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Cutoff for the Fredrickson-Andersen one spin facilitated model

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Abstract. The Fredrickson-Andersen one spin facilitated model belongs to the class of Kinetically Constrained Spin Models. It is a non attractive process with positive spectral gap. In this paper we give a precise result on the relaxation for this process on an interval $[1, L]$ starting from any initial configuration. A consequence of this result is that this process exhibits cutoff at time $L/(2v)$ with window $O(\sqrt{L})$ for a certain positive constant v. The key ingredient is the study of the evolution of the leftmost empty site in a filled infinite half-line called the front. In the process of the proof, we improve recent results about the front motion by showing that it evolves at speed v according to a uniform central limit theorem.

1. Introduction

The Fredrickson–Andersen one spin facilitated model (FA-1f) model is an interacting particle system that belongs to the class of kinetically constrained spin models (KCSM). These models are Markov processes that were first introduced in the 1980's by physicists to model liquid-glass transitions (see Fredrickson and Andersen, 1984 and Jäckle and Eisinger, 1991) and that have some analogy with the Glauber dynamics of the Ising model. In a KCSM on a graph G, each vertex can either have a particle or be empty. At rate 1, each site tries to update according to a Bernoulli measure, but does so only if a local constraint is satisfied. On the graph $\mathbb{Z}$, the constraint can for example be that the site immediately to the right is empty, in which case we get the *East model*. Another possible constraint is to have either of the adjacent neighboring sites empty, which defines the FA-1f model.

Since the 2000's, the dynamics of KCSM both at and out of equilibrium have been thoroughly studied by the mathematical community. In 2002, Aldous and Diaconis in Aldous and Diaconis (2002) proved that the East model has a positive spectral gap. This result has been generalized to a large class of models in Cancrini et al. (2008). Starting out of equilibrium studies are made more difficult by the lack of attractiveness of the processes, preventing usual monotonicity and coupling arguments. Some results can be found for example in Cancrini et al. (2010) for the East model and

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Mountford and Valle (2019); Blondel et al. (2013) for FA-1f at low density. For the East model and the FA-1f model on $\mathbb{Z}$, an interesting topic of study is the motion of the front, i.e. the rightmost empty site of a configuration filled on an infinite half-line. First, Blondel Blondel (2013) showed that the front has a linear speed in the East model. Based on this work, Ganguly, Lubetzky and Martinelli showed in Ganguly et al. (2015) a central limit theorem for the front, which was followed by the proof that the East model exhibits cutoff thanks to a simple coupling argument. In Blondel et al. (2019), it was shown that the front for the FA-1f process also behaves according to a CLT when the density is below a certain threshold. The aim of this paper is to use this last result to prove a cutoff for the FA-1f process. It will require much more work than for East because the main coupling argument Ganguly et al. (2015, Section 4.2) no longer holds for FA-1f.

The cutoff phenomenon was first exhibited by Aldous and Diaconis in the context of card shuffling Aldous and Diaconis (1986). It consists of a sharp drop of the total variation distance to the equilibrium measure of a Markov process (see Levin et al., 2009 for an introduction to this phenomenon). Since then, examples and counter-examples of cutoff have be shown for a wide variety of processes, but a universal criterion is missing. In 2004, Peres conjectured that a process exhibits cutoff if and only if it satisfies the *product condition* $t_{rel} = o(t_{mix})$, where t_{rel} is the inverse of the spectral gap and t_{mix} is the total variation mixing time of the process. This condition turned out not to be sufficient in general but sufficient for a large class of processes (see Basu et al., 2017 and more recently Salez, 2021). Recently, several cutoff results have been shown for particle systems like the Ising model Lubetzky and Sly (2017), the Asymmetric Simple Exclusion Process Labbé and Lacoin (2019) or a stratified random walk on the hypercube Ben-Hamou and Peres (2018).

In this paper, we will study in detail the relaxation of the FA-1f process depending on the initial configuration. We mainly rely on a result giving a bound on the mixing time for a configuration with many empty sites. The core of our study will thus be to see how quickly any initial configuration can create enough empty sites. For that, we study the big intervals that are initially filled with particles, and interpret the endpoint of those as fronts going inward at speed v. Thanks to another result referred to as "Zeros Lemma", we will be able to show that the configuration becomes suitable for relaxation after roughly the time it takes for the fronts of the biggest particle cluster to meet. Finally, our main result gives precise bounds on the mixing time of the process for *any* initial configuration, which is a more complete result than just a cutoff statement.

First, we give the definition of the model and our main result in Section 2, then an important coupling with a threshold contact process in Section 3 and results about relaxation to equilibrium in Section 4. In Section 5, we extend the central limit theorem proved in Blondel et al. (2019) in order to fit the actual proof of the main result, which appears in Section 6.

2. Definitions and main result

2.1. *FA-1f processes.* We will encounter three types of FA-1f processes, depending on their state space. Fix $q \in [0, 1]$. The FA-1f process of parameter q on an interval $\Lambda \subset \mathbb{Z}$ is given by the generator:

$$\mathcal{L}f(\sigma) = \sum_{x\in\Lambda} r(x, \sigma)(f(\sigma^x) - f(\sigma)),$$

for any local function f and $\sigma \in \{0, 1\}^{\mathbb{Z}}$, where σ^x is the configuration equal to σ everywhere except at site x. The rate $r(x, \sigma)$ is given by:

$$r(x, \sigma) = (1 - \sigma(x - 1)\sigma(x + 1))(q\sigma(x) + (1 - q)(1 - \sigma(x))).$$

$\sigma(x) = 1$ is to be interpreted as the presence of a particle at x, and $\sigma(x) = 0$ as an empty site. In words, every site makes a flip $0 \to 1$ at rate $1 - q$ and $1 \to 0$ at rate q but only if it satisfies the kinetic constraint $c_x(\sigma) := (1 - \sigma(x - 1)\sigma(x + 1))$ is equal to 1, that is if it has a least one empty neighbor. If Λ has boundaries, we set the configuration to be fixed at 0 at the (outer) boundaries. We now fix some notations and conventions used throughout the paper.

- FA-1f processes on $\mathbb{Z}$ are generally denoted by the letter σ. Their state space is $\Omega := \{0,1\}^{\mathbb{Z}}$. We define $LO = \{\sigma \in \Omega \mid \exists X, \forall x < X, \sigma(x) = 1\}$ the set of configurations equal to one on an infinite left-oriented half line.
- FA-1f processes on $\mathbb{Z}_- := \{-1,-2,...\}$ are generally denoted by the letter η. Their state space is $\Omega_- := \{\eta \in \Omega \mid \eta(0) = 0,\ \forall x \geq 0, \eta(x) = 1\}$. We define $LO_- = LO \cap \Omega_-$.
- For a finite interval $\Lambda = [1, L]$, we define $\Omega_L = \{\sigma \in \Omega \mid \sigma(0) = \sigma(L+1) = 0,\ \forall x \notin [0, L+1], \sigma(x) = 1\}$.

For $\sigma \in \{0,1\}^{\mathbb{Z}}$ and $x \in \mathbb{Z}$, we define the shifted configuration $\theta_x\sigma$ by:

$$\forall y \in \mathbb{Z},\ \theta_x\sigma(y) = \sigma(y+x).$$

For any $\sigma \in LO$, we define the front $X(\sigma)$ (or sometimes simply written X) as the leftmost zero in σ. Note that if the process is defined on an interval $\Lambda \neq \mathbb{Z}$, the front can be an outer boundary of Λ. When $(\sigma_t)_{t\geq 0}$ is a process, we will write X_t instead of $X(\sigma_t)$ when σ_t clear from the context. We also denote by $\tilde{\sigma}$ the configuration seen from the front, namely $\theta_{X(\sigma)}\sigma$.

We also define μ_t^σ (resp. $\tilde{\mu}_t^\sigma$) the law of the FA-1f process (resp. seen from the front) at time t starting from σ. Whether the process takes place in $\mathbb{Z}$ or $\mathbb{Z}_-$ is implicitly given by the configuration σ. We denote by μ_Λ^p the Bernoulli product law of parameter p on $\{0,1\}^\Lambda$. The FA-1f process on Λ with parameter q is reversible with respect to μ_Λ^{1-q}. In the following, we shall write μ instead of μ_Λ^{1-q} when q and Λ are clearly fixed.

Finally we define, for $I = [a,b] \subset \Lambda$ an interval and $\ell \geq 10$ the set:

$$\mathcal{H}(I,\ell) = \mathcal{H}(a,b,\ell) := \{\sigma \in \{0,1\}^\Lambda | \forall x \in [a, b-\ell+1], \exists y \in [x, x+\ell-1],\ \sigma(y) = 0\}.$$

If $J = \bigcup_k I_k$ is a union of disjoint non adjacent intervals, then we define $\mathcal{H}(J,\ell) := \bigcap_k \mathcal{H}(I_k,\ell)$. In words, in a configuration in $\mathcal{H}(I,\ell)$, every site is at distance at most $\ell - 1$ to either the right boundary of I or to an empty site to its right. In such a configuration, two consecutive empty sites are within distance ℓ.

2.2. *Main result.* We denote $\Omega_L^\delta = \{\sigma \in \Omega_L \mid \sigma \in \mathcal{H}(1, L, \delta L)\}$. We also define, for $\sigma \in \Omega_L$,

$$B(\sigma) := \max\{h \geq 0 \mid \exists x \in [0, L-h],\ \restriction \sigma[x+1, x+h] \equiv 1\}$$

the size of the largest component of occupied sites in σ. The most precise result about the relaxation of the FA-1f process in a finite interval proved in this paper is the following. Recall the usual notations for real numbers: $a \wedge b = \max(a,b)$, $a \vee b = \min(a,b)$.

Theorem 2.1. *There exists $\bar{q} < 1$ such that for every $q > \bar{q}$, $\delta \in (0,1)$ and $\varepsilon > 0$, the following holds. There exist three constants $a = a(\varepsilon, q) > 0$ and $0 < \underline{v} = \underline{v}(q) < v = v(q)$ such that, if we first define the following three times for any $\sigma \in \Omega_L$:*

$$t_1(\sigma) = \Big(\frac{B(\sigma)}{\underline{v}}\Big) \vee \Big(\frac{(\log L)^9}{\underline{v}}\Big),$$
$$t_2(\sigma) = \frac{B(\sigma)}{2v} + \frac{a}{\underline{v}\delta}\sqrt{B(\sigma)},$$
$$t_3(\sigma) = \frac{B(\sigma)}{2v} - \frac{a}{v}\sqrt{B(\sigma)},$$

then:

$$\limsup_{L\to+\infty} \sup_{\sigma \in \Omega_L^\delta} ||\mu_{t_1(\sigma)}^\sigma - \mu||_{TV} = 0, \tag{2.1}$$

$$\limsup_{L\to+\infty} \sup_{\sigma\in(\Omega_L^\delta)^c} ||\mu_{t_2(\sigma)}^\sigma - \mu||_{TV} \le \frac{\varepsilon}{\delta^2}. \tag{2.2}$$

Moreover, for any function Φ *such that* $\Phi(L) \underset{L\to+\infty}{\longrightarrow} +\infty$,

$$\liminf_{L\to+\infty} \inf_{\sigma\in\mathcal{H}(1,L,\Phi(L))^c} ||\mu_{t_3(\sigma)}^\sigma - \mu||_{TV} \ge 1-\varepsilon. \tag{2.3}$$

Let us give some comments about this statement. First, the two constants $\underline{v}$ and v have the following interpretations. The constant $\underline{v}$ is the speed of the infection propagation for a threshold contact process defined in the following Section. The constant v corresponds to the speed of the front for the FA-1f process on $\mathbb{Z}$ starting with an infinitely filled half-line, that is with initial configuration in LO. Both $\underline{v}$ and v depend on q that we chose here to be greater to a certain threshold $\bar{q}$ necessary for the threshold contact process to survive. As we will see, the value $\bar{q}$ is roughly equal to 0.76 Brower et al. (1978). The first two equations (2.1) and (2.2) of Theorem 2.1 give an upper bound on the time at which the process is well mixed. The first one works nicely for initial configurations that already have enough empty sites. Indeed, in this case $B(\sigma)$ is smaller than δL, making up for the loss in the constant $1/(2\underline{v})$ instead of the expected $1/(2v)$. The time t_1 has to be bounded from below by $(\log L)^9$ for technical reasons we will see later. This bound is most likely not optimal but still offers a precise upper bound on the mixing time. The second equation handles the configurations that have macroscopic components of occupied sites and has the optimal leading behaviour $\frac{B(\sigma)}{2v}$ one can expect as shown in the last equation (2.3). The last equation gives a lower bound on the mixing time for configurations with at least an interval of size $\Phi(L)$ occupied. Since it is only required that $\Phi(L)$ goes to infinity, this hypothesis is not really restrictive. It is however necessary for our argument (Theorem 5.1) to work. As a consequence of this theorem, we can conclude that the FA-1f process exhibits a cutoff.

Theorem 2.2. *Let* $d(t) = \sup_{\sigma\in\Omega_\Lambda} ||\mu_t^\sigma - \mu||_{TV}$. *Then for all* $q > \bar{q}$ *and* $\varepsilon > 0$, *there exist* $\alpha(\varepsilon,q), \beta(\varepsilon,q) > 0$ *independent of* L, *such that for* L *large enough:*

$$d\Big(\frac{L}{2v} - \alpha(\varepsilon)\sqrt{L}\Big) \ge 1-\varepsilon,$$
$$d\Big(\frac{L}{2v} + \beta(\varepsilon)\sqrt{L}\Big) \le \varepsilon.$$

3. Coupling with a contact process and consequences

3.1. *Graphical construction.* Although we can easily define the FA-1f process through its generator, we also provide a more convenient construction called the graphical construction. For each site $x \in \mathbb{Z}$, define a Poisson process T^x of parameter 1 and an infinite sequence of Bernoulli random variables $(\beta_n^x)_{n\ge1}$ of parameter $1-q$, all of these variables being independent. Given an initial configuration σ_0, at each increment t of a Poisson process, say the n^{th} increment at site x, we check if the constraint $c_x(\sigma)$ is satisfied, that is if the site x has at least one empty neighbor. If it does, then we set $\sigma_t(x) = \beta_n^x$. Otherwise, $\sigma_t(x)$ is unchanged.

