Alea 2, 217–229 (2006)

Strong disorder implies strong localization for directed polymers in a random environment

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Abstract. In this note we show that in any dimension d, the strong disorder property implies the strong localization property. This is established for a continuous time model of directed polymers in a random environment : the parabolic Anderson Model.

1. Introduction

Let $\omega = (\omega(t))_{t\geq 0}$ be the simple continuous time random walk on the d-dimensional lattice $\mathbb{Z}^d$, with jump rate $\kappa > 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider an *environment* $B = (B_x(t), t \geq 0, x \in \mathbb{Z}^d)$ made of independent standard Brownian motions B_x defined on another probability space $(H, \mathcal{G}, \mathbf{P})$.
For any $t > 0$ the (random) *polymer measure* μ_t is the probability defined on the path space $(\Omega, \mathcal{F})$ by

$$\mu_t(d\omega) = \frac{1}{Z_t} e^{\beta H_t(\omega) - t\beta^2/2} \mathbb{P}(d\omega),$$

where $\beta \geq 0$ is the inverse temperature, the Hamiltonian is

$$H_t(\omega) = \int_0^t dB_{\omega(s)}(s)$$

and the *partition function* is

$$Z_t = Z_t(\beta) = \mathbb{E}\left[e^{\beta H_t(\omega) - t\beta^2/2}\right],$$

where $\mathbb{E}\,[]$ denotes expectation with respect to $\mathbb{P}$.

Received by the editors January 02, 2006; accepted April 10, 2006.
2000 *Mathematics Subject Classification.* primary 60K35, secondary 60J30, 60H30, 60H15.
Key words and phrases. Directed polymers, Random environment, strong disorder, localization, parabolic Anderson.

Bolthausen (1989) was the first to establish that $(Z_t)_{t\geq 0}$ was a positive martingale, converging almost surely to a finite random variable Z_∞, satisfying a $z$ero-one law: $\mathbf{P}(Z_\infty > 0) \in \{0,1\}$. We shall say that there is *strong disorder* if $Z_\infty = 0$ almost surely, and *weak disorder* if $Z_\infty > 0$ almost surely.
Another martingale argument, based on a supermartingale decomposition of $\log Z_t$, enabled Carmona and Hu (2004), then Comets et al. (2003, 2004), and Rovira and Tindel (2005), to show the equivalence between strong disorder and *weak-localization* :

$$Z_\infty = 0 \; a.s. \qquad \Longleftrightarrow \int_0^\infty \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t))\, dt = +\infty \quad a.s.\,,$$

where ω_1, ω_2 are two independent copies of the random walk ω, considered under the product polymer measure $\mu_t^{\otimes 2}$:

$$\mu_t^{\otimes 2}(d\omega_1, d\omega_2) = \frac{1}{Z_t^2} e^{\beta(H_t(\omega_1)+H_t(\omega_2))-t\beta^2}\, \mathbb{P}^{\otimes 2}(d\omega_1, d\omega_2)\,.$$

Let us define *strong localization* as the existence of a constant $c > 0$ such that

$$\limsup_{t\to+\infty} \sup_x \mu_t(\omega(t) = x) \geq c \quad a.s.$$

This property implies the existence of highly favored sites, in contrast to the simple random walk ($\beta = 0$) for which $\sup_x \mathbb{P}\left(X_t = x\right) \sim Ct^{-d/2} \to 0$. Carmona and Hu (2004), and then Comets et al. (2004), showed that in dimension $d = 1, 2$, for any $\beta > 0$, there was not only strong disorder but also strong localization.

We shall prove in this note the

Theorem 1.1. *In any dimension d, strong disorder implies strong localization.*

This completes the picture since we know now from Comets and Yoshida (2004) that weak disorder implies diffusivity under the polymer measure.

For sake of completeness, let us state yet another localization property. The *free energy* is the limit

$$p(\beta) = \lim_{t\to+\infty} \frac{1}{t} \log Z_t\,,$$

where the limit can be shown to hold almost surely and in every L^p, $p \geq 1$ (see e.g. Comets et al. (2004)). The function $p(\beta)$ is continuous, non increasing on $[0, +\infty[$, $p(\beta) \leq 0$, $p(0) = 0$, so there exists a critical inverse temperature $\beta_c \in [0, +\infty]$ such that:

$$\begin{cases} p(\beta) = 0 & \text{if } 0 \leq \beta \leq \beta_c\,; \\ p(\beta) < 0 & \text{if } \beta > \beta_c\,. \end{cases}$$

When $p(\beta) < 0$ we say that the system has the *very strong disorder* property. We shall prove that (see equation (2.1)):

$$p(\beta) = -\frac{\beta^2}{2} \lim_{t\to+\infty} \frac{1}{t} \int_0^t \mu_s^{\otimes 2}(\omega_1(s) = \omega_2(s))\, ds \quad a.s.$$

Therefore there is *very strong disorder* if and only if there exists a constant $c > 0$ such that almost surely:

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \mu_s^{\otimes 2}(\omega_1(s) = \omega_2(s))\, ds = c.$$

The recent beautiful result of Comets and Vargas (2006), that is $\beta_c = 0$ in dimension $d = 1$, strengthen our belief in the

$$\texttt{Conjecture}: \quad \text{very strong disorder} \iff \text{strong disorder}$$

Proving this conjecture would unify all these notions of disorder and localization.

