ALEA, Lat. Am. J. Probab. Math. Stat. 19, 617–640 (2022)
DOI: 10.30757/ALEA.v19-24

Moderate deviation principles for bifurcating Markov chains: case of functions dependent of one variable

S. Valère Bitseki Penda and Gorgui Gackou

S. Valère Bitseki Penda, IMB, CNRS-UMR 5584, Université Bourgogne Franche-Comté, 9 avenue Alain Savary, 21078 Dijon Cedex, France.
E-mail address: simeon-valere.bitseki-penda@u-bourgogne.fr

Gorgui Gackou, Laboratoire de Mathématiques Blaise Pascal, CNRS-UMR 6620, Université Clermont-Auvergne
E-mail address: gorgui.gackou@uca.fr

Abstract. The main purpose of this article is to establish moderate deviation principles for additive functionals of bifurcating Markov chains. Bifurcating Markov chains are a class of processes which are indexed by a regular binary tree. They can be seen as the models which represent the evolution of a trait along a population where each individual has two offsprings. Unlike the previous results of Bitseki, Djellout & Guillin (2014), we consider here the case of functions which depend only on one variable. So, mainly inspired by the recent works of Bitseki & Delmas (2020) about the central limit theorem for general additive functionals of bifurcating Markov chains, we give here a moderate deviation principle for additive functionals of bifurcating Markov chains when the functions depend on one variable. This work is done under the uniform geometric ergodicity and the uniform ergodic property based on the second spectral gap assumptions. The proofs of our results are based on martingale decomposition recently developed by Bitseki & Delmas (2020) and on results of Dembo (1996), Djellout (2001) and Puhalski (1997).

1. Introduction

First, we give a general definition of a moderate deviation principles. Let $(Z_n)_{n\geq 0}$ be a sequence of random variables with values in S endowed with its Borel σ-field $\mathcal{B}(S)$ and let $(s_n)_{n\geq 0}$ be a positive sequence that converges to $+\infty$. We assume that Z_n/s_n converges in probability to 0 and that $Z_n/\sqrt{s_n}$ converges in distribution to a centered Gaussian law. Let $I : S \to \mathbb{R}^+$ be a lower semicontinuous function, that is for all $c > 0$ the sub-level set $\{x \in S, I(x) \leq c\}$ is a closed set. Such a function I is called *rate function* and it is called *good rate function* if all its sub-level sets are compact sets. Let $(b_n)_{n\geq 0}$ be a positive sequence such that $b_n \to +\infty$ and $b_n/\sqrt{s_n} \to 0$ as n goes to $+\infty$.

Received by the editors May 20th, 2021; accepted January 15th, 2022.
2010 *Mathematics Subject Classification.* 60F10, 60J80.
Key words and phrases. Bifurcating Markov chains; moderate deviation principles; deviation inequalities; binary trees.

Definition 1.1 (Moderate deviation principle, MDP).
We say that $Z_n/(b_n\sqrt{s_n})$ satisfies a moderate deviation principle in S with speed b_n^2 and the rate function I if, for any $A \in \mathcal{B}(S)$,

$$-\inf_{x\in A^\circ} I(x) \leq \liminf_{n\to\infty} \frac{1}{b_n^2}\log\mathbb{P}\big(\frac{Z_n}{b_n\sqrt{s_n}} \in A\big) \leq \limsup_{n\to\infty} \frac{1}{b_n^2}\log\mathbb{P}\big(\frac{Z_n}{b_n\sqrt{s_n}} \in A\big) \leq -\inf_{x\in\bar{A}} I(x),$$

where A° and $\bar{A}$ denote respectively the interior and the closure of A.

Bifurcating Markov chains (BMC, for short) are a class of stochastic processes indexed by regular binary tree. They are appropriate for example in the modeling of cell lineage data when each cell in one generation gives birth to two offspring in the next one. Recently, they have received a great deal of attention because of the experiments of biologists on aging of Escherichia Coli (E. Coli , for short). E. Coli is a rod-shaped bacterium which reproduces by dividing in two, thus producing two daughters: one of type 0 which has the old pole of the mother and the other of type 1 which has the new pole of the mother. The genealogy of the cells may be entirely described by a binary tree. To the best of our knowledge, the term bifurcating Markov chains appears for the first time in the works of Basawa and Zhou (2004). Thereafter, it was Guyon (2007) who had introduced and properly studied the theory of BMC. The first example of BMC, named bifurcating autoregressive process (BAR, for short), were introduced by Cowan and Staudte (1986) in order to study the mechanisms of cell division in Escherichia Coli. Since this work of Cowan and Staudte, the BAR process has been widely studied in the literature and several extensions have been made. In particular, Guyon (2007) have used an extension of BAR process to get statistical evidence of aging in E.Coli.

In this paper, we are interested in moderate deviation principles(MDP, for short) for additive functionals of bifurcating Markov chains. The MDP can be seen as an intermediate behavior between the central limit theorem and large deviation. Usually, the MDP exhibits a simpler rate function inherited from the approximated Gaussian process, and holds for a larger class of dependent random variables than the large deviation principle. Unlike the results given by Bitseki Penda et al. (2014), we treat here the case of functions which depends on one variable only. For this type of additive functionals, the martingale decomposition done by Bitseki Penda et al. (2014) is no longer valid. Indeed, as explained for e.g. in Delmas and Marsalle (2010) Remark 1.7, the error term on the last generation is not negligible. Note that recently, Bitseki Penda and Delmas (2022+) have studied central limit theorem for additive functionals of bifurcating Markov chain. They have studied the case where the functions depend only on the trait of a single individual for BMC. Bitseki Penda and Delmas (2022+) observes three regimes (sub-critical, critical, super-critical), which correspond to a competition between the reproducing rate (a mother has two daughters) and the ergodicity rate for the evolution of the trait along a lineage taken uniformly at random. This phenomenon already appears in the works of Athreya (1969). Here we investigate the moderate deviation principles for MBC depending only on one variable for the two cases: sub-critical and critical regimes. The super-critical regime, which require another way of centering will be done in a future work.

The rest of the paper is organized as follows. In Section 2, we present the model of bifurcating Markov chains. In Section 3, we give some notations and the main assumptions for our results. In Section 4, we set our main results: the sub-critical case in Section 4.1 and the critical case in Section 4.2. Section 5 is dedicated to the proof of the main result in sub-critical case and Section 6 is dedicated to the proof of the main result in Critical case. In Section 7, we illustrate numerically our results. Finally, in Section 8, we give some useful results.

2. The model of bifurcating Markov chain

2.1. *The regular binary tree associated to BMC models.* We denote by $\mathbb{N}$ (resp. $\mathbb{N}^*$) the space of (resp. positive) natural integers. We set $\mathbb{T}_0 = \mathbb{G}_0 = \{\emptyset\}$, $\mathbb{G}_k = \{0,1\}^k = \{0,1\} \times \{0,1\} \times \dots\dots \times$

$\{0,1\}$, the k-times cartesian product of $\{0,1\}$, and $\mathbb{T}_k = \bigcup_{0\leq r\leq k} \mathbb{G}_r$ for $k \in \mathbb{N}^*$, and $\mathbb{T} = \bigcup_{r\in\mathbb{N}} \mathbb{G}_r$. The set $\mathbb{G}_k$ corresponds to the k-th generation, $\mathbb{T}_k$ to the tree up to the k-th generation, and $\mathbb{T}$ the complete binary tree. One can see that the genealogy of the cells is entirely described by $\mathbb{T}$ (each vertex of the tree designates an individual). For $i \in \mathbb{T}$, we denote by $|i|$ the generation of i ($|i| = k$ if and only if $i \in \mathbb{G}_k$) and $iA = \{ij; j \in A\}$ for $A \subset \mathbb{T}$, where ij is the concatenation of the two sequences $i, j \in \mathbb{T}$, with the convention that $\emptyset i = i\emptyset = i$.

2.2. *The probability kernels associated to BMC models.*
Let $(S, \mathscr{S})$ be a measurable space. For any $q \in \mathbb{N}^*$, we denote by $\mathcal{B}(S^q)$ (resp. $\mathcal{B}_b(S^q)$, resp. $\mathcal{C}_b(S^q)$) the space of (resp. bounded, resp. bounded continuous) $\mathbb{R}$-valued measurable functions defined on S^q. For all $q \in \mathbb{N}^*$, we set $\mathscr{S}^{\otimes q} = \mathscr{S} \otimes \ldots \otimes \mathscr{S}$. Let $\mathcal{P}$ be a probability kernel on $(S, \mathscr{S}^{\otimes 2})$, that is: $\mathcal{P}(\cdot, A)$ is measurable for all $A \in \mathscr{S}^{\otimes 2}$, and $\mathcal{P}(x, \cdot)$ is a probability measure on $(S^2, \mathscr{S}^{\otimes 2})$ for all $x \in S$. For any $g \in \mathcal{B}_b(S^3)$ and $h \in \mathcal{B}_b(S^2)$, we set for $x \in S$:

$$(\mathcal{P}g)(x) = \int_{S^2} g(x, y, z)\, \mathcal{P}(x, \mathrm{d}y, \mathrm{d}z) \quad \text{and} \quad (\mathcal{P}h)(x) = \int_{S^2} h(y, z)\, \mathcal{P}(x, \mathrm{d}y, \mathrm{d}z). \tag{2.1}$$

We define $(\mathcal{P}g)$ (resp. $(\mathcal{P}h)$), or simply Pg for $g \in \mathcal{B}(S^3)$(resp. $\mathcal{P}h$ for $h \in \mathcal{B}(S^2)$), as soon as the corresponding integral (2.1) is well defined, and we have that $\mathcal{P}g$ and $\mathcal{P}h$ belong to $\mathcal{B}(S)$. we denote by $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{Q}$ respectively the first and the second marginal of $\mathcal{P}$, and the mean of $\mathcal{P}_0$ and $\mathcal{P}_1$, that is, for all $x \in S$ and $B \in \mathcal{S}$

$$\mathcal{P}_0(x, B) = \mathcal{P}(x, B \times S), \quad \mathcal{P}_1(x, B) = \mathcal{P}(x, S \times B) \quad \text{and} \quad \mathcal{Q} = \frac{(\mathcal{P}_0 + \mathcal{P}_1)}{2}.$$

Now let us give a precise definition of bifurcating Markov chain.

Definition 2.1 (Bifurcating Markov Chains, see Guyon (2007); Bitseki Penda and Delmas (2022+))**.**

We say a stochastic process indexed by $\mathbb{T}$, $X = (X_i, i \in \mathbb{T})$, is a bifurcating Markov chain (BMC) on a measurable space $(S, \mathscr{S})$ with initial probability distribution ν on $(S, \mathscr{S})$ and probability kernel $\mathcal{P}$ on $S \times \mathscr{S}^{\otimes 2}$ if:

- *(Initial distribution.) The random variable $X_\emptyset$ is distributed as ν.*
- *(Branching Markov property.) For any sequence $(g_i, i \in \mathbb{T})$ of functions belonging to $\mathcal{B}_b(S^3)$ and for all $k \geq 0$, we have*

$$\mathbb{E}\Big[\prod_{i\in\mathbb{G}_k} g_i(X_i, X_{i0}, X_{i1}) | \sigma(X_j; j \in \mathbb{T}_k)\Big] = \prod_{i\in\mathbb{G}_k} \mathcal{P}g_i(X_i).$$

Following Guyon (2007), we introduce an auxiliary Markov chain $Y = (Y_n, n \in \mathbb{N})$ on $(S, \mathscr{S})$ with $Y_0 = X_1$ and transition probability $\mathcal{Q}$. The chain $(Y_n, n \in \mathbb{N})$ corresponds to a random lineage taken in the population. We shall write $\mathbb{E}_x$ when $X_\emptyset = x$ (*i.e.* the initial distribution ν is the Dirac mass at $x \in S$).

3. Notations and assumptions

For $f \in \mathcal{B}_b(S)$, we set $\|f\|_\infty = \sup\{|f(x)|, x \in S\}$. For a finite measure λ on $(S, \mathscr{S})$ and $f \in \mathcal{B}(S)$, we set $\langle \lambda, f\rangle = \int f(x)\lambda(dx)$. We will work with the following ergodic property.

Assumption 3.1. *There exists a probability measure μ on $(S, \mathscr{S})$, a positive real number M and $\alpha \in (0, 1)$ such that for all $f \in \mathcal{B}_b(S)$:*

$$|\mathcal{Q}^n f - \langle \mu, f\rangle| \leq M\, \alpha^n \|f\|_\infty \quad \textit{for all } n \in \mathbb{N}. \tag{3.1}$$

We consider the stronger ergodic property based on a second spectral gap.

