal{D}_\infty$ in some appropriate infinitely pointed Gromov–Hausdorff topology,
- the sequence of decorations $(\mathcal{D}_n)_{n\geq 1}$ satisfies the following relative compactness property

$$\inf_{\substack{\theta\subset\mathbb{U}\\ \theta \text{ finite}}} \sup_{u\in\mathbb{U}} \left(\sum_{\substack{v\preceq u\\ v\notin\theta}} \sup_{n\geq 1} \operatorname{diam}(\mathcal{D}_n(v)) \right) = 0, \tag{1.3}$$

  where the relation $v \preceq u$ in the last display means that v is an ancestor of u (or, said in terms of words, that v is a prefix of u),

then we have the convergence $\mathscr{G}(\mathcal{D}_n) \to \mathscr{G}(\mathcal{D}_\infty)$ in the Gromov–Hausdorff topology as $n \to \infty$. The condition (1.3) appearing above is there to ensure that the object $\mathscr{G}(\mathcal{D}_n)$ is well-approximated by a finite number of blocks, in a way that is uniform in $n \geq 1$. In particular, it prevents cases where vanishingly small blocks would get stacked on top of each other to create some macroscopic structure.

With some appropriate assumptions, we can also endow these metric spaces with measures and get a similar statement in the Gromov–Hausdorff–Prokhorov topology. We recall the definition and some properties of those topologies in Section 2.3.1.

Two families of distributions on decorations that appear in the limit. In Section 4, we define two families of random decorations that naturally appear as the limit of some natural random growth models. One of those families contains the random decorations associated to iterative gluing constructions, an example of which we saw in Section 1.1. The other family contains what we call *self-similar* decorations: these are decorations $\mathcal{D}$ for which all the blocks $\mathcal{D}(u)$ for $u \in \mathbb{U}$ are rescaled versions of i.i.d. random metric spaces, where the (random) scaling factors are of some specific product form. Under some assumptions, we prove in Section 4.2 that the distribution of the metric space $\mathscr{G}(\mathcal{D})$ associated to some random self-similar decoration $\mathcal{D}$ can be characterized as the unique fixed point of some contraction in an appropriate space of distributions on metric spaces, in the same spirit as the self-similar random trees studied by Rembart and Winkel in Rembart and Winkel (2018). In some specific instances discussed in Proposition 4.3, a distribution on random decorations belongs to the two families presented above, and that is the case for the limit of some of the discrete models that we study in Section 5.

The fact that metric spaces described using an iterative gluing construction appear in the limit, as in Proposition 1.1, crucially depends on some relation, explored in Sénizergues (2021), between preferential attachment trees and weighted recursive trees (see Section 3 for definitions and properties of those distribution). In particular, in the case of the generalised Rémy algorithm, the distribution of the sequence of trees $(\mathsf{P}_n)_{n\geq 1}$ introduced above can also be expressed as that of a weighted recursive tree using the random weight sequence $(\mathsf{m}_n^{\mathbf{a}})_{n\geq 1} := (\mathsf{M}_n^{\mathbf{a}} - \mathsf{M}_{n-1}^{\mathbf{a}})_{n\geq 1}$ to which we referred in the description of the limiting space $\mathcal{H}$ and which will be defined in (3.1).

1.3. *Scope of our results and their relation to previous work.* We now discuss some of our results and how they are related to the existing literature. As previously stated in the introduction, Proposition 1.1 is only a consequence of the more general Theorem 5.2, which is the main contribution of this paper. Nevertheless, it already encompasses several models that were studied in their own right in the literature, possibly using methods that are different from ours, and that each correspond to choosing certain specific sequences of graphs $(G_n)_{n\geq 1}$:

- Of course, we recover the convergence for the standard Rémy's algorithm whenever the sequence $(G_n)_{n\geq 1}$ is constant equal to the single-edge graph.

- When $(G_n)_{n\geq 1}$ is constant equal to a vertex with a single loop, the model is equivalent to the looptree of the linear preferential attachment tree, and we recover the convergence proved in Curien et al. (2015).
- In Haas and Stephenson (2015), Haas and Stephenson study the case where G_1 is the single-edge graph and the sequence $(G_n)_{n\geq 2}$ is constant equal to the star-graph with $k-1$ branches, for $k \geq 2$. They describe the scaling limit as a fragmentation tree, as introduced in Haas and Miermont (2004). In this case, we improve their convergence which was only in probability and give another construction of the limit.
- In Ross and Wen (2018) Ross and Wen study a model that depends on a integer-valued parameter $\ell \geq 2$ and that corresponds to setting G_n to be a single-edge graph if $n-1$ is a multiple of ℓ, and reduced to a single vertex otherwise. Their result can be recovered from Proposition 1.1 for the GHP-convergence, using Sénizergues (2021, Proposition 5.1 and Remark 5.2) to identify the distribution of the length of the segments used for the line-breaking construction.
- In a recent work of Haas and Stephenson Haas and Stephenson (2021) the authors also study the case where $(G_n)_{n\geq 1}$ is taken as an i.i.d. sequence of rooted trees taken at random from a finite set. They describe the limit (obtained in probability) as a multi-type fragmentation tree as introduced in Stephenson (2018). Again, our result ensures that the convergence is almost sure in the Gromov–Hausdorff–Prokhorov topology and gives another construction of the limit.

In Section 5, we also apply Theorem 5.2 to other models of random growing graphs such as Ford's α-model Ford (2006), Marchal's algorithm Marchal (2008) or their generalisation the $\alpha-\gamma$-growth Chen et al. (2009), possibly started from an arbitrary graph. The same methods apply also for discrete *looptrees* associated to those models (using an appropriate planar embedding) or to planar preferential attachment trees. We obtain the following results:

- We improve the convergence Haas et al. (2008); Haas and Miermont (2012) of Ford trees and α-γ-trees, from a convergence in probability to an almost sure convergence, and also obtain the convergence of their respective discrete looptrees to continuous limits which can be described as the result of iterative gluing constructions, or as self-similar random metric spaces. This is the content of Proposition 5.9.
- We describe in Proposition 5.7 a new iterative gluing construction for α-stable trees and α-stable components, different from the ones appearing in Goldschmidt and Haas (2015); Goldschmidt et al. (2018).
- We obtain Proposition 5.10, which was stated as a conjecture in Curien et al. (2015), and which concerns the scaling limits of looptrees of planar preferential attachment trees with offset δ. We describe the limit as an iterative gluing construction with blocks that are circles of random circumference.

1.4. *Organisation of the paper.* This paper is organised as follows.

We start in Section 2 by developing a framework that allows us to define the gluing of infinitely many metric spaces along the structure of the Ulam tree, as sketched above in Section 1.2. We prove Theorem 2.2 which ensures that this procedure is continuous in some sense with respect to the blocks that we glue together, as soon as they satisfy some relative compactness property. Then in Section 3 we recall the definition of affine preferential attachment trees and weighted recursive trees and the results that were proved for those random trees in the companion paper Sénizergues (2021), and on which our approach strongly relies. In Section 4 we present two families of distributions on the set of decorations, the iterative gluing constructions and the self-similar decorations, which can appear as continuous limits of the models that we study. We derive some properties of those constructions; in particular we derive sufficient conditions for the metric space $\mathscr{G}(\mathcal{D})$ associated to a random decoration $\mathcal{D}$ to be compact almost surely, for decorations $\mathcal{D}$ whose distribution belongs to either of the two families above, and we compute its Hausdorff dimension in some specific cases. The section ends by stating and proving Proposition 4.3, which provide a sufficient condition for a random decoration $\mathcal{D}$ described using an iterative gluing construction to also be self-similar. Last, in Section 5, we apply the preceding results to well-chosen sequences of decorations in order to obtain some scaling limit convergence results for some families of growing random graphs. We start by proving Theorem 5.2, which is a general convergence result that we can apply in all our examples and that expresses the limiting space as the result of an iterative gluing construction. The rest of the section is devoted to applying this theorem to a series of examples of growing random graphs. This includes a proof of Proposition 1.1 for the generalized Rémy algorithm but also Proposition 5.7 for Marchal's algorithm, Proposition 5.9 for the $\alpha-\gamma$-trees and their looptrees, and Proposition 5.10 for the looptrees of affine preferential attachment trees.

Some definitions and results for classical models that are useful to our proofs are recalled or proved in Appendix A.

Contents

2. Gluing metric spaces along the Ulam tree

In this section, we introduce what we call *decorations* on the Ulam tree, which are families of infinitely pointed compact metric spaces that we call *blocks*, indexed by the vertices of the Ulam tree. This structure should be thought of as a plan that specifies how to construct a metric space by gluing together all those decorations onto one another, along the structure of the Ulam tree. We then provide sufficient conditions that ensure that the resulting metric space is compact and depends continuously on the decorations in a sense that we make precise.

2.1. *The Ulam tree. The completed Ulam tree.* Recall the definition of the Ulam tree $\mathbb{U} = \bigcup_{n\geq 0} \mathbb{N}^n$ with $\mathbb{N} = \{1, 2, \dots\}$. Introduce the set $\partial\mathbb{U} = \mathbb{N}^{\mathbb{N}}$, to which we refer as the *leaves* of the Ulam tree, which we see as the infinite rays joining the root to infinity and set $\overline{\mathbb{U}} := \mathbb{U} \cup \partial\mathbb{U}$. On this set, we have a natural genealogical order $\preceq$ defined in such a way that $u \preceq v$ if and only if u is a prefix of v. From this order relation we can define for any $u \in \mathbb{U}$ the *subtree descending from* u as the set $T(u) := \left\{ v \in \overline{\mathbb{U}} \;\middle|\; u \preceq v \right\}$. The collection of sets $\{T(u),\ u \in \mathbb{U}\} \sqcup \{\{u\},\ u \in \mathbb{U}\}$ generates a topology over $\overline{\mathbb{U}}$, which can also be generated using an appropriate ultrametric distance. Endowed with this distance, the set $\overline{\mathbb{U}}$ is then a separable and complete metric space.

In the text, we will consider Borel probability measures on the metric space $\overline{\mathbb{U}}$. For those measures, there is a simple characterization of weak convergence given by the following lemma proved in Sénizergues (2021).

Lemma 2.1. *Sénizergues (2021, Lemma 2.2) Let $(\pi_n)_{n\geq 1}$ be a sequence of Borel probability measures on $\overline{\mathbb{U}}$. Then $(\pi_n)_{n\geq 1}$ converges weakly to a probability measure π on $\overline{\mathbb{U}}$ if and only if for any $u \in \mathbb{U}$,*

$$\pi_n(\{u\}) \to \pi(\{u\}) \quad \textit{and} \quad \pi_n(T(u)) \to \pi(T(u)) \qquad \textit{as } n \to \infty.$$

Plane trees as subsets of $\mathbb{U}$. Classically, a plane tree τ is defined as a finite non-empty subset of $\mathbb{U}$ such that

(i) if $v \in \tau$ and $v = ui$ for some $i \in \mathbb{N}$, then $u \in \tau$,
(ii) for all $u \in \tau$, there exists a number in $\mathbb{N} \cup \{0\}$, denoted by $\deg^+_\tau(u)$, such that for all $i \in \mathbb{N}$, $ui \in \tau$ iff $i \leq \deg^+_\tau(u)$.

We denote $\mathbb{T}$ the set of planes trees.

Elements of notation. We introduce here some elements of notation.

- Elements of $\overline{\mathbb{U}}$ are defined as finite or infinite sequences of integers, which can thought of as words on the alphabet $\mathbb{N}$. We usually use the symbols u or v to denote elements of this space.
- Sometimes we also use a bold letter $\mathbf{i}$ to denote a finite or infinite word $\mathbf{i} = i_1 i_2 \dots$. In this case, for any integer k smaller than the length of $\mathbf{i}$ we also write $\mathbf{i}_k = i_1 \dots i_k$ the word truncated to its k first letters.
- For any two $u, v \in \overline{\mathbb{U}}$, we write $u \wedge v$ the most recent common ancestor of u and v.
- For any $u \in \mathbb{U}$, the height of u is the unique number n such that $u \in \mathbb{N}^n$. We denote it $\mathrm{ht}(u)$ or sometimes also $|u|$.

2.2. *Decorations on the Ulam tree.* We call any function $f : \mathbb{U} \to E$ from the Ulam tree to a space E an *E-valued decoration* on the Ulam tree.

Real-valued decorations. As a first example, a function $\ell : \mathbb{U} \to \mathbb{R}_+$ is a real-valued decoration on the Ulam tree. As this will be useful later on, we introduce the following terminology. We say that ℓ is *non-explosive* if

$$\inf_{\substack{\theta \in \mathbb{T} \\ \theta \text{ finite}}} \sup_{u \in \mathbb{U}} \left(\sum_{\substack{v \preceq u \\ v \notin \theta}} \ell(v) \right) = 0. \tag{2.1}$$

Metric-space-valued decorations. The main objects studied in this paper are metric-space-valued decorations $\mathcal{D} : \mathbb{U} \to \mathbb{M}^{\infty\bullet}$, where the set $\mathbb{M}^{\infty\bullet}$ is the set of non-empty compact metric spaces endowed with an infinite sequence of distinguished points, up to isometry (see below for a proper definition). More precisely,

$$\mathcal{D} : u \mapsto \mathcal{D}(u) = (D_u, d_u, \rho_u, (x_{ui})_{i \geq 1}),$$

where D_u is a set, d_u is a distance function of D_u, and ρ_u and the $(x_{ui})_{i\geq 1}$ are distinguished points of D_u. The point ρ_u is called the *root* of $\mathcal{D}(u)$, and we call $\mathcal{D}(u)$ a *block* of the decoration. In all the paper, the word "decoration" always means "$(\mathbb{M}^{\infty\bullet})$-valued decoration", unless specified otherwise.

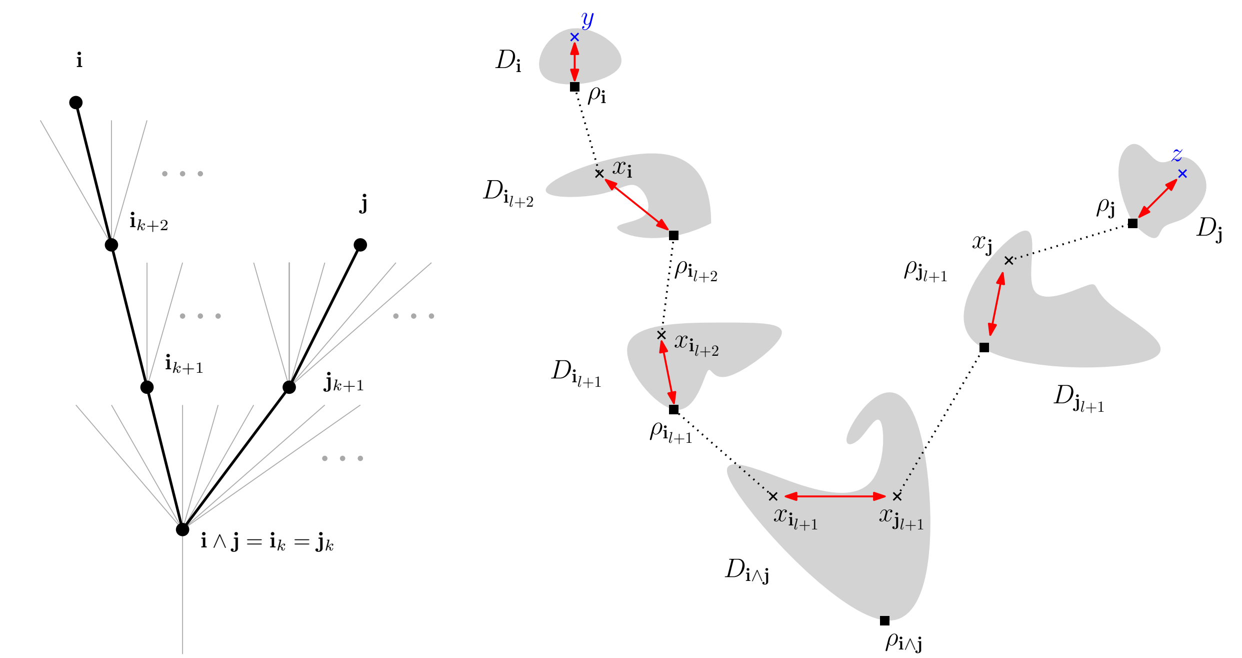
We define a particular element of $\mathbb{M}^{\infty\bullet}$, which we call the *trivial* or *one-point* space $(\{\star\}, 0, \star, (\star)_{i\geq 1})$. For any decoration $\mathcal{D}$, the subset $S \subset \mathbb{U}$ of elements u for which $\mathcal{D}(u)$ is not trivial is called the *support* of the decoration $\mathcal{D}$. In the rest of the paper we will often consider decorations that are supported on finite plane trees.

For $a > 0$, we will use the notation $a \cdot \mathcal{D}$ to denote the decoration created from $\mathcal{D}$ by multiplying all the distances in all the blocks by a factor a.

The gluing operation. We define the *gluing operation* $\mathscr{G}$ on the set of metric-space-valued decorations $(\mathbb{M}^{\infty\bullet})^{\mathbb{U}}$. For any $\mathcal{D} = (\mathcal{D}(u))_{u\in\mathbb{U}}$, we first define a metric space $\mathscr{G}^*(\mathcal{D})$ as

$$\mathscr{G}^*(\mathcal{D}) = \left(\bigsqcup_{u \in \mathbb{U}} D_u \right) / \sim, \tag{2.2}$$

where the equivalence relation $\sim$ is such that for every $u \in \mathbb{U}$ and $i \in \mathbb{N}$ the root ρ_{ui} of D_{ui} is in relation with the distinguished point $x_{ui} \in D_u$. The distance d on the set $\mathscr{G}^*(\mathcal{D})$ is then the



(A) The path between **i** and **j** in the Ulam tree

(B) The contribution from every block along the path

FIGURE 2.3. The distance between two points is computed as the sum of the contributions denoted in red, computed using the distance function from the corresponding block.

one corresponding to the metric gluing of the blocks along the relation $\sim$, in the sense of Burago et al. (2001). This distance is defined as follows: for all $\mathbf{i} = i_1 i_2 \dots i_n$ and $\mathbf{j} = j_1 j_2 \dots j_m$ and points $y \in D_{\mathbf{i}}$, $z \in D_{\mathbf{j}}$,

- if $\mathbf{i} = \mathbf{j}$ then
$$\mathrm{d}(y,z) = \mathrm{d}(z,y) = d_{\mathbf{i}}(y,z),$$

- if $\mathbf{i} \prec \mathbf{j}$ then
$$\mathrm{d}(y,z) = \mathrm{d}(z,y) = d_{\mathbf{i}}(y, x_{\mathbf{j}_{n+1}}) + \sum_{k=n+1}^{m-1} d_{\mathbf{j}_k}(\rho_{\mathbf{j}_k}, x_{\mathbf{j}_{k+1}}) + d_{\mathbf{j}}(\rho_{\mathbf{j}}, z),$$

- and if $\mathbf{i} \wedge \mathbf{j} = \mathbf{i}_l = \mathbf{j}_l$ is different from $\mathbf{i}$ and $\mathbf{j}$ we let
$$\mathrm{d}(y,z) = \mathrm{d}(z,y) = d_{\mathbf{i}\wedge\mathbf{j}}(x_{\mathbf{i}_{l+1}}, x_{\mathbf{j}_{l+1}}) + \sum_{k=l+1}^{n-1} d_{\mathbf{i}_k}(\rho_{\mathbf{i}_k}, x_{\mathbf{i}_{k+1}}) + d_{\mathbf{i}}(\rho_{\mathbf{i}}, z)$$
$$+ \sum_{k=l+1}^{m-1} d_{\mathbf{j}_k}(\rho_{\mathbf{j}_k}, x_{\mathbf{j}_{k+1}}) + d_{\mathbf{j}}(\rho_{\mathbf{j}}, z).$$

This last configuration is illustrated in Figure 2.3. We then set
$$\mathscr{G}(\mathcal{D}) = \overline{\mathscr{G}^*(\mathcal{D})} \tag{2.3}$$
its metric completion for the distance d. We also let $\mathscr{L}(\mathcal{D}) = \mathscr{G}(\mathcal{D}) \setminus \mathscr{G}^*(\mathcal{D})$ be its *set of leaves*.

Whenever the associated function $\ell : \mathbb{U} \to \mathbb{R}_+$ defined as $u \mapsto \ell(u) = \operatorname{diam}(D_u)$ is non-explosive as defined in (2.1), it is easy to see that the object $\mathscr{G}(\mathcal{D})$ defined above is compact and it can be approximated by gluing only finitely many blocks of the decoration.

Remark that if $\mathcal{D}$ is supported on a plane tree τ, then for any $u \in \tau$ the result of the gluing operation does not depend on the distinguished points $(x_{ui})_{i \geq \deg_\tau^+(u)+1}$ of $\mathcal{D}(u)$ with index greater than $\deg_\tau^+(u) + 1$.

Identification of the leaves. Suppose that $\mathcal{D}$ is such that $\mathscr{G}(\mathcal{D})$ is compact. We can define a continuous map

$$\iota_{\mathcal{D}} : \partial\mathbb{U} \to \mathscr{G}(\mathcal{D}), \tag{2.4}$$

that maps every leaf of the Ulam-Harris tree to a point of $\mathscr{G}(\mathcal{D})$. For any $\mathbf{i} = i_1 i_2 \cdots \in \partial\mathbb{U}$, the map is defined as

$$\iota_{\mathcal{D}}(\mathbf{i}) := \lim_{n\to\infty} x_{\mathbf{i}_n} \in \mathscr{G}(\mathcal{D}).$$

It is easy to see that the sequence $(x_{\mathbf{i}_n})_{n\geq 1}$ is indeed a Cauchy sequence from the boundedness of $\mathscr{G}(\mathcal{D})$ and so the limit exits by completeness. One can then check that the map is continuous because of the compactness assumption. Indeed, for any sequence $(u_n)_{n\geq 1} \in (\partial\mathbb{U})^{\mathbb{N}}$ such that $u_n \to u$ as $n \to \infty$ in $\partial\mathbb{U}$, if $\iota_{\mathcal{D}}(u_n)$ does not converge to $\iota_{\mathcal{D}}(u)$ then it can be shown that there exists a sequence $(\sigma_n)_{n\geq 1}$ for which the points $(\iota_{\mathcal{D}}(u_{\sigma_n}))_{n\geq 1}$ are all at a distance that is bounded below by some positive value from each other. This would contradict the compactness of $\mathscr{G}(\mathcal{D})$.

Measure-valued decorations. Let $\mathcal{D}$ be a metric-space-valued decoration. Suppose that we have a family $\boldsymbol{\nu} : u \mapsto (\nu_u)_{u\in\mathbb{U}}$ such that ν_u is a Borel measure on $\mathcal{D}(u)$, for all $u \in \mathbb{U}$. Then we can define a corresponding measure ν on $\mathbb{U}$, so that for all $u \in \mathbb{U}$,

$$\nu(\{u\}) := \nu_u(D_u). \tag{2.5}$$

We define the *support* of $\boldsymbol{\nu}$ as the support of the corresponding measure ν on $\mathbb{U}$.

In this setting, we can define in a natural way a measure on $\mathscr{G}(\mathcal{D})$ from $\boldsymbol{\nu}$ by seeing $\sum_{u\in\mathbb{U}} \nu_u$ as a measure on $\mathscr{G}(\mathcal{D})$, identifying every block as a subspace of $\mathscr{G}(\mathcal{D})$. In this case, we write

$$\mathscr{G}(\mathcal{D}, \boldsymbol{\nu}) \tag{2.6}$$

for the corresponding measured metric space. In the case where $\mathscr{G}(\mathcal{D})$ is compact, then the function $\iota_{\mathcal{D}} : \partial\mathbb{U} \to \mathscr{G}(\mathcal{D})$ appearing in (2.4) is well-defined and continuous so that if μ denotes a measure on $\partial\mathbb{U}$, then we can consider the push-forward measure $(\iota_{\mathcal{D}})_*\mu$ on $\mathscr{G}(\mathcal{D})$. In this case we write

$$\mathscr{G}(\mathcal{D}, \mu) = (\mathscr{G}(\mathcal{D}), (\iota_{\mathcal{D}})_*\mu), \tag{2.7}$$

which is a measured metric space. We can now state the main result of Section 2.

Theorem 2.2. *Suppose that $(\mathcal{D}_n)_{n\geq 1}$ is a sequence of decorations and that there exists a decoration $\mathcal{D}_\infty$ such that for every $u \in \mathbb{U}$,*

$$\mathcal{D}_n(u) \underset{n\to\infty}{\longrightarrow} \mathcal{D}_\infty(u),$$

for the infinitely pointed Gromov–Hausdorff–Prokhorov topology and such that the associated real-valued decoration ($\ell\colon u \mapsto \sup_{n\geq 1} \operatorname{diam}(\mathcal{D}_n(u))$) is non-explosive. Then, the following conclusions hold:

(i) *We have the convergence*

$$\mathscr{G}(\mathcal{D}_n) \longrightarrow \mathscr{G}(\mathcal{D}_\infty) \quad \text{as } n \to \infty \text{ for the Gromov–Hausdorff topology.}$$

(ii) *Suppose that for all $n \geq 1$, we have $\boldsymbol{\nu}_n = (\nu_{u,n})_{u\in\mathbb{U}}$, measures over $\mathcal{D}_n$ such that the corresponding measures $(\nu_n)_{n\geq 1}$ are probabilities on $\mathbb{U}$ and converge weakly in $\overline{\mathbb{U}}$ as $n \to \infty$ to some probability measure ν_∞ that only charges $\partial\mathbb{U}$, then we have the convergence*

$$\mathscr{G}(\mathcal{D}_n, \boldsymbol{\nu}_n) \longrightarrow \mathscr{G}(\mathcal{D}_\infty, \nu_\infty) \quad \text{as } n \to \infty,$$

for the Gromov–Hausdorff–Prokhorov topology.

Point (i) of this theorem states that the convergence of a global structure defined as $\mathscr{G}(\mathcal{D}_n)$, for some decoration $\mathcal{D}_n$, can be obtained by proving the convergence of every $\mathcal{D}_n(u)$, for all $u \in \mathbb{U}$ (convergence of finite dimensional marginals) with the additional assumption that they satisfy some relative compactness property which is here expressed as the non-explosion condition. Point (ii) ensures that if we consider probability measures on our decorations and that those measures converge nicely, then we can improve the convergence to a convergence in the Gromov–Hausdorff–Prokhorov topology on measured metric spaces. We only treat the case where the measure gets "pushed to the leaves" because it is the only case that arises in our applications. A more general statement where ν is not carried on $\partial\mathbb{U}$ could be proven under the appropriate assumptions.