Eventually, let us end this rather lengthy introduction by making clearer the connection with the parabolic Anderson model (see Carmona and Molchanov (1994) or Cranston et al. (2002)). The point to point partition functions

$$Z_t(x,y) = \mathbb{E}_x\left[e^{\beta H_t(\omega) - t\beta^2/2}\,\mathbf{1}_{(\omega(t)=y)}\right]$$

satisfy the stochastic partial differential equation (see Section 2)

$$dZ_t(0,x) = LZ_t(0,.)(x)\,dt + \beta Z_t(0,x)\,dB_x(t)\,,$$

where $L = \kappa\Delta$ is the generator of the simple random walk ω with jump rate κ, that is Δ is the discrete Laplacian.

Let us explain now the structure of this paper. Section 2 is devoted to the study of the partition function as a martingale, and we prove that its asymptotics are governed by the asymptotics of the overlap $I_t = \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t))$.
An important fact is that I_t itself is a semimartingale. In Section 3 we establish a decomposition of I_t which is not its canonical semimartingale decomposition (this decomposition can be obtained via the parabolic Anderson equation(1)). In fact this decomposition looks a lot like a renewal equation involving the overlap for the simple random walk : it is the basic ingredient of our proof of the main result, since it is in this decomposition that we inject our knowledge of the behaviour of the overlap for simple random walk.

2. The partition function

Without loss in generality we can work on the canonical path space Ω made of $\omega : \mathbb{R}^+ \to \mathbb{Z}^d$, càdlàg, with a finite number of jumps in each finite interval $[0,t]$. We endow Ω with the canonical sigma-field $\mathcal{F}$ and the family of laws $(\mathbb{P}_x, x \in \mathbb{Z}^d)$ such that under $\mathbb{P}_x$, $(\omega(t))_{t\geq 0}$ is the simple random walk starting from x, with generator $L = \kappa\Delta$. With these notations, we consider, attached to each path $\omega \in \Omega$, the exponential martingale

$$M_t^\omega = \exp(\beta H_t(\omega) - t\beta^2/2) = 1 + \beta\int_0^t M_s^\omega\,dB_{\omega(s)}(s)\,,$$

with respect to the filtration $\mathcal{G}_t = \sigma(B_x(s), s \leq t, x \in \mathbb{Z}^d)$. We have $Z_t = \mathbb{E}\left[M_t^\omega\right]$ and thus the

Proposition 2.1. *The process $(Z_t)_{t\geq 0}$ is a continuous positive $\mathcal{G}_t$ martingale with quadratic variation*

$$d\langle Z,Z\rangle_t = Z_t^2\beta^2\,I_t\,dt\,, \quad \textit{with} \quad I_t = \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t))\,.$$

Proof: We know that linear combinations of martingales are martingales. This extends easily to probability mixtures of martingales. Indeed, let $0 \leq s \leq t$ and let U be positive bounded and $\mathcal{G}_s$-measurable. Then, by Fubini-Tonelli's theorem :

$$\begin{aligned}
\mathbf{E}[Z_t U] &= \mathbf{E}[\mathbb{E}\,[M_t^\omega]U] = \mathbb{E}\,[\mathbf{E}[M_t^\omega U]] \\
&= \mathbb{E}\,[\mathbf{E}[M_s^\omega U]] && (M^\omega \text{ is a martingale}) \\
&= \mathbf{E}[\mathbb{E}\,[M_s^\omega]U] = \mathbf{E}[Z_s U]\,.
\end{aligned}$$

Observe that if ω_1, ω_2 are paths, then we can compute the quadratic covariation

$$d\langle M^{\omega_1}, M^{\omega_2}\rangle_t = M_t^{\omega_1} M_t^{\omega_2} \beta^2\, \mathbf{1}_{(\omega_1(t)=\omega_2(t))}\, dt.$$

Therefore, we have formally:

$$\begin{aligned}
d\langle Z, Z\rangle_t &= d\left\langle \int \mathbb{P}(d\omega_1) M^{\omega_1}, \int \mathbb{P}(d\omega_2) M^{\omega_2} \right\rangle_t \\
&= \int \mathbb{P}^{\otimes 2}(d\omega_1, d\omega_2) d\langle M^{\omega_1}, M^{\omega_2}\rangle_t \\
&= \beta^2 Z_t^2 \frac{1}{Z_t^2} \int \mathbb{P}^{\otimes 2}(d\omega_1, d\omega_2) M_t^{\omega_1} M_t^{\omega_2}\, \mathbf{1}_{(\omega_1(t)=\omega_2(t))}\, dt \\
&= Z_t^2 \beta^2\, I_t\, dt.
\end{aligned}$$

