Alea 2, 119–123 (2006)

First passage of reflected strictly stable processes

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Abstract. Suppose that X is a strictly stable process under $\mathbb{P}$ and denote its running maximum to time $t \geq 0$ by $\overline{X}_t$. Suppose that for $z \geq 0$, Y under P_z has the same law as the process $\{(z \vee \overline{X}_t) - X_t : t \geq 0\}$ under $\mathbb{P}$. Let $\sigma^{[0,1]}$ be the first passage time of the process Y over the level 1. We give in explicit terms the law of $Y_{\sigma^{[0,1]}}$ under P_z for any $z \geq 0$.

1. Introduction

Let $X = \{X_t : t \geq 0\}$ be a strictly stable process on $\mathbb{R}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ (and corresponding expectation operators $\{\mathbb{E}_x : x \in \mathbb{R}\}$). That is to say X is a member of the family of Lévy processes whose characteristic exponent $\Psi(\theta) := -\log \mathbb{E}_0(e^{i\theta X_1})$ is given by

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha(1 - i\beta\mathrm{sgn}(\theta)\tan(\pi\alpha/2)) & \text{for } \alpha \in (0,1) \cup (1,2), \beta \in [-1,1], c > 0 \\ c|\theta| + i\mathrm{d}\theta & \text{for } \alpha = 1, \mathrm{d} \in \mathbb{R}, c > 0 \end{cases}$$

for all $\theta \in \mathbb{R}$.

Following a number of articles through the 1960s concerning exit problems of symmetric stable processes, Rogozin (1972) established, with an elementary proof, the law of the overshoot distribution for the above class of processes when exiting the interval $[0, 1]$. We shall state Rogozin's result shortly, however before doing so, let us note the following facts about strictly stable processes.

Strictly stable processes are called so on account of a scaling property which, for our purposes, can be stated as follows. For all $t \geq 0$

$$X_t \stackrel{d}{=} t^{1/\alpha} X_1.$$

Received by the editors November 2, 2005; accepted February 21, 2006.
2000 *Mathematics Subject Classification.* 60G51, 60G51.
Key words and phrases. Strictly stable processes, reflected Lévy processes, overshoot distribution, first passage problem.

In particular, this means that $\mathbb{P}(X_t \geq 0)$ is a constant which we denote ρ. Zolotarev (1986) computes this constant in the following form

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan(\beta\tan(\pi\alpha/2)).$$

One consequence of this formula is that for $\alpha \in (0,1)$, ρ ranges over $[0,1]$ with the boundary points $\rho = 1$ and $\rho = 0$ corresponding to the cases that X is an ascending and descending subordinator respectively, that is to say $\beta = 1, -1$ respectively. When $\alpha \in (1,2)$, ρ ranges over $[1-1/\alpha, 1/\alpha]$ with the boundary points $\rho = 1-1/\alpha$ and $\rho = 1/\alpha$ corresponding to the cases that X is spectrally positive or spectrally negative processes respectively (again $\beta = 1, -1$ respectively). When $\alpha = 1$ then $\rho \in (0,1)$ without restriction. Finally when $\alpha = 2$ then necessarily $\rho = 1/2$; this is the case of a Gaussian process.

Theorem 1.1 (Rogozin)**.** *Let*

$$\tau^{[0,1]} = \inf\{t > 0 : X_t \notin [0,1]\}$$

(i) Two sided jumps: *For $\alpha \in (0,2)$, $\rho \in (1-1/\alpha, 1/\alpha)$, $x \in [0,1]$ and $y \in [0,\infty]$,*

$$\mathbb{P}_x(X_{\tau^{[0,1]}} \in (1, 1+y]) = \frac{\sin\pi\alpha\rho}{\pi}(1-x)^{\alpha\rho}x^{\alpha(1-\rho)}\int_0^y t^{-\alpha\rho}(t+1)^{-\alpha(1-\rho)}(t+1-x)^{-1}dt,$$

$$\mathbb{P}_x(X_{\tau^{[0,1]}} \in [-y, 0)) = \frac{\sin\pi\alpha(1-\rho)}{\pi}(1-x)^{\alpha\rho}x^{\alpha(1-\rho)}\int_0^y t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho}(t+x)^{-1}dt.$$

