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Component sizes of the random graph outside the scaling window

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Abstract. We provide simple proofs describing the behavior of the largest component of the Erdős-Rényi random graph $G(n,p)$ outside of the scaling window, $p = \frac{1+\epsilon(n)}{n}$ where $\epsilon(n) \to 0$ but $\epsilon(n)n^{1/3} \to \infty$.

1. Introduction

Consider the random graph $G(n,p)$ obtained from the complete graph on n vertices by retaining each edge with probability p and deleting each edge with probability $1-p$. We denote by $\mathcal{C}_j$ the j-th largest component. Let $\epsilon(n)$ be a non-negative sequence such that $\epsilon(n) \to 0$ and $\epsilon(n)n^{1/3} \to \infty$. The following theorems describe the behavior of the largest component when $p = \frac{1+\epsilon(n)}{n}$ is outside the "scaling-window". The theorems, up to some logarithmic errors, were proved first by Bollobás (1984) using enumerative methods. The logarithmic errors were removed later by Luczak (1990).

Theorem 1.1. **[Subcritical phase]** *If $p(n) = \frac{1-\epsilon(n)}{n}$, then for any $\delta > 0$ and integer $\ell > 0$ we have*

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon(n)^{-2}\log(n\epsilon(n)^3)} - 1\Big| \geq \delta\Big) \to 0\,,$$

as $n \to \infty$.

Theorem 1.2. **[Supercritical phase]** *If $p(n) = \frac{1+\epsilon(n)}{n}$, then for any $\delta > 0$ we have*

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_1|}{2n\epsilon(n)} - 1\Big| \geq \delta\Big) \to 0\,,$$

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and for any integer $\ell > 1$ we have

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon^{-2}(n)\log(n\epsilon^3)} - 1\Big| \geq \delta\Big) \to 0\,,$$

as $n \to \infty$.

The proofs of these theorems in Bollobás (1984) and Luczak (1990) are quite involved and use the detailed asymptotics from Wright (1977), Bollobás (1984) and Bender et al. (1990) for the number of graphs on k vertices with $k+\ell$ edges. The proofs we present here are simple and require no hard theorems. The main advantage, however, of these proofs is their robustness. In a companion paper (see Nachmias and Peres (2007)) we use similar methods to analyze critical percolation on random regular graphs. In this case, the enumerative methods employed in Bollobás (1984) and Luczak (1990) are not available.

The phase transition in the Erdős-Rényi random graphs $G(n,p)$ occurs when $p = \frac{c}{n}$. Namely, if $c > 1$, then with high probability (w.h.p) $|\mathcal{C}_1|$ is linear in n, and if $c < 1$, then w.h.p. $|\mathcal{C}_1|$ is logarithmic in n. When $c \sim 1$ the situation is more delicate. Luczak et al. (1994) prove that for $p = \frac{1+\lambda n^{-1/3}}{n}$, the law of $n^{-2/3}|\mathcal{C}_1|$ converges to a positive non-constant distribution which was identified by Aldous (1997) as the longest excursion length of Brownian motion with some variable drift. See Nachmias and Peres (2005) for a recent account of the case $p = \frac{1+\lambda n^{-1/3}}{n}$ with simple proofs.

Thus, $|\mathcal{C}_1|$ is not concentrated and is roughly of size $n^{2/3}$ if $p = \frac{1+\lambda n^{-1/3}}{n}$. However, if $\epsilon(n)$ a sequence such that $n^{1/3}\epsilon(n) \to \infty$ and $p = \frac{1+\epsilon(n)}{n}$, then as stated in Theorems 1.1 and 1.2, the size $|\mathcal{C}_1|$ of the largest component in $G(n,p)$ is concentrated.

2. The exploration process

We recall an exploration process, due to Martin-Löf (1986) and Karp (1990), in which vertices will be either *active, explored* or *neutral*. After the completion of step $t \in \{0,1,\ldots,n\}$ we will have precisely t explored vertices and the number of the active and neutral vertices is denoted by A_t and N_t respectively.

Fix an ordering of the vertices $\{v_1,\ldots,v_n\}$. In step $t=0$ of the process, we declare vertex v_1 *active* and all other vertices *neutral*. Thus $A_0 = 1$ and $N_0 = n-1$. In step $t \in \{1,\ldots,n\}$, if $A_{t-1} > 0$, let w_t be the first active vertex; if $A_{t-1} = 0$, let w_t be the first neutral vertex. Denote by η_t the number of neutral neighbors of w_t in $G(n,p)$, and change the status of these vertices to *active*. Then, set w_t itself *explored*.