From this construction, we can define the *standard coupling* which simply consists in taking the same set of random variables $(T^x)_{x\in\mathbb{Z}}$ and $(\beta_n^x)_{n\ge1}$ for different initial configurations. Note that this coupling is not monotone: one can have $\sigma_0 \le \sigma_0'$ and $\sigma_t \not\le \sigma_t'$ even if $(\sigma_t)_{t\ge0}$ and $(\sigma_t')_{t\ge0}$ follow the standard coupling (see Figure 3.1). This is an important reason why this dynamics can be difficult to study.

Using the graphical construction, it is standard to show that there is a finite speed of propagation:

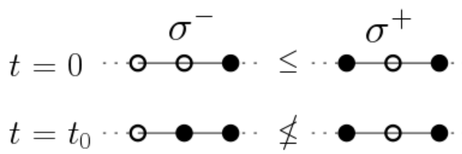

FIGURE 3.1. An example of non monotonicity via the standard coupling. The middle site could update in σ_- but not in σ_+.

Proposition 3.1 (Blondel et al. (2019, Lemma 2.5)). *Let $(\sigma_t)_{t\geq 0}$ be a FA-1f process. For $t \geq 0$ and two sites $x < y$, we define the events:*

$$F(x, y, t) = \{\text{there is a sequence of successive rings linking } x \text{ and } y \text{ in a time interval of length } t\},$$

$$\tilde{F}(x, y, t) = \{\exists z \in [x, y] \text{ s.t. } F(x, z, t) \cap F(y, z, t)\}.$$

Then there exists a constant $\bar{v} > 0$ such that if $|x - y| \geq \bar{v}t$, then

$$\mathbb{P}(F(x, y, t)) \leq \mathbb{P}(\tilde{F}(x, y, t)) \leq e^{-|x-y|}.$$

This implies a maximum speed for the front:

Corollary 3.2. *For all $\sigma \in LO$ or $\eta \in LO_-$, and $c \geq \bar{v}$, $\mathbb{P}(X_t - X_0 < -ct) \leq e^{-ct}$.*

3.2. *Contact process.* In Harris (1974), Harris introduced the contact process on $\mathbb{Z}$. It is a process on $\{0, 1\}^{\mathbb{Z}}$ that allows sites to flip from 0 (infected site) to 1 at rate 1, and from 1 to 0 at a rate depending on the amount of empty adjacent sites. Based on this process, we introduce the same *threshold contact process* as in Blondel et al. (2019), denoted here by $(\zeta_t)_{t\geq 0}$ and given by the following generator:

$$\forall f \text{ local}, \forall \zeta \in \Omega,\ \mathcal{L}' f(\zeta) = \sum_{x\in\mathbb{Z}} r'(x, \zeta)(f(\zeta^x) - f(\zeta)),$$

where $r(x, \zeta) = (1 - \zeta(x-1)\zeta(x+1))q\zeta(x) + (1-q)(1-\zeta(x))$. Note that we took here the convention that 0 is an infected site, which is not the most common one.

This process differs from both FA-1f and Harris' process in its constraint: a site is free to flip from 0 to 1 at rate p but has to have at least one empty neighbor to flip from 1 to 0 at rate q. This threshold contact process is the one well suited to a coupling with FA-1f as we will explain later. Let $\bar{q}$ be the same parameter than in Blondel et al. (2019, Appendix B) such that whenever $q \geq \bar{q}$, then the threshold contact process starting from a single infected site had a positive probability of surviving and will create infected sites in an interval scaling linearly with time.

Let $\eta_0 \in LO_-$ and $(\eta_t)_{t\geq 0}$ be a FA-1f process starting from η_0. Let $\zeta_0 = \delta_{X_0}$ where X_0 is the front of η_0 and δ_{X_0} is the configuration (on $\mathbb{Z}$) equal to 0 in site X_0 and equal to 1 everywhere else. Define $(\zeta_t)_{t\geq 0}$ the contact process starting from configuration ζ_0, evolving with respect to the standard coupling with $(\eta_t)_{t\geq 0}$. Then $\forall t \geq 0, \forall x \in \mathbb{Z}_-, \eta_t(x) \leq \zeta_t(x)$. Indeed, this inequality holds for $t = 0$. Then, for each ring of a site $x \in \mathbb{Z}_- \setminus \{0\}$, assuming $\forall y \in \mathbb{Z}_-, \eta_{t^-}(y) \leq \zeta_{t^-}(y)$, there are three cases:

- The kinetic constraint at x is not satisfied in η_{t^-}, in which case it is not satisfied in ζ_{t^-} either. In this case, the value $\eta_t(x)$ does not change and the only possible change for $\zeta_t(x)$ is $0 \to 1$, which preserves the order.
- The kinetic constraint at x is satisfied in η_{t^-} but not in ζ_{t^-}. In this case, a $0 \to 1$ flip is possible for both configurations, and a $1 \to 0$ flip is only possible for $\eta_t(x)$, which does not break the inequality.
- The constraint is satisfied for both configurations, and then by construction of the coupling, $\eta_{t^-}(x) = \zeta_{t^-}(x)$, which again does not change the inequality.

Note also that the behavior of ζ on $\mathbb{Z}_+$ influences the behavior on $\mathbb{Z}_-$ only through the $x = 0$ site. Since $\forall t, \eta_t(0) = 0$, we clearly have $\eta_t(0) \leq \zeta_t(0)$.

Thanks to this observation, and to the known results on the contact process Blondel et al. (2019, Appendix B), we can now state a first important lemma which provides a minimal speed for the front.

Lemma 3.3. *Let $q > \bar{q}$. There exists $\underline{v} > 0$ and $A, B > 0$ such that for every $\eta \in LO_-$, and $t \geq 0$,*

$$\mathbb{P}(X_t - X_0 > -\underline{v}t) \leq Ae^{-Bt}$$

The proof of the result is based on the comparison explained above. Hence, it is identical to the proof of Corollary 4.2 in Blondel et al. (2019).

3.3. *Zeros Lemmas.* Next, we state a crucial lemma based on the coupling explained above and on results for surviving contact processes. It is the same result as Blondel et al. (2019, Corollary 4.3), only extended to the FA-1f process on $\mathbb{Z}_-$ and $[1, L]$, and it follows from the same coupling argument.

Lemma 3.4. *Let $q > \bar{q}$. There exist $c_1, c_2 > 0$ such that for any $\eta_0 \in \Omega_-$ and $x \leq 0$ such that $\eta_0(x) = 0$,*

$$\forall t \geq 0,\ \mathbb{P}(\eta_t \notin \mathcal{H}(x - \underline{v}t, (x + \underline{v}t) \wedge 0, \ell) \leq c_1 t \exp(-c_2(t \wedge \ell)).$$

For $L > 0$, if $\sigma_0 \in \Omega_L$ and $x \in [0, L+1]$ is such that $\sigma_0(x) = 0$, then similarly,

$$\forall t \geq 0,\ \mathbb{P}(\sigma_t \notin \mathcal{H}((x - \underline{v}t) \vee 0, (x + \underline{v}t) \wedge (L+1), \ell) \leq c_1 t \exp(-c_2(t \wedge \ell)).$$

This last lemma can be used with x being the front of a configuration $\eta \in LO_-$ (or its boundary). This idea leads to a result we will now refer to as the "Zeros Lemma". We give here two versions of this result. The first one matches Lemma 4.4 in Blondel et al. (2019) while the second one is more specific to our needs later on.

Lemma 3.5 (Zeros Lemma)**.** *Let $q > \bar{q}$. Let $s, \ell, M, L > 0$ and $\eta \in LO_-$.*

- *If $L + M \leq 2\underline{v}s$, there exists $c > 0$ depending only on q such that:*
$$\mathbb{P}_\eta(\tilde{\eta}_s \notin \mathcal{H}(L, (L+M) \wedge (-X_s), \ell)) \leq (L+M)^2 \exp(-c(L \wedge \ell)).$$
- *If $L + M > 2\underline{v}s$ and $\tilde{\eta}_0 \in \mathcal{H}(0, (L+M) \wedge (-X_0), 2\underline{v}s)$, then there exists $c > 0$ such that:*
$$\mathbb{P}_\eta(\tilde{\eta}_s \notin \mathcal{H}(L, (L+M) \wedge -X_s, \ell)) \leq \frac{s^2}{L} \exp(-c(L \wedge \ell)) + Ms \exp(-c(s \wedge \ell)).$$

Proof: It is identical to the proof in Blondel et al. (2019), just taking into account the boundary. ☐

Lemma 3.6 (Zeros Lemma II)**.** *Let $q > \bar{q}$. Let $s, \ell > 1$ and $\eta \in LO_-$. Assume $\eta \in \mathcal{H}(X_0, y, \ell)$ for some $y \in [X_0, 0]$, where X_0 denotes the leftmost zero of η. Then, there exists $C > 0$ depending only on q such that if $2\underline{v}s \geq \ell$,*

$$P_\eta(\eta_s \notin \mathcal{H}(X_s, y, \ell)) \leq C(1 + |X_0|)\frac{s^2}{\ell} \exp(-c(s \wedge \ell)).$$

Remark 3.7. From now on and all throughout the proofs of this paper, the letters c, C, c', C' always denote generic positive constants that may differ from line to line. Consequently, any equation such as

$$\text{“ } \mathbb{P}(A_t) \leq Ce^{-ct} \text{ ”}$$

should be understood as

$$\text{“ } \exists C, c > 0,\ \mathbb{P}(A_t) \leq Ce^{-ct} \text{ ”}.$$

The dependency of these constants on the various parameters appearing in this paper should be explicit or clear from the context.

Proof: In the same way as the proof of Blondel et al. (2019, Lemma 4.4), we use Lemma 3.4 at intermediate times. Let us fix s and define:

$$\Delta = \frac{\ell}{2(\bar{v} - \underline{v})} \wedge s,$$
$$n = \left\lceil \frac{(s-\Delta)(\bar{v} - \underline{v})}{2\underline{v}\Delta} \right\rceil,$$
$$\Delta' = \frac{s-\Delta}{n} \text{ if } n > 0,$$
$$s_i = i\Delta' \text{ for } i \in [0, n].$$

For $i \in [0, n]$, by Lemma 3.4 and Markov property applied at time s_i, there exist $c, C > 0$ such that:

$$\mathbb{P}_\eta(\eta_s \notin \mathcal{H}(X_{s_i} - \underline{v}(s - s_i), 0 \wedge (X_{s_i} + \underline{v}(s - s_i)), \ell/2)) \leq C(s - s_i)e^{-c((s-s_i)\wedge(\ell/2))} \tag{3.1}$$

In words, each position of the fronts on the times s_i provides an interval at time s where it is likely to find enough empty sites. Moreover, we can control the front evolution during the intervals $[s_i, s_{i+1}]$ with Corollary 3.2 and Lemma 3.3 to find that:

$$\mathbb{P}_\eta(0 \leq X_{s_i} - X_{s_{i+1}} \leq \bar{v}\Delta') \geq 1 - e^{-c\Delta'}, \tag{3.2}$$
$$\mathbb{P}_\eta(0 \leq X_s - X_{s_n} \leq \bar{v}\Delta) \geq 1 - e^{-c\Delta}. \tag{3.3}$$

Our choice of Δ and Δ' ensures that under these events, the intervals at time s appearing in (3.1) overlap, that is to say:

$$\bigcup_{i=0}^{n} [X_{s_i} - \underline{v}(s - s_i), X_{s_i} + \underline{v}(s - s_i)] \supset [X_s + \ell/2, X_0 + \underline{v}s].$$

We now take into account the zeros of the initial configuration to make this interval go all the way to the boundary. Let $x_0 = X_0 < x_1 < \cdots < x_m \leq y$ be empty sites in η_0 between X_0 and y. We choose them so that for all i, $x_{i+1} - x_i \leq \ell$ and $y - x_m \leq \ell$, and their cardinality m is minimal. This way we can have $m \leq \frac{2|X_0|}{\ell}$.

By Lemma 3.4, we have for all i,

$$\mathbb{P}_\eta(\eta_s \notin \mathcal{H}(x_i - \underline{v}s, 0 \wedge (x_i + \underline{v}s), \ell/2)) \leq Cse^{-c(s\wedge(\ell/2))} \tag{3.4}$$

Now since $2\underline{v}s \geq \ell$, then $(x_{i+1} - \underline{v}s) - (x_i + \underline{v}s) \leq 0$ which guarantees that the intervals $[x_i - \underline{v}s, x_i + \underline{v}s]$ overlap.

We can now conclude from equations (3.1), (3.2), (3.3) and (3.4):

$$\begin{aligned}\mathbb{P}_\eta(\tilde{\eta}_s \notin \mathcal{H}(0, -X_s + y, \ell)) &\leq C(n+1)se^{-c(s\wedge(\ell/2))} + e^{-c\Delta} + ne^{-c\Delta'} + Cmse^{-c(s\wedge(\ell/2))} \\ &\leq C(1 + |X_0|)\frac{s^2}{\ell}e^{-c(s\wedge\ell)}.\end{aligned}$$

In the last line, we bounded $n \leq C\frac{s}{\ell}$ with C depending only on q. Also note that we artificially added a factor $s > 1$ to the last three terms of the first line so we have a nicer expression in the end. While non optimal, this result will be precise enough for our needs. □

4. Relaxation results

We give here several results of relaxation starting out of equilibrium. They build upon the results of Blondel et al. (2013). First, let us recall a proposition about the relaxation of the FA-1f process on a finite interval with zero boundary conditions.