Assumption 3.2. *There exists a probability measure μ on $(S,\mathscr{S})$, a positive real number M, $\alpha \in (0,1)$, a finite non-empty set J of indices, distinct complex eigenvalues $\{\alpha_j,\ j\in J\}$ of the operator $\mathcal{Q}$ with $|\alpha_j| = \alpha$, non-zero complex projectors $\{\mathcal{R}_j,\ j\in J\}$ defined on $\mathbb{C}\mathcal{B}_b(S)$, the $\mathbb{C}$-vector space spanned by $\mathcal{B}_b(S)$, such that $\mathcal{R}_j \circ \mathcal{R}_{j'} = \mathcal{R}_{j'} \circ \mathcal{R}_j = 0$ for all $j \neq j'$ (so that $\sum_{j\in J}\mathcal{R}_j$ is also a projector defined on $\mathbb{C}\mathcal{B}_b(S)$) and a positive sequence $(\beta_n, n\in\mathbb{N})$ converging to 0, such that for all $f \in \mathcal{B}_b(S)$, with $\theta_j = \alpha_j/\alpha$:*

$$\Big|\mathcal{Q}^n(f) - \langle \mu, f\rangle - \alpha^n \sum_{j\in J} \theta_j^n\, \mathcal{R}_j(f)\Big| \leq M\, \beta_n \alpha^n \|f\|_\infty \quad \textit{for all } n\in\mathbb{N}. \tag{3.2}$$

Without loss of generality, we shall assume that the sequence $(\beta_n, n\in\mathbb{N})$ in Assumption 3.2 is non-increasing and bounded from above by 1. This assumption will be used when $\alpha = 1/\sqrt{2}$. For $f\in\mathcal{B}_b(S)$, $\tilde f$ and $\hat f$ will denote the functions defined by:

$$\tilde f = f - \langle f, \mu\rangle \quad \text{and} \quad \hat f = \tilde f - \sum_{j\in J} \mathcal{R}_j(f). \tag{3.3}$$

For a finite set $A\subset\mathbb{T}$ and a function $f\in\mathcal{B}_b(S)$, we set:

$$M_A(f) = \sum_{i\in A} f(X_i).$$

Let $\mathfrak{f} = (f_\ell, \ell\in\mathbb{N})$ be a sequence of elements of $\mathcal{B}_b(S)$. We will assume in the sequel that

$$\sup_{\ell\in\mathbb{N}}\{\|f_\ell\|_\infty\} = c_\infty < +\infty, \tag{3.4}$$

in such a way that (3.1) and (3.2) are uniformly satisfied by the sequence $\mathfrak{f}$. We set for $n\in\mathbb{N}$ and $i\in\mathbb{T}_n$:

$$N_{n,i}(\mathfrak{f}) = \sum_{\ell=0}^{n-|i|} N_{n,i}^{\ell}(f_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-|i|} M_{i\mathbb{G}_{n-|i|-\ell}}(\tilde f_\ell), \tag{3.5}$$

where $i\mathbb{G}_{n-|i|-\ell} = \{ij; j\in\mathbb{G}_{n-|i|-\ell}\}\subset\mathbb{G}_{n-\ell}$. We deduce that

$$\sum_{i\in\mathbb{G}_k} N_{n,i}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-k} M_{\mathbb{G}_{n-\ell}}(\tilde f_\ell),$$

which gives for $k=0$ that

$$\boxed{N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n} M_{\mathbb{G}_{n-\ell}}(\tilde f_\ell).}$$

To study the asymptotics of $N_{n,\emptyset}(\mathfrak{f})$, it is convenient to write for $n\geq k\geq 1$:

$$N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{r=0}^{k-1} M_{\mathbb{G}_r}(\tilde f_{n-r}) + \sum_{i\in\mathbb{G}_k} N_{n,i}(\mathfrak{f}). \tag{3.6}$$

Asymptotic normality for $N_{n,\emptyset}(\mathfrak{f})$ have been studied by Bitseki Penda and Delmas (2022+). Our aim in this paper is to complete this result by studying moderate deviation principles for $N_{n,\emptyset}(\mathfrak{f})$. More precisely, given a sequence $(b_n, n\in\mathbb{N})$ such that:

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{\sqrt{|\mathbb{G}_n|}} = 0,$$

our aim is to prove that $b_n^{-1} N_{n,\emptyset}(\mathfrak{f})$ satisfies a moderate deviation principle with speed b_n^2 and rate function I defined by

$$I(x) = \sup_{\lambda\in\mathbb{R}} \{\lambda x - \tfrac{1}{2}\lambda^2 \Sigma(\mathfrak{f})\} = \begin{cases} \frac{1}{2}\Sigma(\mathfrak{f})^{-1} x^2 & \text{if } \Sigma(\mathfrak{f}) \neq 0 \\ +\infty & \text{if } \Sigma(\mathfrak{f}) = 0, \end{cases} \tag{3.7}$$

where

$$\Sigma(\mathfrak{f}) = \begin{cases} \Sigma^{\text{sub}}(\mathfrak{f}) = \Sigma_1^{\text{sub}}(\mathfrak{f}) + 2\Sigma_2^{\text{sub}}(\mathfrak{f}) & \text{if } 2\alpha^2 < 1 \\ \Sigma^{\text{crit}}(\mathfrak{f}) = \Sigma_1^{\text{crit}}(\mathfrak{f}) + 2\Sigma_2^{\text{crit}}(\mathfrak{f}) & \text{if } 2\alpha^2 = 1, \end{cases}$$

with

$$\begin{aligned} \Sigma_1^{\text{sub}}(\mathfrak{f}) &= \sum_{\ell\geq 0} 2^{-\ell} \langle \mu, \tilde{f}_\ell^2\rangle + \sum_{\ell\geq 0,\, k\geq 0} 2^{k-\ell} \langle \mu, \mathcal{P}\left((\mathcal{Q}^k \tilde{f}_\ell)\otimes^2\right)\rangle, \\ \Sigma_2^{\text{sub}}(\mathfrak{f}) &= \sum_{0\leq \ell<k} 2^{-\ell}\langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell}\tilde{f}_\ell\rangle + \sum_{\substack{0\leq\ell<k\\ r\geq 0}} 2^{r-\ell}\langle \mu, \mathcal{P}\left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}\tilde{f}_\ell\right)\rangle, \\ \Sigma_1^{\text{crit}}(\mathfrak{f}) &= \sum_{k\geq 0} 2^{-k}\langle \mu, \mathcal{P} f_{k,k}^*\rangle = \sum_{k\geq 0} 2^{-k} \sum_{j\in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f_k)\otimes_{\text{sym}} \overline{\mathcal{R}}_j(f_k))\rangle, \\ \Sigma_2^{\text{crit}}(\mathfrak{f}) &= \sum_{0\leq\ell<k} 2^{-(k+\ell)/2}\langle \mu, \mathcal{P} f_{k,\ell}^*\rangle, \end{aligned} \tag{3.8}$$

and where for $k, \ell \in \mathbb{N}$:

$$f_{k,\ell}^* = \sum_{j\in J} \theta_j^{\ell-k}\, \mathcal{R}_j(f_k) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f_\ell). \tag{3.9}$$

More precisely, our aim is to prove that

$$\begin{aligned} -\inf_{x\in A^\circ} I(x) &\leq \liminf_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\big(b_n^{-1} N_{n,\emptyset}(\mathfrak{f}) \in A\big) \\ &\leq \limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\big(b_n^{-1} N_{n,\emptyset}(\mathfrak{f}) \in A\big) \leq -\inf_{x\in\bar{A}} I(x), \end{aligned}$$

where A° and $\bar{A}$ denote respectively the interior and the closure of A. In particular, the latter asymptotic result implies that

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left| b_n^{-1} N_{n,\emptyset}(\mathfrak{f}) \right| > \delta \right) = -I(\delta) \quad \forall \delta > 0.$$

We note that $2\alpha^2 < 1$ corresponds to the sub-critical regime and $2\alpha^2 = 1$ to the critical regime. The super-critical regime, that is the case where $2\alpha^2 > 1$, is not treated in this paper. Indeed, for this case, another way to centered the functions is necessary to get moderate deviation principles. This will be done in a future work.

Remark 3.3. Let $f \in \mathcal{B}_b(S)$. If the sequence $\mathfrak{f} = (f_\ell, \ell\in\mathbb{N})$ is defined by: $f_0 = f$ and $f_\ell = 0$ for all $\ell \geq 1$, then we have $N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f})$ and $\Sigma(\mathfrak{f}) = \Sigma_{\mathbb{G}}(f)$, where

$$\Sigma_{\mathbb{G}}(f) = \begin{cases} \Sigma_{\mathbb{G}}^{\text{sub}}(f) = \langle \mu, \tilde{f}^2\rangle + \sum_{k\geq 0} 2^k \langle \mu, \mathcal{P}\left(\mathcal{Q}^k \tilde{f} \otimes^2\right)\rangle & \text{if } 2\alpha^2 < 1 \\ \Sigma_{\mathbb{G}}^{\text{crit}}(f) = \sum_{j\in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f))\rangle & \text{if } 2\alpha^2 = 1, \end{cases}$$

If the sequence $\mathfrak{f} = (f_\ell, \ell\in\mathbb{N})$ is defined by: $f_\ell = f$ for all $\ell \in \mathbb{N}$, then we have $N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = \sqrt{2-2^{-n}}\, |\mathbb{T}_n|^{-1/2}\, M_{\mathbb{T}_n}(\tilde{f})$ and $\Sigma(\mathfrak{f}) = \Sigma_{\mathbb{T}}(f)$, where

$$\Sigma_{\mathbb{T}}(f) = \begin{cases} \Sigma_{\mathbb{T}}^{\text{sub}}(f) = \Sigma_{\mathbb{G}}^{\text{sub}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{sub}}(f) & \text{if } 2\alpha^2 < 1 \\ \Sigma_{\mathbb{T}}^{\text{crit}}(f) = \Sigma_{\mathbb{G}}^{\text{crit}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{crit}}(f) & \text{if } 2\alpha^2 = 1, \end{cases}$$

with

$$\Sigma^{\text{sub}}_{\mathbb{T},2}(f) = \sum_{k\geq 1} \langle \mu, \tilde{f} \mathcal{Q}^k \tilde{f} \rangle + \sum_{\substack{k\geq 1\\ r\geq 0}} 2^r \langle \mu, \mathcal{P}\left(\mathcal{Q}^r \tilde{f} \otimes_{\text{sym}} \mathcal{Q}^{r+k} \tilde{f}\right) \rangle,$$

$$\Sigma^{\text{crit}}_{\mathbb{T},2}(f) = \sum_{j\in J} \frac{1}{\sqrt{2}\,\theta_j - 1} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f)) \rangle.$$

4. The main results

4.1. *The sub-critical cases:* $2\alpha^2 < 1$.
In the sub-critical case, we consider a sequence $(b_n, n \in \mathbb{N})$ such that:

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{\sqrt{|\mathbb{G}_n|}} = 0.$$

Then, we have the following result.

Theorem 4.1. *Let X be a BMC with kernel $\mathcal{P}$ and initial distribution ν such that Assumption 3.1 is in force with $\alpha \in (0, 1/\sqrt{2})$. Let $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ be a sequence of elements of $\mathcal{B}_b(S)$ satisfying (3.4) and Assumption 3.1 uniformly. Then $b_n^{-1} N_{n,\emptyset}(\mathfrak{f})$ satisfies a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7).*

As a direct consequence of Remark 3.3 and Theorem 4.1, we have the following result.

Corollary 4.2. *Let X be a BMC with kernel $\mathcal{P}$ and initial distribution ν such that Assumption 3.1 is in force with $\alpha \in (0, 1/\sqrt{2})$. Let $f \in \mathcal{B}_b(S)$. Then $b_n^{-1}|\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f})$ and $b_n^{-1}|\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f})$ satisfy a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7), with $\Sigma(\mathfrak{f})$ replaced respectively by $\Sigma_{\mathbb{G}}(f)$ and $\Sigma_{\mathbb{T}}(f)$.*

4.2. *The critical cases:* $2\alpha^2 = 1$.
In this critical case, we consider a sequence $(b_n, n \in \mathbb{N})$ such that:

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{\sqrt{n|\mathbb{G}_n|}} = 0 \quad \text{and for some } a \in (0,1), \quad \lim_{n\to\infty} b_n^2 n^2 2^{-n^a} = 0. \tag{4.1}$$

Then, we have the following result.