This again can be made rigorous by writing $N_t = Z_t^2 - \beta^2 \int_0^t Z_s^2 I_s\, ds$ as a probability mixture of martingales:

$$N_t = \int \mathbb{P}^{\otimes 2}(d\omega_1, d\omega_2) (M_t^{\omega_1} M_t^{\omega_2} - \beta^2 \int_0^t M_s^{\omega_1} M_s^{\omega_2}\, \mathbf{1}_{(\omega_1(s)=\omega_2(s))}\, ds)\,.$$

□

The positive martingale Z_t converges almost surely to a positive finite random variable Z_∞. We refer to any of Bolthausen (1989); Comets et al. (2004); Carmona and Hu (2002) for a proof of the following zero-one law.

Proposition 2.2.

$$\mathbf{P}(Z_\infty = 0) \in \{0, 1\}\,.$$

We can now show the equivalence between strong disorder and weak localization.

Proposition 2.3. *The supermartingale* $\log Z_t$ *has the decomposition*

$$\log Z_t = M_t - \frac{1}{2} A_t$$

with $(M_t)_{t \geq 0}$ *a continuous martingale of quadratic variation*

$$\langle M, M\rangle_t = A_t = \beta^2 \int_0^t I_s\, ds\,.$$

Consequently:

- *either* $Z_\infty = 0$ *and* $\int_0^\infty I_s\, ds = +\infty$ *almost surely;*
- *or* $Z_\infty > 0$ *and* $\int_0^\infty I_s\, ds < +\infty$ *almost surely.*

In both cases the free energy is given by

$$p(\beta) = -\frac{\beta^2}{2} \lim_{t \to +\infty} \frac{1}{t} \int_0^t I_s\, ds = -\frac{\beta^2}{2} \lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbf{E}[I_s]\, ds\,. \tag{2.1}$$

Proof: One can even prove (see Carmona and Hu, 2002) that weak disorder is equivalent to the uniform integrability of the martingale $(Z_t)_{t\geq 0}$.

Itô's formula yields :

$$\log Z_t = \int_0^t \frac{dZ_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle Z,Z\rangle_s}{Z_s^2} = M_t - \frac{1}{2}\beta^2 \int_0^t I_s\,ds = M_t - \frac{1}{2}A_t.$$

Therefore,

- On $\{A_\infty = \langle M,M\rangle_\infty < +\infty\}$ the martingale M_t converges almost surely, $M_t \to M_\infty$ so $\log Z_t \to M_\infty - \frac{1}{2}A_\infty$ and $Z_\infty > 0$ almost surely, and $p(\beta) = \lim_{t\to+\infty} \frac{1}{t}\log Z_t = 0$.
- On $\{A_\infty = \langle M,M\rangle_\infty = +\infty\}$, we have almost surely $\frac{M_t}{\langle M,M\rangle_t} \to 0$ so $\frac{\log Z_t}{A_t} \to -\frac{1}{2}$ and $\log Z_t \to -\infty$, so $Z_\infty = 0$. Furthermore, $p(\beta) = \lim_{t\to+\infty} \frac{1}{t}\log Z_t = -\frac{1}{2}\lim_{t\to+\infty}\frac{1}{t}A_t$.

We conclude this proof by taking expectations:

$$p(\beta) = \lim_{t\to+\infty}\frac{1}{t}\mathbf{E}[\log Z_t] = -\frac{1}{2}\lim_{t\to+\infty}\frac{1}{t}\mathbf{E}[A_t] = -\frac{\beta^2}{2}\lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{E}[I_s]\,ds\,.$$

□

The connection with the parabolic Anderson model is contained in the

Proposition 2.4. *The point to point partition functions* $(Z_t(0,x), t\geq 0, x\in\mathbb{Z}^d)$ *satisfy the stochastic partial differential equation*

$$dZ_t(0,x) = LZ_t(0,.)(x)\,dt + \beta\, Z_t(0,x)\,dB_x(t)\,,$$

where $L = \kappa\Delta$ *is the generator of the simple random walk with jump rate* κ, *that is* Δ *is the discrete Laplacian.*

Proof: Let $p_t(x) = \mathbb{P}(X_t = x)$ be the probability function at time t of simple random walk. By Fubini's stochastic theorem and Markov property:

$$\begin{aligned}
Z_t(0,x) &= \int \mathbb{P}(d\omega) M_t^\omega\, \mathbf{1}_{(\omega(t)=x)} \\
&= \int \mathbb{P}(d\omega)\, \mathbf{1}_{(\omega(t)=x)}(1+\beta\int_0^t M_s^\omega dB_{\omega(s)}(s)) \\
&= p_t(x) + \beta\int_0^t \int \mathbb{P}(d\omega)\, \mathbf{1}_{(\omega(t)=x)} M_s^\omega dB_{\omega(s)}(s) \\
&= p_t(x) + \beta\int_0^t \int \mathbb{P}(d\omega) p_{t-s}(\omega(s)-x) M_s^\omega dB_{\omega(s)}(s) \\
&= p_t(x) + \beta\int_0^t Z_s\mu_s(p_{t-s}(\omega(s)-x)dB_{\omega(s)}(s))\,.
\end{aligned}$$