2.3. *Some formal topological arguments.* The aim of this section is to properly define the construction described in the previous section, in a way that can be adapted to *random* decorations without causing any measurability issue. We begin by recalling some topological facts about the Urysohn universal space, and the so-called Hausdorff/Gromov–Hausdorff/Gromov–Hausdorff–Prokhorov topologies.

2.3.1. *Urysohn space and Gromov–Hausdorff–Prokhorov topology. Urysohn universal space.* Let us consider $(\mathcal{U}, \delta)$ the Urysohn space, and fix a point $* \in \mathcal{U}$. The metric space $(\mathcal{U}, \delta)$ is defined as the only Polish metric space (up to isometry) which has the following extension property (see Hušek (2008) for constructions and basic properties of $\mathcal{U}$): given any finite metric space X, and any point $x \in X$, any isometry from $X \setminus \{x\}$ to $\mathcal{U}$ can be extended to an isometry from X to $\mathcal{U}$. This property ensures in particular that any separable metric space can be isometrically embedded into $\mathcal{U}$. In what follows we will use the fact that if (K, d, ρ) is a rooted compact metric space, there exists an isometric embedding of K into U such that ρ is mapped to $*$. It also has a very useful property called compact homogeneity (see Melleray (2007, Corollary 1.2)), which ensures that any isometry φ between two compact subsets K and L of $\mathcal{U}$ can be extended to the whole space $\mathcal{U}$, meaning that there exists a global isometry ϕ such that φ is just the restriction $\varphi = \phi_{|_K}$.

Hausdorff distance, Lévy-Prokhorov distance. For any two compact subsets A and B of the same metric space (E, d), we can define their Hausdorff distance as

$$\mathrm{d}_{\mathrm{H}}^{E}(A, B) = \inf\left\{\epsilon > 0 \,\middle|\, A \subset B^{(\epsilon)}, \quad B \subset A^{(\epsilon)}\right\},$$

where $A^{(\epsilon)}$ and $B^{(\epsilon)}$ are the ϵ-fattening of the corresponding sets. We denote $\mathcal{P}(E)$ the set of Borel probability measures on E. For any two probability measures $\mu, \nu \in \mathcal{P}(E)$, we can define their Lévy-Prokhorov distance as

$$\mathrm{d}_{\mathrm{LP}}^{E}(\mu, \nu) = \inf\left\{\epsilon > 0 \,\middle|\, \forall F \in \mathcal{B}(E), \mu(F) \le \nu(F^{(\epsilon)}) + \epsilon \text{ and } \nu(F) \le \mu(F^{(\epsilon)}) + \epsilon\right\}.$$

Whenever the space E is the Urysohn space, we drop the index E in the notation for those distances.

Equivalence classes of non-empty compact metric spaces. In what follows, we are going to consider equivalence classes of non-empty compact metric spaces, possibly endowed with some extra structure, under isometries. We will be loose on the fact that we are actually considering equivalence classes and do as if the elements of the space that we are considering are themselves metric spaces. We refer to Abraham et al. (2013) for a more detailed account on the spaces introduced in the next paragraphs.

Infinitely pointed Gromov–Hausdorff topology. We write $\mathbb{M}^{k\bullet}$ for the space of all equivalence classes of $(k+1)$-pointed metric spaces. We can define the Gromov–Hausdorff distance on $\mathbb{M}^{k\bullet}$ by

$$\begin{aligned}&\mathrm{d_{GH}}^{(k)}((X,d,\rho_0,(\rho_1,\dots,\rho_k)),(X',d',\rho_0,(\rho'_1,\dots,\rho'_k)))\\&\qquad=\inf_{\phi:X\to\mathcal{U},\phi':X'\to\mathcal{U}}\left\{d_{\mathrm{H}}(\phi(X),\phi'(X))\vee\max_{0\le i\le k}\delta(\phi(\rho_i),\phi'(\rho'_i))\right\},\end{aligned}$$

where, as previously, the infimum is over all isometric embeddings ϕ and ϕ' of X and X' into the Urysohn space $(\mathcal{U},\delta)$. We write $\mathbb{M}^{\infty\bullet}$ for the space of all (equivalence classes of) ∞-pointed measured metric spaces. We can define the infinitely pointed Gromov–Hausdorff distance on $\mathbb{M}^{\infty\bullet}$ by

$$\begin{aligned}&\mathrm{d_{GH}}^{(\infty)}((X,d,\rho_0,(\rho_i)_{i\ge1}),(X',d',\rho_0,(\rho'_i)_{i\ge1}))\\&=\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{d_{GH}}^{(k)}((X,d,\rho_0,(\rho_1,\dots,\rho_k)),(X',d',\rho_0,(\rho'_1,\dots,\rho'_k)))\\&\le(\operatorname{diam}X+\operatorname{diam}X')<\infty.\end{aligned}$$

By abuse of notation, we will also consider (equivalence classes of) finitely pointed compact metric spaces $(X,\mathrm{d},\rho,(x_i)_{1\le i\le k})$ as elements of $\mathbb{M}^{\infty\bullet}$, by arbitrarily extending the sequence (x_i) with $x_i=\rho$ for all $i\ge k+1$.

Infinitely pointed Gromov–Hausdorff–Prokhorov topology. In some of our applications, we work on $\mathbb{K}^{\infty\bullet}$, which is the corresponding space for elements of $\mathbb{M}^{\infty\bullet}$ endowed with a Borel probability measure. In the same way as before, elements of $\mathbb{K}^{\infty\bullet}$ are 5-tuples $(X,\mathrm{d},\rho,(x_i)_{i\ge1},\mu)$, where $(X,\mathrm{d},\rho,(x_i)_{i\ge1})\in\mathbb{K}^{\infty\bullet}$ and μ is a finite Borel measure on X. Again we set

$$\begin{aligned}&\mathrm{d_{GHP}}^{(k)}((X,d,\rho_0,(\rho_1,\dots,\rho_k),\mu),(X',d',\rho_0,(\rho'_1,\dots,\rho'_k),\mu'))\\&\quad:=\inf_{\phi:X\to\mathcal{U},\phi':X'\to\mathcal{U}}\left\{d_{\mathrm{H}}(\phi(X),\phi'(X))\vee d_{\mathrm{LP}}((\phi)_*\mu,(\phi')_*\mu')\vee\max_{0\le i\le k}d(\phi(\rho_i),\phi'(\rho'_i))\right\},\end{aligned}$$

and similarly

$$\begin{aligned}&\mathrm{d_{GHP}}^{(\infty)}((X,d,\rho_0,(\rho_i)_{i\ge1},\mu),(X',d',\rho_0,(\rho'_i)_{i\ge1},\mu'))\\&\qquad:=\sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{d_{GHP}}^{(k)}((X,d,\rho_0,(\rho_1,\dots,\rho_k),\mu),(X',d',\rho_0,(\rho'_1,\dots,\rho'_k),\mu')).\end{aligned}$$

2.3.2. *Construction in the appropriate ambient space.* In order to ease the definition of our objects and avoid some measurability issues that may arise when working with abstract equivalence classes of metric spaces, we describe a way of only dealing with some particular representatives of those equivalence classes that are compact subsets of the set $\mathcal{U}$. For that matter we define $\mathbb{K}^{\infty\bullet}(\mathcal{U})$ the counterpart of $\mathbb{K}^{\infty\bullet}$ as

$$\mathbb{K}^{\infty\bullet}(\mathcal{U}):=\left\{(K,\delta_{|K},*,(\rho_i)_{i\ge1},\mu)\;\middle|\;\begin{array}{c}*\in K\subset\mathcal{U},\ K\text{ compact},\forall i\ge1,\rho_i\in K,\\ \mu\in\mathcal{P}(\mathcal{U}),\ \operatorname{supp}(\mu)\subset K\end{array}\right\},$$

where $\delta_{|K}$ is the distance on $\mathcal{U}$ restricted to the subset K. We accordingly set

$$\mathrm{d_{HP}}^{(k)}((K,(\rho_1,\dots,\rho_k),\mu),(K',(\rho'_1,\dots,\rho'_k),\mu'))=d_{\mathrm{H}}(K,K')\vee d_{\mathrm{LP}}(\mu,\mu')\vee\max_{1\le i\le k}d(\rho_i,\rho'_i),$$

and

$$\mathrm{d_{HP}}^{(\infty)}(K,\delta_{|_K},*,(\rho_i)_{i\geq 1},\mu),(K',\delta_{|_{K'}},*,(\rho_i')_{i\geq 1},\mu')) = \sum_{k=1}^{\infty}\frac{1}{2^k}\mathrm{d_{HP}}^{(k)}((K,(\rho_1,\dots,\rho_k),\mu),(K',(\rho_1',\dots,\rho_k'),\mu')).$$

We define the projection map $\pi:\mathbb{K}^{\infty\bullet}(\mathcal{U})\longrightarrow\mathbb{K}^{\infty\bullet}$, such that

$$\pi((K,\delta_{|_K},*,(\rho_i)_{i\geq 1},\mu)) = \left[(K,\delta_{|_K},*,(\rho_i)_{i\geq 1},\mu)\right],$$

the corresponding equivalence class in $\mathbb{K}^{\infty\bullet}$. This map is surjective by the properties of Urysohn space and continuous because it is obviously 1-Lipschitz. Using the surjectivity, we know that we can lift any deterministic element of $\mathbb{K}^{\infty\bullet}$ to an element $\mathbb{K}^{\infty\bullet}(\mathcal{U})$.

Actually, since we are dealing with *random variables* that take values in the space $\mathbb{K}^{\infty\bullet}$, we also want to ensure that we can consider versions of those random variables that take values in $\mathbb{K}^{\infty\bullet}(\mathcal{U})$. In fact, since both space are Polish, we can use a theorem of measure theory from Lubin (1974) that ensures that every probability distribution τ on $\mathbb{K}^{\infty\bullet}$ can be lifted to a probability distribution σ on $\mathbb{K}^{\infty\bullet}(\mathcal{U})$ such that its corresponding push-forward measure $\pi_*\sigma$ by the projection π is equal to the probability distribution τ. Hence, whenever we consider a random variable with values in $\mathbb{K}^{\infty\bullet}$, we can always work with a version of that random variable that is embedded in the space $\mathcal{U}$, and whose root coincides with $*$. The same line of reasoning can be made with $\mathbb{M}^{\infty\bullet}$.

From now on, we work with decorations $\mathcal{D}\in(\mathbb{K}^{\infty\bullet}(\mathcal{U}))^{\mathbb{U}}$ by taking a representative for every one of the blocks of the decoration.

Construction embedded in a space. We introduce the following space, in which we will be able to define a representative of the space $\mathscr{G}(\mathcal{D})$ for any decoration $\mathcal{D}$.

$$\ell^1(\mathcal{U},\mathbb{U},*) := \left\{(y_u)_{u\in\mathbb{U}}\in\mathcal{U}^{\mathbb{U}}\;\middle|\;\sum_{u\in\mathbb{U}}\delta(y_u,*)<+\infty\right\}.$$

We endow $\ell^1(\mathcal{U},\mathbb{U},*)$ with the distance $\mathrm{d}((y_u)_{u\in\mathbb{U}},(z_u)_{u\in\mathbb{U}}) = \sum_{u\in\mathbb{U}}\delta(y_u,z_u)$, which makes it a Polish space.

Remark 2.3. If, for each $u\in\mathbb{U}$, we are given an isometry $\phi_u:\mathcal{U}\to\mathcal{U}$ such that $\phi_u(*)=*$, then we can introduce

$$\phi := \prod_{u\in\mathbb{U}}\phi_u : \ell^1(\mathcal{U},\mathbb{U},*)\to\ell^1(\mathcal{U},\mathbb{U},*)$$
$$(y_u)_{u\in\mathbb{U}}\mapsto(\phi_u(y_u))_{u\in\mathbb{U}}$$

and ϕ is an isometry of the space $\ell^1(\mathcal{U},\mathbb{U},*)$.

For each $u\in\mathbb{U}$, we consider a representative of the block $(D_u,d_u,\rho_u,(x_{ui})_{i\geq 1})$ that belongs to $\mathbb{M}^{\infty\bullet}(\mathcal{U})$, meaning that we see D_u as a subset of $\mathcal{U}$ and,

$$(D_u,d_u,\rho_u,(x_{ui})_{i\geq 1}) = \left(D_u,\delta_{|_{D_u}},*,(x_{ui})_{i\geq 1}\right)$$

Then the gluing operation is defined in the following way. Let $\mathbf{i}=i_1i_2...i_n\in\mathbb{U}$. For any such $\mathbf{i}\in\mathbb{U}$, we define,

$$\tilde{D}_{\mathbf{i}} = \left\{(y_u)_{u\in\mathbb{U}}\;\middle|\;y_\emptyset = x_{\mathbf{i}_1}, y_{\mathbf{i}_1}=x_{\mathbf{i}_2},\dots,y_{\mathbf{i}_{n-1}}=x_{\mathbf{i}_n}, y_{\mathbf{i}}\in D_{\mathbf{i}},\text{ and }\forall u\npreceq\mathbf{i}, y_u=*\right\}$$

Remark that each of the subsets $\tilde{D}_{\mathbf{i}}$ is isometric to the corresponding bloc $D_{\mathbf{i}}$. Then we consider

$$\mathscr{G}^*(\mathcal{D}) = \bigcup_{\mathbf{i}\in\mathbb{U}}\tilde{D}_{\mathbf{i}} \tag{2.8}$$

The structure $\mathscr{G}(\mathcal{D})$ is then defined as the closure of $\mathscr{G}^*(\mathcal{D})$ in the space $\ell^1(\mathcal{U},\mathbb{U},*)$. Thanks to Remark 2.3, the resulting space (up to isometry) does not depend on the choice of representative for the different blocks of the decoration.

For convenience, for any plane tree θ we introduce $\mathscr{G}(\theta,\mathcal{D})$ the metric space obtained by only gluing the decorations that are indexed by the vertices in θ, i.e.

$$\mathscr{G}(\theta,\mathcal{D}) := \bigcup_{\mathbf{i}\in\theta} \tilde{D}_{\mathbf{i}} \tag{2.9}$$

We do not need to complete it since it is already compact, as a union of a finite number of compact metric spaces.

Identification of the leaves. Suppose that $\mathcal{D}$ is such that $\mathscr{G}(\mathcal{D})$ is compact. Then, in this setting, the map $\iota_{\mathcal{D}} : \partial\mathbb{U} \to \mathscr{G}(\mathcal{D})$ defined in (2.4) has the following form: for any $\mathbf{i} = i_1 i_2 \dots \in \partial\mathbb{U}$,

$$\iota_{\mathcal{D}}(\mathbf{i}) = (y_u)_{u\in\mathbb{U}} \quad \text{with } y_{\mathbf{i}_n} = x_{\mathbf{i}_{n+1}} \quad \text{for all } n \geq 0,$$
$$y_u = * \quad \text{whenever} \quad u \not\prec \mathbf{i}.$$

2.4. *Proof of Theorem 2.2.* Before proving the theorem, let us state a lemma that ensures that the gluing operation is continuous when considering a finite number of decorations.

Lemma 2.4. *For any θ finite plane tree, and $\mathcal{D}$ and $\mathcal{D}'$ decorations, we have*

$$\mathrm{d_{GH}}\left(\mathscr{G}\left(\theta,\mathcal{D}\right),\mathscr{G}\left(\theta,\mathcal{D}'\right)\right) \leq 2\cdot\sum_{u\in\theta} \mathrm{d_{GH}}^{(\deg_\theta^+(u))}\left(\mathcal{D}(u),\mathcal{D}'(u)\right).$$

Proof: For all $u\in\theta$, thanks to the compact homogeneity of $\mathcal{U}$, we can find an isometry $\phi_u : \mathcal{U}\to\mathcal{U}$ such that $\phi_u(*) = *$ and

$$\mathrm{d_H}(\phi_u\left(D'_u\right), D_u) \vee \max_{1\leq i\leq \deg_\theta^+(u)} \delta(\phi_u(x'_{ui}), x_{ui}) \leq 2\mathrm{d_{GH}}^{(\deg_\theta^+(u))}\left(\mathcal{D}(u),\mathcal{D}'(u)\right). \tag{2.10}$$

Then let $\phi_u = \mathrm{id}_{\mathcal{U}}$, for every $u\notin\theta$, and let $\phi = \prod_{u\in\mathbb{U}}\phi_u$ be the corresponding isometry of $\ell^1(\mathcal{U},\mathbb{U},*)$. Then let us show that we control the Hausdorff distance between

$$\begin{aligned}\mathscr{G}(\theta,\mathcal{D}) &= \bigcup_{\mathbf{i}\in\theta}\left\{(y_u)_{u\in\mathbb{U}} \;\middle|\; y_\emptyset = x_{\mathbf{i}_1},\ y_{\mathbf{i}_1} = x_{\mathbf{i}_2},\ \dots,\ y_{\mathbf{i}_{n-2}} = x_{\mathbf{i}_{n-1}},\ y_{\mathbf{i}_{n-1}} = x_{\mathbf{i}},\ \text{and } \forall u \not\preceq \mathbf{i}, y_u = *\right\},\\ &= \bigcup_{\mathbf{i}\in\theta}\tilde{D}_{\mathbf{i}},\end{aligned}$$

and

$$\begin{aligned}\phi(\mathscr{G}(\theta,\mathcal{D}')) &= \bigcup_{\mathbf{i}\in\theta}\left\{(y_u)_{u\in\mathbb{U}} \;\middle|\; y_\emptyset = \phi_\emptyset(x'_{\mathbf{i}_1}),\dots,y_{\mathbf{i}_{n-1}} = \phi_{\mathbf{i}_{n-1}}(x'_{\mathbf{i}}), y_{\mathbf{i}}\in\phi_{\mathbf{i}}(D'_{\mathbf{i}}),\ \text{and } \forall u\not\prec\mathbf{i}, y_u = *\right\}\\ &= \bigcup_{\mathbf{i}\in\theta}\phi\left(\tilde{D}'_{\mathbf{i}}\right)\end{aligned}$$

Now for any $\mathbf{i} = i_1 i_2 \dots i_n \in\theta$, any $y = (y_u)_{u\in\mathbb{U}}\in\tilde{D}_{\mathbf{i}}$ and $z = (z_u)_{u\in\mathbb{U}}\in\phi\left(\tilde{D}'_{\mathbf{i}}\right)$, we can write

$$\mathrm{d}(y,z) = \delta(y_{\mathbf{i}}, z_{\mathbf{i}}) + \sum_{\ell=1}^{n}\delta(x_{\mathbf{i}_\ell}, \phi_{\mathbf{i}_{\ell-1}}(x'_{\mathbf{i}_\ell})),$$

with $y_{\mathbf{i}} \in D_{\mathbf{i}}$ and $z_{\mathbf{i}} \in \phi(D'_{\mathbf{i}})$. Now using equation (2.10), we get that

$$\begin{aligned}
\mathrm{d_H}\left(\tilde{D}_{\mathbf{i}}, \phi\left(\tilde{D}'_{\mathbf{i}}\right)\right) &\leq \mathrm{d_H}(D_{\mathbf{i}}, \phi_{\mathbf{i}}(D'_{\mathbf{i}})) + \sum_{\ell=1}^{n} \delta(x_{\mathbf{i}_\ell}, \phi_{\mathbf{i}_{\ell-1}}(x'_{\mathbf{i}_\ell})) \\
&\leq 2 \cdot \sum_{u \in \theta} \mathrm{d_{GH}}^{(\deg^+_\theta(u))}\left(\mathcal{D}(u), \mathcal{D}'(u)\right). \qquad (2.11)
\end{aligned}$$

The last inequality is true for any $\mathbf{i} \in \theta$, hence taking a union yields,

$$\mathrm{d_H}\left(\bigcup_{\mathbf{i}\in\theta} \tilde{D}_{\mathbf{i}}, \bigcup_{\mathbf{i}\in\theta} \phi\left(\tilde{D}'_{\mathbf{i}}\right)\right) = \mathrm{d_H}\left(\mathscr{G}(\theta, \mathcal{D}), \phi(\mathscr{G}(\theta, \mathcal{D}'))\right) \leq 2 \cdot \sum_{u\in\theta} \mathrm{d_{GH}}^{(\deg^+_\theta(u))}\left(\mathcal{D}(u), \mathcal{D}'(u)\right),$$

which finishes to prove the lemma. □

We are now ready to proceed to the proof of Theorem 2.2.

Proof of Theorem 2.2: Let $n \geq 1$ be an integer and θ a finite plane tree and $y = (y_u)_{u\in\mathbb{U}} \in \mathscr{G}(\mathcal{D}_n)$. From our construction of $\mathscr{G}(\mathcal{D}_n)$, we know that the indices v for which $y_v \neq *$ are all contained in an infinite ray in $\mathbb{U}$, meaning that there exists $u \in \partial\mathbb{U}$ such that for all $v \nprec u, y_v = *$. Now we can check that

$$\begin{aligned}
\mathrm{d}(y, \mathscr{G}(\theta, \mathcal{D}_n)) = \sum_{\substack{v \prec u \\ v \notin \theta}} \delta(y_v, *) &\leq \sum_{\substack{v \prec u \\ v \notin \theta}} \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(v)), \\
&\leq \sup_{u\in\mathbb{U}} \sum_{\substack{v \prec u \\ v \notin \theta}} \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(v)).
\end{aligned}$$

Since it holds for any $y \in \mathscr{G}(\mathcal{D}_n)$ and the bound on the right-hand side is uniform for all such y, we have

$$\mathrm{d_H}\left(\mathscr{G}(\theta, \mathcal{D}_n), \mathscr{G}(\mathcal{D}_n)\right) \leq \sup_{u\in\mathbb{U}} \sum_{\substack{v \prec u \\ v \notin \theta}} \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(v)).$$

Now we can write

$$\begin{aligned}
\mathrm{d_{GH}}\left(\mathscr{G}(\mathcal{D}_n), \mathscr{G}(\mathcal{D})\right) &\leq \mathrm{d_{GH}}\left(\mathscr{G}(\mathcal{D}_n), \mathscr{G}(\theta, \mathcal{D}_n)\right) + \mathrm{d_{GH}}\left(\mathscr{G}(\theta, \mathcal{D}_n), \mathscr{G}(\theta, \mathcal{D})\right) + \mathrm{d_{GH}}\left(\mathscr{G}(\theta, \mathcal{D}), \mathscr{G}(\mathcal{D})\right) \\
&\leq 2 \sup_{u\in\mathbb{U}} \sum_{\substack{v \prec u \\ v \notin \theta}} \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(v)) + \mathrm{d_{GH}}\left(\mathscr{G}(\theta, \mathcal{D}_n), \mathscr{G}(\theta, \mathcal{D})\right)
\end{aligned}$$

Now using the non-explosion of the function $(u \mapsto \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(u)))$ we can make the first term as small as we want by taking the appropriate θ, and when θ is fixed, the second term vanishes as $n \to \infty$ thanks to Lemma 2.4. This finishes the proof of (i).

Now let us prove point (ii). For simplicity, we write $\mu_n = \sum_{u\in\mathbb{U}} \nu_{u,n}$ and also $\mu_\infty = (\iota_{\mathcal{D}_\infty})_* \nu_\infty$. Let $\epsilon > 0$. From the non-explosion condition we know that we can find a plane tree θ such that

$$\sup_{u\in\mathbb{U}} \left(\sum_{\substack{v \prec u \\ v \notin \theta}} \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(v)) \right) < \epsilon. \qquad (2.12)$$

Now, we construct another finite plane tree θ', such that $\theta \subset \theta'$, by adding only children of vertices of θ. We do so in such a way that

$$\sum_{v\in\theta'\setminus\theta} \nu_\infty\left(T(v)\right) \geq 1 - \epsilon/2.$$

Remark that it is always possible to do so because if we constructed θ' by adding every children of every vertex of θ, the last sum would be 1. Note that from (2.12), for any $v \in \theta' \setminus \theta$ and any $n \geq 1$, we have $\operatorname{diam}(\mathcal{D}_n(v)) < \epsilon$.