Proposition 4.1 (Blondel et al. (2019, Corollary 3.3)). *Let $q > 1/2$, $L > 0$ and f a bounded function with support contained in $[1, L]$ such that $\mu(f) = 0$. If $L \le e^{t^\alpha}$ for some $\alpha < 1/2$, then there exists $c' = c'(\alpha, q) > 0$ such that, if $\sigma_0 \in \mathcal{H}(0, L+1, \sqrt{t})$,*

$$|\mathbb{E}_{\sigma_0}[f(\sigma_t)]| \le \frac{1}{c'} ||f||_\infty e^{-c'\sqrt{t}}$$

In words, this proposition states that if the process starts in $\mathcal{H}(0, L+1, \ell)$, then it is mixed after a time ℓ^2. Note that this holds for $q > 1/2$ which covers a larger regime than $q > \bar{q}$ because it does not rely on the coupling with the threshold contact process. Actually, we can combine this proposition with Lemma 3.4 to get a better relaxation speed for the supercritical regime and have a mixing after a time $\sim \frac{\ell}{2\underline{v}}$ instead of ℓ^2:

Proposition 4.2. *Let $q > \bar{q}$, $0 < \beta < 1/2$, $L > 0$, $\ell > 0$ and f a bounded function with support contained in $[1, L]$ such that $\mu(f) = 0$. If $L \le e^{\ell^\alpha}$ for some $\alpha < \beta$, then there exist $c, C > 0$ depending only on q and α such that, if $\sigma_0 \in \mathcal{H}(0, L+1, \ell)$ and $t = \frac{\ell}{2\underline{v}} + \ell^\beta$,*

$$|\mathbb{E}_{\sigma_0}[f(\sigma_t)]| \le C||f||_\infty L e^{-c\ell^{\beta/2}}.$$

Proof: Set $t_1 = \frac{\ell}{2\underline{v}}$. Let $0 = x_1 < \cdots < x_p = L+1$ a sequence of sites in $[0, L+1]$ such that:

- $\forall 1 \le i \le p,\ \sigma_0(x_i) = 0$,
- $\forall 1 \le i \le p-1,\ x_{i+1} - x_i \le \ell$,
- $p \le \lceil \frac{2L}{\ell} \rceil$.

Thanks to Lemma 3.4, we have

$$\begin{aligned}
\mathbb{P}\Big(\sigma_{t_1} \notin \mathcal{H}(0, L+1, \ell^{\beta/2})\Big) &\le \mathbb{P}\Bigg(\bigcup_{i=1}^{p} \{\sigma_{t_1} \notin \mathcal{H}((x_i - \underline{v}t_1) \vee 0, (x_i + \underline{v}t_1) \wedge (L+1), \ell^{\beta/2}/2)\}\Bigg) \\
&\le \sum_{i=1}^{p} \mathbb{P}\Big(\sigma_{t_1} \notin \mathcal{H}(x_i - \underline{v}t_1, x_i + \underline{v}t_1, \ell^{\beta/2}/2)\Big) \\
&\le Cpt_1 e^{-c(t_1 \wedge (\ell^{\beta/2}/2))} \\
&\le C' L e^{-c'\ell^{\beta/2}}.
\end{aligned}$$

Now we can use Proposition 4.1 (with $\sigma_{t_1} \in \mathcal{H}(0, L+1, \sqrt{\ell^\beta})$, and $L < e^{\ell^\beta}$ with $\beta < 1/2$) and Markov property:

$$\begin{aligned}
|\mathbb{E}_{\sigma_0}[f(\sigma_t)]| &\le \left|\mathbb{E}_{\sigma_0}[f(\sigma_t)\mathbb{1}_{\sigma_{t_1} \in \mathcal{H}(0, L+1, \ell^{\beta/2})}]\right| + ||f||_\infty \mathbb{P}(\sigma_{t_1} \notin \mathcal{H}(0, L, \ell^{\beta/2})) \\
&\le \mathbb{E}_{\sigma_0}[\mathbb{1}_{\sigma_{t_1} \in \mathcal{H}(0, L+1, \ell^{\beta/2})} |\mathbb{E}_{\sigma_{t_1}}[f(\sigma_{t-t_1})]|] + C'||f||_\infty L e^{-c'\ell^{\beta/2}} \\
&\le \frac{1}{c''}||f||_\infty e^{-c''\ell^{\beta/2}} + C'||f||_\infty L e^{-c'\ell^{\beta/2}}.
\end{aligned}$$

□

A notable result comes from the particular case $\ell = L+1$ and $\beta = 1/4$. Indeed, any configuration $\sigma_0 \in \Omega_L$ belongs to $\mathcal{H}(0, L+1, L+1)$.

Corollary 4.3. *There exist $c, C > 0$, such that for any $L > 0$, if $t = \frac{L+1}{2\underline{v}} + (L+1)^{1/4}$,*

$$\sup_{\sigma_0 \in LO_{[0,L]}} ||\mu_t^{\sigma_0} - \mu||_{TV} \le CLe^{-cL^{1/8}}.$$

Remark 4.4. This results proves that the mixing time is always bounded by $\frac{L}{2\underline{v}} + o(L)$. This gives a first scale for t_{mix} but the constant $\frac{1}{2\underline{v}}$ will turn out to be too large in general, as we will see in the final section.

5. Central limit theorem for FA-1f on $\mathbb{Z}_-$

We now aim to prove a central limit theorem for the front of the FA-1f process on $\mathbb{Z}_-$. Our result differs from the one proved in Blondel et al. (2019, Theorem 2.2) in two ways. First, it studies an FA-1f process on the half-line as opposed to $\mathbb{Z}$. This change forces us to take into account the boundary condition in the origin, tough we will see that the front gets far from the origin so this barely makes a difference in the proof. The second change is a *uniformity* result with respect to the initial configuration. This requires to study carefully the original proof and tweak a few key arguments. For a first read, one can take Theorem 5.1 for granted and skip to the next section in order to get to the main result.

Theorem 5.1. *Let $q > \bar{q}$. There exist $v > 0$ and $s \geq 0$ such that for all $\eta \in LO_-$, and all real numbers $a < b$,*

$$\sup_{\eta \in LO_-} \left| \mathbb{P}_\eta\Big(a \leq \frac{X_t - vt}{\sqrt{t}} \leq b\Big) - \mathbb{P}(a \leq N \leq b) \right| \underset{t\to\infty}{\longrightarrow} 0, \tag{5.1}$$

with $N \sim \mathcal{N}(0, s^2)$. The constants v and s are the same as in Blondel et al. (2019, Theorem 2.2).

Remark 5.2. In this theorem, the variance s^2 of the normal law can be zero. For this more favorable case, the law $\mathcal{N}(0,0)$ is simply δ_0, so that we have $\sup_{\eta \in LO_-} \left| \mathbb{P}_\eta\Big(-a \leq \frac{X_t - vt}{\sqrt{t}} \leq a\Big) - 1 \right| \underset{t\to\infty}{\longrightarrow} 0$ for all $a > 0$.

To prove this theorem, we first need to study the convergence of the law behind the front, which is the topic of the following theorem.

Theorem 5.3. *Let $q > \bar{q}$. The process seen from the front has a unique invariant measure ν. This measure is the same as in Blondel et al. (2019, Theorem 2.1). There exist $d^*, c > 0$ such that for all $\eta_0 \in LO_-$, for t large enough,*

$$||\tilde{\mu}_t^{\eta_0} - \nu||_{[0,d^*t]} \leq \exp\Big(-ce^{(\log t)^{1/4}}\Big),$$

where $\tilde{\mu}_t^{\eta_0}$ is the distribution of the configuration seen from the front at time t, starting from η_0 and $|| \cdot - \cdot ||_\Lambda$ is the total variation distance on Λ.

The proof of this result is almost identical to the original proof in Blondel et al. (2019). It relies on a rather technical coupling argument that only requires tiny adjusments due to the presence of a boundary in $\mathbb{Z}_-$. The adapted version of the proof can be found in Appendix A.

5.1. *A Central Limit Theorem.* From Theorem 5.3, we will now get a central limit theorem as in Blondel et al. (2019). Here, we state a general central limit theorem that we will apply to the increments of th front in the next paragraph. It resembles Blondel et al. (2019, Theorem A.1) but adds a uniformity result.

Theorem 5.4. *Let (σ_t) be a Markov process (in continuous time) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)$ the adapted filtration. Let $(X_i)_{i\geq 1}$ be real random variables satisfying the following hypotheses:*

- *(1)* *(a)* $\sup_{\sigma\in\Omega} \sup_{n\in\mathbb{N}} \mathbb{E}_\sigma[X_n^2] < \infty$;
  - *(b) for every $i \geq 1$, X_i is measurable w.r.t. $\mathcal{F}_i$;*
  - *(c) for every $k, n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ measurable such that $\sup_\sigma \mathbb{E}_\sigma[f(X_1, ..., X_n)] < \infty$, for all initial σ, we have the Markov property*

$$\mathbb{E}_\sigma[f(X_k, ..., X_{k+n-1}) \mid \mathcal{F}_{k-1}] = \mathbb{E}_{\sigma_{k-1}}[f(X_1, ..., X_{n-1})]; \tag{5.2}$$

(2) There exists a decreasing function Φ, constants $C, c^* \geq 1$ and $v \in \mathbb{R}$ and a measure ν such that
(a) $\lim_{n\to\infty} e^{(\log n)^2}\Phi(n) = 0$;
(b) for every $i \geq 1$, $\mathbb{E}_\nu[X_i] = v$;
(c) for every k, $f : \mathbb{R} \to \mathbb{R}$ s.t. $e^{-|x|}f^2(x) \in L^1(\mathbb{R})$, we have $\sup_\sigma \mathbb{E}_\sigma[f^2(X_1)] < \infty$ and

$$\sup_\sigma |\mathbb{E}_\sigma[f(X_k)] - \mathbb{E}_\nu[f(X_1)]| \leq C(f)\Phi(k); \tag{5.3}$$

(d) for every k, n and f such that $f : \mathbb{R} \to \mathbb{R}$ s.t. $e^{-|x|}f^2(x) \in L^1(\mathbb{R})$,

$$\sup_\sigma |\mathrm{Cov}_\sigma[f(X_k), f(X_n)] - \mathrm{Cov}_\nu[f(X_1), f(X_{n-k+1})]| \leq C(f)\Phi(k); \tag{5.4}$$

$$\sup_\sigma |\mathrm{Cov}_\sigma[f(X_k), f(X_n)]| \leq C \sup_\sigma \mathbb{E}_\sigma[f(X_1)^2]^{1/2}\ \Phi(n-k); \tag{5.5}$$

(e) for every k, n such that $k \geq c^* n$ and any bounded function $F : \mathbb{R}^n \to \mathbb{R}$,

$$\sup_\sigma |\mathbb{E}_\sigma[F(X_k, ..., X_{k+n-1})] - \mathbb{E}_\nu[F(X_1, ..., X_n)]| \leq C||F||_\infty \Phi(k). \tag{5.6}$$

Then there exists $s \geq 0$ such that

$$\frac{\sum_{i=1}^n X_i - vn}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, s^2).$$

Moreover, this convergence holds uniformly in the initial configuration in the following sense:

$$\forall a < b,\ \sup_\sigma |\mathbb{P}\Big(a \leq \frac{\sum_{i=1}^n X_i - vn}{\sqrt{n}} \leq b\Big) - \mathbb{P}(a \leq N \leq b)| \underset{n\to\infty}{\longrightarrow} 0,$$

where $N \sim \mathcal{N}(0, s^2)$.

Note that the hypothesis 2.(d) differs from the one in Blondel et al. (2019). It is the correct hypothesis and should be corrected in the original article. It was however verified in the application therein, see Blondel et al. (2019, Lemma 7.3).

Proof: We first focus on bounded random variables. Let $(X_i)_{i\geq 1}$, satisfying the hypotheses. Let us define like in Blondel et al. (2019) $Y_i := X_i - \mathbb{E}_\nu[X_1]$, $\ell_n = n^{1/3}$, $S_n = \sum_{i=1}^n Y_i$, $S_{j,n} = \sum_{i=1}^n \mathbb{1}_{|k-j|\leq \ell_n} Y_i$ and finally $\alpha_n = \sum_{i=1}^n \mathbb{E}_\sigma[Y_j S_{j,n}]$. and assume each Y_i is bounded. We follow the same line of arguments, replacing the "Bolthausen Lemma" Blondel et al. (2019, Lemma A.3) with an adapted version:

Lemma 5.5. *Let $(\nu_n^\sigma)_{n\geq 0}$ be a family of probability measures on $\mathbb{R}$ such that:*
(1) $\sup_\sigma \sup_n \int |x|^2 d\nu_n^\sigma(x) < \infty$
(2) $\forall R > 0$, $\lim_{n\to\infty} \sup_\sigma \sup_{|\lambda|\leq R} |\int (i\lambda - x)e^{i\lambda x} d\nu_n^\sigma(x)| = 0$

Then for all $\sigma \in \Omega$, $(\nu_n^\sigma)_{n\geq 0}$ converges to the standard normal law. Furthermore, for all continuous bounded function f:

$$\lim_{n\to\infty} \sup_\sigma |\nu_n^\sigma(f) - \mu(f)| = 0, \tag{5.7}$$

where $\mu = \mathcal{N}(0, 1)$.
Moreover, for all real numbers $a < b$,

$$\lim_{n\to\infty} \sup_\sigma |\nu_n^\sigma(\mathbb{1}_{[a,b]}) - \mu(\mathbb{1}_{[a,b]})| = 0 \tag{5.8}$$

Proof of Lemma 5.5: Suppose that there is no uniformity in (5.7). Then there exist $\varepsilon > 0$, an increasing sequence of integers (k_n) and a sequence (σ_n) such that:

$$|\nu_{k_n}^{\sigma_n}(f) - \mu(f)| > \varepsilon$$

Then the sequence of measures $(\nu_{k_n}^{\sigma_n})$ clearly satisfies the hypothesis of the original Lemma Blondel et al. (2019, Lemma A.3) so should converge to a standard normal law, which is a contradiction with the above equation. □

Looking at the proof of Theorem A.1 in Blondel et al. (2019), the distributions of $\Big(\frac{S_n}{\sqrt{\alpha_n}}\Big)_{n\geq 0}$ satisfy the stronger hypotheses (in particular item 2) of the above Lemma. Consequently, we have the conclusion of Theorem 5.4 for bounded variables.