Denote by $\mathcal{F}_t$ the σ-algebra generated by $\{\eta_1,\ldots,\eta_t\}$. Observe that given $\mathcal{F}_{t-1}$ the random variable η_t is distributed as $\mathrm{Bin}(N_{t-1} - \mathbf{1}_{\{A_{t-1}=0\}}, p)$ and we have the recursions

$$N_t = N_{t-1} - \eta_t - \mathbf{1}_{\{A_{t-1}=0\}}\,, \qquad t \leq n\,, \tag{2.1}$$

and

$$A_t = \begin{cases} A_{t-1} + \eta_t - 1, & A_{t-1} > 0 \\ \eta_t, & A_{t-1} = 0\,, \qquad t \leq n\,. \end{cases} \tag{2.2}$$

As every vertex is either neutral, active or explored,

$$N_t = n - t - A_t\,, \qquad t \le n\,. \tag{2.3}$$

At each time $j \le n$ in which $A_j = 0$, we have finished exploring a connected component. Hence the random variable Z_t defined by

$$Z_t = \sum_{j=1}^{t-1} \mathbf{1}_{\{A_j=0\}}\,,$$

counts the number of components completely explored by the process before time t. Define the process $\{Y_t\}$ by $Y_0 = 1$ and

$$Y_t = Y_{t-1} + \eta_t - 1\,.$$

By (2.2) we have that $Y_t = A_t - Z_t$, i.e. Y_t counts the number of active vertices at step t minus the number of components completely explored before step t.

At each step we marked as explored precisely one vertex. Hence, the component of v_1 has size $\min\{t \ge 1 : A_t = 0\}$. Moreover, let $t_1 < t_2 \ldots$ be the times at which $A_{t_j} = 0$; then $(t_1, t_2 - t_1, t_3 - t_2, \ldots)$ are the sizes of the components. Observe that $Z_t = Z_{t_j} + 1$ for all $t \in \{t_j + 1, \ldots, t_{j+1}\}$. Thus $Y_{t_{j+1}} = Y_{t_j} - 1$ and if $t \in \{t_j + 1, \ldots, t_{j+1} - 1\}$, then $A_t > 0$, and thus $Y_{t_{j+1}} < Y_t$. By induction we conclude that $A_t = 0$ if and only if $Y_t < Y_s$ for all $s < t$, i.e. $A_t = 0$ if and only if $\{Y_t\}$ has hit a new record minimum at time t. By induction we also observe that $Y_{t_j} = -(j-1)$ and that for $t \in \{t_j + 1, \ldots t_{j+1}\}$ we have $Z_t = j$. Also, by our previous discussion for $t \in \{t_j + 1, \ldots t_{j+1}\}$ we have $\min_{s \le t-1} Y_s = Y_{t_j} = -(j-1)$, hence by induction we deduce that $Z_t = -\min_{s\le t-1} Y_s + 1$. Consequently,

$$A_t = Y_t - \min_{s \le t-1} Y_s + 1\,. \tag{2.4}$$

Lemma 2.1. *For all $p \le \frac{2}{n}$ there exists a constant $c > 0$ such that for any integer $t > 0$,*

$$\mathbf{P}\Big(N_t \le n - 5t\Big) \le e^{-ct}\,.$$

Proof. Let $\{\alpha_i\}_{i=1}^t$ be a sequence of i.i.d. random variables distributed as $\mathrm{Bin}(n, p)$. It is clear that we can couple η_i and α_i so $\eta_i \le \alpha_i$ for all i, and thus by (2.1)

$$N_t \ge n - 1 - t - \sum_{i=1}^t \alpha_i\,. \tag{2.5}$$

The sum $\sum_{i=1}^t \alpha_i$ is distributed as $\mathrm{Bin}(nt, p)$ and $p \le \frac{2}{n}$ so by Large Deviations (see Alon and Spencer (2000) section A.14) we get that for some fixed $c > 0$

$$\mathbf{P}\Big(\sum_{i=1}^t \alpha_i \ge 3t\Big) \le e^{-ct}\,,$$

which together with (2.5) concludes the proof. $\square$

3. The subcritical phase

Before beginning the proof of Theorem 1.1 we require some facts about processes with i.i.d. increments. Fix some small $\epsilon > 0$ and let $p = \frac{1-\epsilon}{m}$ for some integer $m > 1$.