(ii) Spectrally negative: *For $\alpha \in (1,2)$, $\rho = 1/\alpha$, $x \in [0,1]$ and $y \in [0,\infty]$,*

$$\mathbb{P}_x(X_{\tau^{[0,1]}} \in (1, 1+y]) = x^{\alpha-1},$$

$$\mathbb{P}_x(X_{\tau^{[0,1]}} \in [-y, 0)) = \frac{\sin\pi(\alpha-1)}{\pi}x^{\alpha-1}(1-x)\int_0^y t^{-(\alpha-1)}(t+1)^{-1}(t+x)^{-1}dt.$$

In fact the main result of Rogozin (1972) was more complete than the version presented here, covering subordinators, the spectrally positive case and Brownian motion. However, as these shall be of no further consequence we have excluded them. (Note however that the spectrally negative case follows from the spectrally positive case by simple change of variables). In addition, Rogozin showed that a number of the identities given above when $y = \infty$ simplify further. For example in case (i) of the above theorem, when $x \in [0,1]$, one may also write

$$\mathbb{P}_x(X_{\tau^{[0,1]}} > 1) = \frac{1}{B(\alpha\rho, \alpha(1-\rho))}\int_0^x t^{\alpha\rho-1}(1-t)^{\alpha(1-\rho)-1}dt$$

where $B(\cdot,\cdot)$ is the Beta function.

The purpose of this note is to combine Rogozin's result with a martingale technique developed in Avram et al. (2004) and compute the distribution of the overshoot of level 1 by the process equal in law to a strictly stable processes reflected in its supremum. (Note that the overshoot distribution of an arbitrary level can be obtained from the latter by rescaling). To be precise, suppose that X is any strictly stable process and define its running maximum at time $t \geq 0$, $\overline{X}_t = \sup_{s \leq t} X_s$. The strong Markov process we are interested in, $Y = \{Y_t : t \geq 0\}$ with probabilities $\{P_z : z \geq 0\}$, is equal in law to the process $\{(z \vee \overline{X}_t) - X_t : t \geq 0\}$ under $\mathbb{P}_0$. Note that Y is $[0,\infty)$-valued with a reflecting barrier at 0 and further, that $Y_0 = z$. Define

$$\sigma^{[0,1]} = \inf\{t > 0 : Y_t \notin [0,1]\}.$$

We are thus interested in characterizing the law of $Y_{\sigma^{[0,1]}}$ under P_z for each $z \in [0,1]$. We shall not consider however the cases that the underlying process X is either an ascending or descending subordinator as the problem is degenerate for these cases. Further, the case that X is spectrally positive is also excluded due to the overshoot being zero. Our main result is stated as follows.

Theorem 1.2. *Suppose that (Y, P_z) is a strictly stable process reflected in its supremum as described above.*

(i) Two sided jumps: *For $\alpha \in (0,2)$, $\rho \in (1-1/\alpha, 1/\alpha)$, $z \in [0,1]$ and $y \in [0,\infty]$,*

$$\begin{aligned} P_z(Y_{\sigma^{[0,1]}} \in (1, 1+y]) &= \frac{\sin \pi\alpha(1-\rho)}{\pi} z^{\alpha\rho}(1-z)^{\alpha(1-\rho)} \int_0^y t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho}(t+1-z)^{-1}dt \\ &\quad + \frac{\int_0^y t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho-1}dt \cdot \int_0^{1-z} t^{\alpha\rho-1}(1-t)^{\alpha(1-\rho)-1}dt}{B(\alpha\rho, \alpha(1-\rho)) \int_0^\infty t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho-1}dt}. \end{aligned}$$

(ii) Spectrally negative: *For $\alpha \in (1,2)$, $\rho = 1/\alpha$, $z \in [0,1]$ and $y \in [0,\infty]$,*

$$\begin{aligned} P_z(Y_{\sigma^{[0,1]}} \in (1, 1+y]) &= \frac{\sin \pi(\alpha-1)}{\pi}(1-z)^{\alpha-1} z \int_0^y t^{-(\alpha-1)}(t+1)^{-1}(t+1-z)^{-1}dt \\ &\quad +(1-z)^{\alpha-1} \frac{\int_0^y t^{-(\alpha-1)}(t+1)^{-2}dt}{\int_0^\infty t^{-(\alpha-1)}(t+1)^{-2}dt}. \end{aligned}$$

