

Quasi-Limiting Behavior of Drifted Brownian Motion

Iddo Ben-Ari and SangJoon Lee

Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009, USA

E-mail address: iddo.ben-ari@uconn.edu

URL: iddo.ben-ari.uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009, USA

E-mail address: sangjoon.lee@uconn.edu

Abstract. A Quasi-Stationary Distribution (QSD) for a Markov process with an almost surely hit absorbing state is a time-invariant initial distribution for the process conditioned on not being absorbed by any given time. An initial distribution for the process is in the domain of attraction of some QSD ν if the distribution of the process a time t , conditioned not to be absorbed by time t converges to ν as t tends to infinity. We study Brownian motion with constant drift on the half line $[0, \infty)$ absorbed at 0. Previous work by [Martínez and San Martín \(1994\)](#) and [Martínez et al. \(1998\)](#) identifies all QSDs and provides a nearly complete characterization for their domains of attraction. Specifically, it was shown that if the distribution a well-defined exponential tail (including the case of lighter than any exponential tail), then it is in the domain of attraction of a QSD determined by the exponent. In this work we expand the discussion regarding the dependence on the initial distribution through

- (1) Obtaining a new approach to existing results, explaining the direct relation between a QSD and an initial distribution in its domain of attraction; and
- (2) Considering a wide class of heavy-tailed initial distributions, where non-trivial limits are obtained under appropriate scaling.

1. Introduction

Here we review the origin and some well-known results of the study of QSDs. In section [1.1](#), we will present the general definition of QSD and related theorems. In section [1.2](#), we will introduce the specific model we work in this paper, and present some previous results on the model.

1.1. Definitions and General Results. Consider $\mathbf{X} = (X_t : t \geq 0)$, a Markov process on $\mathbb{R}_+ = [0, \infty)$ with 0 as a unique absorbing state. Let

$$\tau = \inf\{t \geq 0 : X_t = 0\}.$$

Received by the editors January 31st, 2020; accepted June 4th, 2021.

2010 Mathematics Subject Classification. 60J65, 60F99, 60J35, 60G70.

Key words and phrases. Brownian Motion, Quasistationary Distribution, Domain of Attraction, heavy-tailed distributions.

The authors would like to thank an anonymous referee for improving the presentation of the results.

We will work under the assumption

$$P_x(\tau < \infty) = 1, \text{ for all } x \in \mathbb{R}_+. \quad (1.1)$$

The notation P_x is a shorthand for the distribution of \mathbf{X} with initial distribution, the distribution of X , equal to the Dirac-delta measure at x .

If π is a stationary distribution for \mathbf{X} , then (1.1) guarantees that $\pi = \delta_0$, (Collet et al., 2013 Section 2.2). While this result is not very interesting, the distribution of the process and particularly of X_t *conditioned* on $\{\tau > t\}$, is in general far from trivial. This naturally leads to the following “conditional” analog for a stationary distribution:

Definition 1.1. The probability distribution π is a Quasi-Stationary Distribution (QSD) for \mathbf{X} if

$$P_\pi(X_t \in \cdot \mid \tau > t) = \pi \text{ for all } t > 0.$$

A seemingly more relaxed definition, in the spirit of ergodic theorems for Markov Chains, is the following:

Definition 1.2. A probability distribution π is a Quasi-Limiting Distribution (QLD) for \mathbf{X} if for some μ ,

$$\lim_{t \rightarrow \infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi, \text{ in distribution,} \quad (1.2)$$

where, as usual, P_π and P_μ are shorthand for the distribution of \mathbf{X} with initial distribution equal to π or μ , respectively. QLDs corresponding to an initial distributions which are Dirac-delta (or, more generally, compactly supported initial distributions) are known as Yaglom limits.

Of course, a QSD is a QLD. A partial converse holds under a standard regularity assumption (the Feller property):

Proposition 1.3. *Suppose that for every $t > 0$ and continuous and bounded function f on $(0, \infty)$, the function $x \rightarrow E_x[f(X_t), \tau > t]$ is continuous. Then every QLD for \mathbf{X} is a QSD for \mathbf{X} .*

For the sake of completeness, we provide a proof in Appendix A.1.

We comment that QLDs are very often defined by requiring a *pointwise limit* rather than limit in distribution. That is (1.2) in Definition 1.2 is replaced by

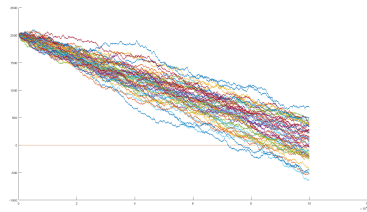
$$\lim_{t \rightarrow \infty} P_\mu(X_t \in A \mid \tau > t) = \pi(A) \text{ for all measurable } A \subseteq (0, \infty). \quad (1.3)$$

With this definition the conclusion of Proposition 1.3 holds without the additional regularity condition we imposed. See Definition 1 and Proposition 1 in Méléard and Villemonais (2012).

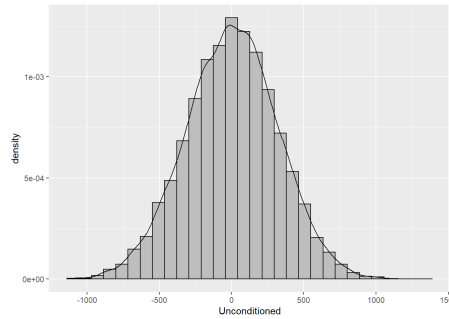
As in the sequel we will only work with processes satisfying the condition in the proposition, we always consider QLDs as QSDs. In light of the above, when μ and π are as in Definition 1.2, we say that μ is in the *domain of attraction* of the QSD π . Of course, the domain of attraction of any QSD contains itself.

Figure 1.1 illustrates the difference between the unconditioned process, the process that is required to be positive only at the given time, and the process that is required to never hit 0 up to the given time.

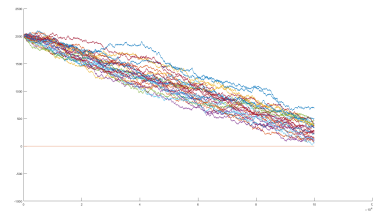
Unlike uniqueness of stationary distribution under irreducibility assumptions, QSDs are in general not unique, and typically a continuum of QSDs exists. Notable exceptions of this are Markov chains on finite state spaces (with a unique absorbing state) and certain diffusion processes on bounded domains absorbed at the boundary. One strategy of finding QSDs is to study the quasi-limiting behavior under different initial distributions. When the class of QSDs is known it is natural to ask what is the domain of attraction of each.



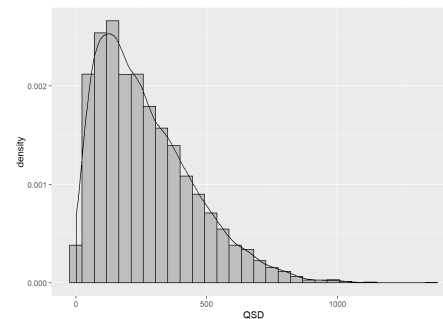
(A) Sample paths of 1-dimensional Brownian Motion with constant negative drift -0.02 , with fixed initial state $X_0 = 2000$



(B) PDF plot of 1-dimensional Brownian Motion with same drift and initial state, at $t = 100000$. Sample size is 10000. As expected, X_{100000} follows a Gaussian distribution.



(C) Sample paths of same processes, conditioned not to be absorbed by $t = 100000$



(D) PDF plot of the same sample processes and same condition. Unlike above, this distribution has exponential tail. Also, the density near 0 drops significantly in this setting.

FIGURE 1.1. Illustration between unconditioned stochastic process and process conditioned not to be absorbed by a given time

The concept of QSD is fairly intuitive and straightforward, as the idea was first introduced as early as [Wright \(1931\)](#), and the terms related to QSD have been crystallized in [Bartlett \(1957\)](#) and [Bartlett \(1960\)](#). Mathematically, [Yaglom \(1947\)](#) first showed an explicit solution to a limiting conditional distribution for the for the subcritical Bienaymé-Galton-Watson branching process. In the discrete setting there are detailed results for some specific models; for example, explicit description of QSDs are known for certain birth-and-death processes ([Collet et al., 2013](#), Theorem 5.4). As for uniqueness, a necessary and sufficient conditions for birth-and-death processes were obtained by [van Doorn \(1991\)](#), and Martínez, San Martín and Villemonais later generalized the result to countable state processes [Martínez et al. \(2014\)](#). For other discrete state space models, [Buiculescu \(1975\)](#) studied QSDs for multi-type Galton-Watson processes, and [Ferrari and Marić \(2007\)](#) discussed QSDs approximated by Fleming-Viot processes. A survey of results provided by [van Doorn and Pollett \(2013\)](#) gives a comprehensive view of the progress on the discrete space models.

Our work is on Brownian Motion with constant drift, which is one among a few models where a lot is known explicitly, in part because it is a Gaussian process. Our main object of interest is the dependence on the initial distribution, and is in continuation to the works of [Martínez and](#)

San Martín (1994) who identified all QSDs for the model, and later identified the domain of attraction for each QSD in Martínez et al. (1998). Rates of convergence to Yaglom limits are also studied by Polak and Rolski (2012), and Oçafrain (2020). Another diffusion - notably also a Gaussian processes - where explicit results are known is the Ornstein-Uhlenbeck process: Lladser and San Martín (2000) classified QSDs through their domains of attraction. Ye (2008) identified the Yaglom limit for fractional-dimensional radial Ornstein-Uhlenbeck processes. As for general theory for diffusion processes, there has been much work and progress on conditions for existence and uniqueness of QSDs and on convergence to the Yaglom limit. Here is a partial list of references: Pinsky (1990) (smooth bounded domains with absorption on the boundary), Cattiaux et al. (2009), Steinsaltz and Evans (2007), Kolb and Steinsaltz (2012), and Hening and Kolb (2019) (uniqueness and convergence for one dimensional diffusions) and Champagnat and Villemonais (2017) (rates of convergence for one-dimensional diffusions).

We close this section with the well-known properties related to QSDs and QLDs.

Theorem 1.4. (Collet et al., 2013, Theorem 2.2) *Suppose that π is a QSD. Then under P_π , τ is exponentially distributed with parameter $\lambda_\pi > 0$.*

Proposition 1.5. *Let the assumption of Proposition 1.3 hold. Let μ is in the domain of attraction of the QSD π . Then for every $\epsilon > 0$,*

$$P_\mu(\tau > t) = o(e^{-(\lambda_\pi - \epsilon)t}),$$

where λ_π is as in Theorem 1.4.

We give an elementary proof in Appendix A.2. See also Proposition 5 in Méléard and Villemonais (2012) for a sharper result under slightly stronger assumptions.