Let $\{\beta_j\}$ be a sequence of random variables distributed as $\mathrm{Bin}(m,p)$. Let $\{W_t\}_{t\geq 0}$ be a process defined by

$$W_0 = 1, \qquad W_t = W_{t-1} + \beta_t - 1\,.$$

Let τ be the hitting time of 0,

$$\tau = \min_t \{W_t = 0\}\,.$$

By Wald's lemma we have that $\mathbf{E}\,\tau = \epsilon^{-1}$. Further information on the tail distribution of τ is given by the following lemma.

Lemma 3.1. *There exists constant $C_1, C_2, c_1, c_2 > 0$ such that for all $T \geq \epsilon^{-2}$ we have*

$$\mathbf{P}(\tau \geq T) \leq C_1\Big(\epsilon^{-2} T^{-3/2} e^{-\frac{(\epsilon^2 - c_1\epsilon^3)T}{2}}\Big)\,,$$

and

$$\mathbf{P}(\tau \geq T) \geq c_2\Big(\epsilon^{-2} T^{-3/2} e^{-\frac{(\epsilon^2 + C_2\epsilon^3)T}{2}}\Big)\,.$$

Furthermore,

$$\mathbf{E}\,\tau^2 = O(\epsilon^{-3})\,.$$

We will use the following proposition due to Spitzer (1956).

Proposition 1. Let $a_0, \ldots, a_{k-1} \in \mathbb{Z}$ satisfy $\sum_{i=0}^{k-1} a_i = -1$. Then there is precisely one $j \in \{0, \ldots, k-1\}$ such that for all $r \in \{0, \ldots, k-2\}$

$$\sum_{i=0}^{r} a_{(j+i) \bmod k} \geq 0\,.$$

Proof of Lemma 3.1. By Proposition 1, $\mathbf{P}(\tau = t) = \frac{1}{t}\mathbf{P}(W_t = 0)$. Since $\sum_{j=1}^{t} \beta_j$ is distributed as a $\mathrm{Bin}(mt,p)$ random variable we have

$$\mathbf{P}(W_t = 0) = \binom{mt}{t-1} p^{t-1}(1-p)^{mt-(t-1)}\,.$$

Replacing $t-1$ with t in the above formula only changes it by a multiplicative constant which is always between 1/2 and 2. A straightforward computation using Stirling's approximation gives

$$\mathbf{P}(W_t = 0) = \Theta\Big\{t^{-1/2}(1-\epsilon)^t\Big(1 + \frac{1}{m-1}\Big)^{t(m-1)}\Big(1 - \frac{1-\epsilon}{m}\Big)^{t(m-1)}\Big\}\,. \tag{3.1}$$

Denote $q = (1-\epsilon)\Big(1 + \frac{1}{m-1}\Big)^{m-1}\Big(1 - \frac{1-\epsilon}{m}\Big)^{m-1}$, then

$$\mathbf{P}(\tau \geq T) = \sum_{t\geq T} \mathbf{P}(\tau = t) = \sum_{t \geq T} \frac{1}{t}\mathbf{P}(W_t = 0) = \Theta\Big(\sum_{t\geq T} t^{-3/2} q^t\Big)\,.$$

This sum can be bounded above by

$$T^{-3/2}\sum_{t\geq T} q^t = T^{-3/2}\frac{q^T}{1-q}\,,$$

and below by

$$\sum_{t=T}^{2T} t^{-3/2} q^t \geq (2T)^{-3/2}\frac{q^T(1-q^T)}{1-q}\,.$$

Observe that as $m \to \infty$ we have that q tends to $(1-\epsilon)e^{\epsilon}$. By expanding e^{ϵ} we find that

$$q = (1-\epsilon)(1+\epsilon+\frac{\epsilon^2}{2}) + \Theta(\epsilon^3) = 1 - \frac{\epsilon^2}{2} + \Theta(\epsilon^3)\,.$$

Using this and the previous bounds on $\mathbf{P}(\tau \geq T)$ we get the first two assertions of the Lemma.