We now give details on how to extend the bounded case to the general case. We no longer assume the $(Y_i)_{i\geq 0}$ to be bounded. Let $a \in \mathbb{R}$ be a real number and $\varepsilon > 0$. Let $N \geq 0$ be an integer that will be fixed later, and define the truncation operator $T^N(x) := \max\{\min(x, N), -N\}$, and the remainder $R^N(x) := x - T^N(x)$.

Then we can estimate:

$$\begin{aligned}\mathbb{P}_\sigma\Big(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\Big) &= \mathbb{P}_\sigma\Big(\frac{\sum_{i=1}^n (T^N(Y_i) - \mathbb{E}_\nu[T^N(Y_i)]) + (R^N(Y_i) - \mathbb{E}_\nu[R^N(Y_i)])}{\sqrt{n}} \geq a\Big) \\ &\leq \mathbb{P}_\sigma\Big(\frac{\sum_{i=1}^n (T^N(Y_i) - \mathbb{E}_\nu[T^N(Y_i)])}{\sqrt{n}} \geq a - \varepsilon\Big) + 2\mathbb{P}_\sigma\Big(\frac{|R^N(Y_i) - \mathbb{E}_\nu[R^N(Y_i)]|}{\sqrt{n}} \geq \varepsilon\Big). \end{aligned} \tag{5.9}$$

We first handle the last term:

$$\begin{aligned}\mathbb{P}_\sigma\Big(\frac{|R^N(Y_i) - \mathbb{E}_\nu[R^N(Y_i)]|}{\sqrt{n}} \geq \varepsilon\Big) &\leq \frac{1}{\varepsilon^2 n}\mathrm{Var}_\sigma\Big(\sum_{i=1}^n R^N(Y_i)\Big) \\ &\leq \frac{1}{\varepsilon^2 n}\Big(\sum_{i=1}^n \mathrm{Var}_\sigma(R^N(Y_i)) + \sum_{i\neq j}\mathrm{Cov}_\sigma(R^N(Y_i), R^N(Y_j))\Big).\end{aligned}$$

- For any i, note that we can easily bound $\mathbb{E}_\sigma[R^N(Y_i)^2]$:

$$\begin{aligned}\mathbb{E}_\sigma(R^N(Y_i)^2) &= \mathbb{E}_\sigma[(Y_i - N)^2 \mathbb{1}_{Y_1 \geq N}] + \mathbb{E}_\sigma[(Y_i + N)^2 \mathbb{1}_{Y_1 \leq -N}] \\ &\leq 4\sqrt{2}(\mathbb{E}_\sigma[Y_i^4] + N^4)^{1/2}\mathbb{P}_\sigma(|Y_i| \geq N)^{1/2} \\ &\leq 4\sqrt{2}(C + N^4)^{1/2}\mathbb{P}_\sigma(|Y_i| \geq N)^{1/2} \\ &\leq C'(C + N^4)^{1/2} e^{-N/4},\end{aligned}$$

  where C does not depend on σ or i. The variables Y_i have a bounded forth moment thanks to hypothesis 2.(c). The last bound comes from the exponential Chebychev inequality and hypothesis 2.(c).
  From this we can conclude that

$$\frac{1}{\varepsilon^2 n}\sum_{i=1}^n \mathrm{Var}_\sigma(R^N(Y_i)) \leq C\varepsilon^{-2}e^{-cN},$$

  with $c, C > 0$ independent of n, N, σ.

- From assumption 2.(d) (equation (5.5)), we get that:

$$\begin{aligned}\sum_{i\neq j}^{n} \mathrm{Cov}_\sigma(R^N(Y_i), R^N(Y_j)) &\leq \sum_{i=1}^{n}\sum_{j=1}^{\infty} \sup_\sigma \mathbb{E}_\sigma[R^N(Y_1)^2]\Phi(|j-i|) \\ &\leq Ce^{-cN}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\Phi(|j-i|) \\ &\leq C'ne^{-cN},\end{aligned}$$

where we used the bound on $\mathbb{E}_\sigma[R^N(Y_1)^2]$ previously found for the second inequality, and the fact that $\sum_j \Phi(j) < \infty$ for the last one.

Putting the last two inequalities into (5.9), we get:

$$\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\right) \leq \mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n (T^N(Y_i) - \mathbb{E}_\nu[T^N(Y_i)])}{\sqrt{n}} \geq a - \varepsilon\right) + C\varepsilon^{-2}e^{-cN}.$$

Let N be such that $C\varepsilon^{-2}e^{-cN} \leq \varepsilon$. We now use our theorem on the bounded variables $\big(T^N(Y_i) - \mathbb{E}_\nu[T_N(Y_i)]\big)$ to find:

$$\limsup_{n\to\infty}\sup_\sigma\left(\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\right) - \mathbb{P}(N \geq a-\varepsilon)\right) \leq \varepsilon,$$

which in turn shows that

$$\limsup_{n\to\infty}\sup_\sigma\left(\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\right) - \mathbb{P}(N \geq a)\right) \leq 0.$$

It now remains to prove the reverse inequality. $\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\right)$ can be bounded below by:

$$\begin{aligned}&\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n T^N(Y_i) - \mathbb{E}_\nu[T^N(Y_i)]}{\sqrt{n}} \geq a+\varepsilon \text{ and } \left|\frac{\sum_{i=1}^n R^N(Y_i) - \mathbb{E}_\nu[R^N(Y_i)]}{\sqrt{n}}\right| \leq \varepsilon\right) \\ &\geq \mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n T^N(Y_i) - \mathbb{E}_\nu[T^N(Y_i)]}{\sqrt{n}} \geq a+\varepsilon\right) - \mathbb{P}_\sigma\left(\left|\frac{\sum_{i=1}^n R^N(Y_i) - \mathbb{E}_\nu[R^N(Y_i)]}{\sqrt{n}}\right| \geq \varepsilon\right)\end{aligned}$$

From here, we bound the last term as we have done previously, and find this time:

$$\limsup_{n\to\infty}\sup_\sigma\left(\mathbb{P}_\sigma\left(\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \geq a\right) - \mathbb{P}(N \geq a)\right) \geq 0,$$

which concludes the proof. □

5.2. *Proof of Theorem 5.1.* The aim of the following four lemmas is to justify the various hypotheses of Theorem 5.4 on the increments $\xi_n := X_n - X_{n-1}$, where X_n denotes the front at time n of an FA-1f process on $\mathbb{Z}_-$ started at an arbitrary configuration $\eta \in LO_-$. They can be proved in the exact same way as in Blondel et al. (2019), with a minor change explained in the proof of Lemma 5.7. In the following, ν denotes the measure defined in Theorem 5.3.

Lemma 5.6. *For $f : \mathbb{Z} \to \mathbb{R}$ such that $e^{-|x|}f(x)^2 \in L^1$, we have*

$$\sup_{\eta\in LO_-} \mathbb{E}_\eta[f(\xi_1)^2] = c(f) < \infty. \tag{5.10}$$

Lemma 5.7. *There exists $\gamma > 0$ such that for $f : \mathbb{Z} \to R$ with $e^{-|x|}f(x)^2 \in L^1(\mathbb{R})$,*

$$\sup_{\eta \in LO_-} |\mathbb{E}_\eta[f(\xi_n)] - \mathbb{E}_\nu[f(\xi_1)]| \leq C(f)e^{-\gamma e^{(\log n)^{1/4}}}. \tag{5.11}$$

Proof: Here, we define as in Blondel et al. (2019), the configuration $\Phi_t(\eta)$ to be equal to η on $[X(\eta), X(\eta) + d^*t]$ and 1 elsewhere (except at the origin). Let X, (resp. $\tilde{X}$) the front of the configuration starting form η (resp. $\Phi_t(\eta)$), both being coupling via the standard coupling. In our case, it is possible that the interval $[X(\eta), X(\eta) + d^*t]$ contains the origin, which can mess up the original argument. To prevent this, let us note the following fact:
if $t \geq 0, \eta \in LO_-$, the event $X \neq \tilde{X}$ is a subset of $\mathcal{A}_t := F(0, d^*t, 1) \cup \{X_t \geq -\underline{v}t\}$.
For $\eta \in LO_-$, f an appropriate function, and $n \geq 0$, we get:

$$\begin{aligned}\mathbb{E}_\eta[f(\xi_1)] - \mathbb{E}_{\Phi_{n-1}(\eta)}[f(\xi_1)] &= \mathbb{E}[(f(X) - f(\tilde{X}))\mathbb{1}_{X \neq \tilde{X}}] \\ &\leq \sqrt{\mathbb{E}\left[(f(X) - f(\tilde{X}))^2\right]}\sqrt{\mathbb{P}(\mathcal{A}_{n-1})}.\end{aligned}$$

Now note that $\mathbb{P}(\mathcal{A}_{n-1}) \leq e^{-\frac{d^*(n-1)}{2}} + Ae^{-B(n-1)} = O(e^{-cn})$.
From there, we can conclude like in Blondel et al. (2019). □

Lemma 5.8. *There exists $\gamma > 0$ such that, for $f : \mathbb{N} \to \mathbb{R}, e^{-|x|}f(x)^2 \in L^1(\mathbb{R})$ and $j < n$ two positive integers,*

(1) $\sup_{\eta \in LO_-} |\mathrm{Cov}_\eta[f(\xi_j), f(\xi_n)]| \leq C(f)e^{-\gamma e^{(\log(n-j))^{1/4}}}$*, and the same holds for the covariance under ν;*
(2) for $j \geq \frac{\bar{v}}{d^*}(n - j)$,

$$\sup_{\eta \in LO_-} |\mathrm{Cov}_\eta[f(\xi_j), f(\xi_n)] - \mathrm{Cov}_\nu[f(\xi_1), f(\xi_{n-j+1})]| \leq C(f)e^{-\gamma e^{(\log j)^{1/4}}}.$$

Lemma 5.9. *For any $k, n \in \mathbb{N}$ such that $d^*(k-1) \geq \bar{v}n$ and any bounded function $F : \mathbb{R}^n \to \mathbb{R}$*

$$\sup_{\eta \in LO_-} |\mathbb{E}_\eta[F(\xi_k, ..., \xi_{k+n-1})] - \mathbb{E}_\nu[F(\xi_1, ..., \xi_n)]| = O\left(||F||_\infty e^{-\gamma e^{(\log k)^{1/4}}}\right).$$

From here, we can apply Theorem 5.4 and conclude exactly like in Blondel et al. (2019).

6. Cut-off for FA-1f

We are now ready to prove the main result. Let $\Lambda = [1, L]$, we fix $q > \bar{q}$ the parameter of the FA-1f process. Every constant introduced in this section implicitly depends on q.

We split the proof into two cases. First, we tackle initial configurations with macroscopic subintervals of occupied sites. In this section, we shall call such an interval a *particle cluster*. To tackle these, we need to study closely the fronts of each particle cluster that are going inward. We will see that after the time it takes for them to meet, the configuration created enough empty sites to relax to equilibrium quickly. Next, we study the initial configurations with no such particle cluster, that is configurations in $\mathcal{H}(0, L+1, \ell)$ for a certain threshold $\ell(L)$. We handle the latter category with softer arguments using the coupling of FA-1f and the contact process.

6.1. *Configurations with macroscopic particle clusters.* Define, for any $L > 0$ and $0 < \delta < 1$,

$$\Omega^\delta = \{\sigma \in \Omega_\Lambda \mid \sigma \in \mathcal{H}(0, L+1, \delta L)\}.$$

For $\sigma \in \Omega_\Lambda$, recall that $B(\sigma) := \max\{h \geq 0 \mid \exists x \in [0, L-h], \restriction \sigma[x+1, x+h] \equiv 1\}$ is the size of the largest particle cluster in σ. The following result holds.

Proposition 6.1. *Let $\delta > 0$, $\sigma_0 \in (\Omega^\delta)^c$ and $\varepsilon > 0$. There exists $a = a(\varepsilon) > 0$ such that, if $t = \dfrac{B(\sigma_0)}{2v} + \dfrac{7av}{\underline{v}\delta}\sqrt{L}$, then:*

$$||\mu_t^{\sigma_0} - \mu||_{TV} \leq \frac{v^2}{\underline{v}^2\delta^2}\varepsilon + \phi(L),$$

with $\phi(L)$ independent of σ_0 and $\lim_{L\to\infty} \phi(L) = 0$.

The proof of this proposition is divided into four lemmas. To make things easier, let us first introduce the sketch of the proof in words. In the case we are studying now, the initial configuration has at least one macroscopic particle cluster of size $\geq \delta L$. If we watch the evolution of the process in such an interval during a short period of time, we see that the two endpoints of this particle cluster behave just like fronts of FA-1f processes in half-lines (one left-oriented, the other one right-oriented). As a consequence, thanks to the Central Limit Theorem 5.1 we can find an explicit time after which the fronts are much closer to each other, at a certain threshold distance d of order $\sqrt{L}$. At this point, the space between the original positions of the fronts and the current one should contain a lot of zeros thanks to the Zeros Lemma 3.5. We repeat this argument for every initial particle cluster, and thanks to appropriate choices of constants and coupling arguments handling the space in between the clusters, we end up with a configuration in $\mathcal{H}(\Lambda, d)$ with high probability. From there, it will only remain to apply the relaxation result from Section 4 to conclude.

Before diving into the proof, we need to set a general framework. The following lemmas aim to explain in detail how a configuration with some particle clusters can turn the smallest of them into an interval with a lot of zeros.

Let us fix an arbitrary set of disjoint intervals $\Lambda_1, \dots, \Lambda_r \subset \Lambda$. We define their endpoints: $\Lambda_k = [a_k, b_k]$. Fix $\varepsilon > 0$, and let $d = 6a\sqrt{L}$, with a a constant depending on ε that will be determined later.

We now define a class of configurations that we consider in our proof. We say that a configuration $\sigma \in \Omega_\Lambda$ is in the class $\mathcal{C}$ if for all $1 \leq k \leq r$, there exist $X_k < Y_k \in \Lambda_k$ such that

(1) $\sigma(X^k) = \sigma(Y^k) = 0$,
(2) $\restriction \sigma(X^k, Y^k) \equiv 1$,
(3) $\sigma \in \mathcal{H}(a_k, X^k, d) \cap \mathcal{H}(Y^k, b_k, d)$.