1.2. *Quasi Stationarity for Drifted BM.* In this section and the sequel we will work under the following:

Assumption 1.6. \mathbf{X} is Brownian Motion (BM) with constant negative drift $-\alpha$, $\alpha > 0$, on \mathbb{R}_+ absorbed at 0.

Analytically, BM with constant drift $-\alpha$ on \mathbb{R}_+ absorbed at 0 is the sub-Markovian process generated by \mathcal{L}_α , which for each u satisfying $u \in C^2(\mathbb{R}_+)$ and $u(0) = 0$,

$$\mathcal{L}_\alpha u = \frac{1}{2}u'' - \alpha u'.$$

The works by Martínez and San Martín (1994) and Martínez et al. (1998) studied QSDs for this class of models. The formal derivation for densities of the QSDs, as presented in their main results, will be given in Appendix A.3.

Theorem 1.7. (Martínez and San Martín, 1994, Proposition 1) *Every QSD for \mathbf{X} is of the form π_γ for some $\gamma \in [0, \alpha)$.*

Theorem 1.8. (Martínez et al., 1998, Theorem 1.3) *The probability measure μ is in the domain of attraction of π_0 if*

$$\liminf_{x \rightarrow \infty} \frac{\ln \mu([x, \infty))}{x} \leq -\alpha.$$

Theorem 1.9. (Martínez et al., 1998, Theorem 1.1) *Let $\rho \in (0, \alpha)$. The probability measure μ is in the domain of attraction of $\pi_{\alpha-\rho}$ if*

$$\lim_{x \rightarrow \infty} \frac{\ln \mu([x, \infty))}{x} = -\rho.$$

We note the following:

- (1) Theorem 1.9 was proved under the assumption that μ has a smooth density.
- (2) The limit condition in Theorem 1.9 is not merely technical. The authors constructed an example (Martinez et al., 1998, Theorem 1.4) with initial distribution with tail which alternates between two exponential decay rates and which is not in the domain of attraction of any QSDs. We comment that the method we develop in this paper can provide a simpler construction of such initial distribution.

1.3. *Organization.* We present our main results in Section 2, split according to the tail of the initial distribution, considering initial distributions in the domain of attraction on QSDs in Section 2.1 and heavy tails in Section 2.2. Our proofs are given split across three sections: In Section 3, we present some general tools we will use. In Section 4 we prove the results from Section 2.1. In Section 5 we prove the results from Section 2.2, along with some concrete examples in Section 5.4.

2. Main Results

In this section we will state our main results, by first proposing a principle which we think can help the readers to envision the general classification of quasi-limiting behavior, and then provide the theorems based on the principle. We recall that we are working under Assumption 1.6.

Our goals are twofold:

- (1) Develop a method that would yield alternate proof to Theorems 1.8 and 1.9, which can be generalized to other models, as well as leading to complete characterization of the domain of attraction of every QSD. Our results are presented in Section 2.1.
- (2) Characterize the asymptotic behavior when the initial distribution has tails which are heavier than exponential. It is not hard to show, see Lemma 2.5, that this class of initial distributions is not in the domain of attraction of any QSD. Our results are presented in Section 2.2.

2.1. *Domain of Attraction of QSDs.* As at its core, the concept of quasi-stationarity concerns conditional probabilities under events with diminishing probabilities, namely the events $\{\tau > t\}$. It is therefore natural to study the rate at their probabilities, $P_\mu(\tau > t)$, tend to zero. One of the nice properties of our model is that through Girsanov theorem and the reflection principle (or formulas for Brownian bridges) a closed form formula for these probabilities is readily available. We have:

Proposition 2.1.

$$P_\mu(X_t \in dy, \tau > t) = \frac{1}{\sqrt{2\pi t}} \int \exp\left(\alpha x - \frac{\alpha^2 t}{2} - \alpha y\right) \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) d\mu(x). \tag{2.1}$$

Our approach to the problem is to obtain for each initial distribution μ a family of probability measures $(\nu_t : t \geq 0)$, such that

Principle 2.2.

$$\boxed{\lim_{t \rightarrow \infty} \nu_t = \delta_\gamma} \implies \boxed{\lim_{t \rightarrow \infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi_\gamma} \tag{2.2}$$

The measure ν_t is defined through its cumulative distribution function F_{ν_t} :

$$F_{\nu_t}(z) = C_t \int_{[0, zt]} e^{-x^2/(2t)} e^{\alpha x} d\mu(x) \tag{2.3}$$

where C_t is the normalization constant. Table 2.1 is the summary of our result; it shows the relation between μ , ν_t and the QLD of μ .

The key idea in the method is to “decouple” the initial distribution from the asymptotic distribution, then identifying the relevant QSD as a member of a one-parameter family selected according to the value of γ . Indeed, in our model, observe that the mapping $\gamma \rightarrow \pi_\gamma$, $\gamma \in [0, \alpha)$ as given in

ρ	$\lim \nu_t$	QLD (= QSD)	Example distributions
$\rho \geq \alpha$	δ_0	π_0 (Theorem 2.3)	Half-normal distribution Delta distribution
$\alpha > \rho > 0$	$\delta_{\alpha-\rho}$	$\pi_{\alpha-\rho}$ (Theorem 2.4)	Exponential distribution with rate $\lambda < \alpha$
$\rho = 0$	δ_α	QLD does not exist: scaling is necessary. See Section 2.2 and Table 2.2	Pareto distribution Half-Cauchy distribution

TABLE 2.1. Domain of attraction classified by parameter $\rho = \lim_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x}$

(A.11) is an explicit function, with the case $\gamma = 0$ is merely a removable singularity and is defined as $\lim_{\gamma \rightarrow 0+} \pi_\gamma$.

We believe that this method has a number of advantages:

- (1) It is more intuitive, simpler and elementary than the previous approach. It lets us understand how the initial distribution actually evolves over time, and at a specific time, which part of the initial distribution have evolved to consist the absolute majority of the process not absorbed.
- (2) The method allows for expanded characterization of the domain of attraction of QSDs.
- (3) Our approach simplifies the analysis for the case of a distribution with alternating exponential tails, given in [Martinez et al. \(1998\)](#), and opens the possibility of studying general compound-tail distributions.

We expect this method to be applicable to other models and we hope it can be adopted as a general framework for classifying domain of attraction of QSDs.

Our Principle 2.2 will be employed in two ways. We first observe that

$$\lim_{t \rightarrow \infty} \nu_t = \begin{cases} \delta_0 & \iff \limsup_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x} \geq \alpha \\ \delta_{\alpha-\rho} & \iff \lim_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x} = \rho < \alpha \end{cases} \tag{2.4}$$

Through application of the approach outlined above we obtain the following results:

Theorem 2.3. *Suppose μ satisfies the following assumption.*

$$\rho := \liminf_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x} \geq \alpha. \tag{2.5}$$

Then

$$P_\mu(X_t \in \cdot | \tau > t) \rightarrow \pi_0.$$

Theorem 2.4. *Suppose μ satisfies the following assumption,*

$$\rho := \lim_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x} \in (0, \alpha) \tag{2.6}$$

and let the sequence of measure $(\nu_t : t \geq 0)$ defined as (2.3). Then

$$\lim_{t \rightarrow \infty} \nu_t = \delta_{\alpha-\rho} \tag{2.7}$$

and moreover,

$$\lim_{t \rightarrow \infty} P_\mu(X_t \in \cdot \mid \tau > t) = \pi_{\alpha-\rho}. \tag{2.8}$$

We will refer to μ satisfying (2.5) as possessing ‘‘Critical and Super-critical’’ tails (with critical being an equality), and will prove Theorem 2.3 in Section 4.1. We will refer to μ that satisfies (2.6) as possessing ‘‘Sub-critical Exponential’’ tails and will prove Theorem 2.4 in Section 4.2.

2.2. *Heavy Tails.* A natural question to ask from Martinez et al. (1998) would be the following: what happens if the initial distribution is too heavy to be in the domain of attraction of any QSDs? A first step in this direction is to look for such initial distributions. In light of Theorems 1.8 and 1.9, the following is not surprising:

Lemma 2.5. *Suppose*

$$\lim_{x \rightarrow \infty} \frac{\ln \mu([x, \infty))}{x} = 0.$$

Then $P_\mu(\tau > t)$ does not decay exponentially. As a consequence $(P_\mu(X_t \in \cdot \mid \tau > t) : t \geq 0)$ is not tight.

Thus, in order to obtain a non-trivial limit, one has to scale X_t as $t \rightarrow \infty$. As we will see, the scaling itself depends on μ . We comment that all of the cases covered in this section correspond to $\nu_t \rightarrow \delta_\alpha$ in (2.2).

The next step is to study long-time behavior under such heavier-tailed distributions, and this is the main topic of this part of the project. In order to do so, we mainly rely on the theory of regularly varying functions (Bingham et al., 1989).

Assumption 2.6. Suppose μ is a probability measure satisfying the following:

- (1) $\mu([x, \infty)) = e^{-F(x)}$, with F smoothly varying (Bingham et al., 1989, Section 1.8) with index parameter $\beta < 1/2$.
- (2) There exists a positive function $R(x, c)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ increasing in c , such that for all $c > 0$

$$\lim_{x \rightarrow \infty} F(x + R(x, c)) - F(x) = c. \tag{2.9}$$

Some comments are in order:

- (1) Probability measures with regularly varying tails falls into the category $\beta = 0$. Some distinguished cases are the Weibull distribution with $0 < k < 1$, which has a uniform decay rate with $\beta = k$, and the Pareto and Cauchy distributions, both having uniform decay rate with $\beta = 0$.
- (2) If F is smooth enough, then

$$R(x, c) = \frac{c}{F'(x)}. \tag{2.10}$$

So when $\beta \neq 0$, $R(x, c)$ is a regular varying function with index $\varphi = 1 - \beta$.

- (3) When $\beta = 0$ it is more natural to replace the identity function on the right-hand side of (2.9) with a strictly increasing continuous and nonnegative function H satisfying $H(0) = 0$.

The main principle we developed to obtain results under Assumption 2.6 is the following.

Principle 2.7.

$$\boxed{\text{Assumption 2.6}} \Rightarrow \boxed{\lim_{t \rightarrow \infty} P_\mu(X_t > R(t, c) \mid \tau > t) = e^{-c}} \tag{2.11}$$

We note that the assumption $\beta < 1$ is vital for this to work, as otherwise the conclusion contradicts the results of previous sections. This is due to the fact that $\beta = 1$ is the critical border where the relation between the survival rate $P_\mu(\tau > t)$ and the initial distribution μ changes. Also, although Lemma 2.5 applies whenever $0 \leq \beta < 1$, Principle 2.7 only applies to $0 < \beta < 1/2$. The remaining half $1/2 \leq \beta < 1$ is left as an open problem; there is a difficulty in estimating the distribution of the surviving processes in these cases. (See Proposition 5.2 and (5.12)) The following theorem is the key result from the above principle.