Before moving to the proof, let us make the following remarks in the special case that $z = 0$ where the formulae tidy up somewhat. For both cases considered, one sees that

$$P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y]) = \frac{\int_0^y t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho-1}dt}{\int_0^\infty t^{-\alpha(1-\rho)}(t+1)^{-\alpha\rho-1}dt}.$$

(Note in case (ii) of the theorem that $\alpha\rho = 1$). An elementary change of variable on both the left and right hand side now show that

$$P_0((Y_{\sigma^{[0,1]}})^{-1} \leq x) = \frac{\int_0^x (1-s)^{-\alpha(1-\rho)} s^{\alpha-1} ds}{\int_0^1 (1-s)^{-\alpha(1-\rho)} s^{\alpha-1} ds}.$$

In other words, $(Y_{\sigma^{[0,1]}})^{-1}$ has a Beta$(1-\alpha(1-\rho), \alpha)$ distribution.

2. Proof

We give the proofs in steps.

Step 1. First apply the Strong Markov Property and note that

$$\begin{aligned} P_z(Y_{\sigma^{[0,1]}} \in (1, 1+y]) &= \mathbb{P}_{1-z}(X_{\tau^{[0,1]}} > 1) P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y]) \\ &\quad + \mathbb{P}_{1-z}(X_{\tau^{[0,1]}} \in [-y, 0)) \end{aligned} \tag{2.1}$$

The two probabilities with respect to $\mathbb{P}_{1-z}$ are given by Theorem 1.1. Hence the crux of the proof is to establish an expression for $P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y])$.

Step 2. Let $\{(t, \epsilon_t) : t \geq 0\}$ be the (killed) Poisson point process of excursions. Here the index t is measured in units of local time at the maximum, denoted by the process $\{L_s : s \geq 0\}$. Let L^{-1} the right inverse of L. If for $t > 0$ we have $L_t^{-1} - L_{t-}^{-1} > 0$ then $\epsilon_t = \{\epsilon_t(s) : s \leq \zeta_t\}$ where $\zeta_t > 0$ is the duration (in real time) of the excursion. See Bertoin (1996) for an account of excursion theory in the context of Lévy processes. Further, under these circumstances, write $\overline{\epsilon}_t$ for the height of the excursion and let $T_t^{[0,1]} = \inf\{s > 0 : \epsilon_t(s) > 1\}$. We have

$$P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y]) = \mathbb{E}_0 \left(\sum_g \mathbf{1}_{(\sup_{h<g} \overline{\epsilon}_h \leq 1)} \mathbf{1}_{(\overline{\epsilon}_g > 1, \epsilon_g(T_g^{[0,1]}) \in (1,1+y])} \right)$$

where the indices g and h are taken over left end points of excursions. The compensation formula now gives

$$\begin{aligned} &P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y]) = \\ &\qquad \mathbb{E}_0 \left(\int_0^\infty \mathbf{1}_{(\sup_{h<L_s} \overline{\epsilon}_h \leq 1)} dL_s \right) n(\overline{\epsilon} > 1, \epsilon(T^{[0,1]}) \in (1, 1+y])) \end{aligned}$$

where n is the excursion measure for X reflected in its maximum and ϵ, $\overline{\epsilon}$ and $T^{[0,1]}$ are the the analogues of ϵ_t, $\overline{\epsilon}_t$ and $T_t^{[0,1]}$ for the generic excursion. Continuing we have,

$$\begin{aligned} P_0(Y_{\sigma^{[0,1]}} \in (1, 1+y]) &= \int_0^\infty \mathbb{P}_0(\sup_{h<t} \overline{\epsilon}_h \leq 1, t < L_\infty) dt \\ &\quad \times n(\epsilon(T^{[0,1]}) \in (1, 1+y]) | \overline{\epsilon} > 1) n(\overline{\epsilon} > 1) \\ &= \int_0^\infty e^{-n(\overline{\epsilon}>1)t} dt \cdot n(\epsilon(T^{[0,1]}) \in (1, 1+y]) | \overline{\epsilon} > 1) n(\overline{\epsilon} > 1) \\ &= n(\epsilon(T^{[0,1]}) \in (1, 1+y]) | \overline{\epsilon} > 1). \qquad (2.2) \end{aligned}$$

