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On fractional Brownian motion limits in one dimensional nearest-neighbor symmetric simple exclusion

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Abstract. A well-known result with respect to the one dimensional nearest-neighbor symmetric simple exclusion process is the convergence to fractional Brownian motion with Hurst parameter 1/4, in the sense of finite-dimensional distributions, of the subdiffusively rescaled current across the origin, and the subdiffusively rescaled tagged particle position.

The purpose of this note is to improve this convergence to a functional central limit theorem, with respect to the uniform topology, and so complete the solution to a conjecture in the literature with respect to simple exclusion processes.

1. Introduction

Informally, the one dimensional nearest-neighbor symmetric simple exclusion process follows a collection of random walks on the lattice $\mathbb{Z}$ which move independently except in that jumps to already occupied sites are suppressed. More precisely, the exclusion model is a Markov process $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ evolving on the configuration space $\Sigma = \{0,1\}^{\mathbb{Z}}$ with generator,

$$(L\phi)(\eta) \;=\; \frac{1}{2}\sum_x \left[\phi(\eta^{x,x+1}) - \phi(\eta)\right]$$

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where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the values at x and $x+1$,

$$\eta^{x,x+1}(z) \; = \; \begin{cases} \eta(z) & \text{when } z \neq x, x+1 \\ \eta(x) & \text{when } z = x+1 \\ \eta(x+1) & \text{when } z = x. \end{cases}$$

A more formal treatment can be found in Liggett (1985). Later, in Section 2, we will also give Harris's description of the model in terms of a "stirring process."

As the process is "mass conservative," that is no birth or death, one expects a family of invariant measures corresponding to particle density. In fact, for each $\rho \in [0,1]$, the product over $\mathbb{Z}$ of Bernoulli measures ν_ρ which independently puts a particle at locations $x \in \mathbb{Z}$ with probability ρ, that is $\nu_\rho(\eta_x = 1) = 1 - \nu_\rho(\eta_x = 0) = \rho$, are invariant (cf. Liggett, 1985).

In this note, we concentrate on the integrated flux across the origin $J(t)$, and the position of a tagged, or distinguished particle $X(t)$, say the first particle to the left of $1/2$, when initially the exclusion process starts in an equilibrium ν_ρ for $0 < \rho < 1$. Both objects are interestingly connected, and have been long well-studied in the literature (see Section 8.4 in Liggett, 1985, Section 6.4 in Spohn, 1991, De Masi and Ferrari, 2002).

Perhaps the most intriguing behavior of the current and tagged particle is their subdiffusive fluctuation behavior, explained physically in part by the enforced ordering of particles with no leapfrogging allowed in the dynamics. It was shown in Arratia (1983), Rost and Vares (1985) and De Masi and Ferrari (2002) that

$$t^{-1/4}J(t) \;\stackrel{d}{\to}\; N(0,\sigma_J^2) \;\text{ and }\; t^{-1/4}X(t) \;\stackrel{d}{\to}\; N(0,\sigma_X^2) \tag{1.1}$$

where $\sigma_J^2 = \sqrt{2/\pi}(1-\rho)\rho$ and $\sigma_X^2 = \sqrt{2/\pi}(1-\rho)\rho^{-1}$. This is in contrast to the diffusive behavior in higher dimensions or when the jump probability is longer range (cf. Chapter 6 in Spohn, 1991, Part III in Liggett, 1999, Sethuraman, 2006).

Given these results, the general belief (see Conjecture 6.5 in Spohn, 1991) is that the process limits with respect to the rescaled current and tagged particle position converge to respective fractional Brownian motions with Hurst parameter $1/4$. In fact, straightforward modifications of the arguments for (1.1) give convergence in the sense of finite-dimensional distributions (a case of Theorem 1.2 Landim and Volchan, 2000 gives a specific statement; see also Jara and Landim, 2006). However, it appears the full functional central limit theorem conjectured in Spohn (1991), with respect to exclusion processes, has not been addressed.

The aim of this article is to complete the proof of this conjecture by supplying path tightness estimates to show, as $\lambda \uparrow \infty$, that both

$$\sigma_J^{-1}\lambda^{-1/4}J(\lambda t) \;\Rightarrow\; \mathbb{B}_{1/4}(t), \text{ and } \; \sigma_X^{-1}\lambda^{-1/4}X(\lambda t) \;\Rightarrow\; \mathbb{B}_{1/4}(t) \tag{1.2}$$

where $\mathbb{B}_{1/4}(t)$ is the standard fractional Brownian motion with parameter $1/4$, and $\Rightarrow$ denotes weak convergence in $D([0,1])$ endowed with the uniform topology.