Theorem 2.8. *Suppose $\mu([x, \infty)) = \exp(-F(x))$ where $F(x)$ is strictly increasing smoothly varying function with index $\beta < 0.5$. Then*

$$\lim_{t \rightarrow \infty} P_\mu \left(X_t > \frac{c}{F'(\alpha t)} \mid \tau > t \right) = e^{-c}. \quad (2.12)$$

Table 2.2 summarizes our results by showing how β relates to some of the well-known distributions, and how they lead to quasi-limiting behavior of such initial distribution. The table also lists a number of concrete cases, all presented in Section 5.4.

Throughout the rest of the paper, we will be using some asymptotic notations; $f(t) \sim g(t)$ if $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} \in (0, \infty)$, and $f(t) \ll g(t)$ if $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 0$.

β	Related result	Example distributions
$\beta > 1$	Theorem 2.3	Half-normal distribution Compactly supported distributions Weibull distribution with shape parameter $k > 1$
$\beta = 1$	$\rho = \lim_{x \rightarrow \infty} -\frac{\ln \mu([x, \infty))}{x}$ Theorem 2.3 if $\rho \geq \alpha$ Theorem 2.4 if $\rho < \alpha$	Exponential distribution Erlang distribution
$\frac{1}{2} \leq \beta < 1$	Unknown	Weibull distribution with shape parameter $\frac{1}{2} \leq k < 1$
$0 < \beta < \frac{1}{2}$	Theorem 2.8	Weibull distribution with shape parameter $k < \frac{1}{2}$, Example 5.5
$\beta = 0$	$\mu([x, \infty))$ is RV with index $-\kappa$: Corollary 5.6 if $\kappa \neq 0$ Corollary 5.8 if $\kappa = 0$	Pareto distribution Half-Cauchy distribution, Example 5.7 Log-Cauchy distribution

TABLE 2.2. Distributions classified by index parameter β of $F(x) = -\ln \mu([x, \infty))$

3. Proofs: Base Formula

In this section, we prove Proposition 2.1, which is the master formula we use throughout this paper. We will also further explain the intuition behind the sequence of new measure ν_t . Finally, we will introduce the variations of Scheffe’s lemma (Scheffé, 1947), which is one of the tool for Chapter 4.

3.1. *Conditional transition density.* When X_t is a drifted Brownian Motion with negative drift α , (such that $X_t + \alpha t$ is a standard BM B_t)

$$P_x(X_t \in dy) = \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) P_x(B_t \in dy). \tag{3.1}$$

We also want to enforce the condition $\tau > t$, where τ is the hitting time at 0. We can apply the reflection principle to compute $P_x(X_t \in dv, \tau > t)$.

$$\begin{aligned} P_x(X_t \in dy, \tau > t) &= \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) P_x(B_t \in dy, \tau > t) \\ &= \underbrace{\exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right)}_{=f(t,x,y)}. \end{aligned} \tag{3.2}$$

Integrating $f(t, x, y)$ with respect to μ gives (2.1). Furthermore, we can get the survival rate from the above formula as well:

$$P_\mu(\tau > t) = \int_0^\infty \int_0^\infty \mu(x) f(t, x, y) dy dx. \tag{3.3}$$

We wrap this section with the principle behind finding the family of probability measures ($\nu_t : t \geq 0$) in (2.2). From (3.3),

$$\begin{aligned} P_\mu(\tau > t) &= \int_0^\infty \int_0^\infty \mu(x) \exp\left(\alpha x - \frac{\alpha^2 t}{2} + \alpha y\right) \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right) dy dx \\ &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty \mu(x) e^{-\frac{x^2}{2t}} e^{\alpha x} \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}}\right) dy dx\right). \end{aligned} \tag{3.4}$$

We substitute $z = tx$,

$$(3.4) = \frac{e^{-\frac{\alpha^2 t}{2}} t}{\sqrt{2\pi t}} \left(\int_0^\infty \mu(tz) e^{-\frac{tz^2}{2}} e^{\alpha tz} \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} (e^{tz} - e^{-tz}) dy dz\right). \tag{3.5}$$

For convenience, we will use x instead of z for (3.5) in later parts.

From the above equations, the natural construction of ν_t would come from the terms that consist the outer integral. Indeed, we will use $\nu_t(x) = \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx}$ in Section 4.2. In Section 4.1, (3.4) will be used with some modification.

3.2. *Scheffe’s Lemma.* From (2.1) and (3.3), we can consider the conditional density

$$\begin{aligned} P_\mu(X_t \in dy \mid \tau > t) &= \frac{P_\mu(X_t \in dy, \tau > t)}{P_\mu(\tau > t)} \\ &= \frac{\int_0^\infty \mu(x) f(t, x, y) dx}{\int_0^\infty \int_0^\infty \mu(x) f(t, x, y) dy dx}. \end{aligned} \tag{3.6}$$

When t is fixed, this is clearly a probability density which we will call $\mu_t(y)$. In order to prove convergence of the probability distributions, we will use the following version of Scheffe’s Lemma (Williams, 1991, p. 55):

Lemma 3.1. *Suppose that f_n, f are probability densities on \mathbb{R}_+ satisfying $\liminf f_n \geq f$, a.e. Then $\int_A f_n(x)dx \rightarrow \int_A f(x)dx$ for any A .*

Proof: Let $dm_n = f_n dx$, and $dm_\infty = f dx$. By Fatou’s lemma, for every A ,

$$\liminf m_n(A) \geq m_\infty(A). \tag{3.7}$$

Now

$$1 - \limsup m_n(A) = \liminf(1 - m_n(A)) = \liminf m_n(A^c).$$

Thus, by (3.7) applied to A^c ,

$$1 - \limsup m_n(A) = \liminf m_n(A^c) \geq m_\infty(A^c) = 1 - m_\infty(A).$$

In other words $\limsup m_n(A) \leq m_\infty(A)$ and the first statement follows. □

4. Proofs: Exponential or Lighter Tails

4.1. *Proof of Theorem 2.3 (Critical and Super-critical Tails).* Throughout this section we will assume that (2.5) holds.

Define

$$f(t, x, y) = ye^{-\alpha y} e^{-\frac{y^2}{2t}} \frac{\sinh(xy)}{xy}$$

and

$$h(t, x) = \int_0^\infty f(t, x, y) dy$$

and let

$$h(x) = \lim_{t \rightarrow \infty} h(t, x) = \int_0^\infty e^{-\alpha y} \frac{\sinh(xy)}{x} dy.$$

Note that $h(x)$ is increasing,

$$h(0) := \lim_{x \searrow 0} h(x) = \int_0^\infty ye^{-\alpha y} dy = \frac{1}{\alpha^2}$$

and $h(x) = \infty$ if and only if $x \geq \alpha$.

For every t , we define two measures on $[0, \infty)$:

$$\begin{aligned} d\gamma(x) &= xe^{\alpha x} d\mu(x) \\ d\nu_t(x) &= e^{-\frac{x^2}{2t}} d\gamma(x) \end{aligned} \tag{4.1}$$

By assumption, there exists a function $\delta(x) \rightarrow 0$ such that

$$\gamma([0, x]) \leq e^{\delta(x)x}$$

without loss of generality, we may also assume δ is decreasing.

Observe that

$$P(X_t \in dy | \tau > t) = \frac{\int f(t, x/t, y) d\nu_t(x)}{\int h(t, x/t) d\nu_t(x)}. \tag{4.2}$$

We will now prove the theorem through the application of Lemma 3.1, where

$$f_t(v) = \frac{\int f(t, x/t, y) d\nu_t(x)}{\int h(t, x/t) d\nu_t(x)}$$

and $f(v) = \alpha^2 ye^{-\alpha y}$.

Proof of Theorem 2.3: Let $\epsilon \in (0, 1)$ and let $\eta_t = \epsilon\alpha t$. We begin by analyzing the behavior of the denominator in the right-hand side of (4.2).

Observe that $h(t, y)$ is bounded on $[0, M] \times \mathbb{R}_+$ and increases as $t \rightarrow \infty$ to

$$h(x) = \int_0^\infty ye^{-\alpha y} \frac{\sinh(xy)}{xy} dy.$$

As a result, the convergence is uniform. From this it follows that

$$\limsup_{t \rightarrow \infty} \frac{\int_{[0, \eta_t]} h(t, x/t) d\nu_t(x)}{\nu_t([0, \eta_t])} \leq h(\epsilon\alpha). \tag{4.3}$$

We turn to evaluation of the interval on $[\eta_t, 0.9\alpha t]$. Since here $\frac{x}{t} \leq 0.9\alpha < \alpha$, $h(t, \frac{x}{t})$ is uniformly bounded by a constant depending only on α . Below C denotes a positive constant depending only on α, ϵ , and whose value may change from line to line.

Integrating by parts,

$$\int_{[\eta_t, 0.9\alpha t]} h\left(t, \frac{x}{t}\right) d\nu_t(x) \leq C \frac{1}{t} \int_{[\eta_t, 0.9\alpha t]} xe^{-\frac{x^2}{2t}} \gamma([\eta_t, x]) dx.$$

Changing variables to $z = \frac{x}{\sqrt{t}}$, the last expression becomes

$$\int_{\sqrt{t}\alpha[\epsilon, 0.9]} ze^{-\frac{z^2}{2}} \gamma([\eta_t, \sqrt{t}z]) dz.$$

Now

$$\gamma([\eta_t, \sqrt{t}z]) \leq \gamma([0, \sqrt{t}z]) \leq \gamma([0, \eta_t]) e^{\delta(\eta_t)(\sqrt{t}z - \sqrt{t}\epsilon)} \leq \gamma([0, \eta_t]) e^{\delta(\eta_t)\sqrt{t}z}.$$

Putting this back in the integral gives an upper bound of the form

$$\gamma([0, \eta_t]) \int_{\sqrt{t}\alpha[\epsilon, 0.9]} ze^{-\frac{z^2}{2}} e^{\delta(\eta_t)\sqrt{t}z} dz.$$

Since $\delta(\eta_t) \rightarrow 0$ as $t \rightarrow \infty$, for all t large enough, we have

$$\delta(\eta_t) \leq \min\left(\frac{\alpha^2\epsilon^2}{4}, \alpha\epsilon\right). \tag{4.4}$$

To obtain an upper bound on the integral, observe that as a function of z ,

$$-\frac{z^2}{2} + \delta(\eta_t)\sqrt{t}z = -\frac{z}{2}(z - 2\delta(\sqrt{t}))$$

is decreasing on $[\delta(\eta_t)\sqrt{t}, \infty)$, and by (4.4), if $z > \frac{\eta_t}{\sqrt{t}} = \epsilon\alpha\sqrt{t}$, then $z > \delta(\eta_t)\sqrt{t}$.