The third assertion follows from the following computation. By (3.1) we have that for some constant $C > 0$

$$\mathbf{E}\,\tau^2 = \sum_{t\geq 1} t^2 \mathbf{P}(\tau = t) = \sum_{t \geq 1} t\mathbf{P}(W_t = 0) \leq C \sum_{t\geq 1} \sqrt{t} q^t\,.$$

Thus, by direct computation (or by Feller (1971), section XIII.5, Theorem 5)

$$\mathbf{E}\,\tau^2 \leq O\Big(\frac{1}{1-q}\Big)^{3/2} = O(\epsilon^{-3})\,.$$

$\square$

Proof of Theorem 1.1. We begin with an upper bound. Recall that component sizes are $\{t_{j+1} - t_j : j \geq 0\}$ where t_j are record minima of the process $\{Y_t\}$. For a vertex v denote by $C(v)$ the connected component of $G(n,p)$ which contains v. We first bound $\mathbf{P}(|C(v_1)| \geq T_1)$ where

$$T_1 = 2(1+\delta)\epsilon^{-2}\log(n\epsilon^3)\,.$$

Recall that $|C(v_1)| = \min_t\{Y_t = 0\}$. Couple $\{Y_t\}$ with a process $\{W_t\}$ as in Lemma 3.1, which has increments distributed as $\mathrm{Bin}(n,p) - 1$ such that $Y_t \leq W_t$ for all t. Define τ as in Lemma 3.1. Since $p = \frac{1-\epsilon}{n}$ and $T_1 \geq \epsilon^{-2}$, Lemma 3.1 gives that

$$\mathbf{P}(\tau \geq T_1) \leq C_1 \epsilon (n\epsilon^3)^{-(1+\delta)(1-c_1\epsilon)} \log^{-3/2}(n\epsilon^3)\,,$$

for some fixed $C > 0$. Our coupling implies that $\mathbf{P}(|C(v_1)| \geq T_1) \leq \mathbf{P}(\tau \geq T_1)$. Denote by X the number of vertices v such that $|C(v)| \geq T_1$. If $|\mathcal{C}_1| \geq T_1$, then $X \geq T_1$. Also, for any two vertices v and u, by symmetry we have that $|C(v)|$ and $|C(u)|$ are identically distributed. We conclude that

$$\begin{aligned}
\mathbf{P}(|\mathcal{C}_1| \geq T_1) &\leq \mathbf{P}(X \geq T_1) \leq \frac{\mathbf{E}\,X}{T_1} = \frac{n\mathbf{P}(|C(v_1)| \geq T_1)}{T_1} \\
&\leq \frac{C_1 n\epsilon (n\epsilon^3)^{-(1+\delta)(1-c_1\epsilon)} \log^{-3/2}(n\epsilon^3)}{2(1+\delta)\epsilon^{-2}\log(n\epsilon^3)} \leq (n\epsilon^3)^{-\delta(1-c_1\epsilon)+c_1\epsilon} \to 0\,.
\end{aligned}$$

We now turn to prove a lower bound. Write

$$T_2 = 2(1-\delta)\epsilon^{-2}\log(n\epsilon^3)\,,$$

and define the stopping time

$$\gamma = \min\{t : N_t \leq n - \frac{\delta\epsilon n}{8}\}\,.$$

Recall that $\{t_j\}$ are times in which $A_{t_j} = 0$ and also that $Y_{t_j} = -(j-1)$ is a record minimum for $\{Y_t\}$. For each integer j let $\{W_t^{(j)}\}$ be a process with increments distributed as $\mathrm{Bin}(n - \frac{\delta\epsilon n}{8}, p)$ and initially $W_0^{(j)} = -(j-1)$. Since $\eta_{t\wedge\gamma}$ is stochastically bounded below by a $\mathrm{Bin}(n-\frac{\delta\epsilon n}{8})$ random variable we can couple such that

$$Y_{(t_j+t)\wedge\gamma} \geq W^{(j)}_{(t\wedge(\gamma-t_j))\vee 0}\,.$$

Define the stopping times $\{\tau_j\}$ by

$$\tau_j = \min\{t : W_t^{(j)} = -j\}\,.$$

Take

$$N = \left\lfloor \epsilon^{-1}(n\epsilon^3)^{(1-\frac{\delta}{8})} \right\rfloor .$$

We will prove that w.h.p. $t_N < \gamma$ and that there exists $k_1 < k_2 < \ldots < k_\ell < N$ such that $\tau_{k_i} \geq T_2$. Note that by our coupling, these two events imply that $|\mathcal{C}_\ell| \geq T_2$. Lemma 2.1 shows that for some $c > 0$ we have

$$\mathbf{P}\Big(\gamma \leq \frac{\delta\epsilon n}{40}\Big) \leq e^{-c\epsilon n}\,. \tag{3.2}$$