Introduce the projection $p_{\theta'} : \ell^1(\mathcal{U}, \mathbb{U}, *) \to \ell^1(\mathcal{U}, \mathbb{U}, *)$, such that for any $(y_u)_{u\in\mathbb{U}}$, the image $(z_u)_{u\in\mathbb{U}} = p_{\theta'}((y_u)_{u\in\mathbb{U}})$ is such that $z_u = y_u$ for any $u \in \theta'$ and $z_u = *$ otherwise. Using (2.12), we can check that for any $n \geq 1$ and for any $y \in \mathscr{G}(\mathcal{D}_n)$, we have

$$\mathrm{d}\left(p_{\theta'}(y), y\right) < \epsilon.$$

This observation suffices to show that for any $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathrm{d}_{\mathrm{GHP}}\left(\left(\mathscr{G}(\mathcal{D}_n), \mu_n\right), \left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_n)\right), (p_{\theta'})_*\mu_n\right)\right) < \epsilon.$$

Then,

$$\begin{aligned}
\mathrm{d}_{\mathrm{GHP}}\left(\left(\mathscr{G}(\mathcal{D}_n), \mu_n\right), \left(\mathscr{G}(\mathcal{D}_\infty), \mu_\infty\right)\right) \leq\ & \mathrm{d}_{\mathrm{GHP}}\left(\left(\mathscr{G}(\mathcal{D}_n), \mu_n\right), \left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_n)\right), (p_{\theta'})_*\mu_n\right)\right) \\
&+ \mathrm{d}_{\mathrm{GHP}}\left(\left(\mathscr{G}(\mathcal{D}_\infty), \mu_\infty\right), \left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_\infty)\right), (p_{\theta'})_*\mu_\infty\right)\right) \\
&+ \mathrm{d}_{\mathrm{GHP}}\left(\left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_n)\right), (p_{\theta'})_*\mu_n\right), \left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_\infty)\right), (p_{\theta'})_*\mu_\infty\right)\right).
\end{aligned}$$

The first two terms of the right-hand side are smaller than ϵ from what precedes, we only have to prove that the last one is also small whenever n is large enough. Remark that for any $\mathcal{D}$, $p_{\theta'}\left(\mathscr{G}(\mathcal{D})\right) = \mathscr{G}(\theta', \mathcal{D})$. Let us fix $n \geq 1$ large enough such that

$$2 \cdot \sum_{u \in \theta'} \mathrm{d}_{\mathrm{GH}}{}^{(\deg_{\theta'}(u))}\left(\mathcal{D}_n(u), \mathcal{D}_\infty(u)\right) < \epsilon,$$

and

$$\sum_{v \in \theta' \setminus \theta} \left|\nu_n(T(v)) - \nu_\infty(T(v))\right| < \epsilon. \tag{2.13}$$

From (2.11) in the proof of Lemma 2.4, we can find an isometry ϕ such that for all $\mathbf{i} \in \theta'$,

$$\mathrm{d}_{\mathrm{H}}\left(\tilde{D}_{\infty,\mathbf{i}}, \phi\left(\tilde{D}_{n,\mathbf{i}}\right)\right) < \epsilon. \tag{2.14}$$

Now because of (2.13), we know that we can find a coupling (X_n, X_∞) of random variables with values in $\overline{\mathbb{U}}$ having respective distributions ν_n and ν_∞, such that with probability $> 1 - \epsilon$, they both fall in the same $T(v)$ for $v \in \theta' \setminus \theta$. From this coupling, we can construct another one between (Y_n, Y_∞) of random variables on respectively $\mathscr{G}(\theta', \mathcal{D}_n)$ and $\mathscr{G}(\theta', \mathcal{D}_\infty)$ such that one has distribution $(p_{\theta'})_*\mu_n$ and the other $(p_{\theta'})_*\mu_\infty$ and such that the probability that there exists $v \in \theta' \setminus \theta$ such that $Y_n \in \tilde{D}_{n,v}$ and $Y_\infty \in \tilde{D}_{\infty,v}$ is greater than $1 - \epsilon$. Using this plus (2.14) shows that the couple $(Y_n, \phi(Y_\infty))$ is at distance at most ϵ with probability at least $1 - \epsilon$. This shows that the Lévy-Prokhorov distance between $(p_{\theta'})_*\mu_\infty$ and $\phi_*((p_{\theta'})_*\mu_n)$ is smaller than ϵ. In this end, we just showed that

$$\mathrm{d}_{\mathrm{GHP}}\left(\left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_n)\right), (p_{\theta'})_*\mu_n\right), \left(p_{\theta'}\left(\mathscr{G}(\mathcal{D}_\infty)\right), (p_{\theta'})_*\mu_\infty\right)\right) < \epsilon,$$

which finishes the proof of the proposition. □

2.5. *Sufficient condition for non-explosion.* We finish this section by proving a result that ensures non-explosion for a specific type of real-valued decorations. Let $(x_n)_{n\geq 1}$ be a sequence of non-negative real numbers. We define a real-valued decoration on the Ulam tree $\ell : \mathbb{U} \to \mathbb{R}_+$ using this sequence and a sequence $(u_n)_{n\geq 1}$ of distinct elements of $\mathbb{U}$ as

$$\begin{aligned}
\ell(u_k) &= x_k \quad \text{for all } k \geq 1, \\
\ell(u) &= 0 \quad \text{for any } u \notin \{u_k \mid k \geq 1\}
\end{aligned}$$

The following lemma ensures the non-explosion of ℓ under some assumptions that are often met in our cases of application.

Lemma 2.5. *If there exist constants $\epsilon > 0$ and $K > 0$ such that for all $n \geq 1$*

$$x_n \leq (n+1)^{-\epsilon+o(1)} \quad \text{as } n \to \infty \qquad \text{and} \qquad \mathrm{ht}(u_n) \leq K \cdot \log n,$$

then the real-valued decoration ℓ defined above is non-explosive.

Proof: Let $i \in \mathbb{N}$. For any $u \in \mathbb{U}$ we have

$$\begin{aligned}\sum_{\substack{v \prec u \\ v \in \{u_k,\ 2^i < k \leq 2^{i+1}\}}} \ell(v) &\leq \# \left\{ k \in [\![2^i+1\,, 2^{i+1}]\!] \mid u_k \prec u \right\} \cdot \left(\max_{2^i < k \leq 2^{i+1}} \ell(u_k) \right) \\ &\leq K \cdot \log 2 \cdot (i+1) \cdot (2^i)^{-\epsilon + o(1)},\end{aligned}$$

where the last display is independent of u. Now, if we consider any sequence of plane trees $(\tau_i)_{i \geq 1}$ such that for every $i \geq 1$ the tree τ_i contains all the vertices $\{u_1, u_2, \dots, u_{2^i}\}$, then we have for any $u \in \mathbb{U}$,

$$\sum_{\substack{v \prec u \\ v \notin \tau_i}} \ell(v) \leq \sum_{j=i}^{\infty} \sum_{\substack{v \prec u \\ v \in \{u_k,\ 2^i < k \leq 2^{i+1}\}}} \ell(v) \leq \sum_{j=i}^{\infty} K \cdot \log 2 \cdot (j+1) \cdot (2^j)^{-\epsilon + o(1)}$$

and the last display converges to 0 as $i \to \infty$, which proves the lemma. □

3. Preferential attachment and weighted recursive trees

In this section we recall some results about preferential attachment trees with fitnesses and weighted recursive trees that are proved in the companion paper Sénizergues (2021) and that we use later on in the paper. The objects of interest in this section are sequences of increasing plane trees $(\mathtt{t}_n)_{n \geq 1}$ where for all $n \geq 1$ the tree $\mathtt{t}_n$ has exactly n vertices. The two particular families of models introduced in Section 3.1 construct such objects through a random iterative procedure that describes how to (randomly) obtain $\mathtt{t}_{n+1}$ from $\mathtt{t}_n$. We give in Section 3.3 some quantitative results about those random sequences of trees, particularly in terms of degrees, height and of the behaviour of some probability measures carried on their vertices.

3.1. *Definitions.* We introduce two families of random increasing sequences of plane trees.

Weighted recursive trees (WRT). For any sequence of non-negative real numbers $(w_n)_{n \geq 1}$ with $w_1 > 0$, the distribution $\mathrm{WRT}((w_n)_{n \geq 1})$ of the *weighted recursive tree with weights* $(w_n)_{n \geq 1}$ is defined on sequences of growing plane trees. A sequence $(\mathtt{T}_n)_{n \geq 1}$ having this distribution is constructed iteratively starting from $\mathtt{T}_1$ containing only one vertex $u_1 = \emptyset \in \mathbb{U}$, in the following manner: the tree $\mathtt{T}_{n+1}$ is obtained from $\mathtt{T}_n$ by adding a vertex u_{n+1}. The parent of this new vertex is chosen to be any of the vertices $u_k \in \mathtt{T}_n$ with probability proportional to w_k, and u_{n+1} is added to the tree so that it is the rightmost child of its parent. Whenever we consider a random sequence of weight $(\mathsf{w}_n)_{n \geq 1}$, the distribution $\mathrm{WRT}((\mathsf{w}_n)_{n \geq 1})$ denotes the law of the random tree obtained by first sampling the sequence $(\mathsf{w}_n)_{n \geq 1}$ and then, conditionally on $(\mathsf{w}_n)_{n \geq 1}$, running the above construction with this sequence of weights.

Preferential attachment trees (PAT). For any sequence $\mathbf{a} = (a_n)_{n \geq 1}$ of real numbers, with $a_1 > -1$ and $a_n \geq 0$ for $n \geq 2$, we define another distribution on growing sequences $(\mathtt{P}_n)_{n \geq 1}$ of plane trees called the *affine preferential attachment tree with sequence of fitnesses* $\mathbf{a}$ which is denoted $\mathrm{PAT}(\mathbf{a})$. The construction goes on as before: $\mathtt{P}_1$ contains only one vertex u_1 and $\mathtt{P}_{n+1}$ is obtained from $\mathtt{P}_n$ by adding a vertex u_{n+1}, whose parent is chosen to be any $u_k \in \mathtt{P}_n$ with probability proportional to $\deg^+_{\mathtt{P}_n}(u_k) + a_k$, where $\deg^+_{\mathtt{P}_n}(\cdot)$ denotes the number of children in the tree $\mathtt{P}_n$. By convention if $n = 1$, the second vertex u_2 is always defined as a child of u_1.

Note that this model features a form of reinforcement by the degree: the more a vertex has been picked as a parent in the past, the more likely it is to be chosen as a parent in the future, as opposed to what happens in weighted recursive trees. We will see nonetheless in what follows that preferential attachment trees can be represented by weighted recursive trees with a *random* sequence of weight.

3.2. *PATs are WRTs with a random weight sequence.* We state here some properties proved in the companion paper Sénizergues (2021) that will be needed for our analysis. In what follows, we consider a sequence $\mathbf{a} = (a_n)$ such that

$$A_n := \sum_{i=1}^{n} a_i = c \cdot n + O\big(n^{1-\epsilon}\big) \quad \text{and} \quad a_n \leq n^{c'+o(1)} \tag{$H_{c,c'}$}$$

for some constants $c > 0$, some $0 \leq c' < \frac{1}{c+1}$ and some $\epsilon > 0$. We consider a sequence $(\mathsf{P}_n)_{n\geq 1}$ with distribution $\mathrm{PAT}(\mathbf{a})$, as described above.

Convergence of degrees and representation theorem. A first result concerns the scaling limit of the degrees of the vertices in their order of creation and the distribution of the sequence of trees conditionally on the limiting sequence; it can be read from Sénizergues (2021, Corollary 1.2 and Theorem 1.4): Under assumption $H_{c,c'}$ we have the following almost sure convergence in the product topology to a random sequence

$$n^{-\frac{1}{c+1}} \cdot \left(\deg^+_{\mathsf{P}_n}(u_1), \deg^+_{\mathsf{P}_n}(u_2), \dots\right) \xrightarrow[n\to\infty]{\text{a.s.}} (\mathsf{m}^{\mathbf{a}}_1, \mathsf{m}^{\mathbf{a}}_2, \dots). \tag{3.1}$$

Furthermore, conditionally on the sequence $(\mathsf{m}^{\mathbf{a}}_k)_{k\geq 1}$, the sequence $(\mathsf{P}_n)_{n\geq 1}$ has distribution $\mathrm{WRT}((\mathsf{m}^{\mathbf{a}}_k)_{k\geq 1})$. This result will be quite important in the rest of the paper so we emphasize that in particular, it implies that

$$\mathrm{PAT}(\mathbf{a}) = \mathrm{WRT}((\mathsf{m}^{\mathbf{a}}_n)_{n\geq 1}), \tag{3.2}$$

as distributions on increasing sequences of plane trees.

Additionally, still under assumption $H_{c,c'}$, we have some quantitative information on the limiting sequence $(\mathsf{m}^{\mathbf{a}}_k)_{k\geq 1}$, namely that

$$\mathsf{M}^{\mathbf{a}}_k := \sum_{i=1}^{k} \mathsf{m}^{\mathbf{a}}_i \underset{k\to\infty}{\sim} (c+1) \cdot k^{\frac{c}{c+1}} \qquad \text{and} \qquad \mathsf{m}^{\mathbf{a}}_k \leq (k+1)^{c'-\frac{1}{c+1}+o_\omega(1)}. \tag{3.3}$$

for a random function $o_\omega(1)$ which only depends on k and tends to 0 as $k \to \infty$. Thanks to Sénizergues (2021, Proposition 2.1) the convergence (3.1) is such that for all n large enough

$$\forall k \geq 1, \quad \deg^+_{\mathsf{P}_n}(u_k) \leq n^{\frac{1}{c+1}} \cdot (k+1)^{c'-\frac{1}{c+1}+o_\omega(1)}, \tag{3.4}$$

also for a random function $o_\omega(1)$ of k.

Distribution of $(\mathsf{M}^{\mathbf{a}}_k)_{k\geq 1}$. For any sequence $\mathbf{a}$ satisfying assumption ($H_{c,c'}$), the process $(\mathsf{M}^{\mathbf{a}}_k)_{k\geq 1}$ appearing in (3.3) is a (possibly time-inhomogeneous) Markov chain. In some very specific cases for the sequence $\mathbf{a}$, the process $(\mathsf{M}^{\mathbf{a}}_k)_{k\geq 1}$ actually has an explicit distribution. In particular if $\mathbf{a} = a, b, b, b\dots$ for some $a > -1$ and $b > 0$ then the sequence $(\mathsf{M}^{\mathbf{a}}_k)_{k\geq 1}$ has the Mittag-Leffer Markov chain distribution $\mathrm{MLMC}(\frac{1}{b+1}, \frac{a}{b+1})$, which we define below.

Let $0 < \alpha < 1$ and $\theta > -\alpha$. The generalized Mittag-Leffler $\mathrm{ML}(\alpha, \theta)$ distribution has pth moment

$$\frac{\Gamma(\theta)\Gamma(\theta/\alpha + p)}{\Gamma(\theta/\alpha)\Gamma(\theta + p\alpha)} = \frac{\Gamma(\theta+1)\Gamma(\theta/\alpha + p + 1)}{\Gamma(\theta/\alpha+1)\Gamma(\theta + p\alpha + 1)} \tag{3.5}$$

and the collection of p-th moments for $p \in \mathbb{N}$ uniquely characterizes this distribution. Then, a Markov chain $(\mathsf{M}_n)_{n\geq 1}$ has the distribution $\mathrm{MLMC}(\alpha, \theta)$ if for all $n \geq 1$,

$$\mathsf{M}_n \sim \mathrm{ML}\left(\alpha, \theta + n - 1\right),$$

and its transition probabilities are characterised by the following equality in law

$$(\mathsf{M}_n, \mathsf{M}_{n+1}) = (B_n \cdot \mathsf{M}_{n+1}, \mathsf{M}_{n+1}),$$

where $B_n \sim \text{Beta}\left(\frac{\theta+k-1}{\alpha} + 1, \frac{1}{\alpha} - 1\right)$ is independent of M_{n+1}.

For any sequence $\mathbf{a} = a, k_1, k_2, \dots k_p, k_1, k_2, \dots k_p, \dots$ that is periodic starting from the second term, with $k_1, k_2, \dots k_p$ being non-negative integers, the Markov chain $(\mathsf{M}_n^{\mathbf{a}})_{n\geq 1}$ also has a rather explicit distribution, see Sénizergues (2021, Proposition 5.1), involving a product of independent Gamma random variables.

3.3. *Properties of weighted recursive trees.* We now recall some results on asymptotic properties of weighted recursive trees that hold under very mild assumptions on their sequence of weights. Thanks to the equality in distribution (3.2) and the quantitative estimate (3.3), these results also apply to preferential attachment trees with fitness sequences $\mathbf{a}$ satisfying ($H_{c,c'}$).

Height. For a sequence $(\mathtt{T}_n)_{n\geq 1}$ with distribution $\mathrm{WRT}((w_n)_{n\geq 1})$, under the assumption that $W_n := \sum_{i=1}^n w_i$ is such that $W_n \leq C \cdot n^\gamma$ for all $n \geq 1$ for some constants $C, \gamma > 0$, there exists $K = K(\gamma)$ such that

$$\mathrm{ht}(\mathtt{T}_n) \leq K \cdot \log n, \tag{3.6}$$

almost surely for all n large enough. This follows from Sénizergues (2021, Lemma 3.15). This applies to sequences $(\mathtt{P}_n)_{n\geq 1}$ that have distribution $\mathrm{PAT}(\mathbf{a})$ whenever $\mathbf{a}$ satisfies ($H_{c,c'}$) thanks to (3.2) and (3.3).

Convergence of measures. For a sequence of trees $(\mathtt{T}_n)_{n\geq 1}$ evolving under the distribution $\mathrm{WRT}((w_n)_{n\geq 1})$ for any weight sequence $(w_n)_{n\geq 1}$, we can define the sequence of probability measures $(\mu_n)_{n\geq 1}$ on $\mathbb{U}$ in such a way that for all $k \in \{1, \dots n\}$ we have $\mu_n(u_k) = \frac{w_k}{W_n}$. The result Sénizergues (2021, Theorem 1.7) ensures that we almost surely have the following convergence in the weak topology

$$\mu_n \underset{n\to\infty}{\longrightarrow} \mu,$$

for some random limiting measure μ on $\overline{\mathbb{U}}$. See Section 2.1 for a reminder of the topology that we consider on $\mathbb{U}$. Furthermore, according to the same result, under the conditions $\sum_{n=1}^\infty w_n = \infty$ and $\sum_{n=1}^\infty \left(\frac{w_n}{W_n}\right)^2 < \infty$, the limiting measure μ is carried on $\partial\mathbb{U}$.

For any fixed $n \geq 1$ we can also consider other measures carried on the tree $\mathtt{T}_n$. The first one ν_n is just the uniform measure on $\mathtt{T}_n$. The second one $\eta_n^{\mathbf{b}}$ depends on a sequence $\mathbf{b} = (b_n)_{n\geq 1}$ of real numbers that satisfies $b_1 > -1$ and $b_n \geq 0$ for all $n \geq 2$, and for which $b_n = O\left(n^{1-\epsilon}\right)$ for some $\epsilon > 0$ and $B_n := \sum_{i=1}^n b_i = O(n)$. The measure $\eta_n^{\mathbf{b}}$ is then carried on $\mathtt{T}_n$ and defined in such a way that for any $1 \leq k \leq n$ we have

$$\eta_n^{\mathbf{b}}(u_k) = \frac{b_k + \deg^+_{\mathtt{T}_n}(u_k)}{B_n + n - 1}. \tag{3.7}$$

We recall the following result from the companion paper.

Proposition 3.1. *Sénizergues (2021, Proposition 2.4) Under the assumptions $\sum_{n=1}^\infty w_n = \infty$ and $\sum_{n=1}^\infty \left(\frac{w_n}{W_n}\right)^2 < \infty$, the sequences $(\mu_n)_{n\geq 1}$, $(\nu_n)_{n\geq 1}$ and $(\eta_n^{\mathbf{b}})_{n\geq 1}$ almost surely converge towards the same limit μ.*

Again, this result also applies to the case of sequences of trees $(\mathtt{P}_n)_{n\geq 1}$ with distribution $\mathrm{PAT}(\mathbf{a})$ for some fitness sequence $\mathbf{a}$ that satisfy ($H_{c,c'}$), thanks to (3.2) and (3.3). Note that for any given $n \geq 1$ the measure $\eta_n^{\mathbf{b}}$ defined in (3.7) with the choice $\mathbf{b} = \mathbf{a}$ corresponds to the "preferential attachment measure" at the n-th step of the procedure, that is the probability measure under which the parent of u_{n+1} is sampled, conditionally on $(\mathtt{P}_1, \mathtt{P}_2, \dots, \mathtt{P}_n)$.

3.4. *Other description of the measure μ for a constant sequence* $\mathbf{a}$. Suppose now that $\mathbf{a}$ is constant from the second term, say $a_1 = a > -1$ and $a_n = b > 0$ for all $n \geq 2$, so that it satisfies ($H_{c,c'}$) with $c = b$ and $c' = 0$. For all $u \in \mathbb{U}$ and all $i \geq 1$, we define using the limiting measure μ the quantities

$$\mathsf{p}_{ui} = \frac{\mu(T(ui))}{\mu(T(u))}, \tag{3.8}$$

which describe how the mass above every vertex u is split into the subtrees above its children. By convention we also set $\mathsf{p}_\emptyset := 1$. In this case we can explicitly describe the law of the $(\mathsf{p}_u)_{u\in\mathbb{U}}$ and hence also the law of μ.

Moreover, let $\ell : \mathbb{U} \to \mathbb{R}_+$ be defined as

$$\forall k \geq 1, \qquad \ell(u_k) = \mathsf{m}_k^{\mathbf{a}} = \lim_{n\to\infty} n^{-1/(b+1)} \cdot \deg^+_{\mathsf{P}_n}(u_k).$$

Remark that this almost surely defines ℓ on all vertices of $\mathbb{U}$ so that for any $u \in \mathbb{U}$ we have

$$\ell(u) := \lim_{n\to\infty} n^{-1/(b+1)} \cdot \deg^+_{\mathsf{P}_n}(u).$$

The values $(\ell(u))_{u\in\mathbb{U}}$ can actually also be expressed from the ones of $(\mathsf{p}_u)_{u\in\mathbb{U}}$. The following proposition describes the joint distribution of those random variables, see Section A.2 in the appendix for the definition of the distributions involved.

Proposition 3.2. *In this setting we have*

$$(\mathsf{p}_i)_{i\geq 1} \sim \mathrm{GEM}\left(\frac{1}{b+1}, \frac{a}{b+1}\right) \quad \text{and} \quad \forall u \in \mathbb{U}\setminus\{\emptyset\},\ (\mathsf{p}_{ui})_{i\geq 1} \sim \mathrm{GEM}\left(\frac{1}{b+1}, \frac{b}{b+1}\right),$$

and they are all independent. Denote for all $u \in \mathbb{U}$,

$$S_u := \Gamma\left(\frac{b}{b+1}\right) \cdot \lim_{i\to\infty} i \cdot \mathsf{p}_{ui}^{\frac{1}{b+1}},$$

the $\frac{1}{b+1}$-diversity of the sequence $(\mathsf{p}_{ui})_{i\geq 1}$. Then for all $u \in \mathbb{U}$,

$$\ell(u) = \left(\prod_{v \preceq u} \mathsf{p}_v\right)^{\frac{1}{b+1}} \cdot S_u.$$

Proof: This result follows *mutatis mutandi* from the proof of Janson (2019, Theorem 1.5) and the adaptation to our case is left to the reader. □

4. Distributions on decorations

In this section, we define two families of distributions on decorations on the Ulam tree that will arise as limits of our discrete models. The first family is that of *iterative gluing constructions* and is described in Section 4.1. Such random decorations are defined by assigning random metric spaces to the vertices $u_1, u_2, \ldots$ of some increasing sequence of plane trees $(\mathtt{T}_n)_{n\geq 1}$ that has a WRT distribution (introduced in Section 3.1). The second family, which we introduce in Section 4.2, contains the so-called *self-similar decorations*, which are decorations $\mathcal{D}$ for which all the blocks $\mathcal{D}(u)$ for $u \in \mathbb{U}$ are rescaled versions of a collection of i.i.d. random metric spaces, where the (random) scaling factors are of some specific product form. In Section 4.3, we state Proposition 4.3 which ensures that some distributions belong to both families.

We use the framework introduced in Section 2.1 and Section 2.2.

4.1. *The iterative gluing construction.* Let $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})_{n\geq 1}$ be a sequence of independent random variables in $\mathbb{M}^{\infty\bullet}$, meaning compact pointed metric spaces endowed with a sequence of points. Let also $(w_n)_{n\geq 1}$ and $(\lambda_n)_{n\geq 1}$ be two sequences of non-negative real numbers, which we call respectively the weights and scaling factors. The model is the following: first sample $(\mathtt{T}_n)_{n\geq 1}$ with distribution $\mathrm{WRT}((w_n)_{n\geq 1})$. Then for all $n \geq 1$, denoting u_n the n-th created vertex in the trees $(\mathtt{T}_n)_{n\geq 1}$, we set

$$\mathcal{D}(u_n) = (D_{u_n}, d_{u_n}, \rho_{u_n}, (x_{u_n i})_{i\in\mathbb{N}}) := (\mathsf{B}_n, \lambda_n \cdot \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1}), \tag{4.1}$$

and for all $u \notin \{u_n \mid n \geq 1\}$, we set

$$\mathcal{D}(u) = (\{\star\}, 0, \star, (\star)_{i\geq 1})\,.$$

We assume that the cumulated sum of the weights W_n does not grow faster to infinity than polynomially, so that (3.6) ensures that the height of $\mathtt{T}_n$ grows at most logarithmically in n. We also assume that there exists $\alpha > 0$ and $p > 1$ with $\alpha p > 1$ such that

$$\lambda_n \leq n^{-\alpha+o(1)} \quad \text{and} \quad \sup_{n\geq 1} \mathbb{E}\left[\mathrm{diam}(B_n)^p\right] < \infty,$$

so that we have almost surely $\mathrm{diam}\,\mathcal{D}(u_n) \leq n^{-\epsilon+o_\omega(1)}$, with $\epsilon = \alpha - \frac{1}{p} > 0$. This is easily derived using Markov's inequality and the Borel-Cantelli lemma. Using Lemma 2.5, the function $(u \mapsto \sup_{n\geq 1} \mathrm{diam}(\mathcal{D}_n(u)))$ is then almost surely non-explosive so $\mathscr{G}(\mathcal{D})$ is almost surely compact, by Theorem 2.2.

If we assume that the sequence $(w_n)_{n\geq 1}$ has an infinite sum, then by Proposition 3.1 the limit μ of the weight measure associated to the trees $(\mathtt{T}_n)_{n\geq 1}$ almost surely exists and is carried on $\partial\mathbb{U}$. Recalling from (2.4) the definition of $\iota_{\mathcal{D}}$, the random metric space $\mathscr{G}(\mathcal{D})$ can then almost surely be endowed with the push-forward probability measure $(\iota_{\mathcal{D}})_*\mu$ and this yields the random measured metric space $\mathscr{G}(\mathcal{D}, \mu)$, as constructed in (2.7).

We call this procedure the *iterative gluing construction* with blocks $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})_{n\geq 1}$, scaling factors $(\lambda_n)_{n\geq 1}$ and weights $(w_n)_{n\geq 1}$. We allow the sequences $(\lambda_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ to be random and in this case we assume that they are independent of the blocks $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})_{n\geq 1}$. Note here that we are a bit loose on terminology and the term *iterative gluing construction* refers to the *construction* that yields $\mathcal{D}$ and $\mathscr{G}(\mathcal{D}, \mu)$, more than to the decoration $\mathcal{D}$ or the object $\mathscr{G}(\mathcal{D}, \mu)$ directly.

The case of distinguished points that are sampled from a measure. A special case of the above construction is obtained when $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1}, \nu_n)_{n\geq 1}$ is a sequence in $\mathbb{K}^{\infty\bullet}$ and that for all $n \geq 1$, conditionally on ν_n, the points $(X_{n,i})_{i\geq 1}$ are i.i.d. with law ν_n and independent of everything else. In this case we still call this distribution the iterative gluing construction with blocks $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n)_{n\geq 1}$, scaling factors $(\lambda_n)_{n\geq 1}$ and weights $(w_n)_{n\geq 1}$. This is the setting studied in Sénizergues (2019).