Therefore we have

$$\begin{aligned} -\frac{z^2}{2} + \delta(\eta_t)\sqrt{t}z &\leq -\frac{(\eta_t/\sqrt{t})^2}{2} + \delta(\eta_t)\sqrt{t} \left(\frac{\eta_t}{\sqrt{t}}\right) \\ &\leq -\frac{\alpha^2\epsilon^2 t}{2} + \frac{\alpha^2\epsilon^2 t}{4} \\ &= -\frac{(\alpha\epsilon)^2 t}{4}. \end{aligned} \tag{4.5}$$

Thus,

$$\begin{aligned} \int_{[\eta_t, 0.9\alpha t]} h\left(t, \frac{x}{t}\right) d\nu_t(x) &\leq C e^{-\frac{(\alpha\epsilon)^2 t}{4}} t^{\frac{3}{2}} \gamma([0, \eta_t]) \\ &\leq C e^{\left(-\frac{(\alpha\epsilon)^2}{4} + \delta(\eta_t)\epsilon\alpha\right)t} t^{\frac{3}{2}} \rightarrow 0. \end{aligned} \tag{4.6}$$

Next we consider the behavior over the interval $[0.9\alpha t, \infty)$. Observe that

$$h(t, x) \leq \frac{\sqrt{2\pi t}}{x} E\left[e^{(x-\alpha)\sqrt{t}Z}\right]$$

where Z is standard Gaussian, and therefore

$$h\left(t, \frac{x}{t}\right) \leq \frac{\sqrt{2\pi t}}{x/t} e^{\frac{x^2}{2t}} e^{\frac{\alpha^2 t}{2}} e^{-\alpha x}.$$

Hence

$$\int_{[0.9\alpha t, \infty)} h\left(t, \frac{x}{t}\right) d\nu_t(x) \leq \sqrt{2\pi t^3} e^{\frac{\alpha^2 t}{2}} \int_{[0.9\alpha t, \infty)} d\mu(x).$$

But $\mu([0.9\alpha t, \infty)) = e^{-0.9\alpha^2 t(1+o(1))}$, and as a result

$$\int_{[0.9\alpha t, \infty)} h\left(t, \frac{x}{t}\right) d\nu_t(x) \rightarrow 0. \tag{4.7}$$

Since $\liminf_{t \rightarrow \infty} \nu_t([0, \eta_t]) > 0$, it follows from (4.3), (4.6) and (4.7), that

$$\limsup_{t \rightarrow \infty} \frac{\int h(t, x/t) d\nu_t(x)}{\nu_t([0, \eta_t])} \leq h(\epsilon\alpha). \tag{4.8}$$

Repeating the argument leading to that gave (4.3) mutatis mutandis, we obtain

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{\int_{[0, \eta_t]} f(t, x/t, y) d\nu_t(x)}{\nu_t([0, \eta_t])} &\geq y e^{-\alpha y} \inf_{x \leq \epsilon\alpha} \frac{\sinh(xy)}{xy} \\ &= y e^{-\alpha y}. \end{aligned} \tag{4.9}$$

It therefore follows from (4.8) and (4.9), that

$$\liminf_{t \rightarrow \infty} \frac{\int f(t, x/t, y) d\nu_t(x)}{\int h(t, x/t) d\nu_t(x)} \geq \frac{y e^{-\alpha y}}{h(\epsilon\alpha)}$$

and this holds for every $\epsilon \in (0, 0.9)$.

Therefore since $\lim_{\epsilon \rightarrow 0} h(\epsilon\alpha) = \int_0^\infty y e^{-\alpha y} dy$, we obtain

$$\liminf_{t \rightarrow \infty} \frac{\int f(t, x/t, y) d\nu_t(x)}{\int h(t, x/t) d\nu_t(x)} \geq \frac{y e^{-\alpha y}}{\int_0^\infty y e^{-\alpha y} dy}$$

and the result follows from Lemma 3.1. □

4.2. *Proof of Theorem 2.4 (Sub-critical Tails).* Throughout this section, we assume that (2.6) holds.

We first split (3.4) into three parts.

$$\begin{aligned}
 P_\mu(\tau > t) &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\underbrace{\int_0^M e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_3(t)} \right. \\
 &\quad \left. + \underbrace{\int_M^{st} e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_1(t)} + \underbrace{\int_{st}^\infty e^{-\frac{x^2}{2t}} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x)}_{=J_2(t)} \right)
 \end{aligned}
 \tag{4.10}$$

Where $h(t, x) = \int_0^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) dy$.

Here, M is chosen such that we have the following inequality

$$\frac{e^{-(\rho+\epsilon)x}}{\rho + \epsilon} \leq \frac{\mu([x, \infty))}{c} \leq \frac{e^{-(\rho-\epsilon)x}}{\rho - \epsilon}
 \tag{4.11}$$

for each $x > M$ and some arbitrary $\epsilon > 0$ (c is the normalizing constant of μ). Also, we choose s such that $s = \alpha - \eta$ for some $\alpha > \eta > 0$ that depend on μ . Finally, since we are only interested in the limiting behavior with respect to t , we write $M < st$ which is always true for large enough t .

Proposition 4.1. *Under assumption (2.6)*

$$\begin{aligned}
 P_\mu(\tau > t) &\sim \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} J_1(t) \\
 &\sim ce^{-\frac{(2\alpha\rho-\rho^2)t}{2}} \left(\frac{1}{\rho} - \frac{1}{2\alpha - \rho} \right)
 \end{aligned}
 \tag{4.12}$$

where c is the constant in (4.11) which only depend on μ .

Proof: We first look at the region for $J_1(t)$. In this interval we have the following.

$$\begin{aligned}
 J_1(t) &= \int_M^{st} e^{-x^2/(2t)} e^{\alpha x} h\left(t, \frac{x}{t}\right) d\mu(x) \\
 &= t \int_{M/t}^s e^{-tx^2/2} e^{\alpha tx} h(t, x) d\mu(tx).
 \end{aligned}
 \tag{4.13}$$

Some observations on $h(t, x)$:

- (1) $h(t, x)$ is bounded in $\mathbb{R}_+ \times [0, s]$ since $s < \alpha$.
- (2) $h(x) = \lim_{t \rightarrow \infty} h(t, x) = \frac{1}{\alpha - x} - \frac{1}{\alpha + x}$ by dominated convergence theorem. Moreover, $h(x)$ is also bounded in $[0, s]$.

We introduce a new sequence of measures $(\nu_t^+, \nu_t^-, t \geq 0)$ defined as

$$\begin{aligned}
 d\nu_t^+(x) &= e^{-\frac{tx^2}{2}} e^{\alpha tx} e^{-(\rho-\epsilon)tx} = \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho+\epsilon)^2 t}{2}} \sqrt{\frac{t}{2\pi}} e^{-\frac{t(x-(\alpha-\rho+\epsilon))^2}{2}} \\
 d\nu_t^-(x) &= e^{-\frac{tx^2}{2}} e^{\alpha tx} e^{-(\rho+\epsilon)tx} = \sqrt{\frac{2\pi}{t}} e^{\frac{(\alpha-\rho-\epsilon)^2 t}{2}} \sqrt{\frac{t}{2\pi}} e^{-\frac{t(x-(\alpha-\rho-\epsilon))^2}{2}}.
 \end{aligned}
 \tag{4.14}$$

For both cases notice that the latter part is a Gaussian density with mean $\alpha - \rho \pm \epsilon$ and variance $1/t$, therefore we have the following convergence of measure:

$$\begin{aligned} \nu_t^+ &\rightarrow \sqrt{\frac{2\pi}{t}} e^{-\frac{(\alpha-\rho+\epsilon)^2 t}{2}} \delta_{\alpha-\rho+\epsilon} \\ \nu_t^- &\rightarrow \sqrt{\frac{2\pi}{t}} e^{-\frac{(\alpha-\rho-\epsilon)^2 t}{2}} \delta_{\alpha-\rho-\epsilon}. \end{aligned} \tag{4.15}$$

Therefore,

$$\begin{aligned} \limsup_{t \rightarrow \infty} J_1(t) &= \limsup_{t \rightarrow \infty} c\sqrt{2\pi t} \int_{M/t}^s h(t, x) d\nu_t^+(x) \\ &= c\sqrt{2\pi t} e^{-\frac{(\alpha-\rho+\epsilon)^2 t}{2}} \left(\frac{1}{\rho - \epsilon} - \frac{1}{2\alpha - \rho + \epsilon} \right) \end{aligned} \tag{4.16}$$

$$\begin{aligned} \liminf_{t \rightarrow \infty} J_1(t) &= \liminf_{t \rightarrow \infty} c\sqrt{2\pi t} \int_{M/t}^s h(t, x) d\nu_t^-(x) \\ &= c\sqrt{2\pi t} e^{-\frac{(\alpha-\rho-\epsilon)^2 t}{2}} \left(\frac{1}{\rho + \epsilon} - \frac{1}{2\alpha - \rho - \epsilon} \right) \end{aligned} \tag{4.17}$$

and since ϵ is arbitrary, we conclude that

$$J_1(t) \sim c\sqrt{2\pi t} e^{-\frac{(\alpha-\rho)^2 t}{2}} \left(\frac{1}{\rho} - \frac{1}{2\alpha - \rho} \right). \tag{4.18}$$

For the second interval $x \in (st, \infty)$ we first study some bound for $h(t, x/t)$. We start from the obvious

$$h\left(t, \frac{x}{t}\right) \leq \int_0^\infty \exp\left(-\frac{y^2}{2t} + \alpha y + \frac{xy}{t}\right). \tag{4.19}$$

We can rewrite the exponent as

$$\begin{aligned} -\frac{y}{2\sqrt{t}} \left(\frac{y}{\sqrt{t}} + 2\alpha\sqrt{t} - \frac{2x}{\sqrt{t}} \right) &= -\frac{1}{2} \frac{y}{\sqrt{t}} \left(\frac{y}{\sqrt{t}} + 2\varphi \right) \\ &= -\frac{1}{2} (w - \varphi)(w + \varphi), \end{aligned} \tag{4.20}$$

where $\varphi = \left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}} \right)$, and $w = \frac{y}{\sqrt{t}} + \varphi$. Therefore, after changing variables $y \rightarrow w$, we obtain

$$\begin{aligned} h(t, x) &\leq \sqrt{t} e^{\frac{\varphi^2}{2}} \int_\varphi^\infty e^{-\frac{w^2}{2}} dw \\ &= \sqrt{t} e^{\frac{\alpha^2 t}{2}} e^{\frac{x^2}{2t}} e^{-\alpha x} L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right), \end{aligned} \tag{4.21}$$

where $L(z) = \int_z^\infty e^{-\frac{w^2}{2}} dw$.