By bounding the increments of $\{Y_t\}$ above by variables distributed as $\mathrm{Bin}(n,p)-1$ we learn by Wald's Lemma (see Durrett (1996)) that $\mathbf{E}\,[t_{j+1}-t_j] \leq \epsilon^{-1}$ for any $j \geq 0$, hence $\mathbf{E}\,t_N \leq \epsilon^{-2}(n\epsilon^3)^{(1-\frac{\delta}{8})}$. We conclude that

$$\mathbf{P}(t_N \geq \frac{\delta\epsilon n}{40}) \leq \frac{40\epsilon^{-2}(n\epsilon^3)^{(1-\frac{\delta}{8})}}{\delta\epsilon n} = \frac{40}{\delta}(n\epsilon^3)^{-\frac{\delta}{8}}\,, \tag{3.3}$$

which goes to 0 as $\epsilon n^{-1/3}$ tends to ∞.

Next, we take $m = n - \frac{\delta\epsilon n}{8}$ in Lemma 3.1 and note that $p = \frac{(1-\epsilon)(1-\frac{\delta\epsilon}{8})}{m} \geq \frac{1-(1+\frac{\delta}{8})\epsilon}{m}$. Hence, Lemma 3.1 gives that for any j

$$\mathbf{P}(\tau_j \geq T_2) \geq c_2\epsilon(n\epsilon^3)^{-(1+\frac{\delta}{8})^2(1-\delta)(1+C_2\epsilon)}\log^{-3/2}(\epsilon^3 n) \geq \epsilon(n\epsilon^3)^{-(1-\frac{\delta}{4})}\,.$$

Let X be the number of $j \leq N$ such that $\tau_j \geq T$. Then we have

$$\mathbf{E}\,X \geq N\epsilon(n\epsilon^3)^{-(1-\frac{\delta}{4})} \geq C(n\epsilon^3)^{\frac{\delta}{8}} \to \infty\,,$$

hence by Large Deviations (see Alon and Spencer (2000), section A.14), for any fixed integer $\ell > 0$ we have

$$\mathbf{P}\Big(X < \ell\Big) \leq e^{-c(n\epsilon^3)^{\frac{\delta}{8}}}\,, \tag{3.4}$$

for some $c = c(\ell) > 0$. By our coupling we have that

$$\Big\{|\mathcal{C}_\ell| < T_2\Big\} \subset \Big\{X < \ell\Big\} \cup \Big\{t_N > \gamma\Big\}\,.$$

This together with (3.2), (3.3) and (3.4) gives

$$\mathbf{P}(|\mathcal{C}_\ell| < T_2) \leq O\Big(\frac{(n\epsilon^3)^{-\frac{\delta}{8}}}{\delta}\Big)\,.$$

$\square$

4. The supercritical phase

In this section we denote $\xi_t = \eta_t - 1$. We first prove some Lemmas.

Lemma 4.1. *If $p = \frac{1+\epsilon}{n}$, then for all $t \leq 3\epsilon(n)n$*

$$\mathbf{E}\,A_t = O(\epsilon t + \sqrt{t})\,, \tag{4.1}$$

and

$$\mathbf{E}\, Z_t = O(\epsilon t + \sqrt{t})\,. \tag{4.2}$$

Proof. Write $T = 3\epsilon n$. We will use (2.4). First observe that since η_t can always be bounded above by a $\mathrm{Bin}(n,p)$ random variable, we can bound $\mathbf{E}\,[\xi_t \mid \mathcal{F}_{t-1}] \le \epsilon$ for all t. Hence, the process $\{\epsilon j - Y_j\}_{j=0}^t$ is a submartingale for any t. Denote by γ the stopping time $\gamma = \min\{t : N_t \le n - 15\epsilon n\}$. By Doob's maximal L^2 inequality we have

$$\mathbf{E}\,[\max_{j \le t\wedge\gamma} (\epsilon j - Y_j)^2] \le 4\mathbf{E}\,[(\epsilon(t\wedge\gamma) - Y_{t\wedge\gamma})^2]\,. \tag{4.3}$$