Note that X^k and Y^k are uniquely determined if $\sigma \in \mathcal{C} \setminus \mathcal{H}(\Lambda_k, d)$. If $\sigma \in \mathcal{H}(\Lambda_k, d)$, there can be multiple choices of X^k and Y^k, but this case will be excluded in the following. Let us introduce a few functions on $\mathcal{C}$ (see Figure 6.2). Fix $\sigma \in \mathcal{C}$.

- $\kappa(\sigma) = \{k \mid \sigma \notin \mathcal{H}(\Lambda_k, d)\}$ is the set of indices such that $Y^k - X^k > d$. We write κ if the configuration σ is clear from the context.
- $p(\sigma) = \#\kappa$ is the cardinality of κ,
- $\ell(\sigma) = \min_{k\in\kappa}(Y^k - X^k)$,
- $t(\sigma) = \frac{\ell(\sigma)}{2v} - \frac{2a}{v}\sqrt{\ell(\sigma)}$. We set $t(\sigma) = 0$ if $p(\sigma) = 0$.

Having set that, we define the random variable $F(\sigma) = \sigma_{t(\sigma)}$. That is, $F(\sigma)$ is the result of the FA-1f process with initial configuration $\sigma \in \mathcal{C}$ after a time $t(\sigma)$. The aim of the random function F is to turn the smallest particle cluster $[X^k, Y^k]$ of size $\geq d$ into an interval in $\mathcal{H}(\cdot, d)$. Note that this construction depends on the family of intervals $\Lambda_1, ..., \Lambda_r$ that is for now arbitrary.

Let $\sigma \in \mathcal{C}$ and $(\sigma_t)_{t\geq 0}$ be a FA-1f process starting from σ. Define X_t^k (resp. Y_t^k) as the position of the rightmost (resp. leftmost) zero in $[0, M^k]$ (resp. $[M^k, L+1]$) in σ_t, with $M^k := \frac{1}{2}(X^k + Y^k)$. We have $X_0^k = X^k$ and $Y_0^k = Y^k$. As long as X_t^k and Y_t^k do not meet, we think of X_t^k as the "front" of a configuration on the (right oriented) half-line. Let $\mathcal{R}_t = \{\forall k \in \kappa,\ \forall s < t,\ X_s^k < M^k - 1 \text{ and } Y_s^k > M^k + 1\}$ the event that none of the "fronts" reaches the midpoint of the interval. In particular,

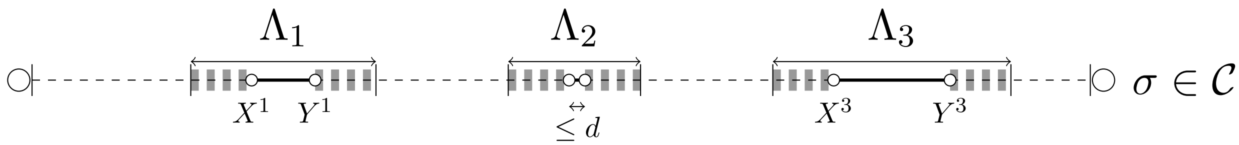

FIGURE 6.2. A configuration in the set $\mathcal{C}$. Dashed zones represent zones in $\mathcal{H}(.,d)$. Here, we have $\kappa(\sigma) = \{1, 3\}$, $p(\sigma) = 2$ and $\ell(\sigma) = Y^1 - X^1$.

under $\mathcal{R}_t$, the front have not met. The following Lemma localizes these fronts at time $t(\sigma)$ using Theorem 5.1.

Lemma 6.2. *There exists $a > 0$ depending only on ε and q such that, with the notations previously introduced, and $d = 6a\sqrt{L}$, the following holds. For all $\sigma \in \mathcal{C}$ such that $p(\sigma) \geq 1$,*

$$\mathbb{P}\Big(\forall k \in \kappa,\ X^k_{t(\sigma)} \in [X^k + vt(\sigma) - a\sqrt{\ell(\sigma)}, X^k + vt(\sigma) + a\sqrt{\ell(\sigma)}],$$
$$Y^k_{t(\sigma)} \in [Y^k + vt(\sigma) - a\sqrt{\ell(\sigma)}, Y^k + vt(\sigma) + a\sqrt{\ell(\sigma)}], \mathcal{R}_{t(\sigma)}\Big) \geq 1 - r\varepsilon + r\phi(L) \quad (6.1)$$

with $\phi(L)$ independent of σ such that $\lim_{L\to\infty} \phi(L) = 0$.
In particular, for k_0 such that $Y^{k_0} - X^{k_0} = \ell(\sigma)$, the event above implies that:

$$0 \leq Y^{k_0}_{t(\sigma)} - X^{k_0}_{t(\sigma)} \leq d$$

From now on, we call

$$I_k = [X^k + vt(\sigma) - a\sqrt{\ell(\sigma)}, X^k + vt(\sigma) + a\sqrt{\ell(\sigma)}]$$

and

$$J_k = [Y^k + vt(\sigma) - a\sqrt{\ell(\sigma)}, Y^k + vt(\sigma) + a\sqrt{\ell(\sigma)}].$$

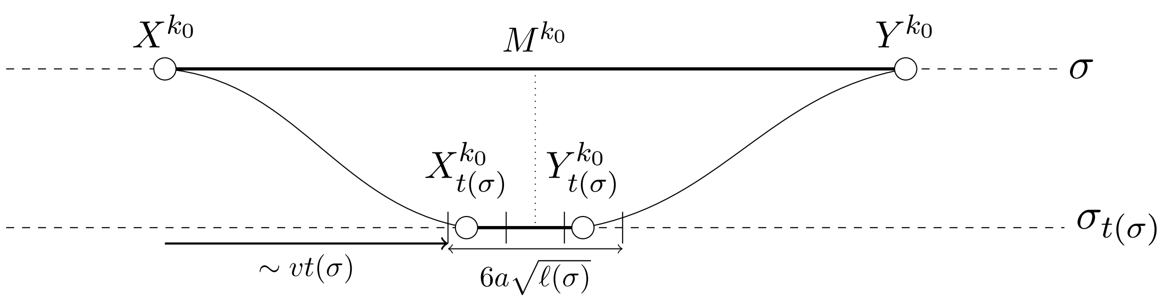

FIGURE 6.3. The expected behaviour of the smallest cluster of σ during time $t(\sigma)$.

Proof: Define $\mathcal{R}^k_t = \{\forall s \leq t,\ X^k_s < M^k - 1 \text{ and } Y^k_s > M^k + 1\}$ such that $\mathcal{R}_t = \bigcap_{1\leq k\leq p(\sigma)} \mathcal{R}^k_t$. The key remark for what follows is that if the event $\mathcal{R}^k_t$ occurs, the fronts X^k_s, Y^k_s behave like independent fronts of FA-1f processes on $\mathbb{Z}_-$ for all $s \leq t$. To justify that, let us define auxiliary processes $(\hat{\sigma}^k_t)_{t\geq 0}$ and $(\check{\sigma}^k_t)_{t\geq 0}$ on $\mathbb{Z}_-$ as follows.

- $\forall x \leq 0,\ \hat{\sigma}^k_0(x) = \begin{cases} \sigma(-x) & \text{if } -x \leq X^k, \\ 1 & \text{if } -x > X^k. \end{cases}$
- $(\hat{\sigma}^k_t)_{t\geq 0}$ is the FA-1f process on $\mathbb{Z}_-$ starting from $\hat{\sigma}^k_0$ constructed with the standard coupling with respect to $(\sigma_t(\text{-}.))_{t\geq 0}$. We denote by $\hat{X}^k_t$ its front at time t.

- $\forall x \leq 0,\ \check{\sigma}_0^k(x) = \begin{cases} \sigma(x+L+1) & \text{if } x+L+1 \geq Y^k, \\ 1 & \text{if } x+L+1 < Y^k. \end{cases}$
- $(\check{\sigma}_t^k)_{t\geq 0}$ is the FA-1f process on $\mathbb{Z}_-$ starting from $\check{\sigma}_0^k$ constructed with the standard coupling with respect to $(\theta_{L+1}\sigma_t)_{t\geq 0}$. We denote by $\check{Y}_t^k$ its front at time t.

Let $\hat{\mathcal{R}}_t^k = \{\forall s \leq t, \hat{X}_s^k > -M^k + 1\}$, and $\check{\mathcal{R}}_t^k = \{\forall s \leq t, \check{Y}_s > M^k - L\}$. Then, for all intervals $B, B' \subset \Lambda$, we have:

$$\mathbb{P}_\sigma\left(X_t^k \in B, Y_t^k \in B' \text{ and } \mathcal{R}_t^k\right) = \mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -B \text{ and } \hat{\mathcal{R}}_t^k\right)\mathbb{P}_{\check{\sigma}_0^k}\left(\check{Y}_t^k \in \theta_{-L-1}(B') \text{ and } \check{\mathcal{R}}_t^k\right). \quad (6.2)$$

We apply this with the intervals I_k and J_k. The two factors in the right hand side are similar so we only study $\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k \text{ and } \hat{\mathcal{R}}_t^k\right)$. Note that

$$\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k \text{ and } \hat{\mathcal{R}}_t^k\right) = \mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k\right) - \mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k \text{ and } (\hat{\mathcal{R}}_t^k)^c\right)$$

Thanks to Theorem 5.1, we have, for $N \sim \mathcal{N}(0, s^2)$,

$$\begin{aligned}\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k\right) &= \mathbb{P}_{\hat{\sigma}_0^k}\left(\frac{\hat{X}_t^k - \hat{X}_0^k - vt(\sigma)}{\sqrt{\ell(\sigma)}/(2v)} \in [-2va, 2va]\right) \\ &\underset{L\to\infty}{\longrightarrow} \mathbb{P}\left(N \in [-2va, 2va]\right) \end{aligned} \quad (6.3)$$

since $\ell(\sigma) \geq d \to \infty$ as $L \to \infty$. This convergence is furthermore uniform in σ. We choose a such that $\mathbb{P}\left(N \in [-2va, 2va]\right) \geq \sqrt{1-\varepsilon}$. Note that this choice only depends on ε and q (because the variance s^2 depends on q). This proves that

$$\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k\right) = \sqrt{1-\varepsilon} + \phi_1(L) \quad (6.4)$$

with $\phi_1(L)$ independent of σ and $\lim_{L\to\infty} \phi_1(L) = 0$. We handle the term in $\check{Y}_t^k$ the same way and find the same kind of equation. Note here that if $s^2 = 0$, our estimate is even better as we would get $\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{X}_t^k \in -I_k\right) = 1 - \phi_1(L)$.

Next, we show that if the fronts end up in the expected intervals at time $t(\sigma)$, then $\hat{\mathcal{R}}_{t(\sigma)}^k$ is very likely. We denote $\hat{M}^k = -M^k$ and we divide the time interval $[0, t(\sigma)]$ into intermediate times $0 < s_1 < ... < s_n = t(\sigma)$ with $n = \left\lceil \frac{2\bar{v}t(\sigma)}{a\sqrt{\ell(\sigma)} - 2} \right\rceil$ such that $\Delta := s_{i+1} - s_i \leq \frac{a}{2\bar{v}}\sqrt{\ell(\sigma)} - \frac{1}{\bar{v}}$. Notice that:

$$\begin{aligned}&\{\exists s < t(\sigma), \hat{X}_s^k \leq \hat{M}^k + 1\} \subset \\ &\qquad \{\exists i, \hat{X}_{s_i}^k \leq \hat{M}^k + \frac{a}{2}\sqrt{\ell(\sigma)}\} \cup \{\exists s < t(\sigma), \hat{X}_s^k \leq \hat{M}^k + 1 \text{ and } \forall i, \hat{X}_{s_i}^k > \hat{M}^k + \frac{a}{2}\sqrt{\ell(\sigma)}\}.\end{aligned}$$

First, with Corollary 3.2 and with our choice of Δ, we have that the second event is unlikely because of finite speed of propagation:

$$\begin{aligned}&\mathbb{P}_\sigma\left(\exists s < t(\sigma), \hat{X}_s^k \leq \hat{M}^k + 1 \ \text{ and } \forall i, \hat{X}_{s_i}^k > \hat{M}^k + \frac{a}{2}\sqrt{\ell(\sigma)}\right) \\ &\qquad \leq \sum_{i=0}^{n-1} \mathbb{P}_\sigma\left(\exists s \in [s_i, s_{i+1}], |\hat{X}_s^k - \hat{X}_{s_{i+1}}^k| \geq \frac{a}{2}\sqrt{\ell(\sigma)} - 1\right) \\ &\qquad \leq \sum_{i=0}^{n-1} \mathbb{P}_\sigma\left(\exists s \in [s_i, s_{i+1}], |\hat{X}_s^k - \hat{X}_{s_{i+1}}^k| \geq \bar{v}(s_{i+1} - s_i)\right) \\ &\qquad \leq Ce^{-c\sqrt{\ell(\sigma)}} \leq Ce^{-c\sqrt{d}} \quad \text{for some } C, c > 0.\end{aligned}$$

Then, by the Markov property at time s_i, the probability that $\hat{X}^k_{s_i} \leq \hat{M}^k + \frac{a}{2}\sqrt{\ell(\sigma)}$ while $\hat{X}^k_{t(\sigma)}$ is to the right of that is bounded by the probability that the front is going backward for a time at least Δ. This gives, thanks to Lemma 3.3:

$$\mathbb{P}_\sigma\left(\hat{X}^k_{t(\sigma)} \in -I_k \text{ and } \exists i, \hat{X}^k_{s_i} \leq \hat{M}^k + \frac{a}{2}\sqrt{\ell(\sigma)}\right) \leq \sum_{i=0}^{n-1} \mathbb{P}_\sigma(\hat{X}^k_{t(\sigma)} > \hat{X}^k_{s_i}) \leq nAe^{-B\Delta} \leq Ae^{-B'\sqrt{d}}.$$

Combining the previous inequalities shows that:

$$\mathbb{P}_\sigma\left(\hat{X}^k_{t(\sigma)} \in -I_k \text{ and } (\hat{\mathcal{R}}^k_t)^c\right) \leq Ce^{-c\sqrt{d}}, \tag{6.5}$$

with $C, c > 0$ independent of σ, L, ε.