L has some nice properties:

- (1) $L(z)$ is strictly decreasing and bounded above by $\sqrt{2\pi}$.
- (2) When z is negative, $L(z) < \sqrt{2\pi}$.
- (3) When z is positive,

$$L(z) \leq \min\left(\frac{e^{-\frac{z^2}{2}}}{z}, \sqrt{\frac{\pi}{2}}\right). \tag{4.22}$$

(4) More specifically, if $z \geq 1$ then

$$L(z) \leq e^{-\frac{z^2}{2}}. \tag{4.23}$$

Using the bound above we get the following.

$$\begin{aligned} J_2(t) &\leq \sqrt{t} \int_{st}^{\infty} e^{\frac{\alpha^2 t}{2}} L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right) d\mu(x) \\ &\leq c\sqrt{2\pi t} e^{\frac{\alpha^2 t}{2}} e^{-\rho st} \\ &= c\sqrt{2\pi t} e^{t\left(\frac{\alpha^2}{2} - \rho(\alpha - \eta)\right)}. \end{aligned} \tag{4.24}$$

We want $J_2(t) = o(J_1(t)) = o\left(\sqrt{t}e^{\frac{(\alpha-\rho)^2 t}{2}}\right)$. Indeed, if we pick $\eta = \rho/4$,

$$\begin{aligned} \frac{(\alpha - \rho)^2}{2} - \left(\frac{\alpha^2}{2} - \rho(\alpha - \eta)\right) &= \frac{\rho^2}{2} - \rho\eta \\ &= \frac{\rho^2}{4} > 0 \end{aligned} \tag{4.25}$$

therefore we get the desired asymptotic.

For the last interval $x \in [0, M]$, we use the fact that for any $\epsilon > 0$, we can fix t_0 such that for each $t > t_0$, $M/\sqrt{t} < \epsilon$. And for such t , we have

$$\begin{aligned} J_3(t) &= \int_0^M e^{-\frac{x^2}{2t}} e^{\alpha x} \int_0^{\infty} e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh\left(\frac{xy}{t}\right) dy d\mu(x) \\ &\leq \sqrt{t} e^{\frac{\alpha^2 t}{2}} \int_0^M \mu(x) L\left(\sqrt{t}\alpha - \frac{x}{\sqrt{t}}\right) d\mu(x). \end{aligned} \tag{4.26}$$

And since L is decreasing,

$$(4.26) \leq \sqrt{t} e^{\frac{\alpha^2 t}{2}} \int_0^M L(\sqrt{t}\alpha - \epsilon) d\mu(x). \tag{4.27}$$

Finally using (4.23) and that μ is a probability measure,

$$\begin{aligned} (4.27) &\leq \sqrt{t} \int_0^M e^{\alpha\epsilon\sqrt{t}} e^{-\frac{\epsilon^2}{2}} d\mu(x) \\ &\leq \sqrt{t} e^{\alpha\epsilon\sqrt{t} - \frac{\epsilon^2}{2}} \\ &= o\left(\sqrt{t} e^{\frac{(\alpha-\rho)^2 t}{2}}\right) = o(J_1(t)). \end{aligned} \tag{4.28}$$

□

We now turn to computing the limiting density.

$$\begin{aligned} P_\mu(X_t \in dy, \tau > t) &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty \underbrace{e^{-\frac{x^2}{2t}} e^{\alpha x} e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh\left(\frac{xy}{t}\right)}_{=g(x,y,t)} d\mu(x) \right) \\ &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\underbrace{\int_0^M g(x,y,t) d\mu(x)}_{=K_3(t,y)} + \underbrace{\int_M^{st} g(x,y,t) d\mu(x)}_{=K_1(t,y)} + \underbrace{\int_{st}^\infty g(x,y,t) d\mu(x)}_{=K_2(t,y)} \right) \end{aligned} \tag{4.29}$$

where M, s are the same as (4.10).

Proposition 4.2. *Under assumption (2.6),*

$$\begin{aligned}
 P_\mu(X_t \in dy, \tau > t) &\sim \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} K_1(t, y) \\
 &\sim ce^{-\frac{(2\alpha\rho - \rho^2)t}{2}} e^{-\alpha y} \sinh((\alpha - \rho)y)
 \end{aligned}
 \tag{4.30}$$

where c is the constant in (4.11) which only depends on μ .

Proof: Using similar estimation method and sequence of measures $(\nu_t^+, \nu_t^-, t \geq 0)$ as before, we can see that for each $y \in \mathbb{R}_+$

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} K_1(t, y) &= \limsup_{t \rightarrow \infty} c\sqrt{2\pi t} \int_{M/t}^s e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) d\nu_t^+(x) \\
 &= c\sqrt{2\pi t} e^{\frac{(\alpha - \rho + \epsilon)^2 t}{2}} e^{-\alpha y} \sinh((\alpha - \rho + \epsilon)y)
 \end{aligned}
 \tag{4.31}$$

$$\begin{aligned}
 \liminf_{t \rightarrow \infty} K_1(t, y) &= \liminf_{t \rightarrow \infty} c\sqrt{2\pi t} \int_{M/t}^s e^{-\frac{y^2}{2t}} e^{-\alpha y} \sinh(xy) d\nu_t^-(x) \\
 &= c\sqrt{2\pi t} e^{\frac{(\alpha - \rho - \epsilon)^2 t}{2}} e^{-\alpha y} \sinh((\alpha - \rho - \epsilon)y)
 \end{aligned}
 \tag{4.32}$$

and therefore

$$K_1(t, y) \sim c\sqrt{2\pi t} e^{\frac{(\alpha - \rho)^2 t}{2}} e^{-\alpha y} \sinh((\alpha - \rho)y).
 \tag{4.33}$$

For $K_2(t, y)$ we use the upper bound in (4.11) to get the following estimate,

$$\begin{aligned}
 K_2(t, y) &\leq e^{-\frac{y^2}{2t}} e^{-\alpha y} \int_{st}^\infty e^{-\frac{x^2}{2t}} e^{(\alpha - \rho + \epsilon)x} e^{\frac{xy}{t}} dx \\
 &= \sqrt{t} e^{\frac{(\alpha - \rho + \epsilon)^2 t}{2}} e^{(-\rho + \epsilon)y} L\left(\sqrt{t}s - \sqrt{t}(\alpha - \rho + \epsilon) - \frac{y}{\sqrt{t}}\right).
 \end{aligned}
 \tag{4.34}$$

Since $s - (\alpha - \rho + \epsilon) > 0$ for small enough ϵ , the argument for L above is strictly positive and increasing. Therefore by (4.23),

$$(4.34) \leq \sqrt{t} e^{\frac{(\alpha - \rho)^2 t}{2}} \exp\left(-\frac{(s - (\alpha - \rho))^2 t}{2} + (2(\alpha - \rho) - s)\epsilon t\right) e^{(s - \alpha + 2\epsilon)y}.
 \tag{4.35}$$

Again, $s - (\alpha - \rho) > 0$ and ϵ is arbitrarily small so the middle term above is exponentially decaying. We conclude that

$$(4.35) = o\left(\sqrt{t} e^{\frac{(\alpha - \rho)^2 t}{2}}\right) = o(K_1(t, y)).
 \tag{4.36}$$

Finally for $K_3(t, y)$ we can directly apply the dominated convergence theorem.

$$\begin{aligned}
 K_3(t, y) &\sim \int_0^M e^{\alpha x} e^{-\alpha y} \sinh(0) d\mu(x) \\
 &= o(1) = o(K_1(t, y)).
 \end{aligned}
 \tag{4.37}$$

□

We can now prove Theorem 2.4.

Proof of Theorem 2.4: The fact that ϵ is arbitrarily small in (4.15) proves the first part of the theorem. The second part follows from Proposition 4.1, Proposition 4.2 and Lemma 3.1. □

5. Proofs: Heavy Tails

In Section 5.1 we will prove Lemma 2.5 to see that adequate scaling is necessary to obtain a non-trivial quasi-limiting behavior. In Section 5.2 we obtain the scaling by estimating the tail distribution of the surviving process, Proposition 5.2. In Section 5.3 we will use this to prove Theorem 2.8. Finally, in Section 5.4 we present a number of concrete applications to Theorem 2.8.

5.1. *Proof of Lemma 2.5.*

Proof: Pick $b > 0$ such that $\sinh(\alpha b) > \frac{1}{4}e^{\alpha b}$ then by Proposition 2.1 we have:

$$\begin{aligned} P_\mu(\tau > t) &\geq P_\mu(X_0 > \alpha t, X_t > b, \tau > t) \\ &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \int_\alpha^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_b^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} (e^{xy} - e^{-xy}) dy dx \\ &\geq \frac{e^{-\frac{\alpha^2 t}{2}}}{4\sqrt{2\pi t}} \int_\alpha^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_b^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} e^{xy} dy dx \\ &= \frac{t}{4\sqrt{2\pi}} \int_\alpha^\infty \mu(tx) L\left(\frac{b}{\sqrt{t}} + \sqrt{t}(\alpha - x)\right) dx \\ &\geq \frac{1}{8} \mu([t\alpha, \infty)). \end{aligned}$$

This implies that $P_\mu(\tau > t)$ is at least as heavy as the tail distribution of μ . By Proposition 1.5, any initial distribution μ that has heavier-than-exponential tail distribution cannot converge to a QSD. □

5.2. *Distribution of the surviving processes.* The method we develop here works for a large class of distributions μ , yet both scaling and limit distributions may depend on the choice of μ .

Recall that we work under the Assumption 2.6. We can write the density of μ as follows,

$$\text{If } \beta > 0 \text{ then } \mu(x) = F'(x) \exp(-F(x)) = F'(x)\mu([x, \infty)). \tag{5.1}$$

Note that by (Bingham et al., 1989, Proposition 1.8.1), $F'(x)$ is smooth varying with index $\beta - 1$.

We turn to the tail distribution. By the above assumption, μ has a continuous density, which we also denote by μ .

$$\begin{aligned} P_\mu(X_t > a_t, \tau > t) &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \left(\int_0^\infty e^{-\frac{x^2}{2t}} e^{\alpha x} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} \left(e^{\frac{xy}{t}} - e^{-\frac{xy}{t}} \right) dy d\mu(x) \right) \\ &= \frac{e^{-\frac{\alpha^2 t}{2}}}{\sqrt{2\pi t}} \int_0^\infty \mu(tx) e^{-\frac{tx^2}{2}} e^{\alpha tx} \int_{a_t}^\infty e^{-\frac{y^2}{2t}} e^{-\alpha y} (e^{xy} - e^{-xy}) dy dx \\ &= \frac{t}{\sqrt{2\pi}} \left(\underbrace{\int_0^\infty \mu(tx) L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx}_{=J_1(t)} - \underbrace{\int_0^\infty \mu(tx) e^{2\alpha tx} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x\right) dx}_{=J_2(t)} \right). \end{aligned} \tag{5.2}$$

We first notice that from the second term J_2 ,

$$\begin{aligned}
 e^{2\alpha t x} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x\right) dx &\leq e^{2\alpha t x} \frac{e^{-\frac{a_t^2/t + t\alpha^2 + tx^2 + 2a_t\alpha + 2a_t x + 2\alpha t x}{2}}}{a_t/\sqrt{t} + \sqrt{t}\alpha + \sqrt{t}x} \\
 &= \frac{e^{-\frac{t(\alpha-x)^2}{2}} e^{-a_t^2/(2t)} e^{-a_t(\alpha+x)}}{a_t/\sqrt{t} + \sqrt{t}\alpha + \sqrt{t}x}.
 \end{aligned}
 \tag{5.3}$$

If $a_t \gg \epsilon\sqrt{t}$ then the term $e^{-a_t^2/(2t)}$ will let J_2 decay faster (in exponential sense) than $\mu(tx)$. In fact, unless $a_t = o(\sqrt{t})$ and $x \in (\alpha - t^{-1/2+\epsilon}, \alpha + t^{-1/2+\epsilon})$, J_2 decays exponentially faster than $\mu(tx)$.