The plan of the paper is to give some preliminary representations and estimates in Section 2, and then deduce the limits (1.2), through certain maximal inequalities and discrete time process approximations, in Corollaries 3.5 and 4.3 in Sections 3 and 4 respectively.

Throughout, unless otherwise clear, $P = P_{\nu_\rho}$ and $E = E_{\nu_\rho}$ denote the process measure and expectation starting under the equilibrium ν_ρ. Also, as standard, $\lfloor x \rfloor$ denotes the integer part of x.

2. Representations of current and tagged particle

In this section, we state a convenient construction of the exclusion process through a "stirring process" first introduced by Harris (1972), and discuss some representations of the current and tagged particle position.

2.1. *Stirring process.* The stirring process $\xi_t^i \in \mathbb{Z}$ for $i \in \mathbb{Z}$ is defined as follows. At time $t = 0$, a particle is put at each site and we define $\xi_0^i = i$ for each $i \in \mathbb{Z}$. To each bond $(x, x+1)$ with $x \in \mathbb{Z}$, we associate a Poisson process (clock) with parameter $1/2$. When the clock rings at bond $(x, x+1)$, the particles at these sites interchange their positions. Then, ξ_t^i is the position at time t of the particle which was at i at time 0. Given an initial configuration η, the simple exclusion process, in terms of the stirring process, is

$$\eta_t(x) \; = \; 1\big\{x \in \{\xi_t^i : \eta(i) = 1\}\big\},$$

that is, in words, $\eta_t(x) = 1$ if and only if there is an $i \in \mathbb{Z}$ so that $\xi_t^i = x$ and $\eta(i) = 1$.

2.2. *Current representations.* Let $N_+(t), N_-(t)$ be counting processes, with infinitesimal rates $(1/2)\eta_s(0)(1-\eta_s(1))$, $(1/2)\eta_s(1)(1-\eta_s(0))$, which count the number of particles which cross $0 \to 1$ and $1 \to 0$ up to time t respectively. Then, the current across the bond $(0,1)$ up to time t is given by

$$J(t) \; = \; N_+(t) - N_-(t).$$

As $N_+(t) - \frac{1}{2}\int_0^t \eta_s(0)(1-\eta_s(1))ds$ and $N_-(t) - \frac{1}{2}\int_0^t \eta_s(1)(1-\eta_s(0))ds$ are martingales, we note the useful decomposition

$$J(t) \; = \; M(t) + A(t)$$

where $M(t) = N_+(t) - N_-(t) - \frac{1}{2}\int_0^t (\eta_s(0) - \eta_s(1))ds$ is a martingale, and $A(t) = \frac{1}{2}\int_0^t (\eta_s(0) - \eta_s(1))ds$ (cf. (III.2.37) in Liggett, 1999).

In terms of the stirring process, following the development in De Masi and Ferrari (2002), define

$$K^+(t) \; = \; \sum_{i \le 0} 1\{\xi_t^i > 0\}; \quad K^-(t) \; = \; \sum_{i > 0} 1\{\xi_t^i \le 0\}.$$

In words, $K^+(t), K^-(t)$ are the number of stirring particles starting to the left and right of the point $1/2$ and sitting at the right and left of $1/2$ at time t respectively. Denote

$$U_i(t) \; = \; \begin{cases} 1\{\xi_t^i > 0\} & \text{when } i \le 0 \\ 1\{\xi_t^i \le 0\} & \text{when } i > 0. \end{cases}$$

As in the stirring process all sites are always occupied, each crossing of the bond $(0,1)$ in one direction corresponds to a simultaneous crossing in the opposite direction. Then, $K^+(t) - K^-(t)$ is constant in t, and since $K^+(0) = K^-(0) = 0$, $K^+(t) = K^-(t) := K(t)$, for all $t \ge 0$. Hence,

$$J(t) = \sum_{i \le 0} 1\{\xi_t^i > 0\}\eta(i) - \sum_{i > 0} 1\{\xi_t^i \le 0\}\eta(i).$$

For $K(t) \ge 1$, let $i_1 < i_2 < \ldots < i_{K(t)} \le 0$ be the random locations for which $\xi_t^{i_k} > 0$, and $0 < j_1 < j_2 < \ldots < j_{K(t)}$ be the random locations for which $\xi_t^{j_k} \le 0$.