The process $\{Y_t\}$ is stochastically bounded above by the process $\{X_t\}$ which has i.i.d. increments distributed as $\mathrm{Bin}(n,p)-1$ random variables. By definition, conditioned on the event $j<\gamma$, the random variable η_j can be stochastically bounded below by a $\mathrm{Bin}(n-15\epsilon n, p)$ random variable. Thus, the process $\{Y_{t\wedge\gamma}\}$ is stochastically bounded below by the process $\{\widetilde{X}_{t\wedge\gamma}\}$, where $\{\widetilde{X}_t\}$ has i.i.d. increments distributed as $\mathrm{Bin}(n-15\epsilon n,p)-1$ random variables. Hence we can couple such that

$$Y_{t\wedge\gamma}^2 \mathbf{1}_{\{Y_{t\wedge\gamma}\ge 0\}} \le X_{t\wedge\gamma}^2\,, \quad Y_{t\wedge\gamma}^2 \mathbf{1}_{\{Y_{t\wedge\gamma}< 0\}} \le \widetilde{X}_{t\wedge\gamma}^2\,.$$

It is an immediate computation to verify that $\mathbf{E}\, X_{t\wedge\gamma}^2 = O(\epsilon^2 t^2 + t)$ and that $\mathbf{E}\, \widetilde{X}_{t\wedge\gamma}^2 = O(\epsilon^2t^2+t)$ and thus $\mathbf{E}\, Y_{t\wedge\gamma}^2 = O(\epsilon^2t^2+t)$. We use this and the Cauchy-Schwarz inequality to bound the right hand side of (4.3),

$$\mathbf{E}\,[(\epsilon(t\wedge\gamma) - Y_{t\wedge\gamma})^2] = O(\epsilon^2t^2+t)\,.$$

Lemma 2.1 implies that for n large enough,

$$\mathbf{P}\Big(N_T \le n - 15\epsilon n\Big) \le e^{-3c\epsilon n} \le \frac{1}{n^2}\,, \tag{4.4}$$

and as $\{N_t\}$ is a decreasing sequence we deduce that $\mathbf{P}(\gamma \le T) \le n^{-2}$. Hence for any $t \le T$

$$\begin{aligned}\mathbf{E}\,[(\epsilon t - Y_t)^2] &\le \mathbf{E}\,[(\epsilon(t\wedge\gamma) - Y_{t\wedge\gamma})^2 \mathbf{1}_{\{t<\gamma\}}] + O(n^2)\mathbf{P}(t \ge \gamma) \\ &= O(\epsilon^2t^2+t)\,.\end{aligned}$$

We deduce by (4.3) and Jensen inequality that for any $t \le T$

$$\mathbf{E}\,[\min_{j\le t}(Y_j - \epsilon j)] = O(\epsilon t + \sqrt{t})\,,$$

hence $\mathbf{E}\,[\min_{j\le t} Y_j] = O(\epsilon t + \sqrt{t})$ and so by (2.4) we obtain (4.1). Inequality (4.2) follows immediately from the relation $Z_t = A_t - Y_t$. □

Lemma 4.2. *If $p = \frac{1+\epsilon}{n}$, then for all $t \le 3\epsilon(n)n$*

$$\mathbf{E}\, N_t = n(1-p)^t + O(\epsilon^2 n)\,, \tag{4.5}$$

and

$$\mathbf{E}\, \xi_t = \epsilon - \frac{t}{n} + O(\epsilon^2)\,. \tag{4.6}$$

Proof. Observe that by (2.1) we have that

$$\mathbf{E}\,[N_t \mid \mathcal{F}_{t-1}] = (1-p)N_{t-1} - (1-p)\mathbf{1}_{\{A_{t-1}=0\}}\,.$$

By iterating this relation we get that $\mathbf{E}\,N_t = n(1-p)^t + O(\mathbf{E}\,Z_t)$ which by Lemma 4.1 yields (4.5) (observe that for $t = 3\epsilon n$ we have $\epsilon t \geq \sqrt{t}$ by our assumption on ϵ). Since

$$E[\xi_t \mid \mathcal{F}_{t-1}] = pN_{t-1} - p\mathbf{1}_{\{A_{t-1}=0\}} - 1\,,$$

by taking expectations and using (4.5) we get

$$\begin{aligned}\mathbf{E}\,\xi_t &= (1+\epsilon)(1-\frac{1+\epsilon}{n})^t - 1 + O(\epsilon^2)\\ &= (1+\epsilon)(1-(1+\epsilon)t/n) - 1 + O(\epsilon^2) = \epsilon - \frac{t}{n} + O(\epsilon^2)\,,\end{aligned}$$