Furthermore, when we define $J_{1,A}(t), J_{2,A}(t)$ to be integrated over some sub-interval A of \mathbb{R}_+ instead of the entire \mathbb{R}_+ as follows:

$$\begin{aligned}
 J_{1,A}(t) &= \int_A \underbrace{\mu(tx) L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right)}_{=f(t,x)} dx \\
 J_{2,A}(t) &= \int_A \mu(tx) e^{2\alpha t x} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha + \sqrt{t}x\right) dx
 \end{aligned}
 \tag{5.4}$$

since $P_\mu(X_0 \in \cdot, X_t \in \cdot, \tau > t) \geq 0$ always, we can claim that $J_{2,A} = O(J_{1,A})$ on the same sub-interval $A \in \mathbb{R}_+$.

For the first term J_1 , we split the integration.

$$J_1(t) = \underbrace{\int_0^{\alpha+a_t/t-\eta_t} f(t,x) dx}_{=J_{1,1}(t)} + \underbrace{\int_{\alpha+a_t/t-\eta_t}^{\alpha+a_t/t+\epsilon_t} f(t,x) dx}_{=J_{1,2}(t)} + \underbrace{\int_{\alpha+a_t/t+\epsilon_t}^\infty f(t,x) dx}_{=J_{1,3}(t)}
 \tag{5.5}$$

where η_t, ϵ_t is to be picked depending on μ .

The goal now is to get an accurate asymptotic on the survival rate.

Proposition 5.1. *Suppose μ satisfies Assumption 2.6. Then for any $\eta_t \gg t^{\beta-1}$,*

$$\log J_{1,1}(t) \ll \log \mu([t\alpha + a_t, \infty)).
 \tag{5.6}$$

Proof: Suppose $\eta_t \gg t^{\frac{\beta-1}{2}}$. Then we have the following estimate

$$\begin{aligned}
 J_{1,1}(t) &\leq L(\sqrt{t}\eta_t) \int_0^{\alpha+a_t/t-\eta_t} \mu(tx) dx \\
 &\leq L(\sqrt{t}\eta_t) \\
 &\leq \exp\left(-\frac{t(\eta_t)^2}{2}\right) \\
 &\ll \exp\left(\frac{t^{-\beta}}{2}\right) \sim \mu([t\alpha + a_t, \infty)).
 \end{aligned}
 \tag{5.7}$$

Now suppose $t^{\frac{\beta-1}{2}} \gg \eta_t \gg t^{\beta-1}$. Pick $t^{(\beta-1)/2} \ll \eta_t^1 = t^{r_1}$ such that by (5.7),

$$\underbrace{\int_0^{\alpha+a_t/t-\eta_t^1} \mu(tx) L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{t}x\right) dx}_{=J_{1,1,1}(t)} \ll \mu([t\alpha + a_t, \infty)).$$

Now we want to pick $t^{r_2} = \eta_t^2 \ll \eta_t^1$ such that

$$\underbrace{\int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{tx}\right) dx}_{=J_{1,1,2}(t)} \ll \mu([t\alpha + a_t, \infty)).$$

Using integration by parts,

$$\begin{aligned} J_{1,1,2}(t) &= \int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{tx}\right) dx \\ &= -\frac{1}{t}\mu([tx, \infty))L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{tx}\right)\Big|_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \\ &\quad + \frac{1}{\sqrt{t}} \int_{\alpha+a_t/t-\eta_t^1}^{\alpha+a_t/t-\eta_t^2} \exp\left(\frac{-t(x - (\alpha + a_t/t))^2}{2}\right) \mu([tx, \infty))dx \\ &\leq \frac{1}{t} \left(-\mu(t\alpha + a_t - t\eta_t^2, \infty)L(\sqrt{t}\eta_t^2) + \mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^1)\right) \\ &\quad + \frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^2). \end{aligned} \tag{5.8}$$

Since both $\mu([x, \infty))$ and $L(x)$ are decreasing functions, the driving term of (5.8) is the last one. And since $\mu(x, \infty) = \exp(-F(x))$ where F is an increasing regularly varying function with index β ,

$$\begin{aligned} \frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^2) &\sim \frac{1}{t}\mu(t\alpha + a_t - t\eta_t^1, \infty)L(\sqrt{t}\eta_t^2) \\ &\leq \frac{1}{t} \exp(-F((t\alpha + a_t) - t^{1+r_1})) \exp\left(-\frac{t^{1+2r_2}}{2}\right) \\ &\sim \frac{1}{t}\mu(t\alpha + a_t, \infty) \exp\left(t^{\beta+r_1} - \frac{t^{1+2r_2}}{2}\right). \end{aligned} \tag{5.9}$$

If $\beta + r_1 < 1 + 2r_2$ we get the desired asymptotic. That is, we need $r_2 > \frac{(\beta - 1) + r_1}{2}$, and combining with $t^{(\beta-1)/2} \ll \eta_t^1$ we can pick

$$\eta_t^2 \gg t^{\frac{(\beta-1)+r_1}{2}} \sim t^{\frac{3(\beta-1)}{4}}$$

to get

$$\begin{aligned} \int_0^{\alpha+a_t/t-\eta_t^2} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{tx}\right) dx &= J_{1,1,1}(t) + J_{1,1,2}(t) \\ &\ll \mu([t\alpha + a_t, \infty)). \end{aligned} \tag{5.10}$$

Recursively, we can pick $\eta_t^n \gg t^{(\beta-1)(1-(1/2)^n)}$ such that

$$J_{1,1,n}(t) = \int_{\alpha+a_t/t-\eta_t^{n-1}}^{\alpha+a_t/t-\eta_t^n} \mu(tx)L\left(\sqrt{t}\alpha + \frac{a_t}{\sqrt{t}} - \sqrt{tx}\right) dx \ll \mu([t\alpha + a_t, \infty)).$$

So for sufficiently large n we have

$$\begin{aligned} \eta_t &= \eta_t^n \gg t^{(\beta-1)(1-(1/2)^n)} \gg t^{\beta-1} \\ J_{1,1}(t) &= \sum_{i=1}^n J_{1,1,i}(t) \ll \mu([t\alpha + a_t, \infty)) \end{aligned}$$

which completes the proof. □

Proposition 5.2. *Suppose μ satisfies 2.6 and $\beta > 0$. If $a_t \gg \sqrt{t}$,*

$$P_\mu(X_t > a_t, \tau > t) \sim \frac{t}{\sqrt{2\pi}} J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty)). \tag{5.11}$$

Proof: Pick η_t and ϵ_t as follows:

$$t^{\beta-1} \ll \eta_t \ll 1, \quad \epsilon_t = t^{-b}, \quad \beta < b < 0.5. \tag{5.12}$$

This choice yields the following asymptotic

$$\eta_t \rightarrow 0, \quad \eta_t \ll a_t/t, \quad \epsilon_t \ll a_t/t, \quad \sqrt{t}\epsilon_t \rightarrow \infty, \quad F'(t\alpha + a_t)\epsilon_t \ll 1/t. \tag{5.13}$$

For $J_{1,2}(t)$, we first observe that the interval $(\alpha + a_t/t - \eta_t, \alpha + a_t/t + \epsilon_t)$ close in to $\alpha + a_t/t$. Moreover, while L does vary between 0 and $\sqrt{\pi}/2$ within the interval, μ does not vary much from $\mu(t(\alpha + a_t/t))$ inside the interval, and therefore we can use the intermediate value theorem. Also, we split the integration to get the following bound for $J_{1,2}(t)$.

$$\begin{aligned} J_{1,2}(t) &\sim \mu(t(\alpha + a_t/t)) \\ &\times \left(\int_{\alpha+a_t/t-\eta_t}^{\alpha+a_t/t} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx + \int_{\alpha+a_t/t}^{\alpha+a_t/t+\epsilon_t} L\left(\frac{a_t}{\sqrt{t}} + \sqrt{t}\alpha - \sqrt{t}x\right) dx \right) \\ &\leq \mu(t\alpha + a_t) \left(\frac{1}{\sqrt{t}} \int_0^{\sqrt{t}\eta_t} L(y)dy + \int_0^{\epsilon_t} \sqrt{2\pi}dx \right) \\ &\leq \mu(t\alpha + a_t) \left(\frac{1}{\sqrt{t}} \int_0^\infty L(y)dy + \sqrt{2\pi}\epsilon_t \right) \\ &\sim \sqrt{2\pi}\mu(t\alpha + a_t)\epsilon_t. \end{aligned}$$

Note that the first integration is essentially the expected value of a half-normal distribution, and second integration is estimated using the fact that L is bounded above.

To estimate $J_{1,3}(t)$, since $\sqrt{t}\epsilon_t \rightarrow \infty$, it follows that $L(\sqrt{t}\epsilon_t) \rightarrow \sqrt{2\pi}$ and we can use IVT to get the sharp estimate

$$\begin{aligned} J_{1,3}(t) &\sim \sqrt{2\pi} \int_{\alpha+a_t/t+\epsilon_t}^\infty \mu(tx)dx \\ &\sim \sqrt{2\pi} \frac{1}{t} \mu([t\alpha + a_t, \infty)). \end{aligned} \tag{5.14}$$

Proposition 5.1 shows that $J_{1,1}(t) = o(J_{1,3}(t))$.