Define $B_k^+ = \eta(i_k)$ and $B_k^- = \eta(j_k)$ and $A_k(t) = B_k^+ - B_k^-$. Then, with the convention that the sum from 1 to 0 is equal to 0, we have the representation

$$J(t) = \sum_{k=1}^{K(t)} A_k(t).$$

We now state some known facts and consequences.

(a) Clearly, given $K(t)$, and the random locations $\{i_k\}$ and $\{j_k\}$ with $1 \leq k \leq K(t)$ the variables $\{A_k(t)\}$ are independent, identically distributed, mean 0, and

$$P(A_k(t) = 1) = P(A_k(t) = -1) = \rho(1-\rho), \quad \text{and} \quad P(A_k(t) = 0) = 1 - 2\rho(1-\rho).$$

(b) Also $K(t)$ is a sum of negatively correlated $0,1$ valued random variables. Moreover, by Lemma 4.12 Liggett (1985), for all finite $T \subset \mathbb{Z}$, and all $A \subset \mathbb{Z}$,

$$P\Big(\bigcap_{i \in T} (\xi_t^i \in A)\Big) \ \leq \ \prod_{i \in T} P\Big(\xi_t^i \in A\Big). \tag{2.1}$$

(c) From basic considerations, $E[K(t)] = E(z(0,t)_+)$, where $z(0,t)$ is a symmetric random walk on $\mathbb{Z}$ starting at the origin. Then, $E[K(t)] \leq \sqrt{t}$, and

$$\lim_{t \to \infty} \frac{E[K(t)]}{\sqrt{t}} \ = \ \frac{1}{\sqrt{2\pi}}.$$

(d) We have also some p-moment estimates. Denote $\|V\|_p = (E[V^p])^{1/p}$ for simplicity.

Lemma 2.1. *For integers $p \geq 1$, there is a constant $C_0 = C_0(p) < \infty$ so that for all $t \geq 1$,*

$$\|K(t)\|_p \ \leq \ C_0 \sqrt{t}.$$

Proof. First, as $K(t) = \sum_{i \leq 0} U_i(t)$, and by (2.1), we have that

$$\|K(t)\|_p \ \leq \ \Big\| \sum_{i \leq 0} \tilde{U}_i \Big\|_p$$

where $\tilde{U}_i \overset{d}{=} U_i(t)$ and $\{\tilde{U}_i\}$ are $0,1$ valued independent random variables.

To further estimate, we use Rosenthal inequality (Theorem 1.5.9 in De la Pena and Giné, 1999) on independent and nonnegative random variables $\{\beta_i\}$: For $p \geq 1$, there exists a constant $C_1 = C_1(p) < \infty$ such that

$$\Big\| \sum \beta_i \Big\|_p \ \leq \ C_1 \max \Big\{ \sum E[\beta_i], \Big(\sum E[\beta_i^p] \Big)^{1/p} \Big\}.$$

Now, applying Rosenthal's inequality, for a positive integer p, noting that $\tilde{U}_i^p = \tilde{U}_i$, we have,

$$\Big\| \sum_{i \leq 0} \tilde{U}_i \Big\|_p \ \leq \ C_1 \Big(\sum_{i \leq 0} E[\tilde{U}_i] + \Big[\sum_{i \leq 0} E[\tilde{U}_i^p] \Big]^{1/p} \Big) \ \leq \ C_1 \Big(E[K(t)] + (E[K(t)])^{1/p} \Big).$$

The result follows from properties of $E[K(t)]$ listed in part (c), and that $p \geq 1$. ∇

2.3. *Tagged particle representations.* Consider a distinguished particle in the exclusion system. One representation for its displacement $Z(t) = X(t) - X(0)$ is through the "Lagrangian frame," $\zeta_t = \theta_{Z(t)}\eta_t$ where $\theta_y\eta$ is the shifted configuration $(\theta_y\eta)(x) = \eta_{x+y}$. Then, ζ_t is a Markov process on $\Sigma' = \{\zeta \in \Sigma : \zeta(0) = 1\}$ with generator

$$\begin{aligned}(\mathcal{L}\phi)(\zeta) &= \frac{1}{2}\sum_{x\neq -1,0}\big(\phi(\zeta^{x,x+1}) - \phi(\zeta)\big)\\ &\quad + \frac{1}{2}\sum_{i=-1,1}(1-\zeta_i)\big(\phi(\tau_i\zeta) - \phi(\zeta)\big)\end{aligned}$$

where $\tau_k\zeta$ is the configuration obtained by displacing the tagged particle k steps and then shifting the frame,

$$(\tau_k\zeta)(x) = \begin{cases} \zeta(x+k) & \text{when } x \neq 0, -k\\ \zeta(0) & \text{when } x = 0\\ \zeta(k) & \text{when } x = -k.\end{cases}$$