Theorem 4.3. *Let X be a BMC with kernel $\mathcal{P}$ and initial distribution ν such that Assumption 3.2 is in force with $\alpha = 1/\sqrt{2}$. Let $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ be a sequence of elements of $\mathcal{B}_b(S)$ satisfying (3.4) and Assumption 3.2 uniformly. Then $b_n^{-1} n^{-\frac{1}{2}} N_{n,\emptyset}(\mathfrak{f})$ satisfies a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7).*

As a direct consequence of Remark 3.3 and Theorem 4.3, we have the following result.

Corollary 4.4. *Let X be a BMC with kernel $\mathcal{P}$ and initial distribution ν such that Assumption 3.2 is in force with $\alpha = 1/\sqrt{2}$. Let $f \in \mathcal{B}_b(S)$. Then $b_n^{-1}(n|\mathbb{G}_n|)^{-1/2} M_{\mathbb{G}_n}(\tilde{f})$ and $b_n^{-1}(n|\mathbb{T}_n|)^{-1/2} M_{\mathbb{T}_n}(\tilde{f})$ satisfy a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7), with $\Sigma(\mathfrak{f})$ replaced respectively by $\Sigma_{\mathbb{G}}(f)$ and $\Sigma_{\mathbb{T}}(f)$.*

5. Proof of Theorem 4.1

5.1. *A quick overview of our strategy.*
Let $(p_n, n \in \mathbb{N})$ be a non-decreasing sequence of elements of $\mathbb{N}^*$ such that:

$$p_n < \frac{n}{2}.$$

When there is no ambiguity, we write p for p_n.

Let $i, j \in \mathbb{T}$. We write $i \preccurlyeq j$ if $j \in i\mathbb{T}$. We denote by $i \wedge j$ the most recent common ancestor of i and j, which is defined as the only $u \in \mathbb{T}$ such that if $v \in \mathbb{T}$ and $v \preccurlyeq i$, $v \preccurlyeq j$ then $v \preccurlyeq u$. We also define the lexicographic order $i \leq j$ if either $i \preccurlyeq j$ or $v0 \preccurlyeq i$ and $v1 \preccurlyeq j$ for $v = i \wedge j$. Let $X = (X_i, i \in \mathbb{T})$ be a *BMC* with kernel $\mathcal{P}$ and initial measure ν. For $i \in \mathbb{T}$, we define the σ-field:

$$\mathcal{F}_i = \{X_u; u \in \mathbb{T} \text{ such that } u \leq i\}.$$

By construction, the σ-fields $(\mathcal{F}_i;\ i \in \mathbb{T})$ are nested as $\mathcal{F}_i \subset \mathcal{F}_j$ for $i \leq j$.

We define for $n \in \mathbb{N}$, $i \in \mathbb{G}_{n-p_n}$ and $\mathfrak{f} \in F^{\mathbb{N}}$ the martingale increments:

$$\Delta_{n,i}(\mathfrak{f}) = N_{n,i}(\mathfrak{f}) - \mathbb{E}\left[N_{n,i}(\mathfrak{f})\middle|\, \mathcal{F}_i\right] \quad \text{and} \quad \Delta_n(\mathfrak{f}) = \sum_{i \in \mathbb{G}_{n-p_n}} \Delta_{n,i}(\mathfrak{f}). \tag{5.1}$$

Thanks to (3.5), we have:

$$\sum_{i \in \mathbb{G}_{n-p_n}} N_{n,i}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{k=n-p_n}^{n} M_{\mathbb{G}_k}(\tilde{f}_{n-k}).$$

Using the branching Markov property, and (3.5), we get for $i \in \mathbb{G}_{n-p_n}$:

$$\mathbb{E}\left[N_{n,i}(\mathfrak{f})\middle|\, \mathcal{F}_i\right] = \mathbb{E}\left[N_{n,i}(\mathfrak{f})\middle|\, X_i\right] = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i}\left[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)\right].$$

We deduce from (3.6) with $k = n - p_n$ that:

$$N_{n,\emptyset}(\mathfrak{f}) = \Delta_n(\mathfrak{f}) + R_0(n) + R_1(n), \tag{5.2}$$

with

$$R_0(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}) \quad \text{and} \quad R_1(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}\left[N_{n,i}(\mathfrak{f})\middle|\, \mathcal{F}_i\right].$$

Our goals will be achieved if we prove the following:

$$\forall \delta > 0, \quad \lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}(|b_n^{-1} R_0(n)| > \delta) = -\infty; \tag{5.3}$$

$$\forall \delta > 0, \quad \lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}(|b_n^{-1} R_1(n)| > \delta) = -\infty; \tag{5.4}$$

$$b_n^{-1}\Delta_n(\mathfrak{f}) \quad \text{satisfies a MDP on } S \text{ with speed } b_n^2 \text{ and rate function } I. \tag{5.5}$$

Note that (5.3) and (5.4) mean that $R_0(n)$ and $R_1(n)$ are negligible in the sense of moderate deviations in such a way that from (5.2), $N_{n,\emptyset}(\mathfrak{f})$ and $\Delta_n(\mathfrak{f})$ satisfy the same moderate deviation principle (see Dembo and Zeitouni (1998), chap. 4).

5.2. *Proof of* (5.3).
Using the Chernoff inequality, we have, for all $\lambda > 0$,

$$\mathbb{P}(b_n^{-1} R_0(n) > \delta) \leq \exp(-\lambda \delta b_n |\mathbb{G}_n|^{1/2}) \, \mathbb{E}\left[\exp\left(\lambda \sum_{k=0}^{n-p-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k})\right)\right]. \tag{5.6}$$

For all $\ell \in \{0, \ldots, n-p-1\}$, we set

$$\mathbb{I}_\ell = \mathbb{E}\left[\exp\left(\lambda \sum_{k=0}^{n-p-\ell-2} M_{\mathbb{G}_k}(\tilde{f}_{n-k})\right) \exp\left(\lambda M_{\mathbb{G}_{n-p-\ell-1}}\left(\sum_{r=0}^{\ell} g_{p,r,\ell}\right)\right)\right],$$

where $g_{p,r,\ell} = 2^r \mathcal{Q}^r \tilde{f}_{p+\ell+1-r}$, with the convention that an empty sum is zero. For all $\ell \in \{0, \ldots, n-p-2\}$, we have the following decomposition:

$$\mathbb{I}_\ell = \mathbb{E}\left[\exp\left(\lambda \sum_{k=0}^{n-p-\ell-2} M_{\mathbb{G}_k}(\tilde{f}_{n-k})\right) \exp\left(\lambda M_{\mathbb{G}_{n-p-\ell-2}}\left(\sum_{r=0}^{\ell} 2\,\mathcal{Q}(g_{p,r,\ell})\right)\right) \mathbb{J}_\ell\right], \tag{5.7}$$

where

$$\mathbb{J}_\ell = \mathbb{E}\left[\exp\left(\lambda \sum_{i \in \mathbb{G}_{n-p-\ell-2}} \sum_{r=0}^{\ell} (g_{p,r,\ell}(X_{i0}) + g_{p,r,\ell}(X_{i1}) - 2\mathcal{Q}(g_{p,r,\ell})(X_i))\right) \Big| \mathcal{H}_{n-p-\ell-2}\right].$$

Using branching Markov property, we get

$$\mathbb{J}_\ell = \prod_{i \in \mathbb{G}_{n-p-\ell-2}} \mathbb{E}_{X_i}\left[\exp\left(\lambda \sum_{r=0}^{\ell} (g_{p,r,\ell}(X_{i0}) + g_{p,r,\ell}(X_{i1}) - 2\mathcal{Q}(g_{p,r,\ell})(X_i))\right)\right].$$

Using (3.1) and (3.4), we get

$$\left|\sum_{r=0}^{\ell} (g_{p,r,\ell}(X_{i0}) + g_{p,r,\ell}(X_{i1}) - 2\mathcal{Q}(g_{p,r,\ell})(X_i))\right| \leq 2Mc_\infty \sum_{r=0}^{\ell} (2\,\alpha)^r.$$

Using Lemma 8.2 and the latter inequality, we get, for all $i \in \mathbb{G}_{n-p-\ell-2}$,

$$\mathbb{E}_{X_i}\left[\exp\left(\lambda \sum_{r=0}^{\ell} (g_{p,r,\ell}(X_{i0}) + g_{p,r,\ell}(X_{i1}) - 2\mathcal{Q}(g_{p,r,\ell})(X_i))\right)\right] \leq \exp\left(2\lambda^2 M^2 c_\infty^2 (1+\alpha)^2 a_\ell^2\right),$$

with $a_\ell = \sum_{r=0}^{\ell} (2\alpha)^r$. The latter inequality implies that

$$\mathbb{J}_\ell \leq \exp\left(2\lambda^2 M^2 c_\infty^2 (1+\alpha)^2 a_\ell^2 |\mathbb{G}_{n-p-\ell-2}|\right). \tag{5.8}$$

From (5.7) and (5.8), it follows that

$$\mathbb{I}_\ell \leq \exp\left(2\lambda^2 M^2 c_\infty^2 (1+\alpha)^2 a_\ell^2 |\mathbb{G}_{n-p-\ell-2}|\right) \mathbb{I}_{\ell+1}. \tag{5.9}$$

Using the recurrence (5.9) for all $\ell \in \{0, \dots, n-p-2\}$ for the first inequality, (3.2) and (3.4) for the second inequality, we are led to

$$\mathbb{E}\left[\exp\left(\lambda \sum_{k=0}^{n-p-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k})\right)\right] = \mathbb{I}_0$$
$$\leq \exp\left(2\lambda^2 M^2 c_\infty^2 (1+\alpha)^2 \sum_{\ell=0}^{n-p-2} a_\ell^2 |\mathbb{G}_{n-p-\ell-2}|\right) \mathbb{I}_{n-p-1}$$
$$\leq \exp\left(2\lambda^2 M^2 c_\infty^2 (1+\alpha)^2 \sum_{\ell=0}^{n-p-2} a_\ell^2 |\mathbb{G}_{n-p-\ell-2}| + \lambda\, c_\infty\, M \sum_{r=0}^{n-p-1} (2\alpha)^{r+1}\right).$$

We have

$$\sum_{\ell=0}^{n-p-2} a_\ell^2 |\mathbb{G}_{n-p-\ell-2}| \leq \begin{cases} 6\,|\mathbb{G}_{n-p-1}| & \text{if } 2\alpha \leq 1 \\ \frac{2\alpha^2}{(2\alpha-1)^2(1-2\alpha^2)}\,|\mathbb{G}_{n-p-1}| & \text{if } 1 < 2\alpha < \sqrt{2} \\ \frac{1}{(2\alpha-1)^2}(n-p-1)|\mathbb{G}_{n-p-1}| & \text{if } 2\alpha^2 = 1 \\ \frac{1}{(2\alpha-1)^2(2\alpha^2-1)}(2\alpha)^{2(n-p-1)} & \text{if } 2\alpha^2 > 1 \end{cases}$$

Moreover, we have that there is a positive constant C_α such that

$$\sum_{r=0}^{n-p-1} (2\alpha)^{r+1} \leq (n-p)\mathbf{1}_{\{2\alpha=1\}} + \frac{2\alpha|(2\alpha)^{n-p}-1|}{|2\alpha-1|}\mathbf{1}_{\{2\alpha\neq 1\}} \leq C_\alpha(n-p+(2\alpha)^{n-p}).$$

It follows from (5.6) that for all $\lambda > 0$, there exists a positive constant c_α such that

$$\mathbb{P}(b_n^{-1}R_0(n) > \delta) \leq \exp\left(-\lambda\delta b_n|\mathbb{G}_n|^{1/2} \;+\; c_\alpha\lambda^2|\mathbb{G}_{n-p}| \;+\; c_\alpha\lambda\left((2\alpha)^{n-p} + n - p\right)\right). \tag{5.10}$$

Taking $\lambda = 2^{-1}\, c_\alpha^{-1}\, \delta\, b_n\, |\mathbb{G}_p|\, |\mathbb{G}_n|^{-1/2}$ in (5.10), we get

$$\mathbb{P}(b_n^{-1}R_0(n) > \delta) \leq \exp\left(-\frac{\delta^2 b_n^2|\mathbb{G}_p|}{4c_\alpha} + \frac{\delta b_n 2^{-\frac{n}{2}+p}((2\alpha)^{n-p}+n-p)}{2}\right)$$
$$\leq \exp\left(-\frac{\delta^2 b_n^2|\mathbb{G}_p|}{4c_\alpha}\left(1 - \frac{2c_\alpha(n-p)}{\delta b_n|\mathbb{G}_n|^{1/2}} - \frac{2c_\alpha(2\alpha^2)^{n/2}}{\delta b_n(2\alpha)^p}\right)\right).$$

Using $p < n/2$ and distinguishing the cases $2\alpha \leq 1$ and $1 < 2\alpha < \sqrt{2}$, we get

$$\lim_{n\to\infty} \frac{2c_\alpha(n-p)}{\delta b_n|\mathbb{G}_n|^{1/2}} + \frac{2c_\alpha(2\alpha^2)^{n/2}}{\delta b_n(2\alpha)^p} = 0.$$

It follows that for n large, there is a positive constant $c_{\alpha,\delta}$ such that

$$\mathbb{P}\left(b_n^{-1}R_0(n) > \delta\right) \leq \exp\left(-\frac{c_{\alpha,\delta}\, b_n^2\, |\mathbb{G}_p|\, \delta^2}{4c_\alpha}\right).$$

Doing the same thing for the sequence $-\mathfrak{f}$ instead of $\mathfrak{f}$, we conclude that

$$\mathbb{P}\left(\left|b_n^{-1}R_0(n)\right| > \delta\right) \leq 2\, \exp\left(-\frac{c_{\alpha,\delta}\, b_n^2\, |\mathbb{G}_p|\, \delta^2}{4c_\alpha}\right). \tag{5.11}$$

In (5.11), taking the log, dividing by b_n^2 and letting n goes to infinity, we get the result.