For $J_{1,2}(t)$, we combine (5.1) and (5.13) to get the following asymptotic comparison

$$\begin{aligned} J_{1,2}(t) &\leq \sqrt{2\pi}\mu(t\alpha + a_t)\epsilon_t \\ &\ll \frac{\sqrt{2\pi}}{t} \mu([t\alpha + a_t, \infty)) \sim J_{1,3}(t). \end{aligned} \tag{5.15}$$

Finally from the choice of ϵ_t we have $b < 0.5$, and therefore

$$\begin{aligned} J_{2,3}(t) &\leq \int_{\alpha+a_t/t+\epsilon_t}^\infty \mu(tx) e^{-\frac{t(\alpha-x)^2}{2}} e^{-a_t^2/(2t)} e^{-a_t(\alpha+x)} dx \\ &\leq e^{-\frac{t\epsilon_t^2}{2}} \mu([t\alpha + a_t, \infty)) = o(J_{1,3}(t)). \end{aligned} \tag{5.16}$$

We conclude that

$$P_\mu(X_t > a_t, \tau > t) \sim (1 + o(1)) \frac{t}{\sqrt{2\pi}} J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty)). \tag{5.17}$$

□

We can extend this proposition to the cases where F is slowly varying. In such cases, we expect the tail distribution $\mu(x, \infty)$ itself to be smoothly varying.

Corollary 5.3. *Suppose $\mu([x, \infty)) = G(x)$, where G is smoothly varying function with index $-\kappa < 0$. Then $P_\mu(X_t > a_t, \tau > t) \sim \frac{t}{\sqrt{2\pi}} J_{1,3}(t) \sim \mu([t\alpha + a_t, \infty))$.*

Proof: It suffices to show that $J_{1,2}(t) = o(J_{1,3}(t))$. The smooth varying condition yields the following relation (Bingham et al., 1989, 1.8.1')

$$t\mu(t\alpha + a_t) \sim \mu([t\alpha + a_t, \infty)). \tag{5.18}$$

Since we have $\epsilon_t \ll 1$,

$$J_{1,2}(t) \leq \mu(t\alpha + a_t)\epsilon_t = o\left(\frac{1}{t}\mu([t\alpha + a_t, \infty))\right) = o(J_{1,3}(t)) \tag{5.19}$$

so we have the desired asymptotic. □

5.3. *Proof of Theorem 2.8 (Heavy-tailed initial distributions).* Proposition 5.2 and Corollary 5.3 show why the second part of Assumption 2.6 is necessary. We need the right a_t that will yield nontrivial result on the limit

$$\lim_{t \rightarrow \infty} P_\mu(X_T > a_t \mid \tau > t) = \lim_{t \rightarrow \infty} \frac{P_\mu(X_t > a_t, \tau > t)}{P_\mu(\tau > t)}. \tag{5.20}$$

Due to Proposition 5.2 this boils down to comparing $\mu(t\alpha, \infty)$ and $\mu([t\alpha + a_t, \infty))$.

Proof of Theorem 2.8: If μ satisfies Assumption 2.6, setting $a_t = R(t, c)$ gives the following

$$\begin{aligned} \mu([t\alpha + a_t, \infty)) &= \exp(-F(t\alpha + R(t, c))) \\ &\sim \exp(-(F(t\alpha) + c)) \\ &= e^{-c}\mu([t\alpha, \infty)). \end{aligned} \tag{5.21}$$

We make few comments on the observation (2.10). If smooth enough, F has the Taylor expansion

$$F(t\alpha + R(t, c)) = F(t\alpha) + F'(t\alpha)R(t, c) + o(F'(t))$$

therefore by choosing $R(t, c) = \frac{c}{F'(t\alpha)}$, we get $F(t\alpha + R(t, c)) - F(t\alpha) = c + o(F'(t))$. Since F has index $\beta < 1$, $F'(t) = o(1)$ so condition (2.9) is satisfied.

We further observe that with the choice $R(t, c) = \frac{c}{F'(t\alpha)}$,

$$\begin{aligned} F'(t\alpha + R(t, c)) &= F'(t\alpha) + F''(t\alpha)R(t, c) + o(F''(t)) \\ &= F'(t\alpha) + \frac{cF''(t\alpha)}{F'(t\alpha)} + o(F''(t)) \\ &= F'(t\alpha) + o(1). \end{aligned} \tag{5.22}$$

Therefore we get $F'(t\alpha + R(t, c)) \sim F'(t\alpha)$, and consequently,

$$\begin{aligned} \mu(t\alpha + R(t, c)) &= F'(t\alpha + R(t, c)) \exp(-F(t\alpha + R(t, c))) \\ &\sim F'(t\alpha) \exp(-(F(t\alpha) + c)) \\ &= e^{-c}\mu(t\alpha). \end{aligned} \tag{5.23}$$

Putting together Proposition 5.2, Corollary 5.3, (5.21), and (5.23) completes the proof. □

5.4. *Some Concrete Examples.* We present some concrete results here.

Corollary 5.4. *Suppose $\mu([x, \infty)) = e^{-x^\beta}$ with $\beta \in (0, 0.5)$. Then*

$$\lim_{t \rightarrow \infty} P_\mu \left(\frac{X_t}{t^{1-\beta}} > c \mid \tau > t \right) = \exp(-\beta \alpha^{\beta-1} c) \tag{5.24}$$

that is, the limiting distribution is exponential with parameter $\beta \alpha^{\beta-1}$.

Proof: From proposition 5.2 we get

$$P_\mu(X_t > a_t, \tau > t) \sim \mu([t\alpha + a_t, \infty)). \tag{5.25}$$

Pick $a_t = c \cdot t^{1-\beta}$. Then by the generalized binomial theorem,

$$(t\alpha + a_t)^\beta = (t\alpha)^\beta + c\beta\alpha^{\beta-1} + o(1).$$

Note that $F'(t\alpha) = \beta(t\alpha)^{\beta-1}$. By substituting $\bar{c} = ct^{1-\beta}((\alpha t)^\beta)^\prime = c\beta\alpha^{\beta-1}$, Theorem 2.8 gives us the desired result. \square

Example 5.5. If μ is a Weibull distribution with scale parameter $\lambda > 0$ and shape parameter $0 < \beta < 0.5$, the limiting distribution of $P_\mu \left(\frac{X_t}{t^{1-\beta}} > c \mid \tau > t \right)$ is exponential distribution with rate $\beta \left(\frac{\alpha}{\lambda} \right)^{\beta-1}$.

Corollary 5.6. *Suppose $\mu([x, \infty)) = G(x)$, where G is smoothly varying function with index $-\kappa < 0$. Then*

$$\lim_{t \rightarrow \infty} P_\mu \left(\frac{X_t}{t} > c \mid \tau > t \right) = \left(\frac{\alpha + c}{\alpha} \right)^{-\kappa} \tag{5.26}$$

that is, the limiting distribution is Lomax (shifted Pareto) distribution with shape parameter κ and scale parameter α .

Proof: Since $G(x) = \exp(\log(G(x)))$ and $\log(G(x))$ is a slowly varying function ($\beta = 0$), the natural choice for $R(t, c)$ would be $a_t = R(t, c) = tc$. Indeed, by the uniform convergence theorem of regular varying function, (Bingham et al., 1989, Theorem 1.5.2)

$$\lim_{t \rightarrow \infty} \frac{G(t\alpha + tc)}{G(t)} = (\alpha + c)^{-\kappa}. \tag{5.27}$$

Therefore we have

$$\frac{P_\mu(X_t > tc, \tau > t)}{P_\mu(\tau > t)} \sim \frac{(\alpha + c)^{-\kappa} G(t)}{\alpha^{-\kappa} G(t)} \tag{5.28}$$

which gives us the desired result. \square

Example 5.7. If μ is a Half-Cauchy distribution (Cauchy distribution supported on \mathbb{R}^+), the limiting distribution of $P_\mu \left(\frac{X_t}{t} > c \mid \tau > t \right)$ is Lomax distribution with shape parameter 1 and scale parameter α .

Note that when $\beta = 0$, μ is a distribution with regular or slowly varying tail. In such cases it is often more convenient to work with the asymptotic result $P_\mu(X_t > \bar{R}(t, c), \tau > t) \sim \mu([t\alpha + \bar{R}(t, c), \infty))$ directly to find the right scaling factor \bar{R} . We conclude this section with showing the quasi-limiting behavior of μ which itself has slowly varying tail.

Corollary 5.8. *Suppose $\mu([x, \infty)) \sim \frac{1}{\ln x}$ as $x \rightarrow \infty$. Then*

$$\lim_{t \rightarrow \infty} P_\mu \left(\frac{\ln X_t}{\ln t} > c \mid \tau > t \right) = \begin{cases} 1 & c \leq 1; \\ \frac{1}{c} & c > 1. \end{cases}$$

that is, the limiting distribution is Pareto distribution with shape parameter 1 and scale parameter 1.

Proof: $\mu([x, \infty)) \sim \exp(-\ln \ln x)$ so we can apply Corollary 5.3. Since we have $\bar{R}(t, c) = t^c$,

$$\begin{aligned} \frac{P_\mu(X_t > t^c, \tau > t)}{P_\mu(\tau > t)} &\sim \frac{\ln(t\alpha)}{\ln(t\alpha + t^c)} \\ &\sim \begin{cases} \frac{\ln t + \ln \alpha}{\ln t + \ln \alpha} \rightarrow 1 & c < 1 \\ \frac{\ln t + \ln \alpha}{\ln t + \ln(\alpha + 1)} \rightarrow 1 & c = 1 \\ \frac{\ln t + \ln \alpha}{c \ln t} \rightarrow \frac{1}{c} & c > 1 \end{cases} \end{aligned} \tag{5.29}$$

which gives us the desired result. □

Notice that in our last example with super-heavy tail initial distribution, the scaled limiting distribution does not depend on the drift parameter α of the BM.

Appendix A. Appendix

A.1. Proof of Proposition 1.3.

Proof: Suppose π is a QLD for μ . Then for every and continuous function f we have $E_\mu[f(X_t)|\tau > t] \rightarrow \int f d\pi$. That is, $E_\mu[f(X_t), \tau_t] \sim P_\mu(\tau > t) \int f d\pi$, provided $\int f d\pi \neq 0$. Fix such f and let $t_1, t_2 > 0$. Then by the Markov property,

$$E_\mu[f(X_{t_1+t_2}), \tau > t_1 + t_2] = E_\mu[h_{t_2}(X_{t_1}), \tau > t_1]. \tag{A.1}$$

where $h_{t_2}(x) = E_x[f(X_{t_2}), \tau > t_2]$. By our assumption $h_{t_2}(\cdot)$ is continuous and bounded, and therefore, $E_\mu[h_{t_2}(X_{t_1})|\tau > t_1] \rightarrow \int h_{t_2} d\pi$. On rewriting (A.1) we have

$$E_\mu[f(X_{t_1+t_2})|\tau > t_1 + t_2] = E_\mu[h_{t_2}(X_{t_1})|\tau > t_1] \times \frac{P_\mu(\tau > t_1)}{P_\mu(\tau > t_1 + t_2)}.$$

By our assumption, as $t_1 \rightarrow \infty$ the lefthand side converges to the positive limit $\int f d\pi$ and the first expression on the righthand side converges to $\int h_{t_2} d\pi$. Therefore the ratio on the righthand side converges to a nonzero limit we denote by $R(t_2)$. This limit is independent of the choice of f . Therefore

$$\int f \pi = E_\pi[f(X_{t_2}), \tau > t_2]R(t_2) = E_\pi[f(X_{t_2})|\tau > t_2]R(t_2)P_\pi(\tau > t_2). \tag{A.2}$$

Taking $f \equiv 1$, we obtain $R(t_2) = \frac{1}{P_\pi(\tau > t_2)}$, and plugging this in back into (A.2) gives proves the claim. □

A.2. Proof of Proposition 1.5.