Define $\mathcal{N}_+(t), \mathcal{N}_-(t)$ as the counting processes, with infinitessimal rates $(1/2)(1-\zeta_s(1))$, $(1/2)(1-\zeta_s(-1))$, which count the number of frame shifts of size 1 and -1 respectively up to time t. Then,

$$X(t) - X(0) = \mathcal{N}_+(t) - \mathcal{N}_-(t).$$

Similar to the current representation, $\mathcal{N}_+(t) - \frac{1}{2}\int_0^t(1-\zeta_s(1))ds$ and $\mathcal{N}_-(t) - \frac{1}{2}\int_0^t(1-\zeta_s(-1))ds$ are martingales, and

$$X(t) - X(0) = \mathcal{M}(t) + \mathcal{A}(t)$$

where $\mathcal{M}(t) = \mathcal{N}_+(t) - \mathcal{N}_-(t) - \frac{1}{2}\int_0^t(\zeta_s(-1) - \zeta_s(1))ds$ is a martingale and $\mathcal{A}(t) = \frac{1}{2}\int_0^t(\zeta_s(-1) - \zeta_s(1))ds$ (cf. Proposition III.4.1 in Liggett, 1999).

On the other hand, with respect to the stirring process and a configuration η drawn from ν_ρ, following Dürr et al. (1985) and the exposition in De Masi and Ferrari (2002), for $k \geq 1$ let $Y_k(t)$ be the position of the kth particle of η_t to the right of $1/2$; for $k \leq 0$ let $Y_k(t)$ be the position of the $(|k|+1)$th particle of η_t to the left of $1/2$. Then, at time t, the tagged particle, initially the 0th labeled particle, is the $J(t)$th particle,

$$X(t) = Y_{J(t)}(t),$$

where $J(t)$ was defined in Subsection 2.2.

It will also be useful to note, under the invariant measure ν_ρ, that $Y_n(t) \stackrel{d}{=} Y_n(0)$,

$$Y_n(t) = \begin{cases} Y_1(t) + \sum_{i=1}^{n-1} d_i(t) & \text{for } n \geq 2\\ Y_0(t) - \sum_{i=n}^{-1} d_i(t) & \text{for } n \leq -1\end{cases}$$

where $d_i(t) = Y_{i+1}(t) - Y_i$ is the spacing between the ith and $(i+1)$th particles, and also, $\{d_i(t) : i \neq 0\}$, $Y_1(t)$ and $|Y_0(t)| + 1$ have independent Geometric(ρ) distributions.

3. Tightness and fBM limit for the current

In this section, we prove the following theorem which is the main vehicle in the article. At the end of the section, we state as Corollary 3.5 the fractional Brownian motion invariance principle for the current.

Theorem 3.1. *Under initial distribution ν_ρ, the stochastic process $\lambda^{-1/4}J(\lambda t)$ is tight in $D([0,1])$ endowed with the uniform topology.*

Proof. The proof follows from showing that the discretized version $\lambda^{-1/4}J(\lfloor\lambda t\rfloor)$ is tight in $D([0,1])$ in the uniform norm (Proposition 3.3), and then that the difference with the desired process is negligible (Proposition 3.4). ∇

The first step is to state a useful maximal inequality.

Lemma 3.2. *For integers $m \geq 1$, and even integers $p \geq 2$,*

$$E[J^p(m)] \ \leq \ C_2 m^{p/4}$$

where $C_2 = C_2(p)$ is a constant. Moreover, for even integers $p \geq 6$, there is a constant $C_3 = C_3(p)$ such that

$$E\Big[\max_{1\leq i\leq m} J^p(i)\Big] \ \leq \ C_3 m^{p/4}.$$

Proof. We recall Marcinkiewicz inequality (Lemma 1.4.13 in De la Pena and Giné, 1999): For $p \geq 1$, there exists a constant $C_4 = C_4(p)$ such that for centered, independent L^p random variables $\{\beta_i\}$,

$$E\Big|\sum \beta_i\Big|^p \ \leq \ C_4 E\Big[\sum \beta_i^2\Big]^{p/2}.$$

Denote by $\mathcal{F}$ the σ-algebra generated by $K(t)$, and the random locations $\{i_k\}$ and $\{j_k\}$ with $1 \leq k \leq K(t)$ introduced in Subsection 2.2. Then, by conditioning first on $\mathcal{F}$, taking into account that $A_k(m)^2 \leq 1$ for all k, m, and properties in part (a) Subsection 2.2, we have