Combining Equations (6.2), (6.4) and (6.5), since $d = 6a\sqrt{L}$, we get

$$\mathbb{P}_\sigma\left(X^k_t \in I_k,\ Y^k_t \in J_k \text{ and } \mathcal{R}^k_t\right) \geq 1 - \varepsilon + \phi(L),$$

with $\phi(L)$ going to 0 as $L \to \infty$, uniformly in σ. By a union bound over all boxes Λ_k for $k \in \kappa$, we get the expected result since the number of boxes is bounded by r. □

We have now localised the fronts in $F(\sigma)$, so we can use Lemma 3.5 to show that behind those fronts, with high probability, there are a lot of zeros (see Figure 6.4).

Lemma 6.3. *Let $\sigma \in \mathcal{C}$ such that $p(\sigma) \geq 1$. With the previous notations, for all $k \in \kappa$,*

$$\mathbb{P}_\sigma\Big(X^k_{t(\sigma)} \in I_k,\ Y^k_{t(\sigma)} \in J_k,\ \mathcal{R}_{t(\sigma)} \text{ and } \sigma_{t(\sigma)} \notin \left(\mathcal{H}(X^k, X^k_{t(\sigma)}, d) \cap \mathcal{H}(Y^k_{t(\sigma)}, Y^k, d)\right)\Big) \leq C\exp(-c\sqrt{L})$$

for some $C, c > 0$ independent of every other parameter.

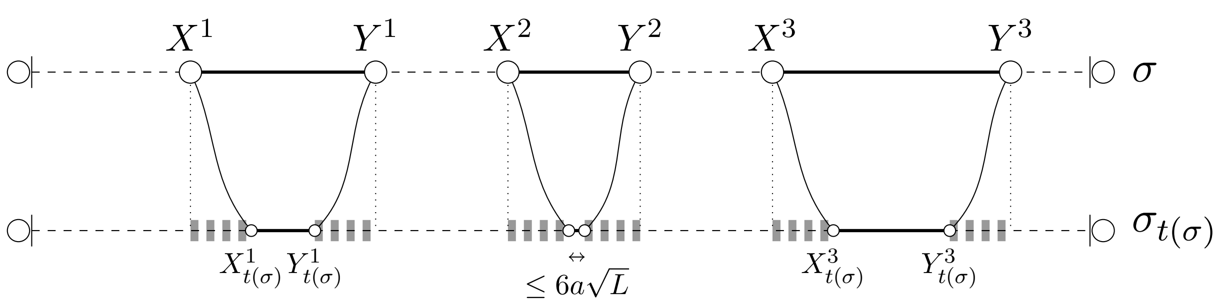

FIGURE 6.4. The expected behaviour of the fronts during time $t(\sigma)$. Dashed zones are in $\mathcal{H}(., d)$. We see that at time $t(\sigma)$, the configuration lost at least one particle cluster.

Proof: Given the event $\mathcal{R}_{t(\sigma)}$, we can use the previous coupling argument to show that:

$$\mathbb{P}_\sigma\left(X^k_{t(\sigma)} \in I_k,\ Y^k_{t(\sigma)} \in J_k,\ \mathcal{R}_{t(\sigma)} \text{ and } \sigma_{t(\sigma)} \notin \left(\mathcal{H}(X^k, X^k_{t(\sigma)}, d) \cap \mathcal{H}(Y^k_{t(\sigma)}, Y^k, d)\right)\right) = \mathbb{P}_{\hat{\sigma}^k_0}\left(\hat{X}^k_t \in -I_k,\ \hat{\sigma}^k_{t(\sigma)} \notin \mathcal{H}(\hat{X}^k_t, \hat{X}^k_0, d), \hat{\mathcal{R}}_{t(\sigma)}\right) \mathbb{P}_{\check{\sigma}^k_0}\left(\check{Y}^k_t \in \theta_{-L}(J_k),\ \check{\sigma}^k_{t(\sigma)} \notin \mathcal{H}(\check{Y}^k_t, \check{Y}^k_0, d), \check{\mathcal{R}}_{t(\sigma)}\right). \tag{6.6}$$

By Lemma 3.6 (with $y = \hat{X}_0^k$), we have that:

$$\mathbb{P}_{\hat{\sigma}_0^k}\left(\hat{\sigma}_{t(\sigma)}^k \notin \mathcal{H}(\hat{X}_t^k, \hat{X}_0^k, d)\right) \leq CL^{5/2}\exp(-c\sqrt{L}).$$

We can get rid of the $L^{5/2}$ factor by taking c smaller. The same inequality holds for $\check{\sigma}^k$ and therefore we get the claimed result. □

Now, we show that if an interval already contains enough zeros, it most likely will also at time $t(\sigma)$. This will ensure that once a cluster has disappeared and has left a lot of empty sites, the region will keep as many empty sites during any other iteration of F. It is actually a very general result that only relies on Lemma 3.4.

Lemma 6.4. *Let $\sigma \in \Omega_\Lambda$, $x, y \in \Lambda$, $d > 0$ such that $\sigma \in \mathcal{H}(x, y, d)$. Then:*

$$\mathbb{P}_\sigma(F(\sigma) \notin \mathcal{H}(x, y, d)) \leq CL\exp(-c\sqrt{L}), \tag{6.7}$$

with c, C independent of d, x, y, σ.

Proof: We simply use the comparison with a supercritical contact process (Lemma 3.4) for every initial zero in $\upharpoonright \sigma(x, y)$ with $\ell = d/2$. □

So far, we have proved that with high probability after a time $t(\sigma)$, every front moves by a distance $\sim \dfrac{\ell(\sigma)}{2v}$ (Lemma 6.2), while creating zeros behind it (Lemma 6.3). This results in one box k_0 initially "blocked" ($Y^{k_0} - X^{k_0} > d$) to become filled with zeros. Meanwhile, every box Λ_k that satisfies $\sigma \in \mathcal{H}(\Lambda_k, d)$ stays in $\mathcal{H}(\Lambda_k, d)$ after a time $t(\sigma)$ (Lemma 6.4). As illustrated in Figure 6.5, all of this combines into a nice result about $F(\sigma)$.

Lemma 6.5. *For any $\sigma \in \mathcal{C}$ such that $p(\sigma) \geq 1$,*

$$\mathbb{P}\left(F(\sigma) \in \mathcal{C} \text{ and } p(F(\sigma)) < p(\sigma)\right) \geq 1 - r\varepsilon - r\phi_2(L)$$

with $\phi_2(L)$ independent of σ (but depending on r) and $\lim_{L\to\infty} \phi_2(L) = 0$.

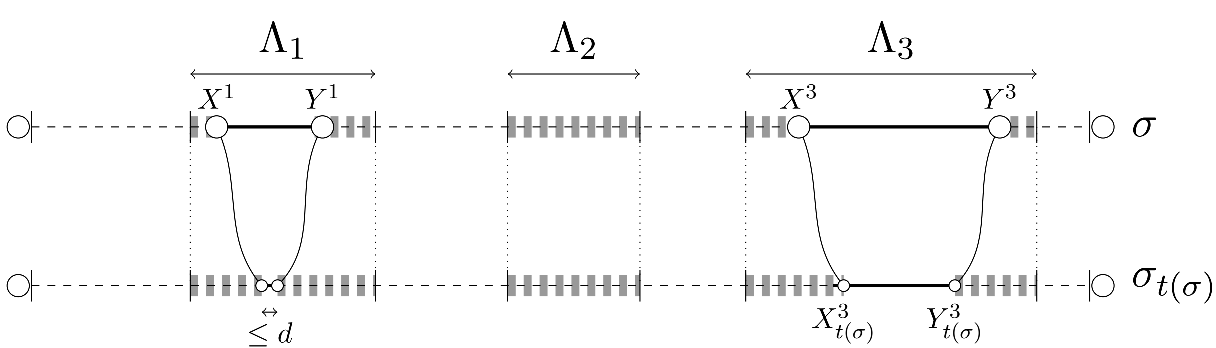

FIGURE 6.5. The expected behaviour of the whole configuration during time $t(\sigma)$. Here, the interval Λ_2 kept its zeros and the particle cluster in Λ_1 shrunk to a size $\leq d$.

Before getting to the proof of Proposition 6.1, let us make a final remark. On the event appearing in Lemma 6.2, we can bound the size of every particle cluster in $F(\sigma)$:

$$\forall k \in \kappa,\ Y_{t(\sigma)}^k - X_{t(\sigma)}^k \leq Y^k - X^k - \ell(\sigma) + 2a\sqrt{\ell} \tag{6.8}$$

Note that it is possible that in one iteration of F, two or more particle clusters go from a size $> d$ to a size $\leq d$.

Let us now prove Proposition 6.1.

Proof of Proposition 6.1: Fix $\delta > 0$, $\sigma_0 \in (\Omega^\delta)^c$ and $\varepsilon > 0$. We also define a as in Lemma 6.2 and $d = 6a\sqrt{L}$. Throughout the proof, $\phi(L)$ will denote a function that may depend on δ but not on σ such that $\lim_{L\to\infty} \phi(L) = 0$. Let $\Lambda_1, ...\Lambda_r$ be the particle clusters of σ_0 of size $\geq \underline{v}\frac{1}{v}B(\sigma_0)$ (with their neighboring sites that are empty in σ). More precisely, the intervals $\Lambda_k = [a_k, b_k]$ are such that $b_k - a_k \geq \frac{\underline{v}}{v}B(\sigma_0)$, $\sigma_0(a_k) = \sigma_0(b_k) = 0$ and $\restriction \sigma_0(a_k, b_k) \equiv 1$. Defining the Λ_k defines the class $\mathcal{C}$ and we have of course $\sigma_0 \in \mathcal{C}$. Note already that even though r now depends on σ, it can simply be bounded by $\frac{v}{\underline{v}\delta}$, which only depends on δ. We aim to prove that after a time $\tau_1 = \frac{B(\sigma_0)}{2v} + 3ar\sqrt{\ell(\sigma_0)}$, the configuration is in $\mathcal{H}(\Lambda, d)$. To do that, we first focus on the particle clusters.
We define now the iterations of F starting from σ_0. For all $1 \leq k \leq r$, $\sigma^{(i)} := F(\sigma^{(i-1)})$ with $\sigma^{(0)} := \sigma_0$. If we ever have $\sigma^{(i)} \notin \mathcal{C}$, then set for example $F(\sigma^{(i+1)}) = \sigma^{(i)}$ and $t(\sigma^{(i)}) = 0$. We will not encounter this case later on.
By Lemma 6.5, we have by induction that

$$\mathbb{P}\left(p(\sigma^{(r)}) = 0\right) \geq 1 - r^2\varepsilon + \phi(L).$$

Note that $p(\sigma^{(r)}) = 0$ simply means that $\sigma^{(r)} \in \bigcap_{k=1}^r \mathcal{H}(\Lambda_k, d)$.
Let $T = \sum_{i=0}^{r-1} t(\sigma^{(i)})$, that is the time at which we reach the configuration $\sigma^{(r)}$. Our goal now is to bound T from above. To do that, we use Equation (6.8) and by induction we get

$$\mathbb{P}\left(T \leq \frac{B(\sigma_0)}{2v} + 2ar\sqrt{\ell(\sigma_0)}\right) \geq 1 - r^2\varepsilon + \phi(L).$$

Now using the exact same argument as in Lemma 6.4, as long as $T \leq \frac{B(\sigma_0)}{2v} + 2ar\sqrt{\ell(\sigma_0)}$ and $\sigma_T \in \bigcap_{k=1}^r \mathcal{H}(\Lambda_k, d)$, then with high probability these zeros will remain at time $\tau_1 = \frac{B(\sigma_0)}{2v} + 3ar\sqrt{\ell(\sigma_0)}$. This leads to the result we were looking for, that at time τ_1, with high probability, every initial particle cluster contains a lot of zeros:

$$\mathbb{P}\left(\sigma_{\tau_1} \in \bigcap_{k=1}^r \mathcal{H}(\Lambda_k, d)\right) \geq 1 - r^2\varepsilon + \phi(L). \tag{6.9}$$

We can now handle the rest of Λ. Initially, we have zeros in $\Lambda \setminus \bigcup_{k=1}^r \Lambda_k$ that are spaced at most $\frac{\underline{v}}{v}B(\sigma_0)$ apart. This threshold was precisely chosen so that we can again use the supercritical contact process to ensure that at time $t_0 = \frac{B(\sigma_0)}{2v}$, we have:

$$\mathbb{P}_{\sigma_0}\left(\sigma_{t_0} \notin \mathcal{H}(\Lambda \setminus \bigcup_{k=1}^r \Lambda_k, d)\right) \leq Crt_0 e^{-c\sqrt{L}}.$$

Again, since $t_0 \leq \tau_1$, this property persists at time t_1. Putting this together with equation (6.9) finally gives:

$$\liminf_{L\to\infty} \mathbb{P}\left(\sigma_{\tau_1} \in \mathcal{H}(\Lambda, d)\right) \geq 1 - r^2\varepsilon. \tag{6.10}$$

Here, we used the fact that r is bounded (with L). More precisely, we have $r \leq \frac{vL}{\underline{v}B(\sigma_0)} \leq \frac{v}{\underline{v}\delta}$.

From there, it only remains to apply Proposition 4.2 with $\ell = d$ and $\beta = 1/4$ to find that for any local function f such that $\mu(f) = 0$, at time $t := \tau_1 + \frac{6a\sqrt{L}}{2\underline{v}} + (6a\sqrt{L})^{1/4}$,

$$\limsup_{L\to\infty} |\mathbb{E}[f(\sigma_t)]| \leq r^2\varepsilon.$$

Finally, we can bound this time replacing L with $B(\sigma_0)/\delta$ to get a consistent expression:

$$t \leq \frac{B(\sigma_0)}{2v} + \frac{\alpha\sqrt{B(\sigma_0)}}{\sqrt{\delta}},$$

with α depending only on ε and q. The fact that all the generic constants c, C used in the proof do not depend on σ_0 and that the CLT used is uniform in the initial configuration guarantees that the bound above is also uniform in σ_0. □

6.2. *Conclusion.* We can now prove Theorem 2.1.