Remark 5.1. Let $f \in \mathcal{B}_b(S)$. Since we will use frequently this type of inequality, we give here a general procedure to upper-bound the probability $\mathbb{P}(||\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}(\tilde{f})| > \delta)$. From Chernoff inequality, we have, for all $\lambda > 0$,

$$\mathbb{P}\left(\left||\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}(\tilde{f})\right| > \delta\right) \leq \exp\left(-\lambda\delta|\mathbb{G}_{n-p}|\right)\mathbb{E}\left[\exp\left(\lambda M_{\mathbb{G}_{n-p}}(\tilde{f})\right)\right]. \tag{5.12}$$

For all $m \in \{0, \dots, n-p\}$, we set

$$\mathbb{I}_m = \mathbb{E}\left[\exp\left(2^m\lambda M_{\mathbb{G}_{n-p-m}}(\mathcal{Q}^m\tilde{f})\right)\right].$$

Using the branching Markov property, we have

$$\mathbb{I}_m = \mathbb{E}\left[\exp\left(2^{m+1}\lambda M_{\mathbb{G}_{n-p-m-1}}(\mathcal{Q}^{m+1}\tilde{f})\right)\mathbb{J}_m\right],$$

where

$$\mathbb{J}_m = \prod_{i\in\mathbb{G}_{n-p-m-1}} \mathbb{E}_{X_i}\left[\exp\left(2^m\lambda\left(\mathcal{Q}^m\tilde{f}(X_{i0}) + \mathcal{Q}^m\tilde{f}(X_{i1}) - 2\mathcal{Q}^{m+1}\tilde{f}(X_i)\right)\right)\right].$$

Using (3.1) and Lemma 8.2, we have the following upper-bound:

$$\mathbb{J}_m \leq \exp\left(\lambda^2\|f\|_\infty^2 M^2(1+\alpha)^2(2\alpha^2)^m|\mathbb{G}_{n-p}|\right).$$

This implies that

$$\mathbb{I}_m \leq \exp\left(\lambda^2\|f\|_\infty^2 M^2(1+\alpha)^2(2\alpha^2)^m|\mathbb{G}_{n-p}|\right)\mathbb{I}_{m+1}. \tag{5.13}$$

Using the recurrence relation (5.13) and (3.1) (to upper-bound $\mathbb{I}_{n-p}$), we are led to

$$\mathbb{E}\left[\exp\left(\lambda M_{\mathbb{G}_{n-p}}(\tilde{f})\right)\right] = \mathbb{I}_0 \leq \exp\left(\lambda^2\|f\|_\infty^2 M^2(1+\alpha)^2 a_{\alpha,n}|\mathbb{G}_{n-p}| + \lambda\|f\|_\infty M(2\alpha)^{n-p}\right), \tag{5.14}$$

where $a_{\alpha,n} = \sum_{m=0}^{n-p-1}(2\alpha^2)^m$. We set $a_\alpha = \lim_{n\to\infty} a_{\alpha,n}$, which is finite since $2\alpha^2 < 1$. Taking $\lambda = \delta/(2\|f\|_\infty^2 M^2(1+\alpha)^2 a_\alpha)$ in (5.12) and from (5.14), we are led to

$$\begin{aligned}\mathbb{P}\left(|\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}(\tilde{f}) > \delta\right) &\leq \exp\left(-\frac{\delta^2|\mathbb{G}_{n-p}|}{4\|f\|_\infty^2 M^2(1+\alpha)^2 a_\alpha} + \frac{\delta(2\alpha)^{n-p}}{2\|f\|_\infty M(1+\alpha)^2 a_\alpha}\right)\\ &= \exp\left(-\frac{\delta^2|\mathbb{G}_{n-p}|}{4\|f\|_\infty^2 M^2(1+\alpha)^2 a_\alpha}\left(1 - \frac{2\|f\|_\infty M\alpha^{n-p}}{\delta}\right)\right).\end{aligned}$$

Finally, since we can do the same thing for $-f$ instead of f, we conclude that

$$\mathbb{P}\left(\left||\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}(\tilde{f})\right| > \delta\right) \leq 2\exp\left(-\frac{\delta^2|\mathbb{G}_{n-p}|}{4\|f\|_\infty^2 M^2(1+\alpha)^2 a_\alpha}\left(1 - \frac{2\|f\|_\infty M\alpha^{n-p}}{\delta}\right)\right). \tag{5.15}$$

5.3. *Proof of* (5.4).
We set $g_p = \sum_{\ell=0}^{p} 2^{p-\ell}\mathcal{Q}^{p-\ell}\tilde{f}_\ell$ in such a way that using the definition of $R_1(n)$, we have

$$\mathbb{P}\left(b_n^{-1}|R_1(n)| > \delta\right) = \mathbb{P}\left(|\mathbb{G}_{n-p}|^{-1}\left|M_{\mathbb{G}_{n-p}}(g_p)\right| > \delta b_n|\mathbb{G}_n|^{-1/2}|\mathbb{G}_p|\right).$$

Using (3.1) and (3.4), we have

$$\|g_p\|_\infty \leq \begin{cases} c_\infty M(p+1) & \text{if } 2\alpha \leq 1\\ c_\alpha c_\infty M & \text{if } 1 < 2\alpha < \sqrt{2}.\end{cases}$$

Applying (5.15) to g_p and $\delta b_n|\mathbb{G}_n|^{-1/2}|\mathbb{G}_p|$, we get, for n going to infinity and for some positive constant $C_{\alpha,\delta}$,

$$\mathbb{P}\left(b_n^{-1}|R_1(n)| > \delta\right) \leq \begin{cases} 2\exp\left(-C_{\alpha,\delta}\,\delta^2\,b_n^2\,|\mathbb{G}_p|p^{-2}\right) & \text{if } 2\alpha \leq 1\\ 2\exp\left(-C_{\alpha,\delta}\,\delta^2 b_n^2\,(2\alpha^2)^{-p}\right) & \text{if } 1 < 2\alpha < \sqrt{2}.\end{cases}$$

Finally, (5.4) follows by taking the log, dividing by b_n^2 and letting n goes to infinity in the latter inequality.

5.4. *Proof of* (5.5)*: Moderate deviations principle for* $b_n^{-1}\Delta_n(\mathfrak{f})$.
First we study the bracket of $\Delta_n(\mathfrak{f})$:

$$V(n) = \sum_{i\in\mathbb{G}_{n-p_n}} \mathbb{E}\left[\Delta_{n,i}(\mathfrak{f})^2|\mathcal{F}_i\right].$$

Using (3.5) and (5.1), we write:

$$V(n) = |\mathbb{G}_n|^{-1} \sum_{i\in\mathbb{G}_{n-p_n}} \mathbb{E}_{X_i}\left[\left(\sum_{\ell=0}^{p_n} M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)\right)^2\right] - R_2(n) = V_1(n) + 2V_2(n) - R_2(n), \tag{5.16}$$

with:

$$V_1(n) = |\mathbb{G}_n|^{-1} \sum_{i\in\mathbb{G}_{n-p_n}} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i}\left[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)^2\right],$$
$$V_2(n) = |\mathbb{G}_n|^{-1} \sum_{i\in\mathbb{G}_{n-p_n}} \sum_{0\leq\ell<k\leq p_n} \mathbb{E}_{X_i}\left[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) M_{\mathbb{G}_{p_n-k}}(\tilde{f}_k)\right],$$
$$R_2(n) = \sum_{i\in\mathbb{G}_{n-p_n}} \mathbb{E}\left[N_{n,i}(\mathfrak{f})|X_i\right]^2.$$

Lemma 5.2. *Under the Assumptions of Theorem 4.1, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(R_2(n) > \delta\right) = -\infty. \tag{5.17}$$

Proof: Using the branching Markov property, we have

$$R_2(n) = |\mathbb{G}_n|^{-1} \sum_{i\in\mathbb{G}_{n-p}} g_p(X_i), \quad \text{with} \quad g_p = \left(\sum_{\ell=0}^{p} 2^{p-\ell} \mathcal{Q}^{p-\ell} \tilde{f}_\ell\right)^2. \tag{5.18}$$

Using (3.1) and (3.4), we get

$$\|g_p\|_\infty \leq c_\infty^2\, M^2 \left(\sum_{\ell=0}^{p} (2\alpha)^{p-\ell}\right)^2 \leq \begin{cases} c_\infty^2\, M^2\, (p+1)^2 & \text{if } 2\alpha \leq 1 \\ c_\infty^2\, M^2\, c_\alpha\, (2\alpha)^{2p} & \text{if } 1 < 2\alpha < \sqrt{2}. \end{cases}$$

This implies that $R_2(n)$ is upper-bounded by a deterministic sequence which converge to 0. As a consequence, we conclude that (5.17) holds. □

Lemma 5.3. *Under the Assumptions of Theorem 4.1, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left|V_1(n) - \Sigma_1^{\text{sub}}(\mathfrak{f})\right| > \delta\right) = -\infty,$$

where

$$\Sigma_1^{\text{sub}}(\mathfrak{f}) = \sum_{\ell\geq 0} 2^{-\ell} \left\langle \mu, \tilde{f}_\ell^2 \right\rangle + \sum_{\ell\geq 0,\, k\geq 0} 2^{k-\ell} \left\langle \mu, \mathcal{P}\left((\mathcal{Q}^k \tilde{f}_\ell)\otimes^2\right)\right\rangle := H_3(\mathfrak{f}) + H_4(\mathfrak{f})$$

Proof: Using (8.2), we get:

$$V_1(n) = V_3(n) + V_4(n), \tag{5.19}$$

with

$$V_3(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{\ell=0}^{p} 2^{p-\ell}\, \mathcal{Q}^{p-\ell}(\tilde{f}_\ell^2)(X_i),$$

$$V_4(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{\ell=0}^{p-1} \sum_{k=0}^{p-\ell-1} 2^{p-\ell+k}\, \mathcal{Q}^{p-1-(\ell+k)} \left(\mathcal{P}\left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2\right)\right)(X_i).$$

The proof is divided into two parts.