Proof: By the Markov property,

$$\begin{aligned} P_\mu(\tau > s + t) &= P_\mu(\tau > t, P_{X_t}(\tau > s)) \\ &= P_\mu(P_{X_t}(\tau > s) | \tau > t)P_\mu(\tau > t). \end{aligned} \tag{A.3}$$

Write $f(x) = P_x(\tau > s)$. π is a QSD and therefore the distribution of τ under P_π is exponential with a parameter $\lambda_\pi > 0$. Since π is the QLD of μ , for arbitrary $\epsilon > 0$ there is some $t_0 = (t_0, \mu, s)$ such that for each $t > t_0$,

$$\left| P_\mu(P_{X_t}(\tau > s) | \tau > t) - \mathbb{E}_\pi(f) \right| < \epsilon e^{-\lambda_\pi s}. \tag{A.4}$$

Since $\mathbb{E}_\pi(f) = P_\pi(\tau > s) = e^{-\lambda_\pi s}$, we have that

$$P_\mu(P_{X_t}(\tau > s) | \tau > t) \leq (1 + \epsilon)e^{-\lambda_\pi s}, \quad t > t_0. \tag{A.5}$$

Choose $s = 1$, and apply (A.5) repeatedly to obtain

$$\begin{aligned}
 P_\mu(\tau > t_0 + 1) &\leq (1 + \epsilon)e^{-\lambda\pi}P_\mu(\tau > t_0) \\
 P_\mu(\tau > t_0 + 2) &\leq (1 + \epsilon)e^{-\lambda\pi}P_\mu(\tau > t_0 + 1) \leq (1 + \epsilon)^2e^{-2\lambda\pi}P_\mu(\tau > t_0) \\
 &\vdots \\
 P_\mu(\tau > t_0 + n) &\leq (1 + \epsilon)^ne^{-n\lambda\pi}P_\mu(\tau > t_0).
 \end{aligned}
 \tag{A.6}$$

Since the choice of ϵ is arbitrary the result follows. □

A.3. *QSDs for BM with constant drift.* Here we provide a formal derivation for densities of the QSDs under assumption 1.6. Recall that a BM with constant drift $-\alpha$ on \mathbb{R}_+ absorbed at 0 is the sub-Markovian process generated by \mathcal{L}_α , which for each u satisfying $u \in C^2(\mathbb{R}_+)$ and $u(0) = 0$,

$$\mathcal{L}_\alpha u = \frac{1}{2}u'' - \alpha u'.$$

The formal adjoint \mathcal{L}_α^* of \mathcal{L}_α , with respect to integration by parts, is given by

$$\mathcal{L}^*v = \frac{1}{2}v'' + \alpha v', \quad v \in C^2(\mathbb{R}_+), v(0) = 0.$$

Observe that for any f in the domain of \mathcal{L}_α ,

$$\begin{aligned}
 \frac{d}{dt}P_x(f(X_t), \tau > t) &= \mathcal{L}_\alpha P_x(f(X_t), \tau > t) \\
 \Rightarrow P_x(f(X_t), \tau > t) &= f(x) + \int_0^t \mathcal{L}_\alpha P_x(f(X_s), \tau > s) ds.
 \end{aligned}
 \tag{A.7}$$

Suppose a probability density function π satisfies $\mathcal{L}_\alpha^*\pi = -\lambda\pi$ for some $\lambda > 0$. Notice that every QSD must be smooth, since if π is a QSD then by definition we have the following density:

$$\begin{aligned}
 \pi(y) &= P_\pi(X_s = y \mid \tau > s) \\
 &= \frac{P_\pi(X_s = y, \tau > s)}{P_\pi(\tau > s)}.
 \end{aligned}
 \tag{A.8}$$

Then with integration by parts we have the following:

$$\begin{aligned}
 E_\pi(f(X_t), \tau > t) &= \int f(x)\pi(x)dx + \int \int_0^t \mathcal{L}_\alpha (E_x(f(X_s), \tau_s)) ds\pi(x)dx \\
 &= \int f(x)\pi(x)dx + \int_0^t \int E_x(f(X_s), \tau > s)\mathcal{L}_\alpha^*\pi(x)dx ds \\
 &= \int f(x)\pi(x)dx - \lambda \int_0^t E_\pi(f(X_s), \tau > s) ds.
 \end{aligned}
 \tag{A.9}$$

Setting $h(t) = E_\pi(f(X_t), \tau > t)$, (A.9) gives

$$h'(t) = -\lambda h(t) \quad \Rightarrow \quad E_\pi(f(X_t), \tau > t) = e^{-\lambda t} \int f(x)\pi(x)dx. \tag{A.10}$$

Therefore by monotone convergence,

$$\begin{aligned}
 P_\pi(\tau > t) &= e^{-\lambda t}, \\
 E_\pi(f(X_t)|\tau > t) &= \int f(x)\pi(x)dx.
 \end{aligned}$$

That is, π is a density of a QSD if and only if $\mathcal{L}_\alpha^*\pi = -\lambda\pi$. We can see that a density for a QSD π is a solution to standard ODE and depends on the parameter λ . The range of λ for which such

a density exists is $\lambda \in (0, \alpha^2/2]$ and for each such λ corresponds a unique density. Fix such λ , set $\gamma = \sqrt{\alpha^2 - 2\lambda}$, and let π_γ be the corresponding density. Then

$$\pi_\gamma(x) = \begin{cases} \frac{\alpha^2 - \gamma^2}{\gamma} e^{-\alpha x} \sinh(\gamma x) & \gamma > 0 \\ \alpha^2 x e^{-\alpha x} & \gamma = 0. \end{cases} \quad (\text{A.11})$$

Acknowledgements

The authors would like to express their deep gratitude to an anonymous referee whose input was invaluable to the presentation of our results and helped correcting errors and omissions.

References

- Bartlett, M. S. On theoretical models for competitive and predatory biological systems. *Biometrika*, **44**, 27–42 (1957). [MR86727](#).
- Bartlett, M. S. *Stochastic population models in ecology and epidemiology*. Methuen's Monographs on Applied Probability and Statistics. Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York (1960). [MR0118550](#).
- Bingham, N. H., Goldie, C. M., and Teugels, J. L. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (1989). ISBN 0-521-37943-1. [MR1015093](#).
- Buiculescu, M. On quasi-stationary distributions for multi-type Galton-Watson processes. *J. Appl. Probability*, **12**, 60–68 (1975). [MR365734](#).
- Cattiaux, P., Collet, P., Lambert, A., Martínez, S., Méléard, S., and San Martín, J. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, **37** (5), 1926–1969 (2009). [MR2561437](#).
- Champagnat, N. and Villemonais, D. Exponential convergence to quasi-stationary distribution for absorbed one-dimensional diffusions with killing. *ALEA Lat. Am. J. Probab. Math. Stat.*, **14** (1), 177–199 (2017). [MR3622466](#).
- Collet, P., Martínez, S., and San Martín, J. *Quasi-stationary distributions. Markov chains, diffusions and dynamical systems*. Probability and its Applications (New York). Springer, Heidelberg (2013). ISBN 978-3-642-33130-5; 978-3-642-33131-2. [MR2986807](#).
- Ferrari, P. A. and Marić, N. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, **12**, no. 24, 684–702 (2007). [MR2318407](#).
- Hening, A. and Kolb, M. Quasistationary distributions for one-dimensional diffusions with singular boundary points. *Stochastic Process. Appl.*, **129** (5), 1659–1696 (2019). [MR3944780](#).
- Kolb, M. and Steinsaltz, D. Quasilimiting behavior for one-dimensional diffusions with killing. *Ann. Probab.*, **40** (1), 162–212 (2012). [MR2917771](#).
- Lladser, M. and San Martín, J. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. *J. Appl. Probab.*, **37** (2), 511–520 (2000). [MR1781008](#).
- Martinez, S., Picco, P., and San Martin, J. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, **30** (2), 385–408 (1998). [MR1642845](#).
- Martínez, S. and San Martín, J. Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.*, **31** (4), 911–920 (1994). [MR1303922](#).
- Martínez, S., San Martín, J., and Villemonais, D. Existence and uniqueness of a quasistationary distribution for Markov processes with fast return from infinity. *J. Appl. Probab.*, **51** (3), 756–768 (2014). [MR3256225](#).
- Méléard, S. and Villemonais, D. Quasi-stationary distributions and population processes. *Probab. Surv.*, **9**, 340–410 (2012). [MR2994898](#).

- Oçafrain, W. Polynomial rate of convergence to the Yaglom limit for Brownian motion with drift. *Electron. Commun. Probab.*, **25**, Paper No. 35, 12 (2020). [MR4095047](#).
- Pinsky, R. G. The lifetimes of conditioned diffusion processes. *Ann. Inst. H. Poincaré Probab. Statist.*, **26** (1), 87–99 (1990). [MR1075440](#).
- Polak, M. and Rolski, T. A note on speed of convergence to the quasi-stationary distribution. *Demonstratio Math.*, **45** (2), 385–397 (2012). [MR2963076](#).
- Scheffé, H. A useful convergence theorem for probability distributions. *Ann. Math. Statistics*, **18**, 434–438 (1947). [MR21585](#).
- Steinsaltz, D. and Evans, S. N. Quasistationary distributions for one-dimensional diffusions with killing. *Trans. Amer. Math. Soc.*, **359** (3), 1285–1324 (2007). [MR2262851](#).
- van Doorn, E. A. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, **23** (4), 683–700 (1991). [MR1133722](#).
- van Doorn, E. A. and Pollett, P. K. Quasi-stationary distributions for discrete-state models. *European J. Oper. Res.*, **230** (1), 1–14 (2013). [MR3063313](#).
- Williams, D. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge (1991). ISBN 0-521-40455-X; 0-521-40605-6. [MR1155402](#).
- Wright, S. Evolution in Mendelian populations. *Genetics*, **16** (2), 97–159 (1931). [DOI: 10.1093/genetics/16.2.97](#).
- Yaglom, A. M. Certain limit theorems of the theory of branching random processes. *Doklady Akad. Nauk SSSR (N.S.)*, **56**, 795–798 (1947). [MR0022045](#).
- Ye, J. Quasi-stationary distributions for the radial Ornstein-Uhlenbeck processes. *Acta Math. Sci. Ser. B (Engl. Ed.)*, **28** (3), 513–522 (2008). [MR2433799](#).