Step 3. Define the probability measure on the space of excursions, $Q(\cdot) = n(\cdot | \overline{\epsilon} > 1)$ and for $\theta \in (0, 1]$, let $\mathcal{G}_\theta = \sigma(\epsilon(s) : s \leq T^{[0,\theta]})$ where $T^{[0,\theta]} = \inf\{s > 0 : \epsilon(s) > \theta\}$. Define the process

$$M_\theta = Q(\epsilon(T^{[0,1]}) \in (1, 1+y] | \mathcal{G}_\theta), = \frac{n(\epsilon(T^{[0,1]}) \in (1, 1+y], \overline{\epsilon} > 1 | \mathcal{G}_\theta)}{n(\overline{\epsilon} > 1 | \mathcal{G}_\theta)} \ \theta \in (0, 1],$$

and note that it is a martingale. Appealing to the Strong Markov Property for excursions we see that

$$n(\epsilon(T^{[0,1]}) \in (1, 1+y], \overline{\epsilon} > 1 | \mathcal{G}_\theta) = \mathbb{P}_{1-\epsilon(T^{[0,\theta]})}(X_{\tau^{[0,1]}} \in [-y, 0))$$

for $\theta \in (0, 1]$. Similarly

$$n(\overline{\epsilon} > 1 | \mathcal{G}_\theta) = \mathbb{P}_{1-\epsilon(T^{[0,\theta]})}(X_{\tau^{[0,1]}} < 0)$$

for $\theta \in (0, 1]$. In the previous two expressions, when $1 - \epsilon(T^{[0,\theta]}) < 0$, we interpret the probabilities on the right hand sides in the literal sense. For example in the first expression the probability is taken as equal to $\mathbf{1}_{(\epsilon(T^{[0,\theta]}) \in [-y,0))}$ on $\{1 - \epsilon(T^{[0,\theta]}) < 0\}$.

Now noting that $M_1 = \mathbf{1}_{(\epsilon(T^{[0,1]})\in(1,1+y])}$, Q-almost surely, it follows by the conservation of martingale expectations that

$$Q(\epsilon(T^{[0,1]}) \in (1, 1+y]) = \lim_{\theta\downarrow 0} Q(M_\theta) = \lim_{\theta\downarrow 0} \frac{\mathbb{P}_{1-\epsilon(T^{[0,\theta]})}(X_{\tau^{[0,1]}} \in [-y, 0))}{\mathbb{P}_{1-\epsilon(T^{[0,\theta]})}(X_{\tau^{[0,1]}} < 0)}. \quad (2.3)$$

Step 4. For the final part, we need the following fact, $\epsilon(0+) := \lim_{\theta\downarrow 0} \epsilon(T^{[0,\theta]}) = 0$. One may deduce from Millar (1977) that this is the case if and only if 0 is regular for $(-\infty, 0)$. However, it is a well known fact that the latter regularity holds for all stable processes which are not subordinators. (Indeed this follows easily from Rogozin's integral test for regularity, see Proposition VI.3.11 in Bertoin (1996)). The proof is now completed by inserting the two sided exit formulae from Rogozin's Theorem into (2.3), taking limits to establish an expression for (2.2) and then plugging this into (2.1). □

The proof also shows that for a general Lévy processes, provided one is in possession of the overshoot distribution for the two sided exit problem and 0 is regular for $(-\infty, 0)$, then one may establish the overshoot distribution for the reflected exit problem explicitly. The only two cases known to the author however are those of a spectrally negative processes, handled in Avram et al. (2004), and the other being the case at hand.

Acknowledgements

I would like to thank Sonia Fourati for introducing me to Rogozin's paper and Ron Doney and Loic Chaumont for bringing Millar (1977) to my attention. I am also greatful to a referee who pointed out that $1/Y_{\sigma^{[0,1]}}$ has a Beta distribution.

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