We conclude by differentiating with respect to t, taking into account that

$$\frac{d}{dt}p_t(x) = Lp_t(x).$$

In other words, we combine

$$p_{t-s}(y) = \mathbf{1}_{(y=0)} + \int_s^t Lp_{u-s}(y)\,du$$

and Fubini's stochastic theorem. (This result is just Feynman-Kac formula combined with time reversal of the continuous time random walk). □

3. Itô's formula for the polymer measure

Let $(P_t^{\otimes n})_{t\geq 0}$ be the semi-group of the Markov process $\boldsymbol{\omega}(t) = (\omega_1(t), \dots, \omega_n(t))$ constructed from n independent copies of the simple random walk $(\omega(t))_{t\geq 0}$: if $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded Borel function, then

$$P_t^{\otimes n} f(x_1, \dots, x_n) = \mathbb{E}_{x_1,\dots,x_n}[f(\omega_1(t), \dots, \omega_n(t))] .$$

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded Borel function, and $t \geq t_0 \geq 0$. Then,*

$$\begin{aligned}
\mu_t^{\otimes n}[f(\boldsymbol{\omega}(t))] = \mu_{t_0}^{\otimes n}\big[P_{t-t_0}^{\otimes n} f(\boldsymbol{\omega}(t_0))\big] \\
+ \beta^2 \sum_{i<j} \int_{t_0}^t \mu_s^{\otimes n}\big[\mathbf{1}_{(\omega_i(s)=\omega_j(s))} P_{t-s}^{\otimes n} f(\boldsymbol{\omega}(s))\big]\, ds \\
- n\beta^2 \sum_i \int_{t_0}^t \mu_s^{\otimes(n+1)}\big[\mathbf{1}_{(\gamma(s)=\omega_i(s))} P_{t-s}^{\otimes n} f(\boldsymbol{\omega}(s))\big]\, ds \\
+ \frac{n(n+1)}{2}\beta^2 \int_{t_0}^t \mu_s^{\otimes n}\big[P_{t-s}^{\otimes n} f(\boldsymbol{\omega}(s))\big]\, I_s\, ds \\
+ \int_{t_0}^t \mu_s^{\otimes n}\left[P_{t-s}^{\otimes n} f(\boldsymbol{\omega}(s))(\beta \sum_i dB_{\omega_i(s)}(s) - n\frac{dZ_s}{Z_s})\right],
\end{aligned}$$

where γ is an extra independent copy of ω.

Proof: Given paths $\omega_1, \dots, \omega_n$, we let

$$U_t = U_t(\omega_1, \dots, \omega_n) = \frac{M_t^{\omega_1} \dots M_t^{\omega_n}}{Z_t^n} .$$

We use the following easy computations of quadratic variations:

$$d\langle M^\gamma, M^\tau\rangle_t = M_t^\gamma M_t^\tau \beta^2\, \mathbf{1}_{(\gamma(t)=\tau(t))}\, dt$$
$$d\langle M^\gamma, Z\rangle_t = \beta^2 M_t^\gamma Z_t \mu_t\big[\mathbf{1}_{(\omega(t)=\gamma(t))}\big]\, dt\,, \qquad d\langle Z, Z\rangle_t = Z_t^2\, \beta^2 I_t dt\,,$$

The classical Itô's formula yields:

$$\begin{aligned}
U_t = U_{t_0} + \int_{t_0}^t U_s \left(\sum_{i=1}^n \beta dB_{\omega_i(s)}(s) - n\frac{dZ_s}{Z_s}\right) \\
+ \beta^2 \int_{t_0}^t U_s \left(\sum_{i<j} \mathbf{1}_{(\omega_i(s)=\omega_j(s))} - n \sum_i \mu_s\big[\mathbf{1}_{(\gamma(s)=\omega_i(s))}\big] + \frac{n(n+1)}{2} I_s\right) ds\,,
\end{aligned}$$

where in the last line μ_s acts on the generic path γ. Since,

$$\mu_t^{\otimes n}[f(\boldsymbol{\omega}(t))] = \int f(\boldsymbol{\omega}(t)) U_t(\boldsymbol{\omega})\, d\mathbb{P}^{\otimes n}(\boldsymbol{\omega})$$

we conclude this proof by applying Fubini's theorem and Markov's property. For example,

$$\begin{aligned}\int f(\boldsymbol{\omega}(t))U_{t_0}(\boldsymbol{\omega})\,d\mathbb{P}^{\otimes n}(\boldsymbol{\omega}) &= \mathbb{E}\left[f(\boldsymbol{\omega}(t))\frac{M_{t_0}^{\omega_1}\dots M_{t_0}^{\omega_n}}{Z(t_0)^n}\right] \\ &= \frac{1}{Z(t_0)^n}\mathbb{E}\left[P_{t-t_0}^{\otimes n}f(\boldsymbol{\omega}(t_0))M_{t_0}^{\omega_1}\dots M_{t_0}^{\omega_n}\right] \\ &= \mu_{t_0}^{\otimes n}\left[P_{t-t_0}^{\otimes n}f(\boldsymbol{\omega}(t_0))\right].\end{aligned}$$