Part I. First we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|V_3(n) - H_3(\mathfrak{f})| > \delta\right) = -\infty. \tag{5.20}$$

Since

$$H_3(\mathfrak{f}) = \sum_{\ell=0}^{p} 2^{-\ell} \left\langle \mu, \tilde{f}_\ell^2 \right\rangle + \sum_{\ell > p} 2^{-\ell} \left\langle \mu, \tilde{f}_\ell^2 \right\rangle \quad \text{and} \quad \lim_{n\to\infty} \sum_{\ell > p} 2^{-\ell} \left\langle \mu, \tilde{f}_\ell^2 \right\rangle = 0,$$

then to get (5.20), it suffices to prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|V_3(n) - H_3^{[n]}(\mathfrak{f})| > \delta\right) = -\infty, \quad \text{where} \quad H_3^{[n]}(\mathfrak{f}) = \sum_{\ell=0}^{p} 2^{-\ell} \left\langle \mu, \tilde{f}_\ell^2 \right\rangle.$$

We set

$$g_p = \sum_{\ell=0}^{p} 2^{-\ell} \mathcal{Q}^{p-\ell} \left(\tilde{f}_\ell^2 - \left\langle \mu, \tilde{f}_\ell^2 \right\rangle\right) \quad \text{and then} \quad V_3(n) - H_3^{[n]}(\mathfrak{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(g_p).$$

Using (3.1) and (3.4), we have, for some positive constant c_α,

$$\|g_p\|_\infty \leq \begin{cases} 4c_\infty^2 c_\alpha M 2^{-p} & \text{if } 2\alpha < 1 \\ 4c_\infty^2 M (p+1)\alpha^p & \text{if } 2\alpha > 1. \end{cases}$$

Using (5.15), we get, for n going to infinity and for some positive constant $C_{\alpha,\delta}$:

$$\mathbb{P}\left(\left|V_3(n) - H_3^{[n]}(\mathfrak{f})\right| > \delta\right) \leq \begin{cases} 2\exp\left(-\delta^2 C_{\alpha,\delta} |\mathbb{G}_{n+p}|\right) & \text{if } 2\alpha \leq 1 \\ 2\exp\left(-\delta^2 C_{\alpha,\delta} p^{-2} |\mathbb{G}_n| (2\alpha^2)^{-p}\right) & \text{if } 1 < 2\alpha < \sqrt{2}. \end{cases}$$

Finally, (5.20) follows from the latter inequality by taking the log and dividing by b_n^2.

Part II. Next, we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|V_4(n) - H_4(\mathfrak{f})| > 2\delta\right) = -\infty. \tag{5.21}$$

Note that $V_4(n) - H_4(\mathfrak{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}(\mathfrak{f}) - H_4(\mathfrak{f}))$, where

$$H_{4,n}(\mathfrak{f}) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k<p\}} \quad \text{and} \quad H_4(\mathfrak{f}) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}, \tag{5.22}$$

with

$$h_{\ell,k}^{(n)} = 2^{k-\ell}\, \mathcal{Q}^{p-1-(\ell+k)} \left(\mathcal{P}\left(\mathcal{Q}^k \tilde{f}_\ell \otimes^2\right)\right) \quad \text{and} \quad h_{\ell,k} = \left\langle \mu, \mathcal{P}\left((\mathcal{Q}^k \tilde{f}_\ell) \otimes^2\right)\right\rangle.$$

Using (3.1) and (3.4), we have

$$|h_{\ell,k}| + |h_{\ell,k}^{(n)}| \leq 2C_\infty^2 M^2 (2\alpha^2)^k 2^{-\ell}. \tag{5.23}$$

Let r_0 large enough such that

$$2c_\infty^2 M^2 \sum_{\ell \vee k > r_0} (2\alpha^2)^k 2^{-\ell} \leq \delta. \tag{5.24}$$

For n going to infinity, we have

$$\begin{aligned} |M_{\mathbb{G}_{n-p}}(H_{4,n}(\mathfrak{f}) - H_4(\mathfrak{f}))| &\leq |M_{\mathbb{G}_{n-p}}(\sum_{\ell \vee k \leq r_0} (h_{\ell,k}^{(n)} - h_{\ell,k}))| + M_{\mathbb{G}_{n-p}}(\sum_{\ell \vee k > r_0} (|h_{\ell,k}| + |h_{\ell,k}^{(n)}|)) \\ &\leq |M_{\mathbb{G}_{n-p}}(\sum_{\ell \vee k \leq r_0} (h_{\ell,k}^{(n)} - h_{\ell,k}))| \, + \, 2\, c_\infty^2 \, M^2 |\mathbb{G}_{n-p}| \sum_{\ell \vee k > r_0} (2\alpha^2)^k 2^{-\ell}, \end{aligned} \tag{5.25}$$

where we used (5.23) for the second inequality. From (5.25), we get

$$|V_4(n) - H_4(\mathfrak{f})| \leq |\mathbb{G}_{n-p}|^{-1} |M_{\mathbb{G}_{n-p}}(g_p)| \, + \, 2\, c_\infty^2 \, M^2 \sum_{\ell \vee k > r_0} (2\alpha^2)^k 2^{-\ell}, \tag{5.26}$$

where $g_p = \sum_{\ell \vee k \leq r_0} (h_{\ell,k}^{(n)} - h_{\ell,k})$. From (5.24) and (5.26), to get (5.21), it suffices to prove that

$$\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left(\left| |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(g_p) \right| > \delta \right). \tag{5.27}$$

Using (3.1) and (3.4) twice, we have, for some positive constant c_α,

$$\|g_p\|_\infty \leq c_\infty^3 M^3 c_\alpha \gamma(r_0) \alpha^{p-1} \quad \text{where} \quad \gamma(r_0) = \begin{cases} r_0 (2\alpha)^{r_0} & \text{if } 2\alpha \leq 1 \\ (2\alpha)^{r_0} & \text{if } 1 < 2\alpha < \sqrt{2}. \end{cases}$$

Using (5.15) with g_p instead of f, we get, for some positive constant $C_{\alpha,\delta}$,

$$\mathbb{P} \left(\left| |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(g_p) \right| > \delta \right) \leq \exp \left(-\delta^2 C_{\alpha,\delta} |\mathbb{G}_{n-p}| \alpha^{-2p} \right).$$

Taking the log, dividing by b_n^2 and letting n goes to infinity in the latter inequality, we get (5.27) and then (5.21). □

Lemma 5.4. *Under the Assumptions of Theorem 4.1, we have*

$$\limsup_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{P} \left(\left| V_2(n) - \Sigma_2^{\text{sub}}(\mathfrak{f}) \right| > \delta \right) = -\infty,$$

where

$$\Sigma_2^{\text{sub}}(\mathfrak{f}) = \sum_{0 \leq \ell < k} 2^{-\ell} \left\langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \right\rangle + \sum_{\substack{0 \leq \ell < k \\ r \geq 0}} 2^{r-\ell} \left\langle \mu, \mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right\rangle$$

$$:= H_5(\mathfrak{f}) + H_6(\mathfrak{f})$$

Proof: Using (8.3), we get:

$$V_2(n) = V_5(n) + V_6(n), \tag{5.28}$$

with

$$V_5(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{0 \leq \ell < k \leq p} 2^{p-\ell} \mathcal{Q}^{p-k} \left(\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \right) (X_i),$$

$$V_6(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p}} \sum_{0 \leq \ell < k < p} \sum_{r=0}^{p-k-1} 2^{p-\ell+r} \, \mathcal{Q}^{p-1-(r+k)} \left(\mathcal{P} \left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell \right) \right) (X_i).$$

Part I. First, we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|V_6(n) - H_6(\mathfrak{f})| > 2\delta\right) = -\infty. \tag{5.29}$$

We have $V_6(n) - H_6(\mathfrak{f}) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n}(\mathfrak{f}) - H_6(\mathfrak{f}))$, where

$$H_{6,n}(\mathfrak{f}) = \sum_{\substack{0\le\ell<k\\ r\ge 0}} h_{k,\ell,r}^{(n)} \, \mathbf{1}_{\{r+k<p\}} \quad \text{and} \quad H_6(\mathfrak{f}) = \sum_{\substack{0\le\ell<k\\ r\ge 0}} h_{k,\ell,r},$$

with

$$h_{k,\ell,r}^{(n)} = 2^{r-\ell}\, \mathcal{Q}^{p-1-(r+k)} \left(\mathcal{P}\left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell\right)\right) \quad \text{and}$$
$$h_{k,\ell,r} = 2^{r-\ell} \left\langle \mu, \mathcal{P}\left(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell\right)\right\rangle.$$

Using (3.1) and (3.4), we have

$$|h_{\ell,k,r}| + |h_{\ell,k,r}^{(n)}| \le 2\, c_\infty^2\, M^2\, 2^{r-\ell}\, \alpha^{k-\ell+2r}. \tag{5.30}$$

Let r_0 large enough such that

$$2\, c_\infty^2\, M^2 \sum_{\substack{0\le\ell<k\\ r\ge 0\\ k\vee r>r_0}} 2^{r-\ell}\, \alpha^{k-\ell+2r} < \delta. \tag{5.31}$$

We set $g_p = \sum_{0\le\ell<k,\ r\ge 0,\ k\vee r\le r_0} (h_{k,\ell,r}^{(n)} - h_{k,\ell,r})$. Using (5.30), we have, for n going to infinity in such a way that $p > r_0$,

$$|V_6(n) - H_6(\mathfrak{f})| \le |\mathbb{G}_{n-p}| \left|M_{\mathbb{G}_{n-p}}(g_p)\right| + 2\, c_\infty^2\, M^2 \sum_{\substack{0\le\ell<k\\ r\ge 0\\ k\vee r>r_0}} 2^{r-\ell}\, \alpha^{k-\ell+2r}. \tag{5.32}$$

From (5.31) and (5.32), to get (5.29), it suffices to prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left| |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(g_p)\right| > \delta\right) = -\infty. \tag{5.33}$$

Using (3.1) and (3.4) twice, we have, for some positive constant c_α,

$$\|g_p\|_\infty \le 2\, c_\alpha\, c_\infty^3\, M^3\, \gamma(r_0)\, \alpha^p,$$

where

$$\gamma(r_0) = \begin{cases} (2\alpha)^{-r_0} & \text{if } 2\alpha < 1 \\ r_0^2 & \text{if } 2\alpha \ge 1. \end{cases}$$

Using (5.15) with g_p instead of f, we get, for some positive constant $C_{\alpha,\delta}$,

$$\mathbb{P}\left(\left||\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(g_p)\right| > \delta\right) \le \exp\left(-\delta^2 C_{\alpha,\delta} |\mathbb{G}_{n-p}| \alpha^{-2p}\right).$$

Taking the log, dividing by b_n^2 and letting n goes to infinity in the latter inequality, we get (5.33) and then (5.29).

Part II. Next, with the finite constant $H_5(\mathfrak{f})$ defined by:

$$H_5(\mathfrak{f}) = \sum_{0\leq \ell<k} 2^{-\ell}\langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell}\tilde{f}_\ell\rangle,$$

we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2}\log \mathbb{P}\left(|V_5(n) - H_5(\mathfrak{f})| > 2\delta\right) = -\infty. \tag{5.34}$$

We set

$$H_{5,n}(\mathfrak{f}) = \sum_{0\leq \ell<k} h_{k,\ell}^{(n)}\mathbf{1}_{\{k\leq p\}}, \quad \text{and} \quad H_5^{[n]}(\mathfrak{f}) = \sum_{0\leq \ell<k} h_{k,\ell}\mathbf{1}_{\{k\leq p\}},$$

with

$$h_{k,\ell}^{(n)} = 2^{-\ell}\mathcal{Q}^{p-k}\left(\tilde{f}_k\mathcal{Q}^{k-\ell}\tilde{f}_\ell\right)\mathbf{1}_{\{k\leq p\}} \quad \text{and} \quad h_{\ell,k} = \left\langle \mu, \tilde{f}_k\mathcal{Q}^{k-\ell}\tilde{f}_\ell\right\rangle.$$

We have the following decomposition:

$$V_5(n) - H_5(\mathfrak{f}) = |\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}\left(H_{5,n}(\mathfrak{f}) - H_5^{[n]}(\mathfrak{f})\right) + \left(H_5^{[n]}(\mathfrak{f}) - H_5(\mathfrak{f})\right).$$

Using (3.1) and (3.4), we have

$$|h_{k,\ell}^{(n)}| + |h_{k,\ell}| \leq 2c_\infty^2 M\alpha^{k-\ell}2^{-\ell},$$

which implies that $\lim_{n\to\infty}|H_5(\mathfrak{f}) - H_5^{[n]}(\mathfrak{f})| = 0$. As a result, to get (5.34), it suffices to prove that

$$\limsup_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(\left||\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}\left(H_{5,n}(\mathfrak{f}) - H_5^{[n]}(\mathfrak{f})\right)\right| > \delta\right) = -\infty. \tag{5.35}$$

Setting $g_p = H_{5,n}(\mathfrak{f}) - H_5^{[n]}(\mathfrak{f})$, we have, using (3.1) and (3.4):

$$\|g_p\|_\infty \leq \begin{cases} c_\alpha 2^{-p} & \text{if } 2\alpha < 1 \\ c_\alpha p\alpha^p & \text{if } 1\leq 2\alpha < \sqrt{2}, \end{cases}$$

for some positive constant c_α. Finally, (5.35), and then (5.34), follows by applying (5.15) to g_p instead of f and by taking the log, dividing by b_n^2 and by letting n goes to infinity. □

As a direct consequence of (5.16) and Lemmas 5.2, 5.3 and 5.4, we have the following result.