4.2. *Self-similar random decorations.* We describe here particular cases of random decorations that exhibit some form of self-similarity. This setting is adapted from the one studied by Rembart and Winkel in Rembart and Winkel (2018), which deals with several models of self-similar random trees. We define a model inspired from theirs. We fix $\beta > 0$ and the law of a couple $((\mathsf{B}, \mathsf{D}, \rho, (X_i)_{i\geq 1}), (P_i)_{i\geq 1})$, where the first coordinate is a random variable in $\mathbb{M}^{\infty\bullet}$ and $(P_i)_{i\geq 1}$ is a random variable in, say, $[0\,,1]^{\mathbb{N}}$. In order to use mimic the notation of Rembart and Winkel (2018), we let $\Xi = \mathbb{M}^{\infty\bullet} \times [0\,,1]^{\mathbb{N}}$. We consider a family

$$(\xi_u)_{u\in\mathbb{U}} = ((\mathsf{B}_u, \mathsf{D}_u, \rho_u, (X_{ui})_{i\geq 1}), (P_{ui})_{i\geq 1})_{u\in\mathbb{U}}$$

of r.v. in Ξ which are i.i.d., with the same law as

$$\xi := ((\mathsf{B}, \mathsf{D}, \rho, (X_i)_{i\geq 1}), (P_i)_{i\geq 1})\,. \tag{4.2}$$

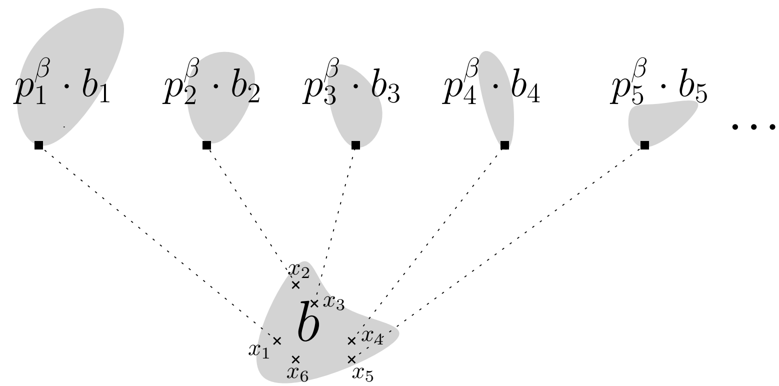


FIGURE 4.4. The function ϕ_β

We set $P_\emptyset = 1$ and

$$\lambda_u = \left(\prod_{v \preceq u} P_v \right)^\beta$$

We then define a random decoration $\mathcal{D}$ as follows: for all $u \in \mathbb{U}$,

$$\mathcal{D}(u) := (\mathsf{B}_u, \lambda_u \cdot \mathsf{D}_u, \rho_u, (X_{ui})_{i\geq 1}). \tag{4.3}$$

We say that $\mathcal{D}$ is a self-similar decoration with exponent β and *base* distribution given by ξ.

We want to show that, under suitable assumptions, the resulting $\mathscr{G}(\mathcal{D})$ is almost surely compact. In our examples, the distribution of $(P_i)_{i\geq 1}$ will always be $\mathrm{GEM}(\alpha, \theta)$ for some parameters $\alpha \in (0\,,1)$ and $\theta > -\alpha$, see Section A.2 in the Appendix for references about this process, but the arguments presented here are still valid in greater generality.

The function ϕ_β. We first define, for any $n \geq 1$, the function

$$\phi_\beta^{(n)} : \Xi \times (\mathbb{M}^\bullet)^{\mathbb{N}} \to \mathbb{M}^\bullet,$$

as follows: $\phi_\beta^{(n)}((b, d, \rho, (x_i)_{i\geq 1}), (p_i)_{i\geq 1}, (b_i, d_i, \rho_i)_{i\geq 1})$ is the metric space obtained after gluing the n first b_i's with distances respectively scaled by p_i^β, by identifying their root ρ_i with the point $x_i \in b$. For any $n \geq 1$ this operation is continuous with respect to the product topology on the starting space, hence it is measurable.

Now we define ϕ_β as $\lim_{n\to\infty} \phi_\beta^{(n)}$ on the set where this limit exists (for the topology of $\mathbb{M}^\bullet$), and constant equal to $(\{\star\}, 0, \star)$ on the complementary set. Since $\mathbb{M}^\bullet$ is Polish, the function ϕ_β is measurable. Remark that the condition for the limit to exists is

$$p_i \cdot \mathrm{diam}(b_i) \underset{i\to\infty}{\longrightarrow} 0.$$

The contraction Φ_β. Consider the set of Borel probability measures $\mathcal{P}(\mathbb{M}^\bullet)$ on the space $\mathbb{M}^\bullet$ and for $p \geq 1$, the subset $\mathcal{P}_p \subset \mathcal{P}(\mathbb{M}^\bullet)$ given by

$$\mathcal{P}_p := \{\eta \in \mathcal{P}(\mathbb{M}^\bullet) \mid \mathbb{E}\left[\mathrm{diam}(\tau)^p\right] < \infty \text{ for } \tau \sim \eta\}. \tag{4.4}$$

We equip $\mathcal{P}_p$ with the *Wasserstein metric* of order $p \geq 1$, which is defined by

$$W_p\left(\eta, \eta'\right) := \left(\inf \mathbb{E}\left[\left|d_{\mathrm{GH}}\left(\tau, \tau'\right)\right|^p\right]\right)^{1/p}, \qquad \eta, \eta' \in \mathcal{P}_p, \tag{4.5}$$

where the infimum is taken over all joint distributions of random variables (τ, τ') on $(\mathbb{M}^\bullet)^2$ with marginal distributions $\tau \sim \eta$ and $\tau' \sim \eta'$. The space $(\mathcal{P}_p, W_p)$ is complete since d_{GH} is a complete metric on $\mathbb{M}^\bullet$, see Bolley (2008); Evans et al. (2006). Convergence in $(\mathcal{P}_p, W_p)$ implies weak convergence in $\mathbb{M}^\bullet$ and convergence of the p-th moment of the diameter.

Then we define the function $\Phi_\beta : \mathcal{P}_p \to \mathcal{P}_p$ where for any $\eta \in \mathcal{P}_p$, the image $\Phi_\beta(\eta)$ is the distribution of

$$\phi_\beta(\xi, (\tau_i)_{i\geq 1}),$$

where the $(\tau_i)_{i\geq 1}$ are i.i.d. random variables with law η, independent of ξ. We now state a result that was stated in the context of trees but remains valid in our case.

Lemma 4.1. *Rembart and Winkel (2018, Lemma 3.4) Let $\beta > 0$, $p \geq 1$ and $\xi = \big((\mathsf{B}, \mathsf{D}, \rho, (X_i)_{i\geq 1}), (P_i)_{i\geq 1}\big)$ a random variable on Ξ such that $\mathbb{E}\left[\mathrm{diam}(\mathsf{B})^p\right] < \infty$ and $\mathbb{E}\left[\sum_{j\geq 1} P_j^{p\beta}\right] < 1$. Then the map $\Phi_\beta \colon \mathcal{P}_p \to \mathcal{P}_p$ associated with ϕ_β is a strict contraction with respect to the Wasserstein metric of order p, i.e.*

$$\sup_{\eta, \eta' \in \mathcal{P}_p, \eta \neq \eta'} \frac{W_p\left(\Phi_\beta(\eta), \Phi_\beta(\eta')\right)}{W_p(\eta, \eta')} < 1. \tag{4.6}$$

Using Banach fixed point theorem, we know that their exists in $\mathcal{P}_p$ a unique fixed point of this function Φ_β.

*Compactness.*The almost sure compactness of our structure $\mathscr{G}(\mathcal{D})$ is ensured by Rembart and Winkel (2018, Proposition 3.5). In fact the distribution of $\mathscr{G}(\mathcal{D})$ is exactly the fixed point of Φ_β, and this fixed point is attractive.

Measure on the leaves. We assume here that the assumptions of Lemma 4.1 and satisfied so that $\mathscr{G}(\mathcal{D})$ is almost surely compact and we further restrict ourselves to the case where $\xi = ((\mathsf{B}, \mathsf{D}, \rho, (X_i)_{i\geq 1}), (P_i)_{i\geq 1})$ is such that the sequence $(P_i)_{i\geq 1}$ satisfies $\sum_{i=1}^\infty P_i = 1$ almost surely. This allows us to define a probability measure μ on $\partial\mathbb{U}$ as

$$\forall u \in \mathbb{U}, \quad \mu(T(u)) := \prod_{v \preceq u} P_v. \tag{4.7}$$

Then we can consider the measured metric space $\mathscr{G}(\mathcal{D}, \mu)$ by endowing $\mathscr{G}(\mathcal{D})$ with the measure $\tilde{\mu} = (\iota_{\mathcal{D}})_* \mu$ using (2.4) and (2.7). Under the condition $\mathbb{P}\left(\exists i \geq 1,\ \mathsf{D}(\rho, X_i) > 0 \text{ and } P_i > 0\right) > 0$, one can check that this measure is almost surely carried on the set of leaves $\mathscr{L}(\mathcal{D})$ of the structure, where

$$\mathscr{L}(\mathcal{D}) = \mathscr{G}(\mathcal{D}) \setminus \mathscr{G}^*(\mathcal{D}),$$

see (2.2) and (2.3) for a reminder on how those sets are defined.

Hausdorff dimension of the leaves. Under some mild hypotheses on the distribution of our blocks, we can compute the almost sure Hausdorff dimension of $\mathscr{L}(\mathcal{D})$.

Proposition 4.2. *Let $\beta > 0$, $p \geq 1$ and $((\mathsf{B}, \mathsf{D}, \rho, (X_i)_{i\geq 1}), (P_i)_{i\geq 1})$ such that $\mathbb{E}\left[\mathrm{diam}(\mathsf{B})^p\right] < \infty$ and $\mathbb{E}\left[\sum_{j\geq 1} P_j^{p\beta}\right] < 1$. Suppose furthermore that almost surely $\sum_{j\geq 1} P_j = 1$ and that*

$$\mathbb{P}\left(\exists i \geq 1,\ \mathsf{D}(\rho, X_i) > 0 \text{ and } P_i > 0\right) > 0.$$

Then the Hausdorff dimension of $\mathscr{L}(\mathcal{D})$ is almost surely

$$\dim_{\mathrm{H}}(\mathscr{L}(\mathcal{D})) = \frac{1}{\beta}.$$

Note that from the equality $\mathscr{G}(\mathcal{D}) = \mathscr{G}^*(\mathcal{D}) \sqcup \mathscr{L}(\mathcal{D})$ and the construction of $\mathscr{G}^*(\mathcal{D})$, the Hausdorff dimension of $\mathscr{G}(\mathcal{D})$ can be obtained as

$$\dim_{\mathrm{H}}(\mathscr{G}(\mathcal{D})) = \dim_{\mathrm{H}}(\mathscr{L}(\mathcal{D})) \vee \sup_{u\in\mathbb{U}} \dim_{\mathrm{H}}(\mathcal{D}(u)).$$

Proof of Proposition 4.2. We prove this result by providing an upper-bound and a lower-bound for the dimension. The upper-bound follows from the proof of Rembart and Winkel (2018, Lemma 4.6), which directly adapts to our setting. For the lower-bound, we provide a direct argument, which crucially uses the assumption that $\sum_{j\geq 1} P_j = 1$ almost surely. Indeed, in this case, the preceding paragraph ensures the existence of a measure $\tilde{\mu}$ on $\mathscr{L}(\mathcal{D})$. Let us show that almost surely, for $\tilde{\mu}$-almost every point x, we have

$$\liminf_{r\to 0} \frac{\log \tilde{\mu}(B(x,r))}{-\log r} \leq -\frac{1}{\beta}, \tag{4.8}$$

which will prove the proposition, using the so-called mass distribution principle (see Mattila (1995, Theorem 8.8) for example). Actually, it is easy to see that, for (4.8) to hold, it is enough to provide a sequence $(r_n)_{n\geq 1}$ tending to 0 such that $\frac{\log r_n}{\log r_{n+1}} \to 1$ and

$$\liminf_{n\to\infty} \frac{\log \tilde{\mu}(B(x,r_n))}{-\log r_n} \leq -\frac{1}{\beta}. \tag{4.9}$$

We prove below that (4.9) holds almost surely for a point L taken under the measure $\tilde{\mu}$ and some random sequence R_n that almost surely satisfies the appropriate property. Using the product definition (4.7) of μ, it is straightforward to see that if $\mathbf{I} = I_1 I_2 \cdots \in \partial\mathbb{U}$ is taken under the measure μ, then (recall the convention that $\mathbf{I}_k$ denotes the word $\mathbf{I}$ truncated to its first k letters) the sequence $(I_1, P_{\mathbf{I}_1}), (I_2, P_{\mathbf{I}_2}), \dots$ is i.i.d. with the same distribution as the couple

$$(I, P_I),$$

where the random sequence $(P_i)_{i\geq 1}$ is the one appearing in the statement of the theorem and I is defined so that for all $j \geq 1$ we have $\mathbb{P}(I = j \mid (P_i)_{i\geq 1}) = P_j$. We can compute $\mathbb{E}[\log P_I] = \mathbb{E}[\mathbb{E}[\log P_I \mid (P_i)_{i\geq 1}]] = \mathbb{E}[\sum_{i=1}^{\infty} P_i \log P_i]$, where "$0 \log 0$" is interpreted as 0.

Let $R_n := \mathrm{d}(\rho_{\mathbf{I}_n}, \iota_{\mathcal{D}}(\mathbf{I}))$ be the distance of the random leaf $L := \iota_{\mathcal{D}}(\mathbf{I})$ to the root $\rho_{\mathbf{I}_n}$ of the block $\mathcal{D}(\mathbf{I}_n)$. Remark that the open ball $\mathrm{B}(L, R_n)$ of centre L and radius R_n only contains points that come from decorations with indices $u \succeq \mathbf{I}_n$ so that $\mu(\mathrm{B}(L, R_n)) \leq \mu(T(\mathbf{I}_n))$.

Now, using (4.7), we can write

$$\log \mu(\mathrm{B}(L, R_n)) \leq \log \mu(T(\mathbf{I}_n)) = \sum_{i=1}^{n} \log P_{\mathbf{I}_i} \underset{n\to\infty}{\sim} n \cdot \mathbb{E}\left[\sum_{i=1}^{\infty} P_i \log P_i\right],$$

almost surely, because of the law of large numbers. Now, we note that it would suffice to prove that we almost surely have

$$\log R_n \underset{n\to\infty}{\sim} n\beta \cdot \mathbb{E}\left[\sum_{i=1}^{\infty} P_i \log P_i\right], \tag{4.10}$$

so that (4.9) would follow for the random leaf L, using the previous display.

The rest of the proof aims at proving (4.10). We write

$$R_n = \mathrm{d}(\rho_{\mathbf{I}_n}, \iota_{\mathcal{D}}(\mathbf{I})) = \sum_{k=n}^{\infty} \left(\prod_{i=1}^{k} P_{\mathbf{I}_i}\right)^{\beta} \mathsf{D}_{\mathbf{I}_k}(\rho_{\mathbf{I}_k}, X_{\mathbf{I}_{k+1}}),$$

using the definition of the distances in $\mathscr{G}(\mathcal{D})$, see Section 2.2. Then fix $\delta > 0$ such that $\mathbb{P}\left(\mathsf{D}(\rho, X_I) > \delta\right) > \delta$, and let $\tau_n = \inf\left\{i \geq n \;\middle|\; \mathsf{D}_{\mathbf{I}_i}(\rho, X_{\mathbf{I}_{i+1}}) > \delta\right\}$. Then we have

$$\mathbb{P}\left(\tau_n \geq n + \sqrt{n}\right) \leq (1-\delta)^{\sqrt{n}},$$

which is summable in n, so using the Borel-Cantelli lemma we almost surely have $n \leq \tau_n \leq n + \sqrt{n}$ for n large enough. Hence for all n large enough

$$R_n \geq \left(\prod_{i=1}^{n+\sqrt{n}} P_{\mathbf{I}_i}\right)^{\beta} \cdot \delta,$$

and this proves the following lower bound

$$\log R_n \geq \beta \sum_{i=1}^{n+\sqrt{n}} \log P_{\mathbf{I}_i} + \log \delta. \tag{4.11}$$

For an upper bound, remark that

$$R_n = \left(\prod_{i=1}^{n} P_{\mathbf{I}_i}\right)^{\beta} \cdot \underbrace{\left(\mathsf{D}_{\mathbf{I}_n}(\rho, X_{\mathbf{I}_{n+1}}) + \sum_{k=n+1}^{\infty} \left(\prod_{i=n+1}^{k} P_{\mathbf{I}_i}\right)^{\beta} \mathsf{D}_{\mathbf{I}_k}(\rho_{\mathbf{I}_k}, X_{\mathbf{I}_{k+1}})\right)}_{R'_n},$$

where R'_n has the same law as R_0, which admits a finite first moment. Using Markov's inequality and the Borel-Cantelli lemma, we get that almost surely for any n large enough we have $R'_n \leq n^p$ for some $p > 1$. Then for all $n \geq 1$ large enough

$$\log R_n = \log \left(\prod_{i=1}^{n} P_{\mathbf{I}_i}\right)^{\beta} + \log R'_n \leq \beta \sum_{i=1}^{n} \log P_{\mathbf{I}_i} + p \log n. \tag{4.12}$$

In the end, using the law of large numbers on the i.i.d. sequence $(\log P_{\mathbf{I}_n})_{n\geq 1}$ we get that almost surely $\sum_{k=1}^{n} \log P_{\mathbf{I}_k} \sim n \cdot \mathbb{E}\left[\log P_I\right] \sim n \cdot \mathbb{E}\left[\sum_{i=1}^{\infty} P_i \log P_i\right]$ as $n \to \infty$. Combining this estimate with (4.11) and (4.12) yields (4.10) and this finishes the proof of the proposition. □

Almost-self-similar decorations. We define a slight variation of this model where we only assume that the random variables $(\xi_u)_{u\in\mathbb{U}\setminus\{\emptyset\}}$ have the same law as ξ, and $\xi_\emptyset = ((\mathsf{B}_\emptyset, \mathsf{D}_\emptyset, \rho_\emptyset, (X_i)_{i\geq 1}), (P_i)_{i\geq 1})$ is independent of the variables $(\xi_u)_{u\in\mathbb{U}\setminus\{\emptyset\}}$ and can possibly have a different law.

In this case, we say that the obtained decoration $\mathcal{D}$ is *almost-self-similar* with exponent β and base distributions $\xi_\emptyset$ and ξ. If $\xi_\emptyset$ satisfies the conditions of Lemma 4.1 as well as ξ, the above arguments still hold and the obtained random metric space has the law of $\phi_\beta(\xi_\emptyset, (\tau_i)_{i\geq 1})$ where the $(\tau_i)_{i\geq 1}$ are i.i.d. with distribution η, where η is the unique fixed point of Φ_β.

4.3. *Some decorations constructed by iterative gluing are also self-similar.* Some random decorations that are described using an iterative gluing construction also belong to the family of almost-self-similar decorations. The following proposition ensures that this is the case for a particular family of iterative gluing constructions that will arise in our examples.

Recall the definition of Mittag-Leffler Markov chains from Section 3.2, and the fact that for a sequence $\mathbf{a} = a, b, b, b, \dots$ with $a > -1$ and $b > 0$, the sequence $(\mathsf{M}_k^{\mathbf{a}})_{k\geq 1}$, defined in (3.3) from its increments $(\mathsf{m}_k^{\mathbf{a}})_{k\geq 1}$, has distribution $\mathrm{MLMC}(\frac{1}{b+1}, \frac{a}{b+1})$. The construction of a random variable with GEM distribution and its *diversity*, as appearing in the statement below, can be found in Appendix A.2.

Proposition 4.3. *Suppose that $\mathcal{D}$ is defined as an iterative gluing construction using*

(i) *a sequence of weights* $(\mathsf{m}_n)_{n\geq 1}$ *defined as the increments of a Mittag-Leffler Markov chain* $(\mathsf{M}_n)_{n\geq 1} \sim \mathrm{MLMC}(\frac{1}{b+1}, \frac{a}{b+1})$*, with* $a > -1$ *and* $b > 0$*,*
(ii) *a sequence of scaling factors taken as* $(\mathsf{m}_n^\gamma)_{n\geq 1}$ *for some* $\gamma > 0$*,*
(iii) *a sequence of independent blocks* $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})$*, with the same distribution for* $n \geq 2$*, such that their diameter admits a* p*-th moment with* $p > 1$*.*

Then $\mathcal{D}$ *is an almost-self-similar decoration with exponent* $\frac{\gamma}{b+1}$ *and base distributions* $\xi_\emptyset$ *and* ξ *such that*

- $\xi_\emptyset \overset{(d)}{=} ((\mathsf{B}_1, (S_\emptyset)^\gamma \cdot \mathsf{D}_1, \rho_1, (X_{1,i})_{i\geq 1}), (P_i)_{i\geq 1})$ *, with* $(P_i)_{i\geq 1} \sim \mathrm{GEM}\left(\frac{1}{b+1}, \frac{a}{b+1}\right)$ *that is independent of* B_1 *and* $S_\emptyset$ *is its* $\frac{1}{b+1}$*-diversity,*
- $\xi \overset{(d)}{=} \left((\mathsf{B}_2, S^\gamma \cdot \mathsf{D}_2, \rho_2, (X_{2,i})_{i\geq 1}), (\tilde{P}_i)_{i\geq 1}\right)$ *, with* $(\tilde{P}_i)_{i\geq 1} \sim \mathrm{GEM}\left(\frac{1}{b+1}, \frac{b}{b+1}\right)$ *that is independent of* B_2 *and* S *is its* $\frac{1}{b+1}$*-diversity.*

Proof: Consider the sequence of trees $(\mathtt{T}_n)_{n\geq 1} \sim \mathrm{WRT}((\mathsf{m}_n))$ associated with this iterative gluing construction as in Section 4.1. Recall the introduction in Section 3.3 of the probability measure μ that arises as the weak limit of the sequence of weight measures $(\mu_n)_{n\geq 1}$. From the equality of distributions (3.2) in Section 3.2, a weighted recursive tree with a random sequence of weights given by $(\mathsf{m}_n^{\mathbf{a}})_{n\geq 1}$, where $\mathbf{a} = a, b, b, b, \ldots$, has the same law as a preferential attachment tree with fitness sequence $\mathbf{a}$. Since our random sequence $(\mathsf{m}_n)_{n\geq 1}$ has the same distribution as $(\mathsf{m}_n^{\mathbf{a}})_{n\geq 1}$, this is also true in our case and so $(\mathtt{T}_n)_{n\geq 1} \sim \mathrm{PAT}(\mathbf{a})$. If we denote for all $u \in \mathbb{U}$ and $i \in \mathbb{N}$,

$$\mathsf{p}_{ui} = \frac{\mu(T(ui))}{\mu(T(u))},$$

then from Proposition 3.2 we have a complete description of the distribution of $(\mathsf{p}_u)_{u\in\mathbb{U}}$ using GEM distributions. For every $u \in \mathbb{U}$, we let S_u be the $\frac{1}{b+1}$-diversity of the sequence $(\mathsf{p}_{ui})_{i\geq 1}$. Denoting

$$\xi_\emptyset := ((\mathsf{B}_1, (S_\emptyset)^\gamma \cdot \mathsf{D}_1, \rho_1, (X_{1,i})_{i\geq 1}), (\mathsf{p}_i)_{i\geq 1}),$$

and for all $k \geq 2$,

$$\xi_{u_k} := ((\mathsf{B}_k, (S_{u_k})^\gamma \cdot \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1}), (\mathsf{p}_{u_k i})_{i\geq 1}),$$

we conclude from assumption (iii) of the proposition and the conclusion of Proposition 3.2 that the $(\xi_u)_{u\in\mathbb{U}}$ are independent and the $(\xi_u)_{u\in\mathbb{U}\setminus\{\emptyset\}}$ are i.i.d.. This entails that the distribution of $\mathcal{D}$ coincides with that of an almost-self-similar decoration with scaling exponent $\frac{\gamma}{b+1}$ with these base distributions. □

Remark 4.4. For a random decoration $\mathcal{D}$ that satisfies the assumptions of Proposition 4.3, the results of the previous section apply and the random measured metric space $\mathscr{G}(\mathcal{D}, \mu)$ is almost surely well-defined and compact, where μ is defined by (4.7). Under the extra condition that $\mathbb{P}(\exists i \geq 1,\ \mathsf{D}_2(\rho_2, X_{2,i}) > 0) > 0$, we can apply Proposition 4.2 to get that the Hausdorff dimension of the set of leaves $\mathscr{L}(\mathcal{D})$ of $\mathscr{G}(\mathcal{D})$ is almost surely $\frac{b+1}{\gamma}$.

5. Application to models of growing random graphs

In this section, we use the framework of random decorations to prove scaling limit convergence results for models of growing random graphs. Every example that we treat is of the following form: we start with a model of objects defined iteratively $(H_n)_{n\geq 1}$ that can be considered as measured metric spaces and we interpret this construction in the framework of random decorations by constructing a sequence of decorations $(\mathcal{D}^{(n)})_{n\geq 1}$ and measures on those decorations $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ such that the distribution of the sequence $\left(\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)})\right)_{n\geq 1}$ coincides with that of $(H_n)_{n\geq 1}$ as measured metric spaces. From there, if we can prove that

- each individual block $\mathcal{D}^{(n)}(u)$, for $u \in \mathbb{U}$, almost surely converges in the scaling limit to some $\mathcal{D}(u)$, where the scaling is the same for all blocks,
- the non-explosiveness condition of Theorem 2.2 is satisfied for the appropriately rescaled decorations,
- the sequence of measures $(\nu^{(n)})_{n\geq 1}$ associated to $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ through (2.5) converges to some measure μ carried on $\partial\mathbb{U}$,

then Theorem 2.2 ensures that the whole sequence $(H_n)_{n\geq 1}$ almost surely converges to $\mathscr{G}(\mathcal{D},\mu)$ in the scaling limit for the Gromov–Hausdorff–Prokhorov topology.

In Section 5.1 we state and prove Theorem 5.2, which ensures that under some specific assumptions that are met in our examples, the proof sketched above goes through. The distribution of the limiting decoration $\mathcal{D}$ is described as an iterative construction, as introduced in Section 4.1. In the following sections, we prove that the assumptions of Theorem 5.2 hold for a number of examples of growing random graphs $(H_n)_{n\geq 1}$.

5.1. *An abstract result that handles all our applications.* We describe a particular type of sequence of decoration $(\mathcal{D}^{(n)})_{n\geq 1}$ for which we can state a general scaling limit result. This setting will be rather abstract but is intended to be general enough to encompass all of our examples.

Consider a sequence $(\mathcal{D}^{(n)})_{n\geq 1}$ of decorations endowed with measures $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ which are constructed on the same probability space as an increasing sequence of trees $(\mathsf{P}_n)_{n\geq 1}$ and a sequence of processes $(\mathcal{A}_k(m), m\geq 0)_{k\geq 1}$ with values in $\mathbb{M}^{\infty\bullet}$ that are jointly independent and independent of $(\mathsf{P}_n)_{n\geq 1}$. Our result relies on the following assumptions:

(A) The sequence $(\mathsf{P}_n)_{n\geq 1}$ evolves as a preferential attachment tree with some sequence of fitnesses $\mathbf{a} = (a_n)_{n\geq 1}$, as described in Section 3.1, that satisfies $(H_{c,c'})$ for some $c>0$ and $0\leq c' < \frac{1}{c+1}$.