□

4. Proof of the main result

We assume that there is strong disorder so almost surely, $Z_\infty = 0$ and $\int_0^\infty I_s\,ds = +\infty$, and we shall show that for a certain $c_0 > 0$, $\limsup_{t\to+\infty} V_t \geq c_0$ almost surely, with $V_t = \sup_x \mu_t(\omega(t) = x)$.

Let $r(t) = \mathbb{P}^{\otimes 2}(\omega_1(t) = \omega_2(t))$ and $R(t) = \int_0^t r(s)\,ds$. In dimension $d = 1, 2$, $R(\infty) = +\infty$ so certainly $\beta^2 R(\infty) > 1$. In dimension $d \geq 3$, $R(\infty) < +\infty$ and Markov's property implies that $L_\infty = \int_0^\infty \mathbf{1}_{(\omega_1(s)=\omega_2(s))}\,ds$ is under $\mathbb{P}^{\otimes 2}$ an exponential random variable of expectation $R(\infty)$. Since, by Fubini's theorem,

$$\begin{aligned}\mathbf{E}\left[Z_t^2\right] &= \mathbb{E}^{\otimes 2}\left[\mathbf{E}\left[e^{\beta(H_t(\omega_1)+H_t(\omega_2))-t\beta^2}\right]\right] \\ &= \mathbb{E}^{\otimes 2}\left[e^{\frac{\beta^2}{2}\operatorname{Var}(H_t(\omega_1)+H_t(\omega_2))-t\beta^2}\right] \\ &= \mathbb{E}^{\otimes 2}\left[e^{\beta^2\int_0^t \mathbf{1}_{(\omega_1(s)=\omega_2(s))}\,ds}\right],\end{aligned}$$

the second moment method yields that if $\beta^2 R(\infty) < 1$, then $sup_t\mathbf{E}\left[Z_t^2\right] = \mathbb{E}^{\otimes 2}\left[e^{\beta^2 L_\infty}\right] < +\infty$, so Z_t is an L^2 bounded martingale, hence $\mathbb{E}\left[Z_\infty\right] = 1$ and $Z_\infty > 0$ almost surely. Birkner (2004) improved this result by using a conditional moment method : if $R(\infty) < +\infty$, then there exists $\beta_c^- > \frac{1}{\sqrt{R(\infty)}}$ such that for $\beta < \beta_c^-$, $Z_\infty > 0$ almost surely. Hence, since we assumed strong disorder, we certainly have $\beta^2 R(\infty) > 1$.

Observe that since $V_t = \sup_x U_t(x)$ with $U_t(x) = \mu_t(\omega(t) = x)$, we have

$$\begin{aligned}I_t = \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t)) &= \sum_x \mu_t^{\otimes 2}(\omega_1(t) = x = \omega_2(t)) \\ &= \sum_x U_t(x)^2 \leq V_t \sum_x U_t(x) = V_t\end{aligned}$$

and $I_t \geq V_t^2$. Therefore we shall show that almost surely, $\limsup_{t\to+\infty} I_t \geq c_0$. It is sufficient to prove that if $J_t = I_t\,\mathbf{1}_{(I_t\geq c_0)}$ then for a constant $c_1 > 0$,

$$\limsup_{t\to+\infty} \frac{\int_0^t J_s\,ds}{\int_0^t I_s\,ds} \geq c_1 \quad \text{almost surely,}$$

(indeed recall that $\int_0^\infty I_s\,ds = +\infty$ almost surely).
We now have to choose $c_0 > 0$. Since $\beta^2 R(\infty) > 1$, there exists $\epsilon_0 \in (0, \frac{1}{16})$ and $t_0 > 0$ such that $\beta^2 R(t_0)(1 - 4\sqrt{\epsilon_0}) > 1$. We let $c_0 = \epsilon_0 \inf_{0\leq t\leq t_0} r(t)$.