$$\begin{aligned} E[J^p(m)] &= E\Big[E\Big[\Big(\sum_{1\leq k\leq K(m)} A_k(m)\Big)^p\Big|\mathcal{F}\Big]\Big] \\ &\leq C_4 E\Big[\Big(\sum_{1\leq k\leq K(m)} A_k(m)^2\Big)^{p/2}\Big] \\ &\leq C_4 E[K^{p/2}(m)]. \end{aligned}$$

The first statement now follows by Lemma 2.1.

For the maximal inequality, we note first, from stationarity of $J(t)$ and the proven first inequality, for all integers $0 \leq i \leq j \leq m$,

$$E\big[(J(j) - J(i))^p\big] \ = \ E\big[J^p(j-i)\big] \ \leq \ C_2(j-i)^{p/4}. \tag{3.1}$$

We now recall a case of Theorem 3.1 in Moricz et al. (1982): Let $S_{i,j} = \sum_{k=i}^j \beta_k$ where $\{\beta_k\}$ are arbitrary random variables. Let also $\mu \geq 1$ and $\alpha > 1$. Suppose for some nonnegative numbers $\{u_k\}$, $E|S_{i,j}|^\mu \leq (\sum_{k=i}^j u_k)^\alpha$ for all $1 \leq i \leq j \leq n$. Then, there is a constant $C_5 = C_5(\mu, \alpha)$ such that

$$E\big[\max\{|S_{1,1}|, \ldots, |S_{1,n}|\}^\mu\big] \ \leq \ C_5\Big(\sum_{k=1}^n u_k\Big)^\alpha. \tag{3.2}$$

Then, as $p/4 > 1$, applying (3.2) with respect to (3.1), the second statement follows. ∇

We now consider the discretized process.

Proposition 3.3. *Under initial distribution ν_ρ, the stochastic process $\lambda^{-1/4}J(\lfloor \lambda t\rfloor)$ is tight in $D([0,1])$ endowed with the uniform topology.*

Proof. According to Billingsley (1968), a well-known tightness condition is to show, for all $\varepsilon > 0$, that

$$\lim_{\delta\to 0}\limsup_{\lambda\to\infty} P\left(\sup_{\substack{|s-t|<\delta\\ s,t\in[0,1]}} \lambda^{-1/4}|J(\lfloor\lambda t\rfloor) - J(\lfloor \lambda s\rfloor)| \geq \varepsilon\right) \ = \ 0$$

which reduces, in our situation of stationary increments, to proving

$$\lim_{\delta\to 0}\limsup_{\lambda\to\infty} \delta^{-1} P\left(\sup_{s\in[0,\delta]} \lambda^{-1/4}|J(\lfloor\lambda s\rfloor)| \geq \varepsilon\right) \ = \ 0.$$

By Chebychev's inequality,

$$P\left(\sup_{s\in[0,\delta]} \lambda^{-1/4}|J(\lfloor\lambda s\rfloor)| \geq \varepsilon\right) \ \leq \ \varepsilon^{-6} E\left(\sup_{s\in[0,\delta]} \lambda^{-6/4}|J^6(\lfloor\lambda s\rfloor)|\right).$$

Also, by the maximal inequality in Lemma 3.2,

$$E\left(\sup_{s\in[0,\delta]} \lambda^{-6/4}|J^6(\lfloor\lambda s\rfloor)|\right) \ = \ E\left(\max_{1\leq i\leq \lfloor\lambda\delta\rfloor} \lambda^{-6/4}|J^6(i)|\right) \ \leq \ C_3\delta^{3/2}$$

which is enough to conclude the proof. ∇

The difference between the discretized and desired process is handled as follows.