Lemma 5.5. *Under the Assumptions of Theorem 4.1, we have*

$$\limsup_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(|V(n) - \Sigma^{\text{sub}}(\mathfrak{f})| > \delta\right) = -\infty$$

We can now state the following result.

Lemma 5.6. *Under Assumptions of Theorem 4.1, we have that $b_n^{-1}\Delta_n(\mathfrak{f})$ satisfies a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7).*

Proof: Since $p < n/2$, we have for all $i \in \mathbb{G}_{n-p}$,

$$|\Delta_{n,i}(\mathfrak{f})| \leq 2c_\infty 2^{-\frac{n}{2}+p} \leq C,$$

where C is a positive constant. This implies that $\Delta_n(\mathfrak{f})$ is a martingale with bounded differences. Using the result of Dembo (1996) (see also Djellout (2002) and Puhalskii (1997)), we get the result from Lemma 5.5. □

5.5. *Completion of the proof of Theorem 4.1.* Finally, using the decomposition (5.2) and the results of sections 5.2, 5.3 and 5.4, we deduce Theorem 4.1.

6. Proof of Theorem 4.3

6.1. *A quick overview of our strategy.*
Let $(p_n, n \in \mathbb{N})$ be a non-decreasing sequence of elements of $\mathbb{N}^*$ such that, for all $\lambda > 0$:

$$p_n < n, \quad \lim_{n\to\infty} p_n/n = 1, \quad \lim_{n\to\infty} n^2\, b_n^2\, 2^{-n+p} = 0 \quad \text{and} \quad \lim_{n\to\infty} n - p_n - \lambda\log(n) = +\infty. \tag{6.1}$$

When there is no ambiguity, we write p for p_n. From (4.1), we can choose $p_n = n - n^a$, $a \in (0,1)$. Let us consider the sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of $\mathcal{B}_b(S)$ which satisfies the Assumption (3.2) (and then Assumption 3.1) uniformly, namely:

$$|\mathcal{Q}^n(\tilde{f}_\ell)| \leq M\alpha^n c_\infty \quad \text{and} \quad |\mathcal{Q}^n(\hat{f}_\ell)| \leq M\beta_n\alpha^n c_\infty \tag{6.2}$$

It follows from (6.2) that there exists a finite constant c_J depending only on $\{\alpha_j, j\in J\}$ such that for all $\ell \in \mathbb{N}$, $n \in \mathbb{N}$, $j_0 \in J$

$$|f_\ell| \leq Mc_\infty, \quad |\tilde{f}_\ell| \leq Mc_\infty, \quad |\langle \mu, f_\ell\rangle| \leq Mc_\infty, \quad |\sum_{j\in J}\theta_j^n \mathcal{R}_j(f_\ell)| \leq 2Mc_\infty \quad \text{and}$$
$$|\mathcal{R}_{j_0}(f_\ell)| \leq c_J M c_\infty.$$

We recall that:

$$N_{n,\emptyset}(\mathfrak{f}) = \Delta_n(\mathfrak{f}) + R_0(n) + R_1(n),$$

with

$$R_0(n) = |\mathbb{G}_n|^{-1/2}\sum_{k=0}^{n-p_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}) \quad \text{and} \quad R_1(n) = \sum_{i\in\mathbb{G}_{n-p_n}} \mathbb{E}\left[N_{n,i}(\mathfrak{f})\middle|\, \mathcal{F}_i\right].$$

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence elements of $\mathbb{N}$ such that :

$$b_n \to \infty \qquad \text{and} \qquad \frac{b_n}{\sqrt{n|\mathbb{G}_n|}} \underset{n\to\infty}{\longrightarrow} 0$$

Our goals will be achieved if we prove the following:

$$\forall\delta > 0, \quad \lim_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}(|b_n^{-1}n^{-1/2}R_0(n)| > \delta) = -\infty; \tag{6.3}$$

$$\forall\delta > 0, \quad \lim_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}(|b_n^{-1}n^{-1/2}R_1(n)| > \delta) = -\infty; \tag{6.4}$$

$$b_n^{-1}n^{-1/2}\Delta_n(\mathfrak{f}) \quad \text{satisfies a MDP on } S \text{ with speed } b_n^2 \text{ and rate function } I. \tag{6.5}$$

6.2. *Proof of* (6.3).
We follow the same lines of the proof of (5.3) with $2\alpha^2 = 1$. First, using Chernoff inequality, we have

$$\mathbb{P}\left(b_n^{-1}n^{-1/2}R_0(n) > \delta\right) \leq \exp\left(-\lambda b_n\delta|\mathbb{G}_n|^{\frac{1}{2}}n^{1/2}\right)\mathbb{E}\left[\exp\left(\lambda\sum_{k=0}^{n-p-1}M_{\mathbb{G}_k}(\tilde{f}_{n-k})\right)\right].$$

Next, taking $\lambda = \frac{b_n\delta(n|\mathbb{G}_n|)^{1/2}}{2c_\alpha(n-p)|\mathbb{G}_{n-p}|}$ and doing the same thing for $-\mathfrak{f}$ instead of $\mathfrak{f}$, we get

$$\mathbb{P}\left(|b_n^{-1}n^{-1/2}R_0(n)| > \delta\right)| \leq 2\exp\left(-\frac{b_n^2\delta^2 n|\mathbb{G}_p|}{4c_\alpha(n-p)}\right).$$

Finally, taking the log, dividing by b_n^2 and letting n goes to infinity, we get the result.

Remark 6.1. We have the following version of Remark 5.1 when $2\alpha^2 = 1$:

$$\mathbb{P}\left(||\mathbb{G}_{n-p}|^{-1}M_{\mathbb{G}_{n-p}}(\tilde{f})| > \delta n^{1/2}\right) \leq 2\exp\left(-\frac{\delta^2 n|\mathbb{G}_{n-p}|}{4\|f\|_\infty^2 M^2(1+\alpha)^2(n-p)}\right). \tag{6.6}$$

6.3. *Proof of* (6.4). With $g_p = \sum_{\ell=0}^{p} 2^{p-\ell}\mathcal{Q}^{p-\ell}\tilde{f}_\ell$, and using the definition of $R_1(n)$, we have for all $\delta > 0$

$$\mathbb{P}\left(|b_n^{-1}R_1(n)| > \delta n^{1/2}\right) = \mathbb{P}\left(|\mathbb{G}_{n-p}|^{-1}|M_{\mathbb{G}_{n-p}(g_p)}| > b_n\delta n^{1/2}|\mathbb{G}_p||\mathbb{G}_n|^{-1/2}\right).$$

So according to (6.2) , we have:

$$\|g_p\|_\infty \leq c_\infty c_\alpha M|\mathbb{G}_p|^{1/2}.$$

By applying (6.6) to g_p and $b_n\delta n^{1/2}|\mathbb{G}_p||\mathbb{G}_n|^{-1/2}$ and using the fact that $2\alpha^2 = 1$, we have:

$$\mathbb{P}\left(|b_n^{-1}R_1(n)| > \delta n^{1/2}\right) \leq 2\exp\left(-\frac{b_n^2\delta^2 n}{4c_\infty^2 c_\alpha^2 M^4(1+\alpha)^2(n-p)}\right).$$

So taking the log and dividing by b_n^2, and letting n goes to infinity, we get the result.

6.4. *Proof of* (6.5)*: Moderate deviations principle for* $b_n^{-1}n^{-1/2}\Delta_n(\mathfrak{f})$.
First we study the bracket of $n^{-\frac{1}{2}}\Delta_n(\mathfrak{f})$ given by $n^{-1}V(n)$, where $V(n)$ is defined in (5.16). We have the following result:

Lemma 6.2. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(|n^{-1}R_2(n)| > \delta\right) = -\infty.$$

Proof: Recall the definition of $R_2(n)$ and g_p given in (5.18). So according to (6.2) and using $2\alpha^2 = 1$, we have $\|g_p\|_\infty \leq c_\infty^2 c_\alpha^2 M^2|\mathbb{G}_p|$. This implies that

$$R_2(n) \leq \frac{c_\infty^2 c_\alpha^2 M^2}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, $R_2(n)$ is upper-bounded by a deterministic sequence which goes to 0. According to Remark 8.1, we get the result. □

Lemma 6.3. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(|n^{-1}V_1(n) - \Sigma_1^{\text{crit}}(\mathfrak{f})| > \delta\right) = -\infty.$$

Proof: Recall the decomposition of $V_1(n)$ given in (5.19) and the definition of $\Sigma_1^{\text{crit}}(\mathfrak{f})$ given in 3.8. The proof is divided into two parts.

Part I. First we prove that

$$\limsup_{n\to\infty}\frac{1}{b_n^2}\log\mathbb{P}\left(|n^{-1}V_3(n)| > \delta\right) = -\infty.$$

Indeed we have

$$n^{-1}V_3(n) = |\mathbb{G}_{n-p}|^{-1}n^{-1}M_{\mathbb{G}_{n-p}}(g_p), \quad \text{where} \quad g_p = \sum_{\ell=0}^{p} 2^{-\ell}\mathcal{Q}^{p-\ell}(\tilde{f}_\ell^2).$$

Since the sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ satisfies (3.4), we have $\|g_p\|_\infty \leq 4c_\infty^2$. This implies that $|n^{-1}V_3(n)| \leq 4c_\infty^2 n^{-1}$. So $n^{-1}V_3(n)$ is upper-bounded by a deterministic sequence which goes to 0. Then applying the remark 8.1 to $n^{-1}V_3(n)$, we get the result.

Part II. Next, we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|n^{-1}V_4(n) - \Sigma_1^{\text{crit}}(\mathfrak{f})| > \delta\right) = -\infty.$$

Recall $H_{4,n}(\mathfrak{f})$ and $f_{k,\ell}$ defined respectively in (5.22) and (3.9). For $k, \ell, r \in \mathbb{N}$, we consider the following $\mathbb{C}$-valued functions defined on S^2:

$$f_{k,\ell,r} = \Big(\sum_{j\in J} \theta_j^r \mathcal{R}_j(f_k)\Big) \otimes_{\text{sym}} \Big(\sum_{j\in J} \theta_j^{r+k-\ell} \mathcal{R}_j(f_\ell)\Big).$$

Recall that $2\alpha^2 = 1$. We set $\bar{H}_{4,n} = \sum_{\ell\geq 0,\, k\geq 0} \bar{h}_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k<p\}}$ with

$$\bar{h}_{\ell,k}^{(n)} = 2^{k-\ell} \alpha^{2k} \mathcal{Q}^{p-1-(\ell+k)} (\mathcal{P} f_{\ell,\ell,k}) = 2^{-\ell} \mathcal{Q}^{p-1-(\ell+k)} (\mathcal{P} f_{\ell,\ell,k}).$$

For $f \in \mathcal{B}_b(S)$, recall $\hat{f}$ defined in (3.3). Then we have $h_{\ell,k}^{(n)} - \bar{h}_{k,\ell}^{(n)} = h_{\ell,k}^{n,1} + h_{\ell,k}^{n,2} + h_{\ell,k}^{n,3}$, where

$$h_{\ell,k}^{n,1} = 2^{k-\ell} \mathcal{Q}^{p-1-(\ell+k)} \mathcal{P}(\mathcal{Q}^k \hat{f}_k \otimes_{\text{sym}} \mathcal{Q}^k \hat{f}_k),$$
$$h_{\ell,k}^{n,2} = 2^{k-\ell} \mathcal{Q}^{p-1-(\ell+k)} \mathcal{P}(\mathcal{Q}^k \hat{f}_k \otimes_{\text{sym}} \mathcal{Q}^k (\sum_{j\in J} \mathcal{R}_j(f_k))),$$
$$h_{\ell,k}^{n,3} = 2^{k-\ell} \mathcal{Q}^{p-1-(\ell+k)} \mathcal{P}(\mathcal{Q}^k (\sum_{j\in J} \mathcal{R}_j(f_k)) \otimes_{\text{sym}} \mathcal{Q}^k (\sum_{j\in J} \mathcal{R}_j(f_k))).$$

This implies that

$$n^{-1}|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}(\mathfrak{f}) - \bar{H}_{4,n}(\mathfrak{f})) = n^{-1}|\mathbb{G}_{n-p}|^{-1} \sum_{u\in\mathbb{G}_{n-p}} \sum_{i=1}^{3} \sum_{\ell\geq 0; k\geq 0} h_{\ell,k}^{n,i}(X_u) \mathbf{1}_{\{\ell+k<p\}}.$$