Let us apply now Itô's formula of Theorem 3.1, between $t-t_0$ and t, to the function $f(x_1,x_2) = \mathbf{1}_{(x_1=x_2)}$:

$$\begin{aligned} I_t = \mu_t^{\otimes 2}(f(\boldsymbol{\omega}(t))) &= N_{t_0,t} + \mu_{t-t_0}^{\otimes 2}\big[P_{t_0}^{\otimes 2} f(\boldsymbol{\omega}(t-t_0))\big] \\ &+ \beta^2 \int_{t-t_0}^t \mu_s^{\otimes 2}\big[P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\omega_1(s)=\omega_2(s))}\big]\,ds \\ &- 2\beta^2 \int_{t-t_0}^t \mu_s^{\otimes 3}\big[P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))(\,\mathbf{1}_{(\gamma(s)=\omega_1(s))} + \mathbf{1}_{(\gamma(s)=\omega_2(s))})\big]\,ds \\ &+ 3\beta^2 \int_{t-t_0}^t \mu_s^{\otimes 2}\big[P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\big]\, I_s\,ds, \end{aligned} \tag{4.1}$$

where

$$N_{t_0,t} = \int_{t-t_0}^t \mu_s^{\otimes 2}\left[P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))(\beta \sum_i dB_{\omega_i(s)}(s) - 2\frac{dZ_s}{Z_s})\right].$$

The following inequalities are standard folklore,and are crucial in our proof: they will be used repeatedly hereafter and we provide a proof in the appendix.

$$0 \le P_t^{\otimes 2} f(x_1,x_2) \le r(t) = P_t^{\otimes 2} f(x,x) \le 1. \tag{4.2}$$

In particular, we have

$$\begin{aligned} I_t \ge N_{t_0,t} &+ \beta^2 \int_{t-t_0}^t r(t-s) I_s\,ds \\ &- 4\beta^2 \int_{t-t_0}^t \mu_s^{\otimes 3}(P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\gamma(s)=\omega_1(s))})\,ds. \end{aligned} \tag{4.3}$$

Indeed, the second and fifth terms of (4.1) are non negative, in the second term we have

$$\begin{aligned} P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\omega_1(s)=\omega_2(s))} &= P_{t-s}^{\otimes 2} f(\omega_1(s),\omega_1(s))\,\mathbf{1}_{(\omega_1(s)=\omega_2(s))} \\ &= r(t-s)\,\mathbf{1}_{(\omega_1(s)=\omega_2(s))}\,, \end{aligned}$$

and finally, the fourth term can be written, thanks to symmetry of f,

$$-4\beta^2 \int_{t-t_0}^t \mu_s^{\otimes 3}(P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\gamma(s)=\omega_1(s))})\,ds\,.$$

Claim 1 :

$$\mu_s^{\otimes 3}(P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\gamma(s)=\omega_1(s))}) \le I_s \inf(\sqrt{I_s r(t-s)}, r(t-s))\,.$$

Indeed with $U_s(x) = \mu_s(\omega(s)=x)$ we have

$$\begin{aligned} \mu_s^{\otimes 3}\big[P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\gamma(s)=\omega_1(s))}\big] &= \sum_x \mu_s^{\otimes 3}\big[P_{t-s}^{\otimes 2} f(x,\omega_2(s))\ \mathbf{1}_{(\gamma(s)=\omega_1(s)=x)}\big] \\ &= \sum_x U_s(x)^2 \mu_s(P_{t-s}^{\otimes 2} f(x,\omega(s))) \end{aligned}$$

and

$$\mu_s(P_{t-s}^{\otimes 2} f(x,\omega(s))) = \sum_y U_s(y) P_{t-s}^{\otimes 2} f(x,y) \le r(t-s) \sum_y U_s(y) = r(t-s)\,.$$

We also have, by Cauchy-Schwarz,

$$\begin{aligned}\mu_s(P_{t-s}^{\otimes 2} f(x,\omega(s))) &\leq \left(\sum_y U_s(y)^2 \sum_y (P_{t-s}^{\otimes 2} f(x,y))^2\right)^{\frac{1}{2}} \\ &= \sqrt{I_s r(2(t-s))} \leq \sqrt{I_s r(t-s)}\,,\end{aligned}$$

since if $\tilde{\omega}(t) = \omega_1(t) - \omega_2(t)$ we have, thanks to Markov property and symmetry,

$$\begin{aligned}r(2t) &= \mathbb{P}\left(\tilde{\omega}(2t) = 0\right) = \sum_y \mathbb{P}_0(\tilde{\omega}(t) = y)\mathbb{P}_y(\tilde{\omega}(t) = 0) = \sum_y \mathbb{P}_0(\tilde{\omega}(t) = y)^2 \\ &= \sum_y P_t^{\otimes 2} f(0,y)^2 = \sum_y P_t^{\otimes 2} f(x,y)^2.\end{aligned}$$

Claim 2 :

$$\begin{aligned}4\beta^2 R(t_0) \int_0^T J_s\, ds + \int_{t_0}^T I_s\, ds \geq \int_{t_0}^T N_{t_0,t}\, dt& \\ + \beta^2(1 - 4\sqrt{\epsilon_0})R(t_0) \int_{t_0}^{T-t_0} I_s\, ds\,.&\end{aligned} \tag{4.4}$$