Proposition 3.4.

$$\lim_{\lambda\to\infty} P\Big(\sup_{0\leq t\leq 1} \lambda^{-1/4}|J(\lambda t) - J(\lfloor \lambda t\rfloor)| > \varepsilon\Big) \ = \ 0.$$

Proof. By stationary increments, noting $(nt) - \lfloor nt\rfloor \leq 1$,

$$\begin{aligned}
&P\Big(\sup_{0\leq t\leq 1} \lambda^{-1/4}|J(\lambda t) - J(\lfloor \lambda t\rfloor)| > \varepsilon\Big)\\
&\qquad\leq \ P\left(\sup_{\substack{0\leq s\leq t\leq \lambda\\ t-s\leq 1}} \lambda^{-1/4}|J(t) - J(s)| > \varepsilon\right)\\
&\qquad\leq \ 3\sum_{i=0}^{\lfloor\lambda\rfloor} P\Big(\sup_{i\leq t\leq i+1} \lambda^{-1/4}|J(t) - J(i)| > \varepsilon/3\Big)\\
&\qquad= \ 3(\lfloor\lambda\rfloor + 1)P\Big(\sup_{0\leq t\leq 1} |J(t)| > \varepsilon\lambda^{1/4}/3\Big)\\
&\qquad\leq \ 6E\left[\sup_{0\leq t\leq 1} |J(t)|^4 I\Big(\sup_{0\leq t\leq 1}|J(t)| > \varepsilon\lambda^{1/4}/3\Big)\right].
\end{aligned}$$

We now show that $E[\sup_{0\leq t\leq 1}|J(t)|^4] < \infty$ to deduce that the last quantity vanishes as $\lambda\uparrow\infty$. Indeed, from the decomposition $J(t) = M(t) + A(t)$ where $M(t)$ is a martingale and $A(t) = (1/2)\int_0^t(\eta_0(s) - \eta_1(s))ds$, as the integrand $|\eta_0 - \eta_1|$ is bounded by 1, we need only bound $E[\sup_{0\leq t\leq 1} M^4(t)]$. By using Doob's inequality and simple computations,

$$E\Big[\sup_{0\leq t\leq 1} M^4(t)\Big] \ \leq \ (4/3)^4 E[M^4(1)] \ \leq \ 4(4/3)^4\Big(E[J^4(1)] + 1\Big).$$

But, $J(t) = N_+(t) - N_-(t)$ is the difference of two counting processes each with rates bounded by 1/2, and so by coupling with respect to dominating Poisson rate(1/2) processes, we have $E[J^4(1)] < \infty$; this can also be seen from computing directly with $J(1) = \sum_{k=1}^{K(1)} A_k(1)$. ∇

Corollary 3.5. *Under initial distribution ν_ρ, with respect to the uniform topology on $D([0,1])$, we have as $\lambda \uparrow \infty$,*

$$\lambda^{-1/4} J(\lambda t) \ \Rightarrow \ \sigma_J \mathbb{B}_{1/4}(t)$$

where $\mathbb{B}_{1/4}(t)$ is the fractional Brownian motion process with index 1/4.

Proof. From Theorem 3.1, we know that $\lambda^{-1/4} J(\lambda t)$ is tight. Also, from the literature (cf. Landim and Volchan, 2000) the finite-dimensional distributions of any limit are Gaussian. Hence, if $W(t)$ is a limit along a subsequence, this limit is a continuous Gaussian process. By Lemma 3.2, the sixth moment of $\lambda^{-1/4} J(\lambda t)$ is uniformly bounded. So, we have uniform integrability for first and second powers of the process (see also De Masi and Ferrari, 2002), and therefore convergence of these moments.

To finish, we just have to compute the limit of covariances

$$\begin{aligned}
&4\mathrm{cov}\Big(\lambda^{-1/4}J(\lambda t), \lambda^{-1/4}J(\lambda s)\Big) \\
&\quad = \ E\Big[\lambda^{-1/4}J(\lambda t) + \lambda^{-1/4}J(\lambda s)\Big]^2 - E\Big[\lambda^{-1/4}J(\lambda t) - \lambda^{-1/4}J(\lambda s)\Big]^2 \\
&\quad = \ E\Big[\lambda^{-1/2}J^2(\lambda t)\Big] + E\Big[\lambda^{-1/2}J^2(\lambda t)\Big] \\
&\qquad\quad + 2\mathrm{cov}\Big(\lambda^{-1/4}J(\lambda t), \lambda^{-1/4}J(\lambda s)\Big) - E\Big[\lambda^{-1/2}J^2(\lambda(t-s))\Big].
\end{aligned}$$

Then, for $t > s$, recalling (1.1),

$$\lim_{\lambda\to\infty} \mathrm{cov}\Big(\lambda^{-1/4}J(\lambda t), \lambda^{-1/4}J(\lambda s)\Big) \ = \ \frac{\sigma_J^2}{2}\Big(\sqrt{t} + \sqrt{s} - \sqrt{t-s}\Big).$$

∇

4. Approximation and fBM limit for the tagged particle

In this section, we approximate $\lambda^{-1/4}X(\lambda t)$ by $\lambda^{-1/4}\rho^{-1}J(\lambda t)$ (Proposition 4.2) by adapting part of the proof of Proposition 2.8 in Dürr et al. (1985). Hence, as a process, $\lambda^{-1/4}X(\lambda t)$ will converge to the same fractional Brownian motion limit as $\lambda^{-1/4}\rho^{-1}J(\lambda t)$ (Corollary 4.3).