Proof of Theorem 2.1: First, we proved Equation (2.2) with Proposition 6.1 in the previous subsection. Next, let $\delta > 0$, $\sigma \in \Omega_L^\delta$ and

$$t_1'(\sigma) = \left(\frac{B(\sigma)}{2\underline{v}} + (B(\sigma))^{1/4}\right) \vee \left(\frac{(\log L)^9}{2\underline{v}} + (\log L)^{9/4}\right).$$

If $B(\sigma) \leq (\log L)^9$, then we use the fact than $\sigma \in \mathcal{H}(\Lambda, (\log L)^9)$ to use Proposition 4.2 with $\ell = (\log L)^9$ and $\beta = 1/4$. This way, we get that for any local function f with support in Λ:

$$|\mathbb{E}_\sigma[f(\sigma_{\frac{(\log L)^9}{2\underline{v}}+(\log L)^{9/4}})]| \leq C||f||_\infty L e^{-c(\log L)^{9/8}},$$

which goes to zero as $L \to \infty$ uniformly in σ.
In this case, $t_1'(\sigma) = \left(\frac{(\log L)^9}{2\underline{v}} + (\log L)^{9/4}\right) \leq t_1(\sigma)$ so we get our result.
If $B(\sigma) > (\log L)^9$, then by using Proposition 4.2 with $\ell = B(\sigma)$ and $\beta = 1/4$, we have:

$$|\mathbb{E}_\sigma[f(\sigma_{\frac{B(\sigma)}{2\underline{v}}+B(\sigma)^{1/4}})]| \leq C||f||_\infty L e^{-cB(\sigma)^{1/8}} \leq C||f||_\infty L e^{-c(\log L)^{9/8}},$$

which again leads to our result.

Finally, let $\Phi : \mathbb{N} \to \mathbb{N}$ such that $\Phi(L) \underset{L\to+\infty}{\longrightarrow} +\infty$. Let $\sigma \in \mathcal{H}(0, L+1, \Phi(L))^c$. Let Λ_1 be the largest particle cluster in σ:

$$\Lambda_1 = [X+1, Y-1] \text{ with } \sigma(X) = \sigma(Y) = 0, \restriction \sigma[X+1, Y-1] \equiv 1 \text{ and } Y - X \geq \Phi(L).$$

Let $M = (X+Y)/2$. By Lemma 6.2, we see that with $t = \frac{B(\sigma)}{2v} - \frac{2a}{v}\sqrt{B(\sigma)}$,

$$\begin{aligned}\mathbb{P}_\sigma\Big(X_t \in [M - 3a\sqrt{B(\sigma)}, M - a\sqrt{B(\sigma)}],\\ Y_t \in [M + a\sqrt{B(\sigma)}, M + 3a\sqrt{B(\sigma)}]\Big) \geq 1 - \varepsilon + \phi(L) - Ce^{-c\sqrt{B(\sigma)}},\end{aligned} \tag{6.11}$$

with $\phi(L)$ going to 0 when $L \to +\infty$ uniformly in σ. Here, $r = 1$, we only have one particle cluster. This event implies that $\restriction \sigma_t[M - a\sqrt{B(\sigma)}, M + a\sqrt{B(\sigma)}] \equiv 1$. Now, note that:

$$\mu(\mathbf{1}_{\restriction \sigma_t[M-a\sqrt{B(\sigma)}, M+a\sqrt{B(\sigma)}]\equiv 1}) = (1-q)^{2a\sqrt{B(\sigma)}} \leq (1-q)^{2a\Phi(L)}, \tag{6.12}$$

which goes to zero when $L \to \infty$.
Putting together Equations (6.11) and (6.12) concludes the proof of Equation (2.3) and thus the proof of Theorem 2.1. □

Now we can conclude the proof of the cutoff.

Proof of Theorem 2.2: The proof of the lower bound is straightforward by taking initial configuration $\mathbf{1}_L$ and applying Equation (2.3).

Let $\delta = \frac{\underline{v}}{v}$ and $\varepsilon > 0$. For every $\sigma \in \Omega_\Lambda^\delta$, we have that $t_1(\sigma) \leq \frac{\delta L}{2\underline{v}} + (\delta L)^{1/4} = \frac{L}{2v} + (\delta L)^{1/4}$. If $\sigma \in (\Omega_\Lambda^\delta)^c$, then $t_2(\sigma) \leq \frac{L}{2v} + 8a\sqrt{L}$. So in every case, for L large enough, if $t = \frac{L}{2v} + 8a\sqrt{L}$, we have:

$$d(t) \leq \varepsilon. \qquad \square$$

Appendix A. Convergence behind the front

Before proving Theorem 5.3, we recall a result that deals with the relaxation for the FA-1f process far from the front.

Proposition A.1 (Blondel et al. (2019, Theorem 5.1))**.** *Let $q > \bar{q}$, $\alpha < 1/2$ and $\delta > 0$. There exists $c > 0$ such that for all $t \geq 0$, for any $M \leq e^{\delta t^\alpha}$, any f with support in $[0, M]$, such that $\mu(f) = 0$ and $||f||_\infty \leq 1$, for all $\sigma_0 \in LO$ such that $\tilde{\sigma}_0 \in \mathcal{H}(\underline{v}t, M + (4\bar{v} - \underline{v})t, \sqrt{t})$, then*

$$|\mathbb{E}\left[f(\theta_{3\bar{v}t}\tilde{\sigma}_t)\right]| \leq e^{-c\sqrt{t}}.$$

The same kind of result holds for a process on $\mathbb{Z}_-$:

Proposition A.2. *Let $q > \bar{q}$, $\alpha < 1/2$ and $\delta > 0$. There exists $c > 0$ such that for any f with support $[0, M]$, $\mu(f) = 0$ and $||f||_\infty \leq 1$, for any $t \geq 0$, $M \leq e^{\delta t^\alpha}$, for all $\eta_0 \in LO_-$ such that $\tilde{\eta}_0 \in \mathcal{H}(\underline{v}t, M + (4\bar{v} - \underline{v})t, \sqrt{t})$ and $X_0 \leq -M - (3\bar{v} - \underline{v})t$, then*

$$|\mathbb{E}\left[f(\theta_{3\bar{v}t}\tilde{\eta}_t)\right]| \leq e^{-c\sqrt{t}}.$$

Proof: The proof is almost identical to that of the previous proposition. However, in this case we have to make sure that the interval $[3\bar{v}t, 3\bar{v}t + M]$ seen from the front does not go out of the domain $\mathbb{Z}_-$. Recall that with probability $1 - O(e^{-Bt})$ we have $X_t - X_0 \leq -\underline{v}t$. Then as long as $-X_0 + \underline{v}t \geq 3\bar{v}t + M$, we have $X_t + 3\bar{v}t + M \leq 0$ with probability $1 - O(e^{-Bt})$. $\square$

We now prove Theorem 5.3. The proof relies on the same coupling as in Blondel et al. (2019). Let us fix $\eta_0 \in LO_-$. We pick $\sigma_0 \in LO$ an arbitrary configuration such that $X(\eta_0) = X(\sigma_0)$. We will now prove that there exist $d^* > 0$ and $c > 0$ such that

$$||\tilde{\mu}_t^{\eta_0} - \tilde{\mu}_t^{\sigma_0}||_{[0,d^*t]} \leq \exp\left(-ce^{(\log t)^{1/4}}\right). \tag{A.1}$$

Throughout the proof, c, C denote generic constants independent of σ_0, η_0. These constants may vary from line to line. Before diving into the complete coupling, we shall define a coupling that comes into play during the proof: the $\Lambda-$maximal coupling. If $\Lambda \subset \mathbb{Z}$ is finite, and μ, μ' are two probability measures on Ω_-, we define the Λ-*maximal coupling* between μ and μ' as follow:

(1) we sample $\restriction (\sigma, \sigma')\Lambda \times \Lambda$ according to the maximal coupling, i.e. the one that achieves the total variation distance of the marginals of μ and μ' on Ω_Λ,
(2) we sample $\restriction \sigma\mathbb{Z}_- \setminus \Lambda$ and $\restriction \sigma'\mathbb{Z}_- \setminus \Lambda$ independently according to their conditional distributions $\mu(.| \restriction \sigma\Lambda)$ and $\mu'(.| \restriction \sigma'\Lambda)$.

Throughout this section, μ denotes the Bernoulli product measure on $\mathbb{Z}_-$: $\mu = \mu_{\mathbb{Z}_-}^{1-q}$.
For $t > 0$, let us define the following quantities:

- $\varepsilon = \frac{\bar{v}}{2(\bar{v} + \underline{v})}$,
- $t_0 = (1 - \varepsilon)t$,
- $\Delta_1 = \exp\left((\log t)^{1/4}\right)$,

- $\Delta_2 = (\log t)^{3/4}$,
- $\Delta = \Delta_1 + \Delta_2$.

The time interval $[0,t]$ is divided in several sub-intervals. First, during a time $t_0 = (1-\varepsilon)t$ (which is the longest phase), we aim to use the Zeros Lemma, which provides a good amount of empty sites behind the front. We then divide the remaining time εt into N steps of length Δ, with $N = \lfloor \frac{\varepsilon t}{\Delta} \rfloor$ and a remainder step of length $\varepsilon t - N\Delta$. Each of these steps of length Δ consists in two steps of respective lengths Δ_1 and Δ_2. We define $t_n = t_0 + n\Delta$ and $s_n = t_n + \Delta_1$, $n \leq N$.

We define the interval $\Lambda_n = [3\bar{v}\Delta_1, \underline{v}s_n - (\bar{v}+\underline{v})\Delta_1]$ and the distance $d_n = \underline{v}t_n - (\bar{v}+\underline{v})\Delta n$. Note that d_n slightly differs from the original proof. Lastly, let us introduce the following notation. For (A_n) a sequence of measurable subsets of $LO \times LO_-$, we write:

$$P_{A_n}(\cdot) := \sup_{(\tilde{\sigma},\tilde{\eta})\in A_n} \mathbb{P}\left(\cdot | \left(\tilde{\sigma}_{t_n}, \tilde{\eta}_{t_n}\right) = (\tilde{\sigma}, \tilde{\eta})\right).$$

Depending on the context, t_n can be replaced by s_n in the definition above.

We are now ready to explain the coupling between σ_t and η_t that will lead to Theorem A.1.

- At time t_0, we sample σ_{t_0} and η_{t_0} independently according to the laws $\mu_{t_0}^{\eta_0}$ and $\mu_{t_0}^{\sigma_0}$.
- Suppose the coupling at time t_n constructed. If the configurations $\tilde{\sigma}_{t_n}$ and $\tilde{\eta}_{t_n}$ coincide on the interval $[1, d_n]$, then we let the evolve according to the standard coupling until time t. If not, we proceed in two steps. With high probability, both configurations will have zeros behind the front. More precisely, let Z_n be the event:

  $$Z_n = \{\tilde{\sigma}_{t_n}, \tilde{\eta}_{t_n} \in \mathcal{H}(\underline{v}\Delta_1, \underline{v}t_n, \sqrt{\Delta_1}) \text{ and } (X(\sigma_{t_n}) - X(\sigma_0)), (X(\eta_{t_n}) - X(\eta_0)) \leq -\underline{v}t_n\}.$$

  Here, the event Z_n differs from the original proof to ensure that the interval $[\underline{v}\Delta_1, \underline{v}t_n]$ (seen from the front) does not go out of bounds.
  From both the Zeros Lemma, Lemma 3.3 and their analogs in Blondel et al. (2019), we have for t large enough,

  $$\mathbb{P}(Z_n) \geq 1 - 2(\underline{v}t_n)^2 e^{-c\sqrt{\Delta_1}} - Ce^{-c' t_n} \geq 1 - C{t_n}^2 e^{-c\sqrt{\Delta_1}}.$$

  The last term of the first line comes from the fact that we ask for the inequalities $X(\sigma_{t_n}) \leq X(\sigma_0) - \underline{v}t_n$ and $X(\eta_{t_n}) \leq X(\eta_0) - \underline{v}t_n$. Note that none of the constants c, C depend on η_0 or σ_0.
  In the following, it will be useful to also introduce the event

  $$Z'_n = \{\tilde{\sigma}_{t_n}, \tilde{\eta}_{t_n} \in \mathcal{H}(0, 3\bar{v}\Delta_1, 2\underline{v}\Delta_1)\}.$$

  We can already note that $Z_n \subset Z'_n$ for t large enough.
- At time s_n, we sample $\tilde{\sigma}_{s_n}$ and $\tilde{\eta}_{s_n}$ according to the Λ_n-maximal coupling between the laws $\tilde{\mu}_{\Delta_1}^{\sigma_{tn}}$ and $\tilde{\mu}_{\Delta_1}^{\eta_{tn}}$. Let $Q_n = \{\tilde{\sigma}_{s_n} = \tilde{\eta}_{s_n} \text{ on } \Lambda_n\}$. First, note that by definition of the maximal coupling and by the Markov property, we have, for $(\sigma, \eta) \in LO \times LO_-$:

  $$\begin{aligned}\mathbb{P}(Q_n^c \,|\sigma_{t_n} = \sigma, \eta_{t_n} = \eta) &= ||\tilde{\mu}_{\Delta_1}^{\sigma} - \tilde{\mu}_{\Delta_1}^{\eta}||_{\Lambda_n} \\ &\leq ||\tilde{\mu}_{\Delta_1}^{\sigma} - \mu||_{\Lambda_n} + ||\tilde{\mu}_{\Delta_1}^{\eta} - \mu||_{\Lambda_n}.\end{aligned}$$

  Since the interval Λ_n is "far from the front", then these two distances are small uniformly in $\tilde{\sigma}_{t_n}, \tilde{\eta}_{t_n}$ as long as the event Z_n is satisfied. Let us justify this claim by applying Proposition A.2.