Observe that when $I_s \leq c_0$ and $t - t_0 \leq s \leq t$, we have $I_s \leq \epsilon_0 r(t-s)$, therefore, from Claim 1 we deduce that,

$$\begin{aligned}\int_{t-t_0}^t \mu_s^{\otimes 3}(P_{t-s}^{\otimes 2} f(\boldsymbol{\omega}(s))\,\mathbf{1}_{(\gamma(s)=\omega_1(s))})\, ds &\leq \int_{t-t_0}^t I_s \sqrt{I_s r(t-s)}\,\mathbf{1}_{(I_s \leq c_0)}\, ds \\ &\quad + \int_{t-t_0}^t r(t-s) I_s\,\mathbf{1}_{(I_s > c_0)}\, ds \\ &\leq \sqrt{\epsilon_0} \int_{t-t_0}^t r(t-s) I_s\, ds \\ &\quad + \int_{t-t_0}^t r(t-s) J_s\, ds.\end{aligned}$$

Plugging this inequality into (4.3) yields

$$I_t \geq N_{t_0,t} + \beta^2(1 - 4\sqrt{\epsilon_0}) \int_{t-t_0}^t r(t-s) I_s\, ds - 4\beta^2 \int_{t-t_0}^t r(t-s) J_s\, ds\,.$$

Given $T \geq t_0$, we are going to integrate this inequality between t_0 and T. On the one hand,

$$\begin{aligned}\int_{t_0}^T dt \int_{t-t_0}^t r(t-s) J_s\, ds &= \int\int \mathbf{1}_{(0 \leq u \leq t_0,\, t_0 - u \leq s \leq T-u)} J_s r(u)\, ds du \\ &\leq R(t_0) \int_0^T J_s\, ds\,.\end{aligned}$$

On the other hand,

$$\int_{t_0}^T dt \int_{t-t_0}^t r(t-s) I_s\, ds \geq \int_{t_0}^{T-t_0} I_s\, ds \int_0^{t_0} r(u)\, du = R(t_0) \int_{t_0}^{T-t_0} I_s\, ds\,.$$

The claim follows immediately.

Claim 3 : let $\mathcal{N}_T = \int_{t_0}^T N_{t_0,t}\, dt$. Then as $T \to +\infty$

$$\frac{\mathcal{N}_T}{\int_0^T I_s\, ds} \to 0 \quad \text{in probability.}$$

Let us defer the proof of this claim. Since $0 \le I_s \le 1$ and $\int_0^\infty I_s\, ds = +\infty$, we have,

$$\lim_{T\to+\infty} \frac{\int_{t_0}^T I_s\, ds}{\int_0^T I_s\, ds} = \lim_{T\to+\infty} \frac{\int_{t_0}^{T-t_0} I_s\, ds}{\int_0^T I_s\, ds} = 1 \quad a.s.$$

Let $c_1 = \frac{\beta^2(1-4\sqrt{\epsilon_0})R(t_0)-1}{4\beta^2 R(t_0)}$. If we divide (4.4) by $\phi_T = \int_0^T I_s\, ds$ and take lim sup as $T \to +\infty$, we obtain that almost surely

$$\begin{aligned}
\limsup_{T\to\infty} \frac{1}{\phi_T}\int_0^T J_s\, ds - c_1 &\ge \limsup_{T\to\infty} \frac{\mathcal{N}_T}{4\beta^2 R(t_0)\phi_T} \\
&\ge \limsup_{T\to+\infty} -\frac{|\mathcal{N}_T|}{4\beta^2 R(t_0)\phi_T} \\
&= -\liminf_{T\to+\infty} \frac{|\mathcal{N}_T|}{4\beta^2 R(t_0)\phi_T} \\
&= 0\,.
\end{aligned}$$

This yields

$$\limsup_{T\to\infty} \frac{\int_0^T J_s ds}{\int_0^T I_s\, ds} \ge c_1 \quad a.s.$$

Proof of Claim 3: By Fubini's theorem,

$$\begin{aligned}
\mathcal{N}_T &= \int_{t_0}^T dt \int_{t-t_0}^t \mu_s^{\otimes 2}\left[P_{t-s}^{\otimes 2} f(\omega_1(s),\omega_2(s))\big(\sum_i \beta dB_{\omega_i(s)}(s) - 2\frac{dZ_s}{Z_s}\big)\right] \\
&= \int_0^T \mu_s^{\otimes 2}\left[G(s,\omega_1(s),\omega_2(s))\big(\sum_i \beta dB_{\omega_i(s)}(s) - 2\frac{dZ_s}{Z_s}\big)\right],
\end{aligned}$$

with

$$0 \le G(s,x_1,x_2) := \int_{(t_0-s)^+}^{(T-s)^+\wedge t_0} P_{t-s}^{\otimes 2} f(x_1,x_2)\, dt \le t_0, \quad \forall x_1,x_2 \in \mathbb{Z}^d.$$