Using Assumption 3.2, (3.4) and the fact that the sequence $(\beta_k, k \in \mathbb{N})$ is decreasing, we can upper bound each function $|h_{\ell,k}^{n,i}|$, $i \in \{1, 2, 3\}$, by $C 2^{-\ell} \beta_k$. This implies that

$$\left|n^{-1}|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{4,n}(\mathfrak{f}) - \bar{H}_{4,n}(\mathfrak{f}))\right| \leq C n^{-1} \sum_{k=0}^{p-1} \beta_k. \tag{6.7}$$

We set

$$H_4^{[n]} = \sum_{\ell\geq 0; k\geq 0} h_{\ell,k} \mathbf{1}_{\{k+\ell<p\}} \quad \text{with} \quad h_{\ell,k} = 2^{-\ell} \langle \mu, \mathcal{P}(f_{\ell,\ell,k}) \rangle.$$

Using Assumption 3.2 and (3.4), we get

$$\left|\bar{h}_{\ell,k}^{(n)} - h_{\ell,k}\right| \leq 2^{-\ell} \left(\frac{1}{\sqrt{2}}\right)^{p-1-(\ell+k)} \|\mathcal{P}(f_{\ell,\ell,k})\|_\infty \leq C \left(\frac{1}{\sqrt{2}}\right)^{p-1} \left(\frac{1}{\sqrt{2}}\right)^{\ell-k}.$$

This implies that

$$\left|n^{-1}|\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\bar{H}_{4,n}(\mathfrak{f}) - H_4^{[n]}(\mathfrak{f}))\right| \leq C n^{-1}. \tag{6.8}$$

Finally, from Bitseki Penda and Delmas (2022+), we have

$$\lim_{n\to\infty} |n^{-1} H_4^{[n]}(\mathfrak{f}) - \Sigma_1^{\text{crit}}(\mathfrak{f})| = 0. \tag{6.9}$$

From (6.7), (6.8) and (6.9), we conclude that $|n^{-1}V_4(n) - \Sigma_1^{\text{crit}}(\mathfrak{f})|$ is upper-bounded by a deterministic sequence which goes to 0. Therefore applying the remark 8.1 to $n^{-1}V_4(n) - \Sigma_1^{\text{crit}}(\mathfrak{f})$, we get the result. □

Lemma 6.4. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|n^{-1}V_2(n) - \Sigma_2^{\text{crit}}(\mathfrak{f})| > \delta\right) = -\infty.$$

Proof: Recall the decomposition given in (5.28). Then following the lines of the proof of Lemma 6.3, we prove that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|n^{-1}V_5(n)| > \delta\right) = -\infty \quad \text{and}$$

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|n^{-1}V_6(n) - \Sigma_2^{\text{crit}}(\mathfrak{f})| > \delta\right) = -\infty,$$

and the result follows. □

As a direct consequence of (5.16) and Lemmas 6.2, 6.3 and 6.4, we have the following result.

Lemma 6.5. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(|V(n) - \Sigma^{\text{crit}}(\mathfrak{f})| > \delta\right) = -\infty.$$

Next, contrary to the sub-critical case, we need to check the exponential Lindeberg condition and Chen-Ledoux type condition, that is conditions **(C2)** and **(C3)** given in Proposition 8.4. Indeed, in the critical case, the martingale $n^{-\frac{1}{2}}\Delta_n(\mathfrak{f})$ does not have bounded differences in such a way that Lemma 6.5 is not longer sufficient to get the moderate deviations principle of $n^{-\frac{1}{2}}\Delta_n(\mathfrak{f})$. In order to check exponential Lindeberg condition, we have the following exponential Lyapunov condition which implies the exponential Lindeberg condition.

Lemma 6.6. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\sum_{i\in\mathbb{G}_{n-p}} n^{-2}\mathbb{E}\left[\Delta_{n,i}(\mathfrak{f})^4|\mathcal{F}_i\right] > \frac{\delta n}{b_n^2}\right) = -\infty \quad \forall \delta > 0.$$

Proof: For all $i \in \mathbb{G}_{n-p}$, we have

$$\mathbb{E}\left[\Delta_{n,i}(\mathfrak{f})^4|\mathcal{F}_i\right] \leq 16(p+1)^3 2^{-2n} \sum_{\ell=0}^{p} \mathbb{E}_{X_i}\left[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4\right], \tag{6.10}$$

where we have used the definition of $\Delta_{n,i}(\mathfrak{f})$, the inequality $(\sum_{k=0}^{r} a_k)^4 \leq (r+1)^3 \sum_{k=0}^{r} a_k^4$ and the branching Markov property. Using (3.1) and (3.4), we can apply Theorem 2.1 given in Bitseki Penda et al. (2014) with $2\alpha^2 = 1$ to get $\mathbb{E}_{X_i}[M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)^4] \leq Cp^2 2^{2(p-\ell)}$. The latter inequality and (6.10) imply that

$$\frac{b_n^2}{n} \sum_{i\in\mathbb{G}_{n-p}} n^{-2}\mathbb{E}\left[\Delta_{n,i}(\mathfrak{f})^4|\mathcal{F}_i\right] \leq Cn^3 2^{-n+p}(n^{-1}b_n^2). \tag{6.11}$$

From (6.1), we have

$$\lim_{n\to\infty} n^3 2^{-n+p}(n^{-1}b_n^2) = 0. \tag{6.12}$$

Finally, the result of the Lemma follows using (6.11), (6.12) and Remark 8.1. □

For Chen-Ledoux type condition, we have the following result.

Lemma 6.7. *Under the assumptions of Theorem 4.3, we have*

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log\left(|\mathbb{G}_n| \sup_{i\in\mathbb{G}_{n-p}} \mathbb{P}_{\mathcal{F}_i}\left(|\Delta_{n,i}(\mathfrak{f})| > b_n\sqrt{n}\right)\right) = -\infty.$$

Proof: For all $i \in \mathbb{G}_{n-p}$, using (5.1) we have

$$\mathbb{P}_{\mathcal{F}_i}\left(|\Delta_{n,i}(\mathfrak{f})| > b_n\sqrt{n}\right) \leq \mathbb{P}_{\mathcal{F}_i}\left(|N_{n,i}(\mathfrak{f})| > b_n\sqrt{n}/2\right) + \mathbb{P}_{\mathcal{F}_i}\left(|\mathbb{E}_{X_i}\left[N_{n,i}(\mathfrak{f})\right]| > b_n\sqrt{n}/2\right), \tag{6.13}$$

with $N_{n,i}(\mathfrak{f})$ defined in (3.5). Following the proof of (5.11), we get

$$\mathbb{P}_{\mathcal{F}_i}\Big(|N_{n,i}(\mathfrak{f})| > \frac{b_n\sqrt{n}}{2}\Big) = \mathbb{P}_{X_i}\Big(|\sum_{\ell=0}^{p} M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell)| > \frac{b_n\sqrt{n|\mathbb{G}_n|}}{2}\Big) \leq C\exp\Big(-\frac{Cb_n^2|\mathbb{G}_{n-p}|n}{p}\Big).$$

Next, for all $\lambda = Cb_n\sqrt{n|\mathbb{G}_n|}\,(p|\mathbb{G}_p|)^{-1}$, we have

$$\begin{aligned}\mathbb{P}_{\mathcal{F}_i}\Big(\mathbb{E}_{X_i}\left[N_{n,i}(\mathfrak{f})\right] > \frac{b_n\sqrt{n}}{2}\Big) &= \mathbb{P}_{X_i}\Big(\sum_{\ell=0}^{p} 2^{p-\ell}\mathcal{Q}^{p-\ell}(\tilde{f}_\ell)(X_i) > \frac{b_n\sqrt{n|\mathbb{G}_n|}}{2}\Big)\\ &\leq \exp\Big(-\frac{\lambda b_n\sqrt{n|\mathbb{G}_n|}}{2}\Big)\mathbb{E}_{X_i}\Big[\exp\Big(\lambda\sum_{\ell=0}^{p} 2^{p-\ell}\mathcal{Q}^{p-\ell}(\tilde{f}_\ell)(X_i)\Big)\Big]\\ &\leq C\exp\Big(-\frac{Cb_n^2|\mathbb{G}_{n-p}|n}{p}\Big),\end{aligned}$$

where we used (8.1) and the branching Markov property for the first equality, Chernoff bound for the first inequality and (3.1) for the last inequality. Doing the same thing for $-\mathfrak{f}$ instead of $\mathfrak{f}$, we get

$$\mathbb{P}_{\mathcal{F}_i}\Big(|\mathbb{E}_{X_i}\left[N_{n,i}(\mathfrak{f})\right]| > \frac{b_n\sqrt{n}}{2}\Big) \leq 2\,C\,\exp\Big(-\frac{Cb_n^2|\mathbb{G}_{n-p}|n}{p}\Big).$$

From the foregoing, we get, using (6.13),

$$|\mathbb{G}_n|\sup_{i\in\mathbb{G}_{n-p}}\mathbb{P}_{\mathcal{F}_i}\left(|\Delta_{n,i}(\mathfrak{f})| > b_n\sqrt{n}\right) \leq C\,|\mathbb{G}_n|\,\exp\Big(-\frac{Cb_n^2|\mathbb{G}_{n-p}|n}{p}\Big).$$

Finally, taking the log and dividing by b_n^2 in the latter inequality, we get the result of Lemma 6.7. ☐

We can now state the following result.

Lemma 6.8. *Under the assumptions of Theorem 4.3, we have that $n^{-1/2}b_n^{-1}\Delta_n(\mathfrak{f})$ satisfies a moderate deviation principle with speed b_n^2 and rate function I defined in (3.7).*

Proof: Applying Theorem 1 in Djellout (2002) (a simplified version is given in Proposition 8.4) to the martingale differences $n^{-1/2}\Delta_{n,i}(\mathfrak{f})$, the proof follows from Lemmas 6.5, 6.6 and 6.7. ☐

6.5. *Completion of the proof of Theorem 4.3.* Finally, using (5.2), (6.3), (6.4) and Lemma 6.8, we deduce Theorem 4.3.

7. Numerical studies

For our numerical illustrations, we consider a BMC $(X_u, u\in\mathbb{T})$ living in $[0,1]$, with transition $\mathcal{P} = \mathcal{Q}\otimes\mathcal{Q}$ given by

$$\mathcal{Q}(x,y) := (1-x)\,\frac{y(1-y)^2}{B(2,3)} + x\,\frac{y^2(1-y)}{B(3,2)}\,,\quad x,y\in[0,1],$$

with $B(\alpha,\beta)$ the normalizing constant of a standard Beta distribution with shape parameters α and β. For simplicity, we choose $X_\emptyset$ such that $\mathcal{L}(X_\emptyset) = Beta(2,2)$, where $Beta(2,2)$ is the standard Beta distribution with shape parameters $(2,2)$. Now, one can prove that this process is stationary, it has an explicit invariant density: the standard Beta distribution with shape parameters $(2,2)$. One can also prove that $\mathbb{E}\left[X_{u0}|X_u\right] = \mathbb{E}\left[X_{u1}|X_u\right] = X_u/5 + 2/5$, (for more details, we refer e.g. to Pitt et al. (2002)). Now, it is not hard to verify that this process satisfies our required assumptions. In particular, using for example Theorem 2.1 in Hairer and Mattingly (2011), one can prove that Assumption 3.1 is satisfied with $\alpha = 1/5$. We are thus in the sub-critical case. First, we will illustrate

Theorem 4.1 (and more precisely Corollary 4.2) with the sequence $\mathfrak{f} = (f, 0, 0, \dots)$ and the function $f(x) = x$. In this case, we have the following exact results:

$$\langle \mu, f\rangle = \frac{1}{2}, \quad \mathcal{Q}^k \tilde{f}(x) = 5^{-k}(x - \tfrac{1}{2}) \quad \forall k \geq 0, \quad \Sigma_{\mathbb{G}}(f) = 6/115 \quad \text{and} \quad I(\delta) = \frac{115}{12}\delta^2.$$

Next, we will illustrate that the range of speed considered in the critical case does not work in this example. For that purpose, we simulate $B = 50000$ samples $(X^{(s)} = (X_u^{(s)}, u \in \mathbb{G}_{12}), s \in \{1, \dots, B\})$ of the bifurcating Markov chain at the n-th generation, with $n = 12$. For each sample $X^{(s)}$, we compute $b_n^{-1} N_{n,\emptyset}^{(s)}(f) = b_n^{-1}|\mathbb{G}_n|^{-1/2} \sum_{u\in\mathbb{G}_n}(X_u^{(s)} - 1/2)$. Finally, for different values of $\delta > 0$, we compute $b_n^{-2} \log(B^{-1} \sum_{s=1}^B \mathbf{1}_{\{|b_n^{-1} N_{n,\emptyset}^{(s)}(f)| > \delta\}})$. This allows us to get empirical values of the rate function. Next in the same graph, we plot the true rate function and the empirical rate function. As we can see in Figure 7.1, the empirical rate function fit well exact rate function, except in the last figure where the empirical rate function is near to 0 since the speed considered is not valid for the subcritical case, but only for the critical. We also stress that the differences observed between empirical and exact rate functions can be explained from the fact that the sample size is not large enough.