  Let $\eta \in LO_-$ be an initial configuration such that $\tilde{\eta} \in \mathcal{H}(\underline{v}\Delta_1, \underline{v}t_n, \sqrt{\Delta_1})$. Assume also that $X(\eta) \leq -\underline{v}t_n$. Let us check the hypotheses of Proposition A.2. Take

  $$M = \underline{v}s_n - (\underline{v} + \bar{v})\Delta_1 - 3\bar{v}\Delta_1.$$

Then $M+(4\bar{v}-\underline{v})\Delta_1 = \underline{v}t_n - \underline{v}\Delta_1 \le \underline{v}t_n$, which guarantees that we do have $\tilde{\eta} \in \mathcal{H}(\underline{v}\Delta_1, M+(4\bar{v}-\underline{v})\Delta_1, \sqrt{\Delta_1})$. Next, since $X(\eta) \le -\underline{v}t_n$, we find:

$$-M-(3\bar{v}-\underline{v})\Delta_1 = -\underline{v}t_n + (\bar{v}+\underline{v})\Delta_1 \ge -\underline{v}t_n \ge X(\eta).$$

Finally, we do have $M \le e^{\Delta_1^\alpha}$ for $\alpha = 1/4$ for example.
We can now apply Proposition A.2 that gives $||\tilde{\mu}^\eta_{\Delta_1} - \mu||_{\Lambda_n} \le Ce^{-c\sqrt{\Delta_1}}$ with C, c independent of η. We can handle the term $||\tilde{\mu}^\sigma_{\Delta_1} - \mu||_{\Lambda_n}$ the same way in order to find in the end:

$$P_{Z_n}(Q_n^c) \le Ce^{-c\sqrt{\Delta_1}}.$$

- Next, if the event Q_n occurs, we define $\tilde{x}$ as the distance between the front and the leftmost zero of $\tilde{\sigma}_{s_n}$ (or $\tilde{\eta}_{s_n}$) located in Λ_n. If there is none, define $\tilde{x}$ as the distance between the front and the right boundary of Λ_n. Let β be a Bernoulli random variable (independent of everything else constructed so far) such that $\mathbb{P}(\beta = 1) = e^{-2\Delta_2}$. The event $\{\beta = 1\}$ has the same probability as the event that two independent Poisson clocks do not ring during a time Δ_2. We now construct the configurations at time t_{n+1}.
  - If $\beta = 1$, we consider that, seen from the front at time s_n, the clocks associated with the sites 0 and $\tilde{x}$ do not ring during Δ_2. First, we sample the configuration at the left of 0 according to the standard coupling, with empty boundary condition. Let ξ_{n+1} be the common increment of the front during this time, namely $\xi_{n+1} = X_{t_{n+1}} - X_{s_n}$. Next, we fix $\tilde{\sigma}_{t_{n+1}}(-\xi_{n+1}) = \tilde{\eta}_{t_{n+1}}(-\xi_{n+1}) := 0$ and $\tilde{\sigma}_{t_{n+1}}(\tilde{x}-\xi_{n+1}) = \tilde{\eta}_{t_{n+1}}(\tilde{x}-\xi_{n+1}) := \tilde{\sigma}_{s_n}(\tilde{x})$ and sample the configurations $\tilde{\sigma}_{t_{n+1}}$ and $\tilde{\eta}_{t_{n+1}}$ according to the maximal coupling on $[1-\xi_{n+1}, \tilde{x}-1-\xi_{n+1}]$, with boundary conditions 0 on 0 and $\tilde{\sigma}_{s_n}(\tilde{x})$ on $\tilde{x}$. To the right of this interval, we sample the configurations according to the standard coupling with the appropriate boundary conditions.
  - If $\beta = 0$, then we let evolve $(\tilde{\sigma}_{s_n}, \tilde{\eta}_{s_n})$ for a time Δ_2 via the standard coupling conditioned to have at least one ring either at $\tilde{x}$ or at 0.

First, let us notice that $\tilde{x}$ is likely to be not too far right inside Λ_n. Indeed, under the event Z_n the law of the configuration in Λ_n is close to the product measure at time s_n. Let $B_n := \{\tilde{x} \le 3\bar{v}\Delta_1 + \frac{\sqrt{\Delta_2}}{2}\}$. Then, by Proposition A.1:

$$\mathcal{P}_{Z_n}\left(B_n^c \cap Q_n\right) \le \mathcal{P}_{Z_n}\left(\upharpoonright(\tilde{\sigma}_{s_n})[3\bar{v}\Delta_1, 3\bar{v}\Delta_1 + \frac{\sqrt{\Delta_2}}{2}] \equiv 1\right) \tag{A.2}$$

$$\le p^{\frac{\sqrt{\Delta_2}}{2}} + O(e^{-c\sqrt{\Delta_1}}). \tag{A.3}$$

Now, denote by Z''_n the event $\left\{\tilde{\sigma}_{s_n}, \tilde{\eta}_{s_n} \in \mathcal{H}\left(\frac{\sqrt{\Delta_2}}{2}, 3\bar{v}\Delta_1, \frac{\sqrt{\Delta_2}}{2}\right)\right\}$, and recall that $Z_n \subset Z'_n$. Using the second case of Zeros Lemma 3.5 and its equivalent Blondel et al. (2019, Lemma 4.4) (with $L = \sqrt{\Delta_2}/2,\ L+M = 3\bar{v}\Delta_1,\ \ell = L$), we find that:

$$P_{Z_n}((Z''_n)^c) \le \frac{2\Delta_1^2}{\sqrt{\Delta_2}}\exp(-c\sqrt{\Delta_2}) + \Delta_1\left(3\bar{v}\Delta_1 - \frac{\sqrt{\Delta_2}}{2}\right)\exp(-c\Delta_2) \tag{A.4}$$

$$\le O(e^{-c\sqrt{\Delta_2}}). \tag{A.5}$$

With the event Z''_n, we are now able the use Proposition 4.1 to couple the configuration at time t_n on the interval $J_n := [1-\xi_{n+1}, \tilde{x}-\xi_{n+1}-1]$.
First, note that $Z''_n \cap B_n \subset \{\tilde{\sigma}_{s_n}, \tilde{\eta}_{s_n} \in \mathcal{H}\left(0, \tilde{x}, \sqrt{\Delta_2}\right)\}$. Let $L = 3\bar{v}\Delta_1 + \frac{\sqrt{\Delta_2}}{2} \le e^{\Delta_2^{1/4}}$. Then by Proposition 4.1, we find that for any $y \in [3\bar{v}\Delta_1, 3\bar{v}\Delta_1 + \frac{\sqrt{\Delta_2}}{2}]$:

$$\mathbb{P}\left(\tilde{\sigma}_{t_{n+1}} \neq \tilde{\eta}_{t_{n+1}} \text{ on } [1-\xi_{n+1}, y-\xi_{n+1}-1] \mid \tilde{x} = y \text{ and } \tilde{\sigma}_{s_n}\ \tilde{\eta}_{s_n} \in \mathcal{H}(0, y, \sqrt{\Delta_2})\right) = O(e^{-c\sqrt{\Delta_2}}). \tag{A.6}$$

Again, this bound does not depend on y or the initial configurations. This in turn shows that:

$$\mathbb{P}_{Z''_n,B_n}(\tilde{\sigma}_{t_{n+1}} \neq \tilde{\eta}_{t_{n+1}} \text{ on } J_n) = O(e^{-c\sqrt{\Delta_2}}). \tag{A.7}$$

Under the event Q_n, the configurations seen from the front at time s_n coincide on the interval Λ_n. On the right part of this interval, namely $[\tilde{x}, \underline{v}s_n - (\bar{v}+\underline{v})\Delta_1]$, we let the configurations evolve according to the standard coupling. The matching of the configurations will then persist on the sub-interval $[\tilde{x} - \xi_{n+1}, \underline{v}s_n - (\bar{v}+\underline{v})\Delta_1 - \xi_{n+1} - \bar{v}\Delta_2]$ w.h.p. thanks to Proposition 3.1. Note that $\underline{v}s_n - (\bar{v}+\underline{v})\Delta_1 - \bar{v}\Delta_2 = \underline{v}t_{n+1} - \bar{v}\Delta \geq d_{n+1}$. Denote by K_n the interval $[\tilde{x} - \xi_{n+1}, d_{n+1}]$. We found that:

$$P_{Q_n}\left(\tilde{\sigma}_{t_{n+1}} \neq \tilde{\eta}_{t_{n+1}} \text{ on } K_n \mid \beta = 1\right) \leq \mathbb{P}(F(0, \bar{v}\Delta_2, \Delta_2) + \mathbb{P}(\xi_{n+1} \geq 0) = O(e^{-c\Delta_2}). \tag{A.8}$$

Finally, notice that if the clock attached to the site 0 (i.e the front at time s_n) does not ring, and we let the configurations at the left of this site evolve according to the standard coupling, we naturally get:

$$P_{Q_n}\left(\tilde{\sigma}_{t_{n+1}} \neq \tilde{\eta}_{t_{n+1}} \text{ on } [0, -\xi_{n+1}] \mid \beta = 1\right) = 0. \tag{A.9}$$

From Equations (A.7), (A.8) and (A.9), we can conclude that:

$$P_{Q_n,Z''_n,B_n}\left(\tilde{\sigma}_{t_{n+1}} \neq \tilde{\eta}_{t_{n+1}} \text{ on } [1, d_{n+1}] \mid \beta = 1\right) = O(e^{-c\sqrt{\Delta_2}}). \tag{A.10}$$

We are now ready to estimate the probability that the coupling "succeeds". Let $M_n = \{\tilde{\sigma}_{t_n} = \tilde{\eta}_{t_n} \text{ on } [1, d_n]\}$. At time t_{n+1}, depending on the coupling at time t_n, we can write:

$$\begin{aligned}\mathbb{P}(M^c_{n+1}) &\leq \mathbb{P}(M^c_{n+1} \cap M^c_n \cap Z_n) + \mathbb{P}(Z^c_n) + \mathbb{P}(M^c_{n+1} \cap M_n) \\ &\leq \mathbb{P}(M^c_{n+1} | M^c_n \cap Z_n)\mathbb{P}(M^c_n) + \mathbb{P}(Z^c_n) + \mathbb{P}(M^c_{n+1} \cap M_n). \end{aligned} \tag{A.11}$$

Let us focus on the term $\mathbb{P}(M^c_{n+1}|M^c_n \cap Z_n)$. We condition on the events that can happen at time s_n:

$$\mathbb{P}(M^c_{n+1}|M^c_n \cap Z_n) \leq P_{Z_n,M^c_n}(M^c_{n+1}|Z''_n \cap Q_n \cap B_n) + P_{Z_n}((Z''_n)^c) + P_{Z_n}(Q^c_n) + P_{Z_n}(B^c_n \cap Q_n).$$

Out of all these terms, only the first one is (a priori) not vanishing when t goes to infinity according to equations (A.2), (A.4), (A). Thanks to equation (A.10), for t large enough:

$$P_{Z_n,M^c_n}(M^c_{n+1}|Z''_n \cap Q_n \cap B_n) \leq 1 - \mathbb{P}(\beta = 1)\left(1 - P_{M^c_n \cap Z_n}\left(M^c_{n+1}|\{\beta = 1\} \cap Z''_n \cap Q_n \cap B_n\right)\right) \tag{A.12}$$

$$\leq 1 - \frac{1}{2}e^{-2\Delta_2}. \tag{A.13}$$

Back to the initial inequality (A.11), we can plug our estimate for the first term, and easily bound the remaining two terms to find:

$$\mathbb{P}(M^c_{n+1}) \leq \left(1 - \frac{1}{2}e^{-2\Delta_2}\right)\mathbb{P}(M^c_n) + Ce^{-c\Delta} \tag{A.14}$$

It only remains to solve this recursive equation with the values of N, Δ_1, Δ_2 we chose. We find:

$$\mathbb{P}(M^c_N) \leq \left(1 - \frac{1}{2}e^{-2\Delta_2}\right)^N + C2^{-N+1}e^{-c\Delta - 2(N-1)\Delta_2} \tag{A.15}$$

$$\leq O(e^{-c\Delta_1}). \tag{A.16}$$

Let us now conclude the proof. At time t_N, with probability $\mathbb{P}(M_N)$ the configurations $\tilde{\sigma}_{t_N}$ and $\tilde{\eta}_{t_N}$ match on the interval $[1, d_N]$. From time t_N to time t, we let the configurations evolve according to the standard coupling. In the same way as we did before, we can estimate the interval on which the configurations are equal at time t given the event M_N.

$$\mathbb{P}(\tilde{\sigma}_t = \tilde{\eta}_t \text{ on } [1, d^*t]) = \mathbb{P}(\tilde{\sigma}_t = \tilde{\eta}_t \text{ on } [1, d^*t] \mid M_N)\mathbb{P}(M_N) \tag{A.17}$$

$$\geq \mathbb{P}(\{X_t \leq X_{t_N}\} \cap F(d_N, d^*t, t - t_N)^c)\mathbb{P}(M_N). \tag{A.18}$$

It only remains to chose a d^* such that the event $F(d_N, d^*t, t-t_N)$ has low probability. Let $d^* = \frac{\underline{v}}{4}$. Then using $t - t_N \leq \Delta$ and replacing ε and d^* by their values, we find:

$$d_N - d^*t \geq \frac{\underline{v}t}{4} - \underline{v}t \geq \bar{v}\Delta \geq \bar{v}(t - t_N), \tag{A.19}$$

which implies $\mathbb{P}(F(d_N, d^*t, t-t_N)) = O(e^{-ct})$. The event $\{X_t \leq X_{t_N}\}$ has probability $1 - O(e^{-c\Delta})$ so we find in the end the announced estimate:

$$\mathbb{P}(\tilde{\sigma}_t \neq \tilde{\eta}_t \text{ on } [1, d^*t]) = O(e^{-c\Delta}). \tag{A.20}$$

The existence of an invariant measure ν comes from the compacity of the set of probability measures on Ω. The uniqueness and the convergence estimate of Theorem 5.3 come from Equation (A.20). □

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