where we used the fact that $(1-x)^t = 1 - tx + O(t^2x^2)$. □

Proof of Theorem 1.2. Write $T = 3\epsilon n$ and $\xi_j^* = \mathbf{E}\,[\xi_j \mid \mathcal{F}_{j-1}]$. The process

$$M_t = Y_t - \sum_{j=1}^{t} \xi_j^*\,,$$

is a martingale. By Doob's maximal L^2 inequality we have that

$$\mathbf{E}\,(\max_{t\leq T} M_t^2)) \leq 4\mathbf{E}\,M_T^2\,.$$

Since M_t has orthogonal increments with bounded second moment, we deduce that $\mathbf{E}\,M_T^2 = O(T)$. Hence, by Jensen's inequality,

$$\mathbf{E}\left[\max_{t\leq T}\Big|Y_t - \sum_{j=1}^{t}\xi_j^*\Big|\right] \leq O(\sqrt{T}) = O(\sqrt{\epsilon n})\,. \tag{4.7}$$

As $\xi_j^* = pN_{j-1} - p\mathbf{1}_{\{A_{j-1}=0\}} - 1$ by (2.3) we have

$$\mathbf{E}\,|\xi_j^* - \mathbf{E}\,\xi_j| = p\mathbf{E}\,|A_{j-1} + \mathbf{1}_{\{A_{j-1}=0\}} - \mathbf{E}\,A_{j-1} - \mathbf{E}\,\mathbf{1}_{\{A_{j-1}=0\}}|\,.$$

By the triangle inequality and Lemma 4.1 we conclude that for all $j \leq T$

$$\mathbf{E}\,|\xi_j^* - \mathbf{E}\,\xi_j| \leq p\cdot O(\epsilon j + \sqrt{j})\,,$$

and hence for any $t \leq T$

$$\mathbf{E}\left[\sum_{j\leq t}|\xi_j^* - \mathbf{E}\,\xi_j|\right] \leq p\cdot O(\epsilon t^2 + t^{3/2}) \leq O(\epsilon^3 n)\,.$$

By the triangle inequality we get

$$\mathbf{E}\left[\max_{t\leq T}\Big|\sum_{j=1}^{t}(\xi_j^* - \mathbf{E}\,\xi_j)\Big|\right] \leq O(\epsilon^3 n)\,. \tag{4.8}$$

Using the triangle inequality, (4.7), (4.8) and Markov's inequality give that for any $a > 0$

$$\mathbf{P}\Big(\max_{t\leq T}\Big|Y_t - \sum_{j=1}^{t}\mathbf{E}\,\xi_j\Big| \geq a\epsilon^2 n\Big) \leq a^{-1}(O(\epsilon) + O((\epsilon^3 n)^{-1/2})) \longrightarrow 0\,. \tag{4.9}$$

Lemma 4.2 implies that for any $b > 0$

$$\sum_{j=1}^{b\epsilon n} \mathbf{E}\,\xi_j = \sum_{j=1}^{b\epsilon n} \Big(\epsilon - \frac{t}{n} + O(\epsilon^2)\Big) = (b - \frac{b^2}{2})\epsilon^2 n + O(\epsilon^3 n)\,. \tag{4.10}$$

By (4.9) and (4.10) we deduce that for $\delta > 0$ small enough, with probability tending to 1, the process Y_t is strictly positive at all times in $[\delta\epsilon n, (2-\delta)\epsilon n]$ and hence

$$\mathbf{P}\Big(|\mathcal{C}_1| \geq 2(1-\delta)\epsilon n\Big) \geq 1 - O\Big(\delta^{-1}(\epsilon + (\epsilon^3 n)^{-1/2})\Big)\,.$$

We also deduce by (4.9) and (4.10) that at time $t = (2+\delta)\epsilon n$ we have $Y_t \leq -\frac{\delta^2}{3}\epsilon^2 n$ and at all times $t \leq \delta\epsilon n$ we have that $Y_t > -\frac{\delta^2}{3}\epsilon^2 n$ with probability tending to 1. Since component sizes are excursion lengths of Y_t above its past minima, we conclude that w.h.p. by time $2(1+\delta)\epsilon n$ we have explored completely at least one component of size at least $2(1-\delta)\epsilon n$. Condition on the time t_0 of the first record minimum after time $2(1-\delta)\epsilon n$. The number of neutral vertices remaining at that time is $n - t_0$. The subgraph of $G(n,p)$ induced on these remaining vertices is distributed as $G(n-t_0,p)$. Since $t_0 \geq 2(1-\delta)\epsilon n$ and $p = \frac{1+\epsilon}{n}$, Theorem 1.1 implies that w.h.p. $G(n-t_0,p)$ has no components of size at least ϵn.