(B) For all $n\geq 1$, the decoration $\mathcal{D}^{(n)}$ is such that for all $k\in\{1,\dots,n\}$,
$$\mathcal{D}^{(n)}(u_k) = \mathcal{A}_k(\deg^+_{\mathsf{P}_n}(u_k)),$$
where $u_1,u_2,\dots,u_n\in\mathbb{U}$ are the vertices of the tree $\mathtt{T}_n$ in order of creation. For all $u\notin\{u_1,\dots,u_n\}$, the associated block is trivial i.e. $\mathcal{D}^{(n)}(u) = (\star,0,\star,(\star)_{i\geq 1})$.

(C) There exists $\gamma>0$ such that for all $k\geq 1$,
$$m^{-\gamma}\cdot\mathcal{A}_k(m) \underset{m\to\infty}{\xrightarrow{\text{a.s.}}} (\mathsf{B}_k,\mathsf{D}_k,\rho_k,(X_{k,i})_{i\geq 1}) \quad \text{in } \mathbb{M}^{\infty\bullet},$$
for some limiting random variable $(\mathsf{B}_k,\mathsf{D}_k,\rho_k,(X_{k,i})_{i\geq 1})$.

(D) There exists a sequence $(c_k)_{k\geq 1}$ such that for all $p>0$ we have
$$\sup_{k\geq 1}\mathbb{E}\left[\sup_{m\geq 1}\left(\frac{\operatorname{diam}(\mathcal{A}_k(m))}{(m+c_k)^\gamma}\right)^p\right]<\infty,$$
and the sequence $(c_k)_{k\geq 1}$ is such that $c_k\leq k^{s+o(1)}$ for some $s<\frac{1}{c+1}$.

(E) The sequence of measures $(\nu^{(n)})_{n\geq 1}$ on $\mathbb{U}$ that corresponds through (2.5) to the sequence of measure-valued decorations $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ almost surely converges towards the probability measure μ on $\partial\mathbb{U}$ that is associated through Proposition 3.1 to the sequence of preferential attachment trees $(\mathsf{P}_n)_{n\geq 1}$.

Remark 5.1. Assumptions (A) and (B) characterize the law of the process $(\mathcal{D}^{(n)})_{n\geq 1}$ from the ones of the processes $\mathcal{A}_k$, for $k\geq 1$, and from the sequence $\mathbf{a}$. On the contrary, the details of the measures $\boldsymbol{\nu}^{(n)}$ on the different blocks are not specified and can be anything as long as the associated measures $\nu^{(n)}$ on $\mathbb{U}$ converge towards μ.

Under all those assumptions we have a scaling limit result. For a sequence $\mathbf{a}=(a_n)_{n\geq 1}$, recall the definition (3.1) of the random sequence $(\mathsf{m}_n^{\mathbf{a}})_{n\geq 1}$. Also recall that for a decoration $\mathcal{D}$ and a non-negative number a, we write $a \cdot \mathcal{D}$ for the decoration where the distance defined on each of the blocks $\mathcal{D}(u)$ for $u \in \mathbb{U}$ is multiplied by a.

Theorem 5.2. *Suppose that the sequence of decorations $\mathcal{D}^{(n)}$ satisfies conditions (**A**) to (**E**). Then we have the following almost sure convergence*

$$\mathscr{G}(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)}) \underset{n\to\infty}{\longrightarrow} \mathscr{G}(\mathcal{D}, \mu)$$

in the Gromov–Hausdorff–Prokhorov topology, where the limit is described as an iterative construction with blocks $(\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1})_{k\geq 1}$, scaling factors $((\mathsf{m}_k^{\mathbf{a}})^\gamma)_{k\geq 1}$ and weights $(\mathsf{m}_k^{\mathbf{a}})_{k\geq 1}$.

Proof: First we check that for all $k \geq 1$ and all $n \geq k$ we have

$$\begin{aligned} n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}(u_k) &= n^{-\frac{\gamma}{c+1}} \cdot \mathcal{A}_k(\deg^+_{\mathsf{P}_n}(u_k)) \\ &= \left(n^{-\frac{1}{c+1}} \deg^+_{\mathsf{P}_n}(u_k)\right)^\gamma \cdot (\deg^+_{\mathsf{P}_n}(u_k))^{-\gamma} \cdot \mathcal{A}_k(\deg^+_{\mathsf{P}_n}(u_k)) \\ &\underset{n\to\infty}{\longrightarrow} (\mathsf{m}_k^{\mathbf{a}})^\gamma \cdot (\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1}), \end{aligned}$$

in the topology of $\mathbb{M}^{\infty\bullet}$, using the assumption (**C**), the convergence (3.1) of degrees in the sequence of increasing trees $(\mathsf{P}_n)_{n\geq 1}$, which is a preferential attachment tree by the assumption (**A**). By (**C**), for any $u \notin \{u_1, u_2, \dots\}$ the block $\mathcal{D}^{(n)}(u)$ is constant equal to the trivial block. So $n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}$ converges to some limiting decoration $\mathcal{D}$ in the product topology on $(\mathbb{M}^{\infty\bullet})^{\mathbb{U}}$.

Now, for any $k \geq 1$ and $n \geq k$ we have

$$\operatorname{diam}\left(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}(u_k)\right) = \left(n^{-\frac{1}{c+1}} \left(\deg^+_{\mathsf{P}_n}(u_k) + c_k\right)\right)^\gamma \cdot \frac{\operatorname{diam}(\mathcal{A}_k(\deg^+_{\mathsf{P}_n}(u_k)))}{(\deg^+_{\mathsf{P}_n}(u_k) + c_k)^\gamma}.$$

We use the symbol $o_\omega(1)$ for random functions of k that almost surely tend to 0 as $k \to \infty$. The first term can be shown to be smaller than some random bound $(k+1)^{-\epsilon + o_\omega(1)}$ uniformly in $n \geq k$, for some $\epsilon > 0$, using (3.4) and the assumption that $c_k \leq k^{s+o(1)}$ for $s < \frac{1}{c+1}$. The second term is bounded above by some $k^{o_\omega(1)}$ thanks to (**D**) using Markov's inequality and the Borel-Cantelli lemma. In the end, we almost surely have the following control

$$\operatorname{diam}\left(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}(u_k)\right) \leq k^{-\epsilon + o_\omega(1)}. \tag{5.1}$$

Thanks to (3.6), the height of P_n is almost surely bounded above by some $K \log n$ for some random variable K, so Lemma 2.5 ensures that the function $\ell : u \mapsto \sup_{n\geq 1} \operatorname{diam}\left(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}(u)\right)$ is almost surely non-explosive. Thanks to Theorem 2.2(i), this ensures the almost sure Gromov–Hausdorff convergence of the spaces $\mathscr{G}(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)})$ towards $\mathscr{G}(\mathcal{D})$. Finally, assumption (**E**) ensures that we are in the conditions of application of Theorem 2.2(ii) and so we can improve the last convergence into the almost sure GHP convergence

$$\mathscr{G}(n^{-\frac{\gamma}{c+1}} \cdot \mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)}) \underset{n\to\infty}{\longrightarrow} \mathscr{G}(\mathcal{D}, \mu).$$

The fact that the limiting measured metric space $\mathscr{G}(\mathcal{D}, \mu)$ is described using an iterative gluing construction follows from the definition of the limiting decoration $\mathcal{D}$ and the fact that conditionally on the sequence $(\mathsf{m}_k^{\mathbf{a}})_{k\geq 1}$, the sequence $(\mathsf{P}_n)_{n\geq 1}$ has distribution $\mathrm{WRT}((\mathsf{m}_k^{\mathbf{a}})_{k\geq 1})$. This finishes the proof. ☐

How to apply this theorem. Theorem 5.2 was designed to encompass all our examples of growing random graphs. Now, for all our sequences of graphs $(H_n)_{n\geq 1}$, the goal will be to provide a sequence $\mathbf{a} = (a_n)_{n\geq 1}$ satisfying ($H_{c,c'}$) for some parameters c and c' and processes $(\mathcal{A}_k)_{k\geq 1}$ so that the decorations $(\mathcal{D}^{(n)})_{n\geq 1}$ characterized by (**A**) and (**B**), endowed with some measures $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ of our choosing, indeed evolve in such a way that $\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)})$ coincides with our process $(H_n)_{n\geq 1}$ as measured metric spaces. Then we need to check that the other assumptions **(C)**, **(D)** and **(E)** are satisfied in order to apply the theorem.

Particular form of processes $\mathcal{A}$. First, remark that for any $k \geq 1$ and $m \geq 0$, all the distinguished points in $\mathcal{A}_k(m)$ that matter for the construction are only the m first ones. All the other ones can be set equal to the root vertex without changing the distribution of $\left(\mathscr{G}(\mathcal{D}^{(n)})\right)_{n\geq 1}$, so we can always suppose that at each step $m \geq 0$, the metric space $\mathcal{A}_k(m)$ is endowed with only m distinguished points in addition to the root and can hence be seen as an element of $\mathbb{M}^{m\bullet}$.

Second, in all our examples, the different processes $\mathcal{A}_k$ for $k \geq 0$ all evolve under the same Markovian transitions, possibly starting from different states $\mathcal{A}_k(0)$ for different values of $k \geq 1$. These transitions are often more naturally defined on *weighted graphs*, in which each of the vertices and edges are given some weight. The dynamics involve taking an element (vertex or edge) at random with probability proportional to the weight of that element and doing some local transformation of the graph at that point, by possibly adding one or several vertices and edges to the graph. The list of distinguished points is then updated by appending some vertex to the end of the existing list.

Almost self-similar limits. Theorem 5.2 describes the limiting space as the result of an iterative construction. In our examples, it will be often the case that Proposition 4.3 applies to the limiting decoration and hence that it is almost-self-similar in the sense of Section 4.2. It happens in particular whenever the sequence $\mathbf{a}$ is of the form $\mathbf{a} = (a, b, b, \dots)$ and all the processes $(\mathcal{A}_k)_{k\geq 2}$ have the same law.

5.2. *The generalised Rémy algorithm.* Recall the construction described in the introduction. Consider $(G_n, o_n)_{n\geq 1}$ a sequence of finite rooted graphs with number of edges given by a sequence $\mathbf{a} = (a_n)_{n\geq 1}$ that satisfies $(H_{c,c'})$ for some $c > 0$ and $c' < \frac{1}{c+1}$. We construct the sequence $(H_n)_{n\geq 1}$ recursively as follows. Let $H_1 = G_1$. Then, for any $n \geq 1$, conditionally on the structure H_n already constructed, take an edge in H_n uniformly at random, split it into two edges by adding a vertex "in the middle" of this edge, and glue a copy of G_{n+1} to the structure by identifying o_{n+1} the root vertex of G_{n+1} with the newly created vertex. Call the obtained graph H_{n+1}. See Figure 1.2 for a realisation of H_5 using the sequence $(G_n)_{n\geq 1}$ of Figure 1.1.

Uniform edge-splitting process. Before decomposing this construction as a process on decorations on the Ulam tree, we introduce a simpler process. For any connected, rooted graph (G, ρ) with at least one edge, we introduce the following process $(\mathcal{A}_G(n))_{n\geq 0}$, called the *uniform edge-splitting* process started from G. The initial value for the process $\mathcal{A}_G(0)$ is just (the set of vertices of) the graph G, endowed the corresponding graph distance, rooted at o, with an empty list of distinguished points. Then $\mathcal{A}_G(n+1)$ is obtained from $\mathcal{A}_G(n)$ by duplicating an edge uniformly at random by adding some point x_{n+1} in its centre. The vertex x_{n+1} is then appended to the list of distinguished points, now becoming of length $n+1$. At every step, the obtained object

$$\mathcal{A}_G(n) = (A_G(n), \mathrm{d}_{\mathrm{gr}}, \rho, (x_i)_{1\leq i\leq n})$$

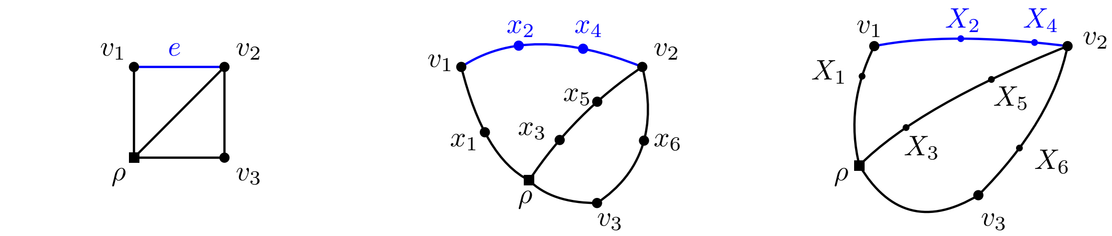
can then be considered as an element of $\mathbb{M}^{n\bullet}$, a metric space with n distinguished points, which we also see as an element of $\mathbb{M}^{\infty\bullet}$ by the usual identification. By construction, $\mathcal{A}_G(n)$ is also a graph, and we will sometimes also consider it as such.

We introduce $\mathcal{C}_G$ a continuous version of G as a random element of $\mathbb{M}^{\bullet\infty}$,

$$\mathcal{C}_G = (C_G, \mathrm{d}, \rho, (X_i)_{i\geq 1}),$$

that is constructed in the following way. If we arbitrarily label $e_1, \dots, e_{|E(G)|}$ the edges of G, then $\mathcal{C}_G$ is obtained from G by replacing each edge e with a segment of length $L(e)$ where the lengths are such that

$$(L(e_1), L(e_2), \dots, L(e_{|E(G)|})) \sim \mathrm{Dir}(1, 1, \dots, 1), \tag{5.2}$$



(A) The initial state $\mathcal{A}_G(0) = G$ (B) A realisation of $\mathcal{A}_G(6)$ (C) The limiting space $\mathcal{C}_G$

FIGURE 5.5. An illustration of the edge-splitting process started from some graph G.

so that the total length is 1. The $(X_i)_{i\geq 1}$ are then obtained conditionally on this construction as i.i.d. points taken under the length measure.

We have the following convergence result for graphs undergoing a uniform edge-splitting process.

Lemma 5.3 (Convergence of the uniform edge-splitting process)**.** *Suppose (G, ρ) is a connected, rooted graph with at least one edge. If we consider the process $(\mathcal{A}_G(n))_{n\geq 0}$ defined as above as a process on pointed metric spaces $(A_G(m), \mathrm{d}_{\mathrm{gr}}, \rho, (x_i)_{1\leq i\leq m})$ then we have the following convergence in $\mathbb{M}^{\infty\bullet}$ as $m \to \infty$,*

$$(A_G(n), \frac{1}{n}\mathrm{d}_{\mathrm{gr}}, \rho, (x_i)_{1\leq i\leq n}) \underset{n\to\infty}{\longrightarrow} \mathcal{C}_G = (C_G, \mathrm{d}, \rho, (X_i)_{i\geq 1}),$$

where $\mathcal{C}_G$ is described as above.

Proof: We give here a sketch of the proof of the statement. For any edge e of the original graph $G = (V, E)$, and any $n \geq 1$, we say that the edges in $\mathcal{A}_G(n)$ that were created through splitting events along this edge e *originate* from e. As illustrated in Figure 5.5, where the blue edges in $\mathcal{A}_G(6)$ originate from e, all the edges originating from e form a path in the graph $\mathcal{A}_G(n)$. Let us denote $L(e, n)$ the number of edges in that path. For an arbitrary labelling $e_1, e_2, \dots, e_{|E|}$ of the edges, it is easy to see that the vector $(L(e_1, n), L(e_2, n), \dots, L(e_{|E|}, n))$ evolves as the weights of different colours in a Pólya urn, as described in Theorem A.1. From this theorem, we then have the almost sure convergence

$$\frac{1}{n}\cdot (L(e_1, n), L(e_2, n), \dots, L(e_{|E|}, n)) \underset{n\to\infty}{\longrightarrow} (L(e_1), L(e_2), \dots, L(e_{|E|})),$$

where the limit has a Dirichlet distribution $\mathrm{Dir}(1, 1, \dots, 1)$. This is enough to prove the convergence of the graphs $\mathcal{A}_G(n)$ to the limiting metric space $\mathcal{C}_G$ as rooted metric spaces. The convergence of the position of the points $x_1, x_2, x_3, \dots$ along their respective path is also obtained by an urn interpretation: whenever a vertex x_i is created along the path originating from an edge e, the number of edges along that path on the left and on the right of x_i evolves also like a (time-changed) Pólya urn and hence once rescaled the position of the point along that edge converges almost surely to some point X_i in the limiting space. Last, we have to prove that the sequence $(X_i)_{i\geq 1}$ is i.i.d. uniform along the length of the limiting structure $\mathcal{C}_G$. This follows from the fact that for any time $n \geq 1$, the labels $x_1, x_2, \dots, x_n$ of the vertices created in the process are exchangeable. $\square$

Construction as a gluing of decorations. Let us now provide a construction of a sequence $(\mathcal{D}^{(n)})_{n\geq 1}$ of decorations, endowed with measures $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$, that satisfies the assumptions of Theorem 5.2 and for which the process $(\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)}))_{n\geq 1}$ coincides with $(H_n)_{n\geq 1}$ endowed with its graph distance and the uniform measure on its vertices. For this, let $(\mathtt{P}_n)_{n\geq 1}$ be a preferential attachment tree with fitnesses $(a_n)_{n\geq 1}$ and let the processes $\mathcal{A}_k$ for $k \geq 1$ be independent with the same law as $\mathcal{A}_{G_k}$, defined in the above paragraph. Now, consider the measures $\boldsymbol{\nu}^{(n)}$ such that for all $u \in \mathbb{U}$, $\nu_u^{(n)}$ charges every point of $\mathcal{D}^{(n)}(u)$ except its root if $u \neq \emptyset$, with the same mass, normalised in such a way that the associated measure $\nu^{(n)}$ on the Ulam tree is a probability measure. It is now an exercise to check that the sequence of graphs $(H_n)_{n\geq 1}$ seen as measured metric spaces has the same distribution as $(\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)}))_{n\geq 1}$.

Applying the theorem. We are now ready to prove Proposition 1.1.

Proof of Proposition 1.1: The result will follow from Theorem 5.2 so we just need to check that assumptions **(A)** to **(E)** are satisfied by the sequences $(\mathcal{D}^{(n)})_{n\geq 1}$ and $(\boldsymbol{\nu}^{(n)})_{n\geq 1}$ of decorations and measures described in the previous paragraph. Assumptions **(A)** and **(B)** are satisfied by construction, so let us verify that the other ones hold as well. The convergence **(C)** is obtained for $\gamma = 1$ by the result of Lemma 5.3, so that for any $k \geq 1$, the limiting block $(\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1})$ has the distribution of $\mathcal{C}_{G_k} = (C_{G_k}, \mathrm{d}, \rho, (X_i)_{i\geq 1})$. The control **(D)** is immediate with $(c_k)_{k\geq 1} = (a_k)_{k\geq 1}$ because the diameter of a graph is smaller than its number of edges so we have the deterministic upper-bound for all $k \geq 1$ and $m \geq 0$, $\mathrm{diam}(\mathcal{A}_k(m)) \leq a_k + m$.

The last point **(E)** is obtained by checking that the measures $\nu^{(n)}$ for $n \geq 1$ have the form (3.7), so that Proposition 3.1 applies. Indeed, let $(b_n)_{n\geq 1}$ be defined such that b_1 is the number of vertices of G_1 and for $n \geq 2$, b_n is the number of vertices minus 1 of the graph G_n. Then the measures $\nu^{(n)}$ on the Ulam tree are probability measures of the form $\nu^{(n)}(u_k) \propto b_n + \deg^+_{\mathsf{P}_n}(u_k)$ for all $k \leq n$ and $\nu^{(n)}(u) = 0$ on other vertices u. Because the graphs G_n for $n \geq 1$ are connected, their number of vertices is smaller than their number of edges minus 1, so that for all $n \geq 1$ we have $b_n \leq a_n$ which is enough to check that $\nu^{(n)}$ is of the form (3.7). In the end, applying Theorem 5.2 yields a proof of Proposition 1.1. $\square$

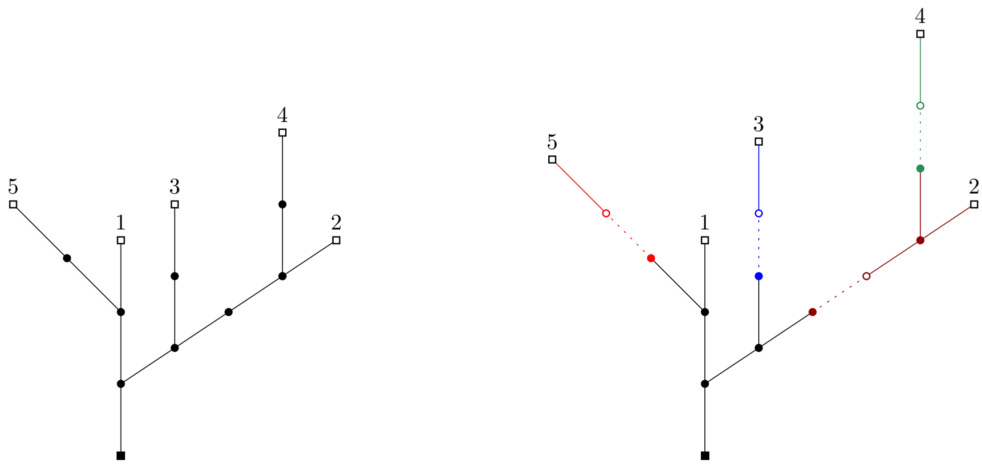
Certain subcases yield an almost self-similar limit. If the sequence $(G_k)_{k\geq 2}$ is constant, then the blocks $(\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1})$ for $k \geq 2$ are i.i.d. and it is easy to check that Proposition 4.3 applies as well as Remark 4.4. In that case, the limiting space is then almost self-similar with exponent $\frac{1}{c+1}$, where $c := |E(G_2)|$ is the common number of edges in the graphs $(G_k)_{k\geq 2}$, and its Hausdorff dimension is $c + 1$.

The Hausdorff dimension $c + 1$ can also be obtained in a more general setting by only assuming that $\mathbf{a}$ satisfies $(H_{c,c'})$ with $c' = 0$. This comes from an application of Sénizergues (2019, Theorem 1) whose assumptions are satisfied here thanks to (3.3) and the properties of the distribution of the limiting blocks.

5.3. *A particular case of the generalised Rémy algorithm using another decomposition.* We consider the former sequence of graphs $(H_n)_{n\geq 1}$ constructed from a particular sequence of graphs $(G_n)_{n\geq 1}$ with G_1 equal to the single-edge graph and constant starting from the second term, equal to a line with two edges, rooted at one end. This time, we use a different decomposition of this model, which will lead to another description of the limit, which will this time be described as an iterative gluing of rescaled i.i.d. Brownian trees.

We decompose this process along a decoration in a slightly different way from before. Indeed, every time that we glue a new copy of the two-edge-line-graph to the structure, we consider that the lower edge is a part of the block to which it attached, and only the upper edge is the newly created block of the decoration (see Figure 5.6).

In this decomposition, every time that an edge belonging to some block is selected by the algorithm, the graph corresponding to that block undergoes a step of the original Rémy algorithm, and hence its number of edges increases by two. In order to stay in a setting where the weight of each block is reinforced by one every time that it is selected, we now see every edge as having a weight $\frac{1}{2}$. From this observation (and Figure 5.6), we take $(a_n)_{n\geq 1} = (\frac{1}{2}, \frac{1}{2}, \dots)$ and independent processes $(\mathcal{A}_k)_{k\geq 1}$ that all have the same distribution as a the standard Rémy algorithm started from a single edge, such that the distinguished points correspond to the added leaves in order of



(A) A realisation of H_5, where the leaves have been labelled by their time of creation

(B) Its decomposition of the second type

FIGURE 5.6. A realisation of the tree H_5 and its decomposition as a decoration

creation. One can check that the corresponding sequence $(\mathscr{G}(\mathcal{D}^{(n)}))_{n\geq 1}$ actually corresponds to the process $(H_n)_{n\geq 1}$.

We also add the measures $\boldsymbol{\nu}^{(n)}$ in the same way as before, that is to say that for all $n \geq 1$, over all $u \in \mathbb{U}$, the measure $\nu_u^{(n)}$ charges every vertex (say, including the leaves, and excluding the root if $u \neq \emptyset$) with the same mass, in such a way that for any $n \geq 1$ the sum over $u \in \mathbb{U}$ of the total mass of the $\nu_u^{(n)}$'s is 1. In this way, the measured metric space $\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)})$ coincides with H_n endowed with its graph distance and its uniform measure on the vertices.

Using Curien and Haas (2013, Theorem 5), for any $k \geq 1$, we have the following almost sure convergence

$$m^{-\frac{1}{2}} \cdot \mathcal{A}_k(m) \underset{m\to\infty}{\to} (\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1}) \qquad \text{in } \mathbb{M}^{\infty\bullet}, \tag{5.3}$$

where the limiting metric space $(\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1})$ has the distribution of (2 times) the Brownian tree, endowed with an i.i.d. sequence of points taken under its mass measure. Condition **(D)** is satisfied thanks to Lemma A.6, proved in the appendix. The same argument as in the previous section shows that the sequence of measures $\nu^{(n)}$ has the form (3.7), so that **(E)** is satisfied thanks to Proposition 3.1 and so Theorem 5.2 applies. This proves the following proposition.

Proposition 5.4. *Under these conditions, we have the following almost sure convergence in Gromov–Hausdorff–Prokhorov topology*

$$(H_n, n^{-\frac{2}{3}} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{H}.$$

The limiting space $\mathcal{H}$ can be constructed by an iterative gluing construction using scaling factors $((\mathsf{m}_n^{\mathsf{a}})^{\frac{1}{2}})_{n\geq 1}$ and weights $(\mathsf{m}_n^{\mathsf{a}})_{n\geq 1}$, where the sequence $(\mathsf{m}_n^{\mathsf{a}})_{n\geq 1}$ has the distribution of the increments of a MLMC $\left(\frac{2}{3}, \frac{1}{3}\right)$ *process, and i.i.d. blocks $((\mathsf{B}_k, \mathsf{D}_k, \rho_k, (X_{k,i})_{i\geq 1}))_{k\geq 1}$ that have the distribution of* 2 *times the Brownian tree endowed with a sequence of i.i.d. leaves taken under its mass measure.*

Remark that Proposition 1.1 describes the limit as an iterative gluing construction using blocks that are (before scaling) all equal to the $[0\,,1]$ interval rooted at 0 endowed with i.i.d. random points. The associated sequence of scaling factors and weights would then be both equal to a sequence $(\mathsf{m}_n)_{n\geq 1}$ having the distribution of the increment of a MLMC$(\frac{1}{3}, \frac{1}{3})$ Mittag-Leffler Markov chain.