8. Appendix

The following Remark is used in the proofs of Lemmas 6.3, 6.4 and 6.2.

Remark 8.1.
We assume that (S, d) is a metric space. Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables valued in S, Z a random variable valued in S and v_n a rate. So if $d(Z_n, Z)$ is upper-bounded by a deterministic sequence which converges to 0, then, for all sequence $(v_n, n \in \mathbb{N})$ converging to ∞, Z_n converges v_n-superexponentially fast in probability to Z, that is for all $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{v_n} \log \mathbb{P}\left(d(Z_n, Z) > \delta\right) = -\infty.$$

The following result is known as Azuma-Bennett-Hoeffding inequality Azuma (1967); Bennett (1962); Hoeffding (1963).

Lemma 8.2. *Let X be a real-valued and centered random variable such that $a \leq X \leq b$ a.s., with $a < b$. Then for all $\lambda > 0$, we have*

$$\mathbb{E}\left[\exp\left(\lambda X\right)\right] \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right).$$

We have the following many-to-one formulas. Ideas of the proofs can be found in Guyon (2007) and Bitseki Penda et al. (2014).

Lemma 8.3. *Let $f, g \in \mathcal{B}(S)$, $x \in S$ and $n \geq m \geq 0$. Assuming that all the quantities below are well defined, we have:*

$$\mathbb{E}_x\left[M_{\mathbb{G}_n}(f)\right] = |\mathbb{G}_n|\, \mathcal{Q}^n f(x) = 2^n\, \mathcal{Q}^n f(x), \tag{8.1}$$

$$\mathbb{E}_x\left[M_{\mathbb{G}_n}(f)^2\right] = 2^n\, \mathcal{Q}^n(f^2)(x) + \sum_{k=0}^{n-1} 2^{n+k}\, \mathcal{Q}^{n-k-1}\left(\mathcal{P}\left(\mathcal{Q}^k f \otimes \mathcal{Q}^k f\right)\right)(x), \tag{8.2}$$

$$\begin{aligned}\mathbb{E}_x\left[M_{\mathbb{G}_n}(f) M_{\mathbb{G}_m}(g)\right] = {} & 2^n \mathcal{Q}^m\left(g \mathcal{Q}^{n-m} f\right)(x) \\ & + \sum_{k=0}^{m-1} 2^{n+k}\, \mathcal{Q}^{m-k-1}\left(\mathcal{P}\left(\mathcal{Q}^k g \otimes_{\text{sym}} \mathcal{Q}^{n-m+k} f\right)\right)(x).\end{aligned} \tag{8.3}$$

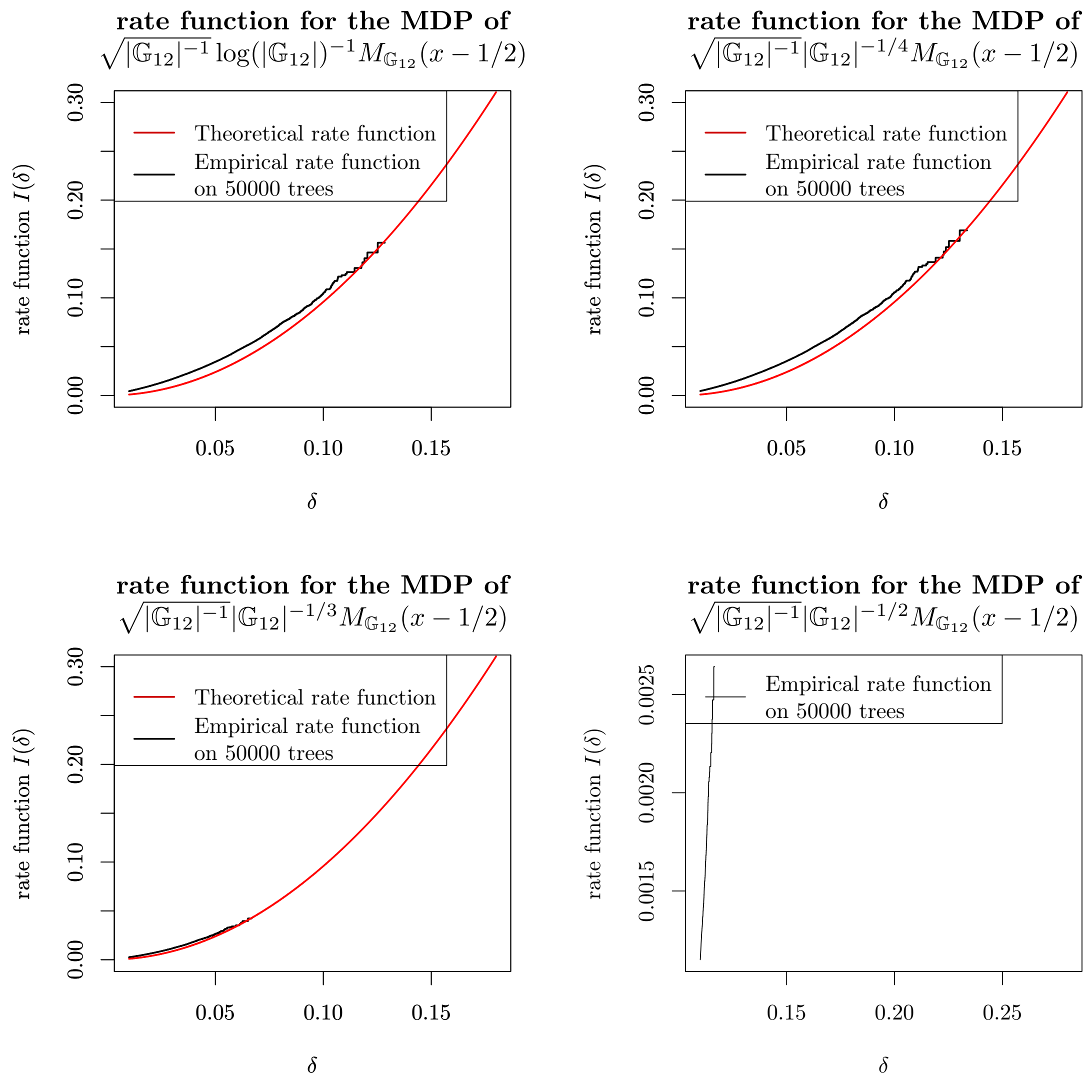

FIGURE 7.1. Exact and empirical rate functions for the moderate deviation principles of $|\mathbb{G}_n|^{-1/2}b_n^{-1}M_{\mathbb{G}_n}(x-1/2)$. In the first three figures, one can see that empirical rate function fit well the exact rate function. The differences can be explained from the fact that the sample size is not large enough to generate enough large deviation events. In the last figure, one can see that the empirical rate function is reduced to 0. This is due to the fact that the speed considered here is valid only in the critical case, not in the subcritical case.

We recall here a simplified version of Theorem 1 in Djellout (2002). We consider the real martingale $(M_n, n \in \mathbb{N})$ with respect to the filtration $(\mathcal{H}_n, n \in \mathbb{N})$ and we denote $(\langle M \rangle_n, n \in \mathbb{N})$ its bracket.

Proposition 8.4. *Let* (b_n) *a sequence satisfying*

$$b_n \quad \textit{is increasing}, \quad b_n \longrightarrow +\infty, \quad \frac{b_n}{\sqrt{n}} \longrightarrow 0,$$

such that $c(n) := \sqrt{n}/b_n$ *is non-decreasing, and define the reciprocal function* $c^{-1}(t)$ *by*

$$c^{-1}(t) := \inf\{n \in \mathbb{N} : c(n) \geq t\}.$$

Under the following conditions:

(**C1**) *there exists* $Q \in \mathbb{R}_+^*$ *such that for all* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log\left(\mathbb{P}\left(\left|\frac{\langle M\rangle_n}{n} - Q\right| > \delta\right)\right) = -\infty,$$

(**C2**) $\limsup_{n\to+\infty} \frac{1}{b_n^2} \log\left(n \operatorname*{ess\,sup}_{1\leq k\leq c^{-1}(b_{n+1})} \mathbb{P}(|M_k - M_{k-1}| > b_n\sqrt{n}\Big|\mathcal{H}_{k-1})\right) = -\infty,$

(**C3**) *for all* $a > 0$ *and for all* $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log\left(\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left(|M_k - M_{k-1}|^2 \mathbf{1}_{\{|M_k - M_{k-1}|\geq a\frac{n}{b_n}\}}\Big|\mathcal{H}_{k-1}\right) > \delta\right)\right) = -\infty,$$

$(M_n/(b_n\sqrt{n}))_{n\in\mathbb{N}}$ *satisfies the MDP in* $\mathbb{R}$ *with the speed* b_n^2/n *and the rate function* $I(x) = \dfrac{x^2}{2Q}$.

Remark 8.5. For all $n \geq 1$, we set $m_n = M_n - M_{n-1}$. Note that, in Proposition 8.4, if the sequence $(m_n)_{n\geq 1}$ is uniformly bounded, we recover a simplified version of the result of Dembo (1996) and if the sequence $(m_n)_{n\geq 1}$ is bounded by a deterministic sequence, we recover a simplified version of the result of Puhalskii (1997).

Acknowledgements

We warmly thank Arnaud Guillin for numerous discussions during the preparation of this article. We also thank the Referee for his useful comments which helped to improve the presentation.

References

Athreya, K. B. Limit theorems for multitype continuous time Markov branching processes. I. The case of an eigenvector linear functional. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **12**, 320–332 (1969). MR254927.

Azuma, K. Weighted sums of certain dependent random variables. *Tohoku Math. J. (2)*, **19**, 357–367 (1967). MR221571.

Basawa, I. V. and Zhou, J. Non-Gaussian bifurcating models and quasi-likelihood estimation. *J. Appl. Probab.*, **41A**, 55–64 (2004). MR2057565.

Bennett, G. Probability Inequalities for the Sum of Independent Random Variables. *J. Amer. Statist. Assoc.*, **57** (297), 33–45 (1962). DOI: 10.2307/2282438.

Bitseki Penda, S. V. and Delmas, J.-F. Central limit theorem for bifurcating Markov chains under pointwise ergodic conditions (2022+). To appear in *Ann. Appl. Probab.*

Bitseki Penda, S. V., Djellout, H., and Guillin, A. Deviation inequalities, moderate deviations and some limit theorems for bifurcating Markov chains with application. *Ann. Appl. Probab.*, **24** (1), 235–291 (2014). MR3161647.

Cowan, R. and Staudte, R. The Bifurcating Autoregression Model in Cell Lineage Studies. *Biometrics*, **42** (4), 769–783 (1986). DOI: 10.2307/2530692.

Delmas, J.-F. and Marsalle, L. Detection of cellular aging in a Galton-Watson process. *Stochastic Process. Appl.*, **120** (12), 2495–2519 (2010). MR2728175.

Dembo, A. Moderate deviations for martingales with bounded jumps. *Electron. Comm. Probab.*, **1**, no. 3, 11–17 (1996). MR1386290.

Dembo, A. and Zeitouni, O. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition (1998). ISBN 0-387-98406-2. MR1619036.

Djellout, H. Moderate deviations for martingale differences and applications to ϕ-mixing sequences. *Stoch. Stoch. Rep.*, **73** (1-2), 37–63 (2002). MR1914978.

Guyon, J. Limit theorems for bifurcating Markov chains. Application to the detection of cellular aging. *Ann. Appl. Probab.*, **17** (5-6), 1538–1569 (2007). MR2358633.

Hairer, M. and Mattingly, J. C. Yet another look at Harris' ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pp. 109–117. Birkhäuser/Springer Basel AG, Basel (2011). MR2857021.

Hoeffding, W. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, **58**, 13–30 (1963). MR144363.

Pitt, M. K., Chatfield, C., and Walker, S. G. Constructing first order stationary autoregressive models via latent processes. *Scand. J. Statist.*, **29** (4), 657–663 (2002). MR1988417.

Puhalskii, A. Large deviations of semimartingales: a maxingale problem approach. I. Limits as solutions to a maxingale problem. *Stochastics Stochastics Rep.*, **61** (3-4), 141–243 (1997). MR1488137.