The first step is to approximate on the integers. For $0 < \epsilon < 1$, $k \geq 1$ and $t \geq 0$, define

$$J_{\epsilon,k}(t) \ = \ \begin{cases} \epsilon|J(t)| & \text{for } |J(t)| \geq t^{1/8} + k \\ 3\rho^{-1}(t^{1/8} + k) & \text{for } |J(t)| < t^{1/8} + k. \end{cases}$$

Recall from Subsection 2.3, the tagged particle representation $X(t) = Y_{J(t)}$.

Proposition 4.1. *For $0 < \epsilon < 1$, we have*

$$\lim_{k\to\infty} P\Big(\sup_{t\in\mathbb{Z}_+} \frac{|Y_{J(t)}(t) - \rho^{-1}J(t)|}{J_{\epsilon,k}(t)} > 1\Big) \ = \ 0.$$

Proof. First, we note, as in Dürr et al. (1985), on the event

$$G_{\epsilon,m} \;=\; \Big\{|Y_n(t)-\rho^{-1}n|\le \epsilon|n| \text{ for all } |n|\ge m\Big\}$$

that $|Y_n(t)-\rho^{-1}n|\le 3\rho^{-1}m$ when $|n|\le m$ because particles cannot cross and so

$$|Y_n(t)| \;\le\; \max\{|Y_{-m}(t)|,|Y_m(t)|\} \;\le\; m(\rho^{-1}+\epsilon)\le 2\rho^{-1}m. \tag{4.1}$$

By the representation from Subsection 2.3, and that $\{d_i(t):i\neq 0\}$, $Y_1(t)$ and $|Y_0(t)|+1$ are independent Geometric(ρ) random variables, we deduce

$$\begin{aligned}
&P\Big(|Y_n(t)-\rho^{-1}n|>|n|\epsilon \text{ for some } |n|\ge m\Big)\\
&\le\; 2\sum_{l\ge m}P\Big(\Big|(Y_1(t)-\rho^{-1})+\sum_{i=1}^{l-1}(d_i(t)-\rho^{-1})\Big|>l\epsilon\Big)\\
&\le\; C_6 m^{-q}
\end{aligned}$$

where $C_6=C_6(\epsilon,\rho,q)$ for any power of $q>0$.

Hence, noting the remark made near (4.1),

$$\begin{aligned}
&P\Big(\text{for some } t\in\mathbb{Z}_+,\ |Y_{J(t)}(t)-\rho^{-1}J(t)|>|J_{\epsilon,k}(t)|\Big)\\
&\le 2\sum_{t\in\mathbb{Z}_+}P\Big(|Y_n(t)-\rho^{-1}n|>\epsilon|n| \text{ for some } |n|\ge t^{1/8}+k\Big)\\
&\le\; C_6\sum_{t\in\mathbb{Z}_+}(t^{1/8}+k)^{-q}.
\end{aligned}$$

With $q>8$, the last expression vanishes as $k\uparrow\infty$. ∇

Proposition 4.2. *For $\delta>0$, we have*

$$\lim_{\lambda\to\infty}P\Big(\sup_{t\in[0,1]}\lambda^{-1/4}|X(\lambda t)-\rho^{-1}J(\lambda t)|>\delta\Big) \;=\; 0.$$

Proof. First, we write

$$\begin{aligned}
|X(\lambda t)-\rho^{-1}J(\lambda t)| \quad&\le\quad |X(\lambda t)-X(\lfloor\lambda t\rfloor)|+|X(\lfloor\lambda t\rfloor)-\rho^{-1}J(\lfloor\lambda t\rfloor)|\\
&\qquad +\rho^{-1}|J(\lfloor\lambda t\rfloor)-J(\lambda t)|
\end{aligned}$$

and note both

$$\lim_{\lambda\to\infty}P\Big(\sup_{t\in[0,1]}\lambda^{-1/4}|X(\lambda t)-X(\lfloor\lambda t\rfloor)|>\delta/3\Big) \;=\; 0$$

and

$$\lim_{\lambda\to\infty}P\Big(\sup_{t\in[0,1]}\lambda^{-1/4}|J(\lambda t)-J(\lfloor\lambda t\rfloor)|>\delta/3\Big) \;=\; 0.$$