Proposition 5.4 proves in particular the non-trivial fact that the two iterative gluing constructions with segments or Brownian trees lead to the same object.

Note that both constructions are also cases of application of Proposition 4.3, so that the limiting space $\mathcal{H}$ can also be described using two different almost self-similar decorations, with exponent $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

5.4. *Marchal algorithm started from an arbitrary seed.* We define Marchal's algorithm started from a rooted connected multigraph G, originally introduced in the case of trees in Marchal (2008) and studied in the case of more general seed graphs G in Goldschmidt et al. (2018). Fix $\alpha \in (1\,,2)$. We let $H_1^\alpha = G$ and for each $n \geq 1$ define H_{n+1}^α recursively. If H_n^α is defined then take a vertex or an edge with probability proportional to their weight, where their weight are defined as

- $\alpha - 1$ for any edge,
- $\deg(v) - 1 - \alpha$ for a vertex v with degree 3 or more,
- 0 for a vertex of degree 2 or less.

Then if an edge was picked, split this edge into 2 edges with a common endpoint and add an edge linking that newly created vertex to a new leaf. Otherwise, attach an edge linking the selected vertex to a new leaf. The obtained graph is then H_{n+1}^α.

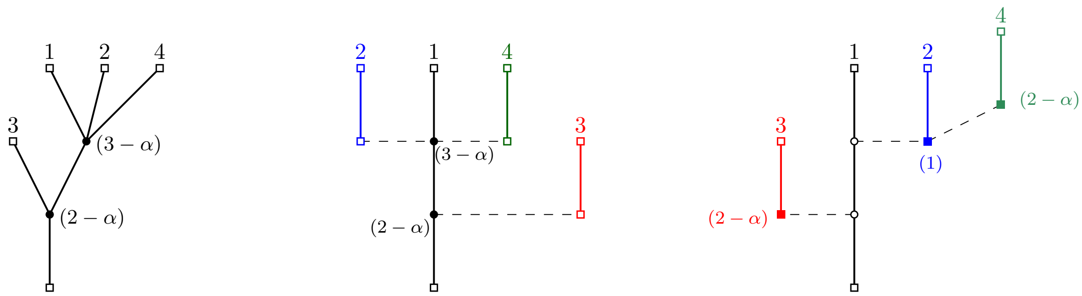
In this case, there are two natural interpretation of this graph process in terms of decorations on the Ulam tree, which give two different descriptions of the limiting object: one of them coincides with the one given in Goldschmidt et al. (2018) and the other one is different.

We describe the difference between the two using Figure 5.7. Consider Marchal's algorithm started from the single-edge graph and label the non-root leaves by their order of appearance. One way of describing the process that allows to apply Theorem 5.2 is the following, which is represented in Figure 5.7b: every time that a new leaf is added, the newly created block consists of just one edge with Marchal weight $\alpha - 1$, whose two extremities have no weight. Consider the block to which the edge or vertex that was selected when performing the step of the algorithm; if a vertex was chosen, then the weight of this vertex is reinforced by 1 within the block; if an edge was chosen, then a new vertex of weight $2 - \alpha$ and a new edge of weight $\alpha - 1$ are created in the block. In both cases, the total Marchal weight of that block is reinforced by 1. Using this description, the limiting blocks are described as segments (or possibly the continuous version of a more elaborate graph for the first one) endowed with a measure that is made of a countable number of atoms.

The second way of describing the process is the following, represented in Figure 5.7c. Every time that a new leaf is added, the newly created block consists one edge with Marchal weight $\alpha - 1$, rooted at a vertex of weight $2 - \alpha$. Consider the block to which the edge or vertex that was selected when performing the step of the algorithm; if a vertex was chosen, then the weight of this vertex is reinforced by $\alpha - 1$ within the block; if an edge was chosen, then a new vertex and a new edge are created in the block; the vertex has no weight and the edge has weight $\alpha - 1$. In both cases, the total Marchal weight of that block is reinforced by $\alpha - 1$. Using this description, in the limit, the blocks (other that the first one) are described as segments endowed with the length measure with an additional atom at their root.

Notation. In what follows, we write $w(G)$ for the sum of the weights of all vertices and edges of a multigraph G. Note that if G has surplus s and ℓ vertices of degree 1 and m vertices of degree 2 then $w(G) = (\ell - 1)\alpha + m(\alpha - 1) + s(\alpha + 1) - 1$. Also we denote by the symbol $-$ the graph with only one edge and two endpoints.

5.4.1. *First decomposition.* In the first decomposition, we take $\mathbf{a} = (w(G), \alpha - 1, \alpha - 1, \dots)$ and $(\mathtt{P}_n)_{n\geq 1}$ taken as a preferential attachment tree $\mathrm{PAT}(\mathbf{a})$. The processes $(\mathcal{A}_k, k \geq 1)$ all follow the same Markov transitions on weighted graphs, starting from the rooted graph G in the case of $\mathcal{A}_1$



(A) The tree H_4^α (B) A decomposition of H_4^α (C) Another decomposition of H_4^α

FIGURE 5.7. Two different decomposition of a tree constructed using Marchal's algorithm as a gluing of a decoration. The filled-in vertices have non-zero Marchal weight, indicated in parenthesis. All the edges have Marchal weight $\alpha - 1$.

and from the single-edge graph, which we denote $-$, in the case of $\mathcal{A}_k$ for any $k \geq 2$. We denote $(\mathcal{A}_G^\alpha(n), n \geq 1)$ such a process started from the seed graph G.

In this setting, the weight of an edge is always $\alpha - 1$ but the weight of vertices evolves with time. At time 0, for any seed graph G, every vertex with degree $d \geq 3$ is given weight $d - 1 - \alpha$ and other vertices have weight 0. The process evolves then in the following way: to obtain $\mathcal{A}_G^\alpha(m+1)$ from $\mathcal{A}_G^\alpha(m)$ we choose at random an edge or a vertex with probability proportional to their weights:

- if we picked a vertex x then $\mathcal{A}_G^\alpha(m+1)$ is obtained from $\mathcal{A}_G^\alpha(m)$ by setting the value of x_{m+1}, its $(m+1)$-st distinguished point, to x, and incrementing the weight of x by 1,
- if we picked an edge, then $\mathcal{A}_G^\alpha(m+1)$ is obtained from $\mathcal{A}_G^\alpha(m)$ by splitting this edge in 2 by adding a new vertex x, giving weight $2 - \alpha$ to this vertex, and setting the value of x_{m+1} to x.

Lemma 5.5. *With this dynamics, for any connected graph G with at least one edge, we have the following almost sure convergence in $\mathbb{M}^{\infty\bullet}$,*

$$m^{-(\alpha-1)} \cdot \mathcal{A}_G^\alpha(m) \underset{n\to\infty}{\longrightarrow} \mathcal{C}_G^\alpha = (C_G^\alpha, \mathrm{d}, \rho, (X_i)_{i\geq 1}), \tag{5.4}$$

where the distribution of the limiting object is described in the paragraph below. Moreover, for any $p \geq 1$, we have

$$\mathbb{E}\left[\left(\sup_{m\geq 0} \frac{\operatorname{diam} \mathcal{A}_G^\alpha(m)}{m^{\alpha-1}}\right)^p\right] < +\infty. \tag{5.5}$$

Limiting block. For any connected multigraph G with at least one edge, let us describe the law of the random metric space

$$\mathcal{C}_G^\alpha = (C_G^\alpha, \mathrm{d}, \rho, (X_i)_{i\geq 1}).$$

It is a continuous version of G meaning that we define it by replacing every edge of G by a segment of some length. We label its edges $e_1, e_2, \dots e_{|E|}$ in arbitrary order and replace each edge e with a segment of length $L(e)$, and the joint distribution of those lengths is characterized by what follows. We let I be the set of vertices of G that have degree at least 3 (which we sometimes refer to as *internal* vertices) and write $I = \{v_1, v_2, ..., v_{|I|}\}$. All the random variables used in the construction are independent.

- We let
$$\left(W_E, W_{v_1}, \dots, W_{v_{|I|}}\right) \sim \mathrm{Dir}\left(|E| \cdot (\alpha - 1), \deg_G(v_1) - 1 - \alpha, \dots, \deg_G(v_{|I|}) - 1 - \alpha\right)$$
- We let
$$(Q_j)_{j\geq 1} \sim \mathrm{PD}\left(\alpha - 1, |E| \cdot (\alpha - 1)\right), \quad \text{and} \quad S \quad \text{its } (\alpha - 1)\text{-diversity, see (A.1)},$$

- The length of the edges are defined as

$$(L(e_1), L(e_2), \dots, L(e_{|E|})) = (W_E)^{\alpha-1} \cdot S \cdot (B_1, B_2, \dots, B_{|E|}),$$

with $(B_1, B_2, \dots, B_{|E|}) \sim \mathrm{Dir}(1, 1, \dots 1)$.
- Conditionally on the lengths, let $(Z_j)_{j\geq 1}$ be independent and uniformly distributed on the total length of the graph.
- We set

$$\nu := \sum_{v\in I} W_v \delta_v + W_E \cdot \sum_{j=1}^{\infty} Q_j \delta_{Z_j},$$

and conditionally on all the rest, the points $(X_i)_{i\geq 1}$ are obtained as i.i.d. samples under the probability measure ν.

Remark 5.6. It is quite natural to consider $\mathcal{C}_G^\alpha$ as a random variable in $\mathbb{K}^{\infty\bullet}$ by keeping track of the probability measure ν that appears in the construction above. As we will discuss in the forthcoming Section 5.4.3, the positions and sizes of the atoms of ν are related, in our decomposition of the graphs H_n^α, to the asymptotic behaviour of the positions and degrees of vertices with high degree.

Proof of Lemma 5.5: The proof of this convergence is based on convergence results for Pólya's urn and Chinese Restaurant Processes, recalled in Section A.1 and Section A.2 of the appendix. We fix a connected graph G with at least one edge and study the evolution of the process $(\mathcal{A}_G^\alpha(n))_{n\geq 0}$. For any internal vertex $v \in I$ and $n \geq 0$, we let $W_v(n)$ be the weight of that vertex in the weighted graph $\mathcal{A}_G^\alpha(n)$, and $W_E(n)$ be the sum of the weight of all the other parts of the graph $\mathcal{A}_G^\alpha(n)$, that is, the sum of the weight of all edges and vertices of degree 2.

From the definition of the process, the vector $\Big(W_E(n), W_{v_1}(n), \dots, W_{v_{|I|}}(n)\Big)$ evolves as the weight of colours in a Pólya urn (whose definition is recalled in Section A.1) with starting proportions $\big(|E|\cdot(\alpha-1), \deg_G(v_1) - 1 - \alpha, \dots, \deg_G(v_{|I|}) - 1 - \alpha\big)$ so thanks to Theorem A.1,

$$\frac{1}{n} \cdot \Big(W_E(n), W_{v_1}(n), \dots, W_{v_{|I|}(n)}\Big) \underset{n\to\infty}{\to} \Big(W_E, W_{v_1}, \dots, W_{v_{|I|}}\Big),$$

with $\Big(W_E, W_{v_1}, \dots, W_{v_{|I|}}\Big) \sim \mathrm{Dir}\,\big(|E|\cdot(\alpha-1), \deg_G(v_1) - 1 - \alpha, \dots, \deg_G(v_{|V|}) - 1 - \alpha\big)$. Also, conditionally on $\Big(W_E, W_{v_1}, \dots, W_{v_{|I|}}\Big)$ the choice at each step of the construction is i.i.d such that any internal vertex v is chosen with probability W_v and something else (edge or vertex of degree 2) is chosen with probability W_E.

Then, we define $y_1, y_2, \cdots$ the vertices of degree 2 created by the process in order of appearance. For any $n \geq 0$ and $i \geq 1$ we let $N_i(n) = \#\{1 \leq k \leq n \mid x_k = y_i\}$, the number of times that y_i appears in the list $x_1, x_2, \dots, x_n$ of distinguished points of $\mathcal{A}_G^\alpha(n)$. Up to a time-change that depends on the sequence $(W_E(n))_{n\geq 0}$, the sequence $(N_i(n), i \geq 1)$ evolves with time like the number of customers sitting at each table in a Chinese Restaurant Process with seating plan $(\alpha - 1, |E|\cdot(\alpha-1))$, so thanks to Theorem A.3, we have the following almost sure convergence in ℓ^1

$$\frac{1}{W_E(n)} \cdot (N_i(n))_{i\geq 1} \underset{n\to\infty}{\to} (P_i)_{i\geq 1},$$

with $(P_i)_{i\geq 1} \sim \mathrm{GEM}\,(\alpha - 1, |E|\cdot(\alpha-1))$.

Conditionally on the sequence $(Q_j)_{j\geq 1}$, which we define as the decreasing rearrangement of the sequence $(P_i)_{i\geq 1}$, Theorem A.4 gives a description of this Chinese Restaurant Process: at any moment, when a new customer enters the restaurant they receive a label in such a way that each label $j \geq 1$ has probability Q_j, independently of the other costumers. If there is already a customer with that label in the restaurant, the new customer joins them, otherwise they sit at a new table.

The total number of edges $L_E(n)$ in the graph $\mathcal{A}_G^\alpha(n)$, up to an additive constant, is equal to the number of vertices created before that time. This number of vertices, in turn, corresponds to the number of tables in the Chinese Restaurant Process defined above. Using Theorem A.3, we get

$$\frac{L_E(n)}{n^{\alpha-1}} = \frac{L_E(n)}{(W_E(n))^{\alpha-1}} \cdot \frac{(W_E(n))^{\alpha-1}}{n^{\alpha-1}} \underset{n\to\infty}{\to} (W_E)^{\alpha-1} \cdot S,$$

where S is the $(\alpha-1)$-diversity of the sequence $(P_i)_{i\geq 1}$. By using Lemma A.5 we also have, for any $p \geq 1$,

$$\mathbb{E}\left[\left(\sup_{n\geq 0} \frac{L_E(n)}{n^{\alpha-1}}\right)^p\right] < \infty.$$

Since $L_E(n)$ is an upper-bound for the diameter of $\mathcal{A}_G^\alpha(n)$, this already proves the claim (5.5).

Now, just looking at the shape of the graph $\mathcal{A}_G^\alpha(n)$, we notice that, up to a time-change that is deduced from the sequence $(L_E(n))_{n\geq 0}$, this graph evolves under the uniform edge-splitting process described in Lemma 5.3. Using the result of this lemma, we deduce that almost surely

$$\frac{1}{L_E(n)}(L(e_1,n), L(e_2,n), \dots, L(e_E,n)) \underset{n\to\infty}{\to} (B_1, B_2, \dots, B_{|E|}),$$

where $(B_1, B_2, \dots, B_{|E|}) \sim \mathrm{Dir}(1,1,\dots 1)$. Also, the vertices created during the process ranked in order of creation, whose positions we denote $(Y_i)_{i\geq 1}$, are in the limit positioned independently and uniformly along the length of the limiting space, independently of $(P_i)_{i\geq 1}$. We define the points $(Z_j)_{j\geq 1}$ as the points such that $((Q_j, Z_j))_{j\geq 1}$ is the non-increasing reordering of $((P_i, Y_i))_{i\geq 1}$ with respect to the first coordinate (there are almost surely no ties), hence by definition $\sum_{i=1}^\infty Q_j \delta_{Z_j} = \sum_{i=1}^\infty P_i \delta_{Y_i}$.

In the end, we just have to justify that the points $(X_i)_{i\geq 1}$ are indeed i.i.d. taken under the measure $\nu = \sum_{v\in I} W_v \delta_v + W_E \cdot \sum_{i=1}^\infty Q_j \delta_{Z_j}$, conditionally on all the rest. This follows from the description of the evolution as n grows of the sequences $\left(W_E(n), W_{v_1}(n), \dots, W_{v_{|I|}}(n)\right)$ and $(N_i(n), i\geq 1)_{n\geq 0}$ when conditioned on their limit. Indeed, at any time $n \geq 0$, conditionally on $\left(W_E, W_{v_1}, \dots, W_{v_{|I|}}\right)$ and $(Q_j)_{j\geq 1}$, the next distinguished vertex x_{n+1} is either any $v \in I$ with probability W_v, or a vertex of degree 2 with probability W_E. If it is a vertex of degree 2, then it corresponds to any of the tables $j \geq 1$ with asymptotic size Q_j with probability Q_j, independently of the other ones. Thanks to the above paragraph, the limiting positions $(Z_j)_{j\geq 1}$ of the vertices corresponding to those tables are i.i.d. along the length of the limiting graph, which concludes the proof. □

Convergence result. Using the result of the previous paragraph, we get the following convergence.

Proposition 5.7. *The graphs H_n^α endowed with the uniform measure on their vertices converge almost surely in the scaling limit for the Gromov–Hausdorff–Prokhorov topology,*

$$(H_n^\alpha, n^{1-1/\alpha} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{H}_G^\alpha.$$

The limiting space $\mathcal{H}_G^\alpha$ is obtained as an iterative gluing construction with blocks

$$(\mathsf{B}_1, \mathsf{D}_1, \rho_1, (X_{1,i})_{i\geq 1}) \overset{(\mathrm{d})}{=} \mathcal{C}_G^\alpha \quad \text{and} \quad \forall n \geq 2,\ (\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1}) \overset{(\mathrm{d})}{=} \mathcal{C}_-^\alpha$$

and sequence of scaling factors $(\mathsf{m}_k^{(\alpha-1)})_{k\geq 1}$ and weights $(\mathsf{m}_k)_{k\geq 1}$, where $(\mathsf{m}_k)_{k\geq 1}$ is obtained as the increments of a $\mathrm{MLMC}(\frac{1}{\alpha}, \frac{w(G)}{\alpha})$.

Proof: The proof of this proposition is another application of Theorem 5.2. Conditions **(A)** and **(B)** are satisfied thanks to the discussion at the beginning of the section, conditions **(C)** and **(D)** are satisfied thanks to Lemma 5.5.

Last, we have to verify that (E) is satisfied for some measure-valued decoration $\boldsymbol{\nu}^{(n)}$ which is such that $\mathscr{G}(\mathcal{D}^{(n)}, \boldsymbol{\nu}^{(n)})$ coincides with $(H_n)_{n\geq 1}$ endowed with its graph distance and its uniform measure on the vertices. A choice for $\boldsymbol{\nu}^{(n)}$ is achieved, as in the other examples, by defining for every $n \geq 1$, for all $u \in \mathbb{U}$, the measure $\nu_u^{(n)}$ as the one that charges every vertex of $\mathcal{D}^{(n)}(u)$, except its root if $u \neq \emptyset$, with the same mass (with the constraint that the associated measure $\nu_u^{(n)}$ is a probability measure).

The slight technical difficulty here is that the total number of vertices $|V(H_n^\alpha)|$ in H_n^α is random, and the convergence of $\nu^{(n)}$ towards μ is not directly ensured by Proposition 3.1. To handle this, we are going to prove that the number of vertices in this algorithm grows asymptotically linearly like a constant multiple of n. Indeed, we define X_n as the total weight of edges in H_n^α and Y_n the total weight of vertices. Every time that an edge is picked by the algorithm, two edges of weight $(\alpha - 1)$ and one vertex of weight $(2 - \alpha)$ are created; when a vertex is picked, one edge of weight $(\alpha - 1)$ is created and the weight of a vertex is reinforced by 1. This indicates that the couple (X_n, Y_n) evolves like a generalised Pólya urn with replacement matrix

$$\begin{bmatrix} 2(\alpha-1) & 2-\alpha \\ \alpha-1 & 1 \end{bmatrix}$$

as described in Section A.1 of the appendix. Using Lemma A.2, the weight of edges almost surely grows like $\alpha(\alpha-1)n$ so the number of edges is asymptotically αn. Since the number of vertices and of edges just differ by a constant, we almost surely have

$$|V(H_n^\alpha)| \underset{n\to\infty}{\sim} n\alpha.$$

Now, consider $u \in \mathbb{U}$, and let us investigate the behaviour of $\nu^{(n)}(T(u))$, where we recall the notation $T(u) = \{v \in \mathbb{U} \mid v \succeq u\}$. We let ν_n be the uniform measure on the vertices of P_n, in such a way that $n\nu_n(T(u))$ is the number of vertices in P_n that are above u. Thanks to Proposition 3.1 the sequence $(\nu_n)_{n\geq 1}$ almost surely converges towards the measure μ associated to $(\mathsf{P}_n)_{n\geq 1}$. We want to show that almost surely, for any choice of $u \in \mathbb{U}$,

$$\nu_n(T(u)) \underset{n\to\infty}{\sim} \nu^{(n)}(T(u)), \qquad \text{and} \qquad \nu^{(n)}(\{u\}) \underset{n\to\infty}{\to} 0, \tag{5.6}$$

which is enough to prove that the sequence $(\nu^{(n)})_{n\geq 1}$ converges to the same limit μ as $(\nu_n)_{n\geq 1}$ almost surely. Note that the second requirement is immediate from the fact that the number of vertices in $\mathcal{D}^{(n)}(u)$ corresponds up to a an additive constant to the out-degree $\deg^+_{\mathsf{P}_n}(u)$, which grows sub-linearly.

Now, the quantity $|V(H_n^\alpha)| \cdot \nu^{(n)}(T(u))$ is the number of vertices in all the blocks $\mathcal{D}^{(n)}(v)$ for $v \succeq u$. By the self-similarity of the process, the reasoning made above for the total weight of edges also applies, and so the number of vertices in the blocks $\mathcal{D}^{(n)}(v)$ for $v \succeq u$ is asymptotically equivalent to the number of times that the algorithm picked an element in those decorations, which corresponds to the number $n\nu_n(T(u))$ of vertices of P_n above u. Hence

$$\alpha n \nu^{(n)}(T(u)) \underset{n\to\infty}{\sim} |V(H_n^\alpha)| \cdot \nu^{(n)}(T(u)) \underset{n\to\infty}{\sim} \alpha n \nu_n(T(u)),$$

which leads to (5.6), and hence finishes the proof. $\square$

5.4.2. *Second decomposition.* Using another decomposition as a gluing of blocks we can retrieve the other description of the limiting space, which appears in Goldschmidt et al. (2018). In that paper, the limiting object $\mathcal{H}_G^\alpha$ is obtained as an iterative gluing construction with blocks (the distribution of which we define below)

$$(\mathsf{B}_1, \mathsf{D}_1, \rho_1, (X_{1,i})_{i\geq 1}) \overset{(\mathrm{d})}{=} \mathcal{C}_G^{\mathrm{len},\alpha} \quad \text{and} \quad \forall n \geq 2,\ (\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1}) \overset{(\mathrm{d})}{=} \mathcal{C}_{\bullet-}^{\mathrm{len},\alpha},$$

and sequence of weights and scaling factors $(\mathsf{m}_k)_{k\geq 1}$, where $(\mathsf{m}_k)_{k\geq 1}$ is obtained as the increments of a MLMC$(1-\frac{1}{\alpha}, \frac{w(G)}{\alpha})$. We use a subscript "len", for *length*, to emphasize the fact that the measure naturally carried on $\mathcal{C}_G^{\mathrm{len},\alpha}$ is defined using the length measure, as we will see below.

Let us describe the random metric space $\mathcal{C}_G^{\mathrm{len},\alpha} = (C_G^{\mathrm{len},\alpha}, \mathrm{d}, \rho, (X_i)_{i\geq 1})$, for any rooted connected multigraph G. As before, we let I be the set of vertices of G that have degree greater than 3 and write $I = \{v_1, v_2, ..., v_{|I|}\}$ and we arbitrarily label its edges $e_1, e_2, \dots e_{|E|}$. As in the previous section, the metric structure of $\mathcal{C}_G^{\mathrm{len},\alpha}$ is given by replacing every edge e of G with a segment of some random length $L(e)$. The joint distribution of those random lengths and of the sequence of distinguished points is given below:

- we define

$$\left(L(e_1), \dots, L(e_{|E|}), L(v_1), \dots, L(v_{|I|})\right) \sim \mathrm{Dir}\left(1, \dots, 1, \frac{\deg_G(v_1) - 1 - \alpha}{\alpha - 1}, \dots, \frac{\deg_G(v_{|I|}) - 1 - \alpha}{\alpha - 1}\right),$$

- and set

$$\nu := \mu_{\mathrm{len}} + \sum_{v\in I} L(v) \cdot \delta_v,$$

  where μ_{len} is the length measure on the structure. Conditionally on all the rest, the sequence $(X_i)_{i\geq 1}$ is i.i.d. with distribution ν.

For the single-edge graph, this yields a segment of unit length endowed with the uniform measure. We also introduce a variant of this one. We define $\mathcal{C}_{\bullet-}^{\mathrm{len},\alpha} = (C_{\bullet-}^{\mathrm{len},\alpha}, \mathrm{d}, \rho, (X_i)_{i\geq 1})$ as follows

- we set

$$(L(e), L(\rho)) \sim \mathrm{Dir}\left(1, \frac{2-\alpha}{\alpha - 1}\right)$$

- and

$$\nu := \mu_{\mathrm{len}} + L(\rho) \cdot \delta_\rho,$$

  where μ_{len} is the length measure on the structure. Conditionally on all the rest, the sequence $(X_i)_{i\geq 1}$ is i.i.d. with distribution ν.

We do not provide another proof of the convergence because the result is already know from Goldschmidt et al. (2018). However, it could be done quite easily using Lemma 5.3 and arguments involving Pólya urns and Theorem A.1. This is left to the reader.

5.4.3. *Consequences for the α-stable tree and the α-stable component.* In the case $G = -$, the limiting space $\mathcal{H}_-^\alpha$ appearing in the statement of Proposition 5.7 has the distribution of $\alpha \cdot \mathcal{T}_\alpha$ where $\mathcal{T}_\alpha$ is the so-called α-stable tree, introduced in Duquesne and Le Gall (2002, 2005) building on the earlier result Le Gall and Le Jan (1998). We refer to Marchal (2008); Haas et al. (2008); Curien and Haas (2013) for different versions of the convergence of the discrete process $(H_n^\alpha)_{n\geq 1}$ started from the graph $-$ to $\alpha \cdot \mathcal{T}_\alpha$, and Goldschmidt and Haas (2015) for a first construction of $\mathcal{T}_\alpha$ as an iterative gluing process, which corresponds to the one described in Section 5.4.2. For more general graphs G, the convergence is studied in Goldschmidt et al. (2018) and several constructions of $\mathcal{H}_G^\alpha$ are given in Goldschmidt et al. (2018, Section 1.2.2 and Section 1.2.3), where one of them coincides with that of Section 5.4.2. The object $\mathcal{H}_G^\alpha$ is related to the so-called α-stable component, which arises as the scaling limit of large components of certain heavy-tailed critical random graphs, as proved in Goldschmidt (2020).