Let us view $\mathcal{N}_T = X_T$ as the value at time T of the continuous martingale

$$X_t = \int_0^t \mu_s^{\otimes 2}\left[G(s,\omega_1(s),\omega_2(s))\big(\sum_{i=1}^2 \beta dB_{\omega_i(s)}(s) - 2\frac{dZ_s}{Z_s}\big)\right].$$

We can compute its quadratic variation :

$$\langle X, X\rangle_T \le 4\beta^2 \int_0^T \mu_s^{\otimes 4}\left[G(s,\omega_1(s),\omega_2(s))G(s,\omega_3(s),\omega_4(s))\big(1_{(\omega_1(s)=\omega_3(s))} + I_s\big)\right] ds,$$

which satisfies

$$\langle X, X\rangle_T \le 8\beta^2 t_0^2 \int_0^T I_s ds. \tag{4.5}$$

Let $\epsilon > 0$, we shall prove that

$$\lim_{T\to\infty} \mathbf{P}\Big(\mathcal{N}_T > \epsilon \int_0^T I_s ds\Big) = 0. \tag{4.6}$$

To this end, define $\delta = \epsilon/(8\beta^2 t_0)$. We have

$$\mathbf{E}\Big[e^{\delta\mathcal{N}_T - \frac{\delta^2}{2}\langle X,X\rangle_T}\Big] = \mathbf{E}\Big[e^{\delta X_T - \frac{\delta^2}{2}\langle X,X\rangle_T}\Big] = 1.$$

(since $\langle X, X\rangle_T$ is bounded, Novikov's criterion for the exponential martingale is obviously satisfied). It follows that

$$\begin{aligned}
1 &\geq \mathbf{E}\Big(1_{(\mathcal{N}_T > \epsilon \int_0^T I_s ds)} e^{\delta\mathcal{N}_T - \frac{\delta^2}{2}\langle X,X\rangle_T}\Big) \\
&\geq \mathbf{E}\Big(1_{(\mathcal{N}_T > \epsilon \int_0^T I_s ds)} e^{(\delta\epsilon - \frac{\delta^2}{2} 8\beta^2 t_0) \int_0^T I_s ds}\Big) \\
&= \mathbf{E}\Big(1_{(\mathcal{N}_T > \epsilon \int_0^T I_s ds)} e^{4\beta^2 t_0 \delta^2 \int_0^T I_s ds}\Big) && \text{by (4.5)} \\
&\geq e^{4\beta^2 t_0 \delta^2 K}\, \mathbf{P}\Big(\mathcal{N}_T > \epsilon \int_0^T I_s ds, \int_0^T I_s ds \geq K\Big),
\end{aligned}$$

for any constant $K > 0$. Consequently, we have

$$\mathbf{P}\Big(\mathcal{N}_T > \epsilon \int_0^T I_s ds\Big) \leq \mathbf{P}\Big(\int_0^T I_s ds < K\Big) + e^{-4\beta^2 t_0 \delta^2 K}.$$

Since $\int_0^T I_s ds \to \infty$ almost surely, we get

$$\limsup_{T\to\infty} \mathbf{P}\Big(\mathcal{N}_T > \epsilon \int_0^T I_s ds\Big) \leq e^{-4\beta^2 t_0 \delta^2 K},$$

for any constant $K > 0$. Then by letting $K \to \infty$ we get (4.6). Considering the martingale $-X$, we prove in the same way that

$$\lim_{T\to\infty} \mathbf{P}\Big(-\mathcal{N}_T > \epsilon \int_0^T I_s ds\Big) = 0. \tag{4.7}$$

and this complete the proof of Claim 3.

Appendix

We provide a proof of (4.2). Recall that $f(x,y) = \mathbf{1}_{(x=y)}$. We let $p_t(x) = \mathbb{P}(\omega(t) = x)$ be the distribution of simple random walk at time t. Then, by translation invariance:

$$\begin{aligned} P_t^{\otimes 2} f(x_1, x_2) &= \mathbb{P}_{x_1,x_2}^{\otimes 2}(\omega_1(t) = \omega_2(t)) \\ &= \mathbb{P}^{\otimes 2}(x_1 + \omega_1(t) = x_2 + \omega_2(t)) \\ &= \sum_z \mathbb{P}(x_1 + \omega_1(t) = z)\mathbb{P}(x_2 + \omega_2(t) = z) \qquad \text{(by independence)} \\ &= \sum_z p_t(z - x_1) p_t(z - x_2) \\ &\le \left(\sum_z p_t(z - x_1)^2\right)^{\frac{1}{2}} \left(\sum_z p_t(z - x_2)^2\right)^{\frac{1}{2}} \qquad \text{(by Cauchy-Schwarz)} \\ &= \sum_z p_t(z)^2 = r(t)\,. \end{aligned}$$

Acknowledgements

We want to thank both referees for a careful reading of the first version of this paper and for suggestions for improvements for the exposition. We are also grateful to A. Camanes for his comments on a very early draft of this paper.

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