Indeed, the second limit is estimated in the proof of Proposition 3.4, and the first limit is similarly argued: Write $X(t)-X(0)=\mathcal{M}(t)+\mathcal{A}(t)$ where $\mathcal{M}(t)$ is a martingale and $\mathcal{A}(t)$ is an additive functional with integrand bounded by $1/2$; note $|X(0)|+1$ is a Geometric(ρ) random variable, and so $E[X^4(0)]\le C_7$; then,

$$\begin{aligned}
E\big[\sup_{0\le t\le 1}X^4(t)\big] \quad&\le\quad 4E\big[\sup_{0\le t\le 1}(X(t)-X(0))^4\big]+4E[X^4(0)]\\
&\le\quad 16(4/3)^4\Big(E\big[M^4(1)\big]+1\Big)+4C_7\\
&\le\quad C_8\Big(E\big[X^4(1)\big]+1\Big);
\end{aligned}$$

and, as $X(t) - X(0) = \mathcal{N}_+(t) - \mathcal{N}_-(t)$ is the difference of two counting processes whose infinitessimal rates are bounded by 1/2, by coupling with respect to dominating Poisson rate(1/2) processes, $E[X^4(1)] < 4E[(X(1) - X(0))^4] + 4C_7 < \infty$.

Hence, we need only show

$$\lim_{\lambda\to\infty} P\Big(\sup_{t\in[0,1]} \lambda^{-1/4}|X(\lfloor\lambda t\rfloor) - \rho^{-1}J(\lfloor\lambda t\rfloor)| > \delta/3\Big) \;=\; 0.$$

This limit is the same as

$$\mathfrak{L} \;:=\; \lim_{\lambda\to\infty} P\Big(\max_{0\le l\le\lfloor\lambda\rfloor} \lambda^{-1/4}|X(l) - \rho^{-1}J(l)| > \delta/3\Big).$$

As $X(l) = Y_{J(l)}(l)$, and

$$\lambda^{-1/4}|Y_{J(l)}(l) - \rho^{-1}J(l)| \;=\; \frac{|Y_{J(l)}(l) - \rho^{-1}J(l)|}{|J_{\epsilon,k}(l)|}\frac{|J_{\epsilon,k}(l)|}{\lambda^{1/4}}$$

for $0 < \epsilon < 1$ and $k \ge 1$, we have

$$\mathfrak{L} \;\le\; \lim_{\lambda\to\infty} P\left(\max_{0\le l\le\lfloor\lambda\rfloor} \frac{|J_{\epsilon,k}(l)|}{\lambda^{1/4}} > \frac{\delta}{3}\right) + P\left(\sup_{t\in\mathbb{Z}_+} \frac{|Y_{J(t)}(t) - \rho^{-1}J(t)|}{J_{\epsilon,k}(t)} > 1\right).$$

With ϵ fixed for the moment, noting for large λ that $3\rho^{-1}(\lambda^{1/8} + k)\lambda^{-1/4} < \delta/6$, the limit $\mathfrak{L}$ is further bounded by

$$\lim_{\lambda\to\infty} P\left(\sup_{t\in[0,1]} \frac{|J(\lambda t)|}{\lambda^{1/4}} \ge \frac{\delta}{3\epsilon}\right) + \lim_{k\to\infty} P\left(\sup_{t\in\mathbb{Z}_+} \frac{|Y_{J(t)}(t) - \rho^{-1}J(t)|}{J_{\epsilon,k}(t)} > 1\right).$$

The second limit vanishes by Proposition 4.1. Also, the first limit, by the invariance principle already proved for $\lambda^{-1/4}J(\lambda t)$ with respect to continuous fractional Brownian motion (Corollary 3.5), vanishes by later taking $\epsilon \downarrow 0$. ∇

As mentioned in the beginning of the section, from Proposition 4.2 and Corollary 3.5, we obtain the fractional Brownian motion limit for the rescaled tagged motion.

Corollary 4.3. *Under initial distribution ν_ρ,*

$$\lambda^{-1/4}X(\lambda t) \;\Rightarrow\; \sigma_X\mathbb{B}_{1/4}(t)$$

in $D([0,1])$ endowed with the uniform topology, where $\mathbb{B}_{1/4}(t)$ is the fractional Brownian motion process with parameter 1/4.

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