In the α-stable tree $\mathcal{T}_\alpha$ and more generally in $\alpha^{-1} \cdot \mathcal{H}_G^\alpha$ for any connected graph G, there is notion of *width* or *local time* of branch-points which is a continuous analogue of the notion of degree in discrete graphs, defined in Miermont (2005, Equation (1)). One of the strengths of the new description of $\mathcal{H}_G^\alpha$ given in Proposition 5.7 is that it implicitly contains the information of the width of every branch-point in the space. Recall from Remark 5.6 that $\mathcal{C}_G^\alpha$ is naturally endowed with a probability measure ν. Every block $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})$ for $n \geq 1$ appearing in Proposition 5.7

can then be seen as endowed with a probability measure ν_n. Then for all $n \geq 1$, the finite measure $\mathsf{m}_n\nu_n$ on the block $(\mathsf{B}_n, \mathsf{m}_n^{(\alpha-1)} \cdot \mathsf{D}_n, \rho_n, (X_{n,i})_{i\geq 1})$, considered as a subset of $\mathcal{H}_G^\alpha$, has an atom at every branch-point along this block and the size of each atom is the width of the corresponding branch-point (computed in $\alpha^{-1} \cdot \mathcal{H}_G^\alpha$). This can be checked using Dieuleveut (2015, Lemma 2.7 and Lemma 2.8) which ensure that the notion of width of a branch-point in the continuous setting indeed arises as a scaling limit of the notion of degree of a vertex in the discrete. We also refer to Goldschmidt et al. (2018, Lemma 2.1) for a similar discussion.

Note that for the two descriptions of $\mathcal{H}_G^\alpha$ as the result of an iterative gluing construction from respectively Proposition 5.7 and Section 5.4.2, we are in a case of application of Proposition 4.3 which then expresses $\mathcal{H}_G^\alpha$ using almost-self-similar decorations. One of those descriptions, in the special case $G = -$ where $\mathcal{H}_-^\alpha$ is a constant times the α-stable tree, could actually already be read from Rembart and Winkel (2018, Theorem 1.5). Using any of those descriptions we can retrieve using Proposition 4.2 the known fact that $\dim_{\mathrm{H}}(\mathcal{H}_G^\alpha) = \dim_{\mathrm{H}}(\mathcal{T}_\alpha) = \frac{\alpha}{\alpha-1}$.

5.5. *Scaling limits for growing trees and/or their looptrees.* The looptree $\mathrm{Loop}(\tau)$ of a plane tree τ is a multigraph constructed from τ as follows: we first place a blue vertex in the middle of every edge of the tree τ. Then, we connect two blue vertices if they correspond to two consecutive edges according to the cyclic ordering around vertices that are not the root. Then $\mathrm{Loop}(\tau)$ is obtained by removing all the vertices and edges that belong to the tree τ, see Figure 5.8a and Figure 5.8c for an example. An informal way of describing that construction is to say that every (non-root) vertex of the tree is replaced by a *loop* that has the same length as the degree of that vertex.

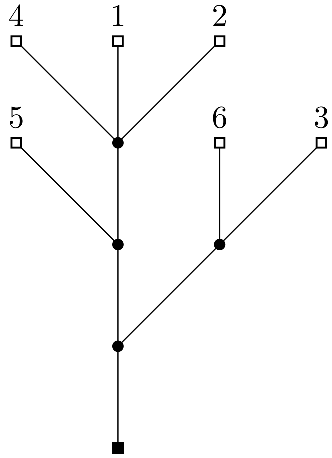
Whenever we work with a model of tree that has degrees that grow to infinity, studying the associated looptrees may allow to pass some information concerning vertices of large degree to the limit, in terms of metric scaling limits. Among the two models that we present here, one of them admits scaling limits for both the tree itself and its associated looptree. For the other one, only the looptree behaves well in this sense.

5.5.1. *The $\alpha - \gamma -$growth model.* Fix $\alpha \in (0\,,1)$ and $\gamma \in (0\,,\alpha]$. The $\alpha - \gamma$-growth model is defined as follows: $T_1^{\alpha,\gamma}$ is a tree with a single edge. Then if $T_n^{\alpha,\gamma}$ is already constructed, take an edge or a vertex at random with probability proportional to

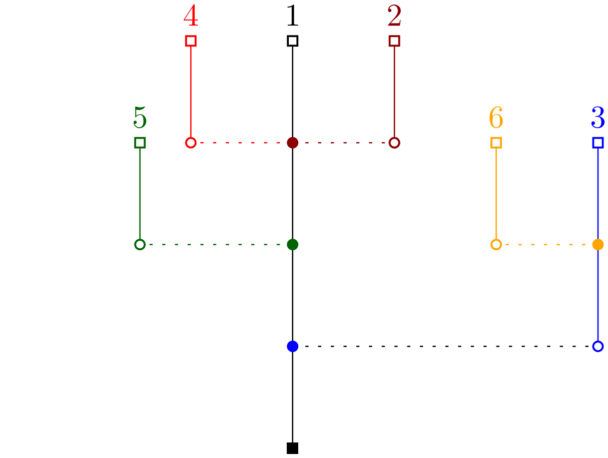
- $1 - \alpha$ for edges that are adjacent to a leaf,
- γ for edges that are not adjacent to a leaf,
- $(d-2)\alpha - \gamma$ for every vertex of degree $d \geq 3$.

Then as in Marchal's algorithm, if an edge is chosen, it is split into two edges by the addition of a new vertex its centre and an edge leading to a new leaf is grafted at the newly created vertex. If a vertex is chosen, we add an edge connecting it to a new leaf. Remark that for $\gamma = 1 - \alpha$, this algorithm corresponds to Marchal's algorithm with parameter $\frac{1}{\gamma}$. We consider a planar variation of this algorithm, where every time that we attach a new leaf to a vertex we attach it in a uniform corner around this vertex. This makes possible to consider $T_n^{\alpha,\gamma}$ as a plane tree and hence also to define its corresponding looptree $\mathrm{Loop}(T_n^{\alpha,\gamma})$.

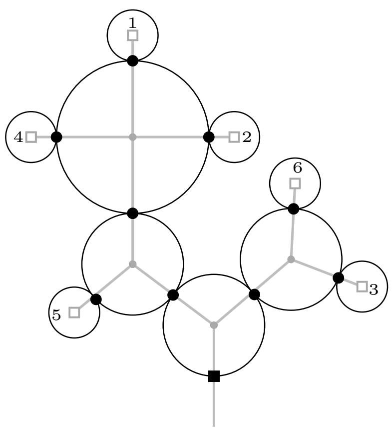
Decomposition along a preferential attachment tree. We use the same decomposition of for trees $T_n^{\alpha,\gamma}$ (illustrated in Figure 5.8) as the one that we used for Marchal's algorithm in Section 5.4.1: every time that a new leaf is added, we create a new block that consists of just one edge with weight $1 - \alpha$ whose two endpoints have no weight. We consider the block that contains the edge or vertex that was selected; if a vertex was chosen then the weight of this vertex is reinforced by α within the block; if an edge was chosen then a new vertex of weight $\alpha - \gamma$ and a new internal edge of weight γ are created in the block. In both cases, the total weight of that block is reinforced by α. For the corresponding looptrees, we use a similar decomposition: for every block in the decomposition of $T_n^{\alpha,\gamma}$, we replace every vertex that is not the root of its block by the corresponding loop in



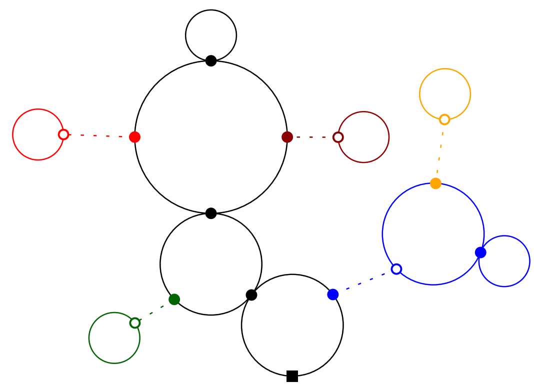
(A) A realisation of $T_6^{\alpha,\gamma}$, with leaves labelled with their time of creation



(B) The corresponding decomposition as a gluing of a decoration



(C) The looptree $\mathrm{Loop}(T_6^{\alpha,\gamma})$ with the tree $T_6^{\alpha,\gamma}$ shown in gray



(D) Its decomposition as a gluing of a decoration

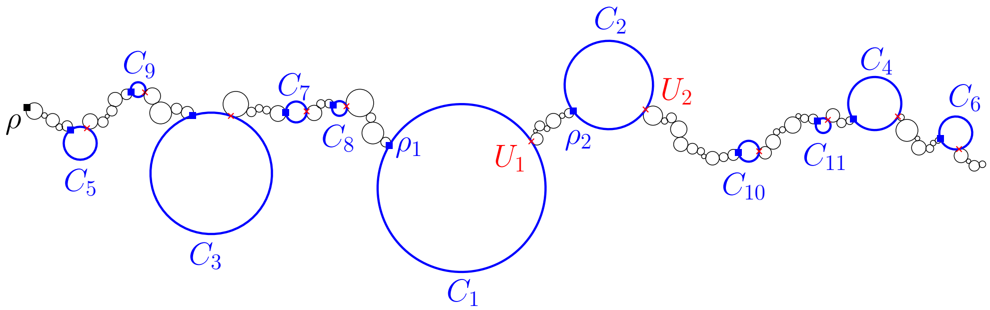
FIGURE 5.8. Decomposition of $T_n^{\alpha,\gamma}$ and its associated looptree

$\mathrm{Loop}(T_n^{\alpha,\gamma})$, as displayed in Figure 5.8. This gives rise to two decorations $\mathcal{D}^{\mathrm{tree},(n)}$ and $\mathcal{D}^{\mathrm{loop},(n)}$ that are constructed jointly and have the same support, as defined in Section 2.2.

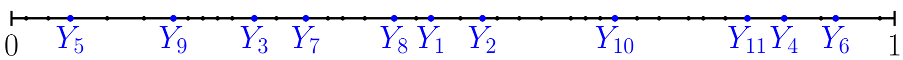
In the end, in order to be in the situation described by Theorem 5.2, we let $\mathbf{a}=(\frac{1-\alpha}{\alpha},\frac{1-\alpha}{\alpha},\frac{1-\alpha}{\alpha},\dots)$. Now let us describe the corresponding Markov processes $\mathcal{A}^{\mathrm{tree}}$ and $\mathcal{A}^{\mathrm{loop}}$ that govern the evolution of the blocks in one decoration and the other; it is quite natural to describe them jointly.

At time 0, the graph $\mathcal{A}^{\mathrm{tree}}(0)$ is a single-edge (with two endpoints) with weight $1-\alpha$, rooted at one end, and $\mathcal{A}^{\mathrm{loop}}(0)$ is a single self-loop on one vertex. Then, we choose at random an edge or a vertex with probability proportional to their weights:

- If we pick a vertex x then $\mathcal{A}^{\mathrm{tree}}(m+1)$ is obtained from $\mathcal{A}^{\mathrm{tree}}(m)$ by setting the value of x_{m+1}^{tree}, its $(m+1)$-st distinguished point, to x, and incrementing the weight of x by α. In the loop corresponding to that vertex in $\mathcal{A}^{\mathrm{loop}}(m)$, an edge is chosen uniformly at random, split in two by the addition of a new vertex y, and this vertex y becomes x_{m+1}^{loop} the $(m+1)$-st distinguished point of $\mathcal{A}^{\mathrm{loop}}(m+1)$.
- If we pick an edge, then $\mathcal{A}^{\mathrm{tree}}(m+1)$ is obtained from $\mathcal{A}^{\mathrm{tree}}(m)$ by splitting this edge in 2 by adding a new vertex x in its centre, giving weight $\alpha-\gamma$ to this vertex, and setting the



(A) The block $\mathcal{B}^{\alpha,\gamma}$ constructed from a countable number of circles $(C_n)_{n\geq 1}$



(B) The points $(Y_n)_{n\geq 1}$ along the segment $[0\,,1]$

FIGURE 5.9. The block $\mathcal{B}^{\alpha,\gamma}$ is constructed by agglomerating countably many circles $(C_n)_{n\geq 1}$, in the order given by the relative position of points $(Y_n)_{n\geq 1}$. Since they are dense in $[0\,,1]$, no two circles are ever adjacent.

value of x_{m+1}^{loop} to x. Among the two edges that result from the edge-splitting step, the one further from the root has weight $1-\alpha$ and the other one has weight γ. In $\mathcal{A}^{\text{loop}}(m)$, the addition of this new vertex corresponds to the creation of a new loop of length 3 in between the two loops that correspond to the two endpoints of the edge that was duplicated. The point x_{m+1}^{loop} of $\mathcal{A}^{\text{loop}}(m+1)$ is defined to be the new vertex of degree two that was created in the process.

Lemma 5.8. *With this dynamics, the processes $\mathcal{A}^{\text{tree}}(m)$ and $\mathcal{A}^{\text{loop}}(m)$ admit an almost sure joint scaling limit in $\mathbb{M}^{\infty\bullet}$ as $m\to\infty$:*

$$m^{-\frac{\gamma}{\alpha}}\cdot\mathcal{A}^{\text{tree}}(m)\underset{n\to\infty}{\to}\mathcal{S}^{\alpha,\gamma},\quad\textit{and}\quad m^{-1}\cdot\mathcal{A}^{\text{loop}}(m)\underset{n\to\infty}{\to}\mathcal{B}^{\alpha,\gamma},$$

where the joint law of the limiting objects $(\mathcal{S}^{\alpha,\gamma},\mathcal{B}^{\alpha,\gamma})$ is defined below. Moreover, for any $p\geq 1$, we have

$$\mathbb{E}\left[\left(\sup_{m\geq 0}\frac{\operatorname{diam}\mathcal{A}^{\text{tree}}(m)}{m^{\frac{\gamma}{\alpha}}}\right)^p\right]<+\infty. \tag{5.7}$$

Joint construction of the limiting blocks. We first define a random sequence $(Y_n)_{n\geq 1}$ on $[0\,,1]$ as follows

- Let $Y_1\sim\text{Beta}(1,\frac{1-\alpha}{\gamma})$.
- Recursively, if $(Y_1,\ldots,Y_n)$ are already defined, then conditionally on those, the point Y_{n+1} is distributed uniformly on $[0\,,\max_{1\leq i\leq n}Y_i]$ with probability $(\max_{1\leq i\leq n}Y_i)$ and as $1-R_n\cdot(1-\max_{1\leq i\leq n}Y_i)$ with complementary probability, with $R_n\sim\text{Beta}(1,\frac{1-\alpha}{\gamma})$ independent of everything else.

Then, if $\gamma = \alpha$, the couple $(\mathcal{S}^{\alpha,\gamma}, \mathcal{B}^{\alpha,\gamma})$ is such that $\mathcal{S}^{\alpha,\gamma} = \mathcal{B}^{\alpha,\gamma}$, which are just defined as the interval $[0\,,1]$, rooted at 0 and endowed with the points $(Y_n)_{n\geq 1}$.

If $\gamma \neq \alpha$, we define the following random variables, independently of the sequence $(Y_n)_{n\geq 1}$.

- We let $(P_i)_{i\geq 1} \sim \mathrm{GEM}(\frac{\gamma}{\alpha}, \frac{1-\alpha}{\alpha})$ and S denote its $\frac{\gamma}{\alpha}$-diversity .
- Define recursively the sequence $(D_k)_{k\geq 1}$ starting from $D_1 = 1$. Conditionally on $(D_1, \dots, D_k)$ we have

$$\begin{aligned} D_{k+1} &= i \quad \text{with probability } P_i \quad \text{for any } i \in \{1, \dots, \max_{1\leq i\leq k} D_i\}, \\ &= 1 + \max_{1\leq i\leq k} D_i \quad \text{with complementary probability.} \end{aligned}$$

- The sequence $(X_k)_{k\geq 1}$ is then defined on the interval $[0\,,S]$ as $(S\cdot Y_{D_k})_{k\geq 1}$.

The block $\mathcal{S}^{\alpha,\gamma}$ is then defined as the interval $[0\,,S]$ rooted at 0 endowed with the sequence $(X_k)_{k\geq 1}$, see Figure 5.9b. In order to construct $\mathcal{B}^{\alpha,\gamma}$, we introduce a sequence $(C_i)_{i\geq 1}$ of circles such that for all $i \geq 1$, (C_i, d_i, ρ_i) is a circle with circumference P_i endowed with its path distance and rooted at some point ρ_i. Conditionally on that, we take on each C_i a uniform point U_i and, independently, a sequence $(V_{i,j})_{j\geq 1}$ of i.i.d. uniform random points on C_i. Then we consider their disjoint union

$$\bigsqcup_{i=1}^{\infty} C_i, \tag{5.8}$$

which we endow with the distance d characterized by

$$\begin{aligned} \mathrm{d}(x,y) &= d_i(x,y) \qquad \text{if} \quad x,y \in C_i, \\ &= d_i(x,U_i) + \sum_{k:\ Y_i<Y_k<Y_j} d_k(\rho_k,U_k) + d_j(\rho_j,y) \qquad \text{if} \quad x\in C_i,\ y\in C_j,\ Y_i<Y_j. \end{aligned}$$

Then $\mathcal{B}^{\alpha,\gamma}$ is defined as the completion of $\bigsqcup_{i=1}^{\infty} C_i$ equipped with this distance, with distinguished points $(V_{N_k,k})_{k\geq 1}$. Its root ρ can be obtained as a limit $\rho = \lim_{i\to\infty} \rho_{\sigma_i}$ for any sequence $(\sigma_i)_{i\geq 1}$ for which $Y_{\sigma_i} \to 0$, see Figure 5.9a.

Proof of Lemma 5.8: Recall that at any time n the object

$$\mathcal{A}^{\mathrm{tree}}(n) = (A^{\mathrm{tree}}(n), \mathrm{d}_n^{\mathrm{tree}}, \rho^{\mathrm{tree}}, (x_i^{\mathrm{tree}})_{1\leq i\leq n})$$

is endowed with a list of distinguished points $x_1^{\mathrm{tree}}, x_2^{\mathrm{tree}}, \dots x_n^{\mathrm{tree}}$, which are not necessarily distinct. Let us drop the superscript for readability. For every $k \geq 1$ let us call z_k the k-th vertex of degree 2 in $\mathcal{A}^{\mathrm{tree}}$ in order of creation. For any $m \geq 1$, denote D_m the unique integer such that $x_m = z_{D_m}$. Also for any $i \geq 1$, denote

$$N_i(n) := \#\left\{k \in [\![1\,,n]\!] \mid x_k = z_i\right\},$$

the number of distinguished points among $x_1, x_2, \dots, x_n$ that are equal to z_i and $K_n = \#\left\{x_i \mid 1 \leq i \leq n\right\}$ the total number of vertices created until time n. Suppose $\gamma < \alpha$ (the case $\gamma = \alpha$ is easier and follows using only a subset of the following arguments), then from the dynamics of $\mathcal{A}^{\mathrm{tree}}$, the numbers $(N_i(n), i \geq 1)$ evolve as the number of customers seated at each table in order of creation in a Chinese Restaurant Process with seating plan $(\frac{\gamma}{\alpha}, \frac{1-\alpha}{\alpha})$. Thanks to Theorem A.3, the following convergences hold almost surely and jointly

$$\left(\frac{N_i(n), i\geq 1}{n}\right) \xrightarrow[n\longrightarrow\infty]{\ell^1} (P_i, i\geq 1) \quad \text{and} \quad \frac{K_n}{n^{\gamma/\alpha}} \underset{n\to\infty}{\longrightarrow} S, \tag{5.9}$$

where $(P_i, i \geq 1) \sim \mathrm{GEM}(\frac{\gamma}{\alpha}, \frac{1-\alpha}{\alpha})$ and S denotes its $\frac{\gamma}{\alpha}$-diversity. Thanks to Theorem A.4, conditionally on $(P_i, i \geq 1)$, the distribution of the sequence $(D_1, D_2, \dots)$ is exactly the one given in the description of $\mathcal{S}^{\alpha,\gamma}$.

Since $\mathcal{A}^{\mathrm{tree}}(n)$ is just a line made of K_n vertices (and so K_n+1 edges), the last convergence is enough to prove that, as rooted metric spaces we have the following almost sure convergence in $\mathbb{M}^\bullet$

$$(A^{\mathrm{tree}}(n), n^{-\frac{\gamma}{\alpha}} \cdot \mathrm{d}_n^{\mathrm{tree}}, \rho^{\mathrm{tree}}) \underset{n\to\infty}{\to} ([0\,,S], \mathrm{d}_{[0,S]}, 0),$$

where $\mathrm{d}_{[0,S]}$ is the usual Euclidian distance on that interval.

Now let us handle the positions of the created vertices along the line. When z_1 the first vertex of degree 2 is created, it is adjacent to one edge of weight $1-\alpha$ on the "leaf-side" and one edge of weight γ on the "root-side". Every time that a new vertex is created, on either side, it reinforces the weight of that side by creating a new edge of weight γ. We recognize the dynamics of a Pólya urn, hence the proportion of the number of vertices on the root-side of z_1 converges almost surely to some random variable $Y_1 \sim \mathrm{Beta}(1, \frac{1-\alpha}{\gamma})$. Recalling (5.9), this means that almost surely

$$\frac{\mathrm{d}_n^{\mathrm{tree}}(\rho, z_1)}{n^{\frac{\gamma}{\alpha}}} = \frac{\mathrm{d}_n^{\mathrm{tree}}(\rho, z_1)}{K_n} \cdot \frac{K_n}{n^{\frac{\gamma}{\alpha}}} \underset{n\to\infty}{\to} Y_1 \cdot S.$$

Conditionally on Y_1, the vertices $z_2, z_3, z_4, \dots$ are inserted independently on the root-side or on the leaf-side of z_1, with respective probability Y_1 and $1-Y_1$. Then, only considering what happens to the root-side of z_1, what we observe is a uniform edge-splitting process (all the edges have weight γ) and so thanks to Lemma 5.3, the positions of all those vertices in order of creation converge and they are in the limit i.i.d. uniform on that segment. Finally, what happens on the leaf-side of z_1 has exactly the same distribution as the whole process starting from only one edge with weight $1-\alpha$. For every $i\geq 1$, the limit $Y_i = \lim_{k\to\infty} \frac{\mathrm{d}_n^{\mathrm{tree}}(\rho, z_i)}{K_n}$ then almost surely exists and one can check that this sequence has the same distribution as the sequence of random variables $(Y_1, Y_2, \dots)$ described in the construction of $\mathcal{S}^{\alpha,\gamma}$. Because the sequence $(x_k)_{k\geq 1}$ is exactly given by $(z_{D_k})_{k\geq 1}$ by construction, using all of the above, we get the almost sure convergence in $\mathbb{M}^{\infty\bullet}$

$$\mathcal{A}^{\mathrm{tree}}(n) = (A^{\mathrm{tree}}(n), n^{-\frac{\gamma}{\alpha}} \cdot \mathrm{d}_n^{\mathrm{tree}}, \rho^{\mathrm{tree}}, (x_i^{\mathrm{tree}})_{1\leq i\leq n}) \underset{n\to\infty}{\to} \mathcal{S}^{\alpha,\gamma} = ([0\,,S], \mathrm{d}_{[0,S]}, 0, (S\cdot Y_{D_i})_{i\geq 1}).$$

Now let us understand what happens for the corresponding string of loops $\mathcal{A}^{\mathrm{loop}}$. Recall that $\mathcal{A}^{\mathrm{loop}}(n)$ is also endowed with distinguished points $x_1^{\mathrm{loop}}, x_2^{\mathrm{loop}}, \dots, x_n^{\mathrm{loop}}$ and let us keep the superscripts for the rest of the proof as to avoid any ambiguity between $\mathcal{A}^{\mathrm{loop}}$ and $\mathcal{A}^{\mathrm{tree}}$. By construction, the loop that corresponds to vertex z_i at time n contains all the $N_i(n)$ vertices $\left\{x_j^{\mathrm{loop}} \;\middle|\; x_j^{\mathrm{tree}} = z_i, j\leq n\right\}$ that are of degree 2, and two other vertices, which are respectively shared with the loop just above and the one just below.

Now, three observations: First, in the limit $n\to\infty$, the size of this loop is such that $(N_i(n)+2)\sim nP_i$ almost surely, so when scaling distances by n^{-1}, a circle of length P_i is going to appear in the limit. Second, after a loop has been created, the addition of vertices around that loop follows exactly the dynamic of a uniform edge-splitting process, and hence thanks to Lemma 5.3, the limiting positions of all the vertices created around the loop, in order of creation, are uniform along the length of the circle. Last, the almost sure convergence $\left(\frac{N_i(n), i\geq 1}{n}\right) \underset{n\longrightarrow\infty}{\overset{\ell^1}{\longrightarrow}} (P_i, i\geq 1)$ and the convergence of the number of loops $\frac{K_n}{n^{\gamma/\alpha}} \to S$ ensure that the total (normalised) length of all the loops created after time t tends to 0 uniformly in n as $t\to\infty$, which ensure the a.s. relative compactness of the sequence $n^{-1}\cdot \mathcal{A}^{\mathrm{loop}}(n)$. From these observations, the identification of the limit is quite straightforward and the details are left to the reader. □

Convergence results. We can now express our scaling limit convergences for our processes.