Thus we have proved that w.h.p. in $G(n,p)$ there exists a unique component of size between $2(1-\delta)\epsilon n$ and $2(1+\delta)\epsilon n$. Condition on this event and on the size of this unique component and consider the graph G_* induced by the complement of this component. This graph has m vertices where

$$|m - (n - 2\epsilon n)| \leq 2\delta\epsilon n\,,$$

and since $p = \frac{1+\epsilon}{n}$ we have that

$$\Big|p - \Big(\frac{1-\epsilon}{m}\Big)\Big| \leq \frac{2\delta\epsilon + O(\epsilon^2)}{m}\,.$$

The graph G_* is distributed as $G(m,p)$ conditioned on the event $\mathcal{A}$ that it does not contain a component of size between $2(1-\delta)\epsilon n$ and $2(1+\delta)\epsilon n$. By Theorem 1.1 we have that $\mathbf{P}(\mathcal{A}) = 1 - o(1)$. Thus for any collection of graphs $\mathcal{B} \subset \mathcal{A}$ we have that $\mathbf{P}^*_{m,p}(\mathcal{B}) = (1+o(1))\mathbf{P}_{m,p}(\mathcal{B})$ where $\mathbf{P}_{m,p}$ is the distribution of $G(m,p)$ and $\mathbf{P}^*_{m,p}$ is the measure $\mathbf{P}_{m,p}$ conditioned on $\mathcal{A}$. Thus, we conclude by Theorem 1.1 that for any integer $\ell > 1$ and $\delta' > 0$

$$\mathbf{P}\Big(\Big|\frac{|\mathcal{C}_\ell|}{2\epsilon^{-2}(n)\log(n\epsilon^3)} - 1\Big| \geq \delta'\Big) \to 0\,,$$

concluding the proof of the theorem. □

Remark. With a little more effort it is possible to show for the supercritical case, that in the exploration process for any fixed $\ell > 1$, the ℓ-th largest component is explored *after* the largest component is explored.

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References

D. Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* **25**, 812–854 (1997).

N. Alon and J. H. Spencer. *The probabilistic method*. Wiley, New York, 2nd edition (2000).

E. A. Bender, E. R. Canfield and B. D. McKay. The asymptotic number of labeled connected graphs with a given number of vertices and edges. *Random Structures Algorithms* **1**, 127–169 (1990).

B. Bollobás. The evolution of random graphs. *Trans. Amer. Math. Soc.* **286**, 257–274 (1984).

R. Durrett. *Probability: Theory and Examples*. Duxbury Press, Belmont, California, 2nd edition (1996).

W. Feller. *An introduction to probability theory and its applications*, volume II. John Wiley & Sons, Inc., New York-London-Sydney, 2nd edition (1971).

R. M. Karp. The transitive closure of a random digraph. *Random Structures Algorithms* **1**, 73–93 (1990).

T. Luczak. Component behavior near the critical point of the random graph process. *Random Structures Algorithms* **1**, 287–310 (1990).

T. Luczak, B. Pittel and J. C. Wierman. The structure of a random graph at the point of the phase transition. *Trans. Amer. Math. Soc.* **341**, 721–748 (1994).

A. Martin-Löf. Symmetric sampling procedures, general epidemic processes and their threshold limit theorems. *J. Appl. Probab.* **23**, 265–282 (1986).

A. Nachmias and Y. Peres. The critical random graph, with martingales. *To appear in Israel J. of Math.* (2005). `http://www.arxiv.org/abs/math.PR/0512201`.

A. Nachmias and Y. Peres. Critical percolation on random regular graphs (2007). Preprint. Available at `http://www.arxiv.org/abs/0707.2839`.

F. Spitzer. A combinatorial lemma and its application to probability theory. *Amer. Math. Soc.* **82**, 323–339 (1956).

E. M. Wright. The number of connected sparsely edged graphs. *Journal of Graph Theory* **1**, 317–330 (1977).