Proposition 5.9. *We have the following joint convergence almost surely in the Gromov–Hausdorff–Prokhorov topology,*

$$(T_n^{\alpha,\gamma}, n^{-\gamma} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{T}^{\alpha,\gamma},$$
$$(\mathrm{Loop}(T_n^{\alpha,\gamma}), n^{-\alpha} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{L}^{\alpha,\gamma}.$$

The limiting objects can be constructed using an iterative gluing construction with i.i.d. blocks using a weight sequence $(\mathsf{m}_n)_{n\geq 1}$ *obtained as the increment of a Mittag-Leffler Markov chain* $\mathrm{MLMC}(\alpha, 1-\alpha)$*. The scaling factors are taken as* $(\mathsf{m}_n^{\gamma/\alpha})_{n\geq 1}$ *for the construction of* $\mathcal{T}^{\alpha,\gamma}$ *and* $(\mathsf{m}_n)_{n\geq 1}$ *for that of* $\mathcal{L}^{\alpha,\gamma}$*, using i.i.d. blocks with the same joint distribution as* $(\mathcal{S}^{\alpha,\gamma}, \mathcal{B}^{\alpha,\gamma})$ *defined above.*

Proof of Proposition 5.9: The proof of this proposition relies once again on Theorem 5.2. Conditions **(A)** and **(B)** are satisfied thanks to the discussion in the first paragraph of the section and condition **(C)** is satisfied thanks to Lemma 5.8. Condition **(D)** is satisfied for $\mathcal{A}^{\mathrm{tree}}$ thanks to the second part of Lemma 5.8; for $\mathcal{A}^{\mathrm{loop}}$, it comes from the fact that the total number of edges in $\mathcal{A}^{\mathrm{loop}}(n)$ is deterministically smaller than $3n+1$, hence also its diameter.

It remains to check that **(E)** is satisfied for some measure-valued decoration $\boldsymbol{\nu}^{\mathrm{tree},(n)}$ and $\boldsymbol{\nu}^{\mathrm{loop},(n)}$ such that $\big(\mathscr{G}(\mathcal{D}^{\mathrm{tree},(n)}, \boldsymbol{\nu}^{\mathrm{tree},(n)}), \mathscr{G}(\mathcal{D}^{\mathrm{loop},(n)}, \boldsymbol{\nu}^{\mathrm{loop},(n)})\big)$ coincides with $(T_n^{\alpha,\gamma}, \mathrm{Loop}(T_n^{\alpha,\gamma}))$ endowed with their graph distance and uniform measure on the vertices. This is, as for all other examples, achieved by charging every vertex of every block $\mathcal{D}^{(n)}(u)$ of the decoration with the same weight, except its root vertex if $u \neq \emptyset$.

Then the proof goes as that of Proposition 5.7: the total weight, of vertices (on one side) and of edges (on the other side) in $T_n^{\alpha,\gamma}$, evolves as a balanced generalised Pólya urn with replacement matrix

$$\begin{bmatrix} \alpha & 1-\alpha \\ \alpha-\gamma & 1-\alpha+\gamma \end{bmatrix}.$$

Using Lemma A.2, the total weight of edges is asymptotically $\frac{1-\alpha}{1-\gamma}n$ almost surely. Since at time n, the total weight of edges adjacent to a leaf is exactly $(1-\alpha)n$, it means that that the total weight of the internal edges is asymptotically $\frac{\gamma(1-\alpha)}{1-\gamma}n$. In the end, the asymptotic number of edges in $T_n^{\alpha,\gamma}$ (and hence also of vertices) is $(1+\frac{1-\alpha}{1-\gamma})n$. This is also true for the number of vertices in $\mathrm{Loop}(T_n^{\alpha,\gamma})$ because by construction, it corresponds to the number of edges in $T_n^{\alpha,\gamma}$.

Going along the same proof as for Proposition 5.7, this is enough to show that asymptotically almost surely, the measures $\nu^{\mathrm{tree},(n)}$ and $\nu^{\mathrm{loop},(n)}$ on $\mathbb{U}$ satisfy

$$\nu^{\mathrm{tree},(n)}(T(u)) \underset{n\to\infty}{\sim} \nu^{\mathrm{loop},(n)}(T(u)) \underset{n\to\infty}{\sim} \nu_n(T(u)),$$

together with

$$\nu^{\mathrm{tree},(n)}(u) \underset{n\to\infty}{\to} 0 \qquad \text{and} \qquad \nu^{\mathrm{loop},(n)}(u) \underset{n\to\infty}{\to} 0 \tag{5.10}$$

for every $u \in \mathbb{U}$, and ν_n being the uniform measure on the preferential attachment tree P_n associated to the construction. This is enough to prove the a.s. convergence of $\nu^{\mathrm{tree},(n)}$ and $\nu^{\mathrm{loop},(n)}$ to μ, which is the weak limit of ν_n. This finishes the proof. □

Here we can apply Proposition 4.3 for both objects $\mathcal{T}^{\alpha,\gamma}$ and $\mathcal{L}^{\alpha,\gamma}$ that appear in the limit, so that they can be described using self-similar decorations. We can further apply Proposition 4.2 to get that their Hausdorff dimension is respectively $\frac{1}{\gamma}$ and $\frac{1}{\alpha}$. The scaling limit convergence of $T_n^{\alpha,\gamma}$ was already obtained (only in probability) in Chen et al. (2009), where the limit was described as a fragmentation tree, for which the Hausdorff dimension was known using Haas and Miermont (2004). The scaling limit convergence of $\mathrm{Loop}(T_n^{\alpha,\gamma})$, however, is new, and so is the limiting object whenever $\gamma \neq 1-\alpha$. When $\gamma = 1-\alpha$ the limiting spaces $(\mathcal{T}^{\alpha,\gamma}, \mathcal{L}^{\alpha,\gamma})$ are constant multiples of respectively the $\frac{1}{\gamma}$-stable tree and its associated $\frac{1}{\gamma}$-stable looptree, thanks to convergence results found in Curien and Kortchemski (2014); Curien and Haas (2013).

5.5.2. *Looptrees constructed using affine preferential attachment.* This last model is very similar to the case of the generalised Rémy algorithm. We mention it separately because it was the object of a conjecture by Curien, Duquesne, Kortchemski and Manolescu Curien et al. (2015), to which we provide a positive answer.

First, for any $\delta > -1$, let us define the model LPAM^{δ} which produces sequences $(T_n^{\delta})_{n\geq 1}$ of plane trees. Start with T_1^{δ} containing a unique vertex connected to a root by an edge (the root is not considered as a real vertex here). Then, if T_n^{δ} is already constructed, take a vertex in the tree at random (the root does not count) with probability proportional to its degree plus δ, then add an edge connected to a new vertex in uniformly chosen corner around this vertex. This yields T_{n+1}^{δ}.

Proposition 5.10. *We have the following almost sure convergence*

$$(\mathrm{Loop}(T_n^{\delta}), n^{-\frac{1}{2+\delta}} \cdot \mathrm{d_{gr}}, \mu_{\mathrm{unif}}) \underset{n\to\infty}{\longrightarrow} \mathcal{L}^{\delta}$$

in the Gromov–Hausdorff–Prokhorov topology. The limiting object can be constructed using an iterative gluing construction with deterministic blocks equal to a circle with unit circumference, using a sequence of weights and scaling factors $(\mathsf{m}_n)_{n\geq 1}$ *obtained as the increment of a Mittag-Leffler Markov chain* $\mathrm{MLMC}\left(\frac{1}{2+\delta}, \frac{1+\delta}{2+\delta}\right)$.

We can apply Proposition 4.3 to the limiting object to describe it using an almost-self-similar decoration. We can further use Proposition 4.2 (or Sénizergues (2019, Theorem 1)) to obtain that the Hausdorff dimension of $\mathcal{L}^{\delta}$ is $2+\delta$ almost surely.

The proof of Proposition 5.10 is really close to that of Proposition 1.1 for the generalised Rémy algorithm, so we omit it.

Appendix A. Some useful definitions and results

This section is devoted to recalling and proving some definitions and results that are used in the paper.

A.1. *Pólya urns. Dirichlet distributions.* For parameters $a_1, a_2, \dots, a_n > 0$, the Dirichlet distribution $\mathrm{Dir}(a_1, \dots, a_n)$ has density

$$\frac{\Gamma(\sum_{i=1}^n a_i)}{\prod_{i=1}^n \Gamma(a_i)} \prod_{j=1}^n x_i^{a_j - 1}$$

with respect to the Lebesgue measure on the simplex $\{(x_1, \dots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$. In the case $n = 2$, a random variable with distribution $\mathrm{Dir}(a_1, a_2)$ can be written as $(B, 1-B)$, and B is said to have distribution $\mathrm{Beta}(a_1, a_2)$.

Convergence and exchangeability.

Theorem A.1. *Consider an urn model with k colours labelled from 1 to k, with initial weights $a_1, \dots, a_k > 0$ respectively. At each step $n \geq 1$, draw a colour with a probability proportional to its weight and increase the weight of this colour by β; we let D_n be the label of the drawn colour. Let $X_n^{(1)}, \dots, X_n^{(k)}$ denote the weights of the k colours after n steps. Then:*

(i) *We have the following convergence*

$$\left(\frac{X_n^{(1)}}{\beta n}, \dots, \frac{X_n^{(k)}}{\beta n}\right) \xrightarrow[n\to\infty]{\text{a.s.}} (X^{(1)}, \dots, X^{(k)})$$

where $(X^{(1)}, \dots, X^{(k)}) \sim \mathrm{Dir}(\frac{a_1}{\beta}, \dots, \frac{a_k}{\beta})$.

(ii) *Conditionally on the limiting proportions* $(X^{(1)}, \dots, X^{(k)})$, *the sequence of draws* $D_1, D_2, \dots$ *is i.i.d. such that*

$$\mathbb{P}\left(D_1 = i \;\middle|\; X^{(1)}, \dots, X^{(k)}\right) = X^{(i)},$$

for all $1 \le i \le k$.

Balanced generalized Pólya urns. Consider the following urn model with two colours, which depends on four positive real numbers $a, b, c, d > 0$. Starting from some initial condition, the weight (X_n, Y_n) of the two colours in the urn evolves in the following way: At each step, draw a colour from the urn with probability proportional to its weight in the urn. If colour 1 is drawn, add a to the weight of colour 1 and b to the weight of colour 2. If colour 2 is drawn, add c to the weight of colour 1 and d to the weight of colour 2. The matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the replacement matrix of the urn.

We suppose that the urn is *balanced*, meaning that $a + b = c + d$, and we call λ_1 the eigenvalue of M with largest modulus, being here equal to $a + b$. Let us now state a lemma that follows from the application of the results of Athreya and Karlin (1968) to our setting.

Lemma A.2. *For any initial weight configuration, we have the following almost sure convergence*

$$\left(\frac{X_n}{n(a+b)}, \frac{Y_n}{n(a+b)}\right) \longrightarrow v_1,$$

where v_1 *is the left eigenvector associated to* λ_1, *normalized to have components that sum to* 1.

A.2. *Chinese Restaurant Processes. Generalized Mittag-Leffler distributions.* Let $0 < \alpha < 1$, $\theta > -\alpha$. The generalized Mittag-Leffler distribution $\mathrm{ML}(\alpha, \theta)$ is characterised by its moments. For $M \sim \mathrm{ML}(\alpha, \theta)$ and any $p \in \mathbb{R}_+$ we have,

$$\mathbb{E}\left[M^p\right] = \frac{\Gamma(\theta)\Gamma(\theta/\alpha + p)}{\Gamma(\theta/\alpha)\Gamma(\theta + p\alpha)} = \frac{\Gamma(\theta+1)\Gamma(\theta/\alpha + p + 1)}{\Gamma(\theta/\alpha + 1)\Gamma(\theta + p\alpha + 1)}.$$

GEM and PD distribution. Let $0 < \alpha < 1$, $\theta > -\alpha$ and for $i \ge 1$, let $B_i \sim \mathrm{Beta}(1 - \alpha, \theta + i\alpha)$ be independent random variables. Then the sequence $(P_i)_{i \ge 1}$ where $P_i = B_i \prod_{k=1}^{i-1}(1 - B_k)$ has the $\mathrm{GEM}(\alpha, \theta)$ distribution. The reordered sequence $(P_i^{\downarrow})_{i \ge 1}$ in non-increasing order is said to have the $\mathrm{PD}(\alpha, \theta)$ distribution. In this setting the limit

$$W := \Gamma(1 - \alpha) \lim_{i \to \infty} i (P_i^{\downarrow})^{\alpha} \tag{A.1}$$

almost surely exists and is said to be the *α-diversity* of the sequence $(P_i)_{i \ge 1}$. It has the $\mathrm{ML}(\alpha, \theta)$ distribution (this can be read from Pitman (2006, Theorem 3.8 and Lemma 3.11)).

Chinese Restaurant Process. Fix two parameters $\alpha \in (0, 1)$ and $\theta > -\alpha$. Let us introduce the so-called Chinese restaurant process with seating plan (α, θ). We refer to Pitman (2006) for a rigorous definition and properties of this process. The process starts with one table occupied by one customer and then evolves in a Markovian way as follows: given that at stage n there are k occupied tables with n_i customers at table i for all $i \le k$, a new customer is placed at any table i with probability $(n_i - \alpha)/(n + \theta)$ and placed at a new table with probability $(\theta + k\alpha)/(n + \theta)$.

Let $N_i(n), i \ge 1$ be the number of customers at table i at stage n. Denote by K_n the number of occupied tables at stage n and D_n the number of the table at which the n-th customer sits. The following theorems follow from Pitman (2006, Chapter 3).

Theorem A.3. *In this setting we have the following convergences*

$$\left(\frac{N_i(n), i \ge 1}{n}\right) \xrightarrow[n \longrightarrow \infty]{\text{a.s. in } \ell^1} (P_i, i \ge 1) \quad \text{and} \quad \frac{K_n}{n^{\alpha}} \xrightarrow[n \to \infty]{\text{a.s.}} W,$$

where $(P_i, i \geq 1)$ *follows a* $\mathrm{GEM}(\alpha, \theta)$*-distribution and* W *is its* α*-diversity, as defined in* (A.1). *The sequence* $(Q_j, j \geq 1) = \left(P_i^{\downarrow}, i \geq 1\right)$, *defined as the non-increasing rearrangement of* $(P_i, i \geq 1)$ *then has the* $\mathrm{PD}(\alpha, \theta)$*-distribution.*

The following result allows us to describe the distribution of the process, conditionally on the limiting size of the tables.

Theorem A.4. *In the setting of the previous theorem, conditionally on the sequence* $(P_i, i \geq 1)$, *the distribution of* $(D_n)_{n \geq 1}$ *can be described as follows:*

(i) $D_1 = 1$ *almost surely.*

(ii) *Conditionally on* $D_1, \ldots, D_n$, *we have*

$$\begin{aligned} D_{n+1} &= k \qquad \text{with probability } P_k, \text{ for any } 1 \leq k \leq \max_{1 \leq i \leq n} D_i, \\ &= 1 + \max_{1 \leq i \leq n} D_i \qquad \text{with complementary probability.} \end{aligned}$$

Also, conditionally on $(Q_j, j \geq 1)$, *the distribution of* $(D_n)_{n \geq 1}$ *can be described as follows:*

(i) *Let* $(I_n)_{n \geq 1}$ *be i.i.d. with distribution* $\mathbb{P}(I_1 = k) = Q_k$,

(ii) $D_1 = 1$ *almost surely,*

(iii) *conditionally on* $I_1, \ldots, I_n$, *we have*

$$D_{n+1} = \begin{cases} 1 + \max_{1 \leq i \leq n} D_i & \text{if } I_{n+1} \notin \{I_1, I_2, \ldots, I_n\}, \\ D_{J_n} \text{ for } J_n = \inf\{k \geq 1 \mid I_{n+1} = I_k\} & \text{otherwise.} \end{cases}$$

For any $\alpha \in (0, 1)$ and $\theta > -\alpha$, the law of this evolving configuration of customers around the different tables using the (α, θ) seating plan is denoted by $\mathbb{P}_{\alpha,\theta}$. The following result is expressed for the canonical process under the probability measure $\mathbb{P}_{\alpha,\theta}$. Its proof is an adaptation of that of Pitman (2006, Theorem 3.8) and uses the same notation.

Lemma A.5. *For every* $p \geq 1$, *we have*

$$\mathbb{E}_{\alpha,\theta}\left[\sup_{n \geq 1}\left(\frac{K_n}{n^\alpha}\right)^p\right] < \infty. \tag{A.2}$$

Proof: Let $f_{\alpha,\theta}(k) := \frac{\Gamma(\theta/\alpha + k)}{\Gamma(\theta/\alpha+1)\Gamma(k)}$, and $\mathcal{F}_n$ the σ-field generated by the n first steps of the process, then

$$M_{\alpha,\theta,n} := \left(\frac{\mathrm{d}\mathbb{P}_{\alpha,\theta}}{\mathrm{d}\mathbb{P}_{\alpha,0}}\right)_{|\mathcal{F}_n} = \frac{f_{\alpha,\theta}(K_n)}{f_{1,\theta}(n)}$$

is a martingale under $\mathbb{P}_{\alpha,0}$ and bounded in L^p, for all $p \geq 1$, see Pitman (2006, Proof of Theorem 3.8). This, in particular, ensure that $M_{\alpha,\theta,n}$ converges almost surely and in every L^p for $p \geq 1$ to some limit $M_{\alpha,\theta}$ and that we have $\frac{\mathrm{d}\mathbb{P}_{\alpha,\theta}}{\mathrm{d}\mathbb{P}_{\alpha,0}} = M_{\alpha,\theta}$.

Using Doob's maximal inequality we have

$$\mathbb{E}_{\alpha,0}\left[\left(\sup_{1 \leq n \leq m} M_{\alpha,\theta,n}\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}_{\alpha,0}\left[(M_{\alpha,\theta,m})^p\right].$$

Introduce $M^*_{\alpha,\theta} := \sup_{n \geq 1} M_{\alpha,\theta,n}$. Passing the limit in the last display, using the monotone convergence theorem and the L^p convergence of $(M_{\alpha,\theta,n})_{n \geq 1}$ we get

$$\mathbb{E}_{\alpha,0}\left[\left(M^*_{\alpha,\theta}\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}_{\alpha,0}\left[(M_{\alpha,\theta})^p\right].$$

Now, from the definition of $f_{\alpha,\theta}$, we can check that there exists a constant $c > 1$ such that for any $k, n \geq 1$ we have

$$\frac{1}{c}\left(\frac{k}{n^\alpha}\right)^{\frac{\theta}{\alpha}} \leq \frac{f_{\alpha,\theta}(k)}{f_{1,\theta}(n)} \leq c\left(\frac{k}{n^\alpha}\right)^{\frac{\theta}{\alpha}}.$$

Hence, using the last display with $\theta = \alpha$ and $k = K_n$ and what we proved above, we get

$$\begin{aligned}\mathbb{E}_{\alpha,\theta}\left[\left(\sup_{n\geq 1}\frac{K_n}{n^\alpha}\right)^p\right] &\leq c\cdot\mathbb{E}_{\alpha,\theta}\left[\left(M^*_{\alpha,\alpha}\right)^p\right] = c\cdot\mathbb{E}_{\alpha,0}\left[\left(M^*_{\alpha,\alpha}\right)^p\cdot M_{\alpha,\theta}\right]\\ &\leq c\cdot\sqrt{\mathbb{E}_{\alpha,0}\left[\left(M^*_{\alpha,\alpha}\right)^{2p}\right]\mathbb{E}_{\alpha,0}\left[M^2_{\alpha,\theta}\right]} < \infty,\end{aligned}$$

where the last inequality uses Cauchy-Schwarz inequality. This finishes the proof of the lemma. □

A.3. *The supremum of the normalised height in Rémy's algorithm.* Let $(T_n)_{n\geq 1}$ be a sequence of trees evolving using Rémy's algorithm. This sequence is a Markov chain in a state space of binary planted trees. Let us denote $H := \sup_{n\geq 1}(n^{-1/2}\,\mathrm{ht}(T_n))$. We prove the following bound on the tail of the distribution of H.

Lemma A.6. *There exists constants C_1 and C_2 such that for all $x > 0$,*

$$\mathbb{P}\left(H > x\right) \leq C_1\exp\bigl(-C_2x^2\bigr).$$

In particular, H admits moments of all orders.

Proof: Let $\tau_x := \inf\left\{n\geq 1 \;\middle|\; \mathrm{ht}(T_n) > xn^{1/2}\right\}$. We write

$$\begin{aligned}\mathbb{P}\left(H > x\right) &= \mathbb{P}\left(\tau_x < +\infty\right)\\ &\leq \mathbb{P}\left(\lim_{n\to\infty} n^{-1/2}\,\mathrm{ht}(T_n) > \frac{x}{2}\right) + \mathbb{P}\left(\tau_x < +\infty, \lim_{n\to\infty} n^{-1/2}\,\mathrm{ht}(T_n) \leq \frac{x}{2}\right)\end{aligned}$$

We know thanks to Curien and Haas (2013) that the trees constructed using Rémy's algorithm converge almost surely in Gromov–Hausdorff topology to Aldous' Brownian tree, so $n^{-1/2}\cdot T_n \to \mathcal{T}$ as $n\to\infty$. By continuity we have $\lim_{n\to\infty} n^{-1/2}\,\mathrm{ht}(T_n) = \mathrm{ht}(\mathcal{T})$. Estimates Kennedy (1976) on the height of the Brownian tree show that the first term of the above sum is smaller than $C_1\exp\bigl(-C_2x^2\bigr)$, for some choice of constants C_1 and C_2. Fix some $N_0\geq 1$ that we will chose later. Then compute

$$\begin{aligned}\mathbb{P}\left(\tau_x < +\infty, \lim_{n\to\infty} n^{-1/2}\,\mathrm{ht}(T_n) \leq \frac{x}{2}\right) &= \sum_{N=1}^{+\infty}\mathbb{P}\left(\tau_x = N\right)\mathbb{P}\left(\mathrm{ht}(\mathcal{T}) \leq \frac{x}{2} \;\middle|\; \tau_x = N\right)\\ &\leq \sum_{N=1}^{N_0}\mathbb{P}\left(\tau_x = N\right) + \sup_{N\geq N_0}\mathbb{P}\left(\mathrm{ht}(\mathcal{T}) \leq \frac{x}{2} \;\middle|\; \tau_x = N\right)\\ &\leq N_0C_1\exp\bigl(-C_2x^2\bigr) + \sup_{N\geq N_0}\mathbb{P}\left(\mathrm{ht}(\mathcal{T}) \leq \frac{x}{2} \;\middle|\; \tau_x = N\right),\end{aligned}$$

where in the last inequality we use the fact that for all N,

$$\mathbb{P}\left(\tau_x = N\right) \leq \mathbb{P}\left(\mathrm{ht}(T_N) \geq xN^{1/2}\right) \leq C_1\exp\bigl(-C_2x^2\bigr),$$

using the results of Addario-Berry et al. (2013).

Now, let us reason conditionally on the event $\{\tau_x = N\}$. On that event, there exists in the tree T_N at least one path of length $\lfloor xN^{1/2}\rfloor$ starting from the root ending at a vertex v. The height $H_n(v)$ of the vertex v at time n evolves under the same dynamics as the number of balls in a triangular urn model with replacement matrix $\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$, and starting proportion $(\lfloor xN^{1/2}\rfloor, 2N+1-\lfloor xN^{1/2}\rfloor)$, see Janson (2006) for definition and results for those urns. Using a theorem of Janson (2006), as $n \to \infty$, we have the almost sure convergence

$$\frac{H_n(v)}{n^{1/2}} \longrightarrow W_N^x,$$

where $W_N^x = \beta_N^x \cdot M_N$ is the product of two independent variables, with

$$\beta_N^x \sim \mathrm{Beta}(\lfloor xN^{1/2} \rfloor, 2N+1-\lfloor xN^{1/2} \rfloor) \qquad \text{and} \qquad M_N \sim \mathrm{ML}\left(\frac{1}{2}, \frac{2N+1}{2}\right). \tag{A.3}$$

Then we write

$$\begin{aligned}
\mathbb{P}\left(W_N^x \le \frac{x}{2}\right) &= \mathbb{P}\left(\beta_N^x \cdot M_N \le \frac{x}{2}\right) \\
&\le \mathbb{P}\left(\beta_N^x \cdot M_N \le \frac{x}{2},\ M_N \ge \frac{3}{2}N^{1/2}\right) + \mathbb{P}\left(M_N \le \frac{3}{2}N^{1/2}\right) \\
&\le \mathbb{P}\left(\beta_N^x \le \frac{1}{3} \cdot x \cdot N^{-1/2}\right) + \mathbb{P}\left(M_N \le \frac{3}{2}N^{1/2}\right).
\end{aligned}$$

We bound the two terms in the last sum using Chebychev's inequality, using

$$\mathbb{E}\left[\beta_N^x\right] = \frac{\lfloor xN^{1/2} \rfloor}{2N+1} \sim \frac{x}{2}N^{-1/2}, \qquad \mathbb{V}\left(\beta_N^x\right) = \frac{\lfloor xN^{1/2} \rfloor (2N+1-\lfloor xN^{1/2} \rfloor)}{(2N+1)^2(2N+2)} \sim \frac{x}{4}N^{-3/2}.$$

We also have

$$\mathbb{E}\left[M_N\right] = \frac{\Gamma\left(N+1/2\right)\Gamma\left(2N+2\right)}{\Gamma\left(2N+1\right)\Gamma\left(N+1\right)} \sim 2N^{1/2},$$

and

$$\mathbb{V}\left(M_N\right) = \frac{\Gamma\left(N+1/2\right)\Gamma\left(2N+3\right)}{\Gamma\left(2N+1\right)\Gamma\left(N+3/2\right)} - \left(\frac{\Gamma\left(N+1/2\right)\Gamma\left(2N+2\right)}{\Gamma\left(2N+1\right)\Gamma\left(N+1\right)}\right)^2 \le 1.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(\beta_N^x \le \frac{1}{3} \cdot x \cdot N^{-1/2}\right) &\le \mathbb{P}\left(\left|\beta_N^x - \mathbb{E}\left[\beta_N^x\right]\right| \ge \frac{1}{8} \cdot x \cdot N^{-1/2}\right) \\
&\le 8x^2 N \mathbb{V}\left(\beta_N^x\right) \\
&\le Cx^3 N^{-1/2}.
\end{aligned}$$

with C a constant that is independent of x and N. We also have

$$\begin{aligned}
\mathbb{P}\left(M_N \le \frac{3}{2}N^{1/2}\right) &\le \mathbb{P}\left(\left|M_N - \mathbb{E}\left[M_N\right]\right| \ge \frac{1}{3}N^{1/2}\right) \\
&\le 3N^{-1}\mathbb{V}\left(M_N\right) \\
&\le C'N^{-1},
\end{aligned}$$

with C' another constant that does not depend on x or N. All this analysis was done conditionally on the event $\{\tau_x = N\}$, so in fact, we have for all $N \ge N_0$,

$$\mathbb{P}\left(\mathrm{ht}(\mathcal{T}) \le \frac{x}{2} \;\middle|\; \tau_x = N\right) \le Cx^3 N^{-1/2} + C'N^{-1} \le Cx^3 N_0^{-1/2} + C'N_0^{-1}$$

Now we just take $N_0 = \exp\left(\frac{C_2}{2}x^2\right)$ and the result follows. □

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