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# Limit theorems for multiple stochastic integrals

**Giovanni Peccati and Murad S. Taqqu**

Equipe Modal'X, Université Paris Ouest - Nanterre La Défense, France
*E-mail address*: giovanni.peccati@gmail.com

Boston University, Departement of Mathematics, 111 Cummington Road, Boston (MA), USA
*E-mail address*: murad@math.bu.edu

**Abstract.** We show that the general stable convergence results proved in Peccati and Taqqu (2007) for generalized adapted stochastic integrals can be used to obtain limit theorems for multiple stochastic integrals with respect to independently scattered random measures. Several applications are developed in a companion paper (see Peccati and Taqqu, 2008a), where we prove central limit results involving single and double Poisson integrals, as well as quadratic functionals associated with moving average Lévy processes.

## 1. Introduction

Let $\{M(B) : B \in \mathcal{Z}\}$ be a square-integrable independently scattered random measure (see Section 4 below) on some measurable space $(Z, \mathcal{Z})$, and fix an integer $d \geq 2$. The aim of this note is to show that one can use the stable convergence results proved in Peccati and Taqqu (2007) to study the asymptotic behavior of sequences of the type

$$I(n) = I_d^M(f_n), \quad n \geq 1, \tag{1.1}$$

where $\{f_n\}$ is a collection of suitably regular (deterministic) functions on $Z^d$, and $I_d^M(f_n)$ is the multiple Wiener-Itô integral (of order $d$) of $f_n$ with respect to $M$ — see Section 4 below for precise definitions.

Our purpose is twofold. We shall show, firstly, that the random variables $I(n)$ appearing in (1.1) are indeed *generalized adapted stochastic integrals* of the type defined and studied in Peccati and Taqqu (2007). This entails that their stable convergence can be characterized by means of the convergence of well-chosen random

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characteristic functions (see Theorem 7 in Peccati and Taqqu, 2007, and Section 6 below). Secondly, we will prove that the convergence of these transforms is equivalent to the convergence in probability of special random Lévy-Khintchine exponents. In particular, we will see that the analytic expressions of such exponents can be obtained by a suitable extension of the techniques developed by Rajput and Rosinski (1989) and Kwapień and Woyczyński (1991).

This paper serves as a bridge between the abstract theory developed in Peccati and Taqqu (2007), and the applications developed in de Blasi et al. (2008), Peccati and Prünster and Peccati and Taqqu (2008a) (see also Peccati and Taqqu, 2008b). In the latter references, special attention is devoted to CLTs involving single and double integrals with respect to a Poisson random measure. These weak convergence results are then applied to obtain CLTs for quadratic functionals associated with random processes having the special form of moving averages of generalized Volterra processes. In particular (see de Blasi et al., 2008; Peccati and Prünster) this kind of limit theorems can be used in order to characterize the asymptotic properties of random hazard rates used in Bayesian survival analysis. To further justify our study, some applications will be sketched at the end of the paper. We also stress that the applications of our techniques are by no means exhausted by those presented in Peccati and Taqqu (2008a), de Blasi et al. (2008) and Peccati and Prünster. For instance, we expect that Theorem 6 and Corollaries 7 and 8 below can be used to obtain CLTs involving vectors of multiple stochastic integrals of order greater than 2.

The paper is organized as follows. In Section 2 we delineate our general setting. In Section 3 we present a brief summary of the results obtained in Peccati and Taqqu (2007), with special attention to the stable convergence of generalized adapted stochastic integrals. In Section 4, the notions of *independently scattered random measure* and of *multiple stochastic integral* are presented in the context of infinitely divisible laws. Section 5 is devoted to the representation of multiple stochastic integrals using the adapted integrals studied in Peccati and Taqqu (2007). In Section 6 we obtain our main stable convergence results, involving multiple integrals of any finite order with respect to general independently scattered random measures. Examples and concluding remarks are collected in Section 7.

## 2. General setting

To obtain our main convergence results we shall rely heavily on the theory developed in Peccati and Taqqu (2007). In that paper we used a decoupling technique, known as *principle of conditioning*, to investigate the stable convergence of *generalized adapted stochastic integrals* with respect to real-valued random fields "with independent increments". This is a framework in which the definition of increment is quite abstract, involving in particular increasing families of projection operators, called *resolutions of the identity.*

We shall see in Section 5 that multiple integrals with respect to independently scattered random measures are indeed generalized adapted integrals of the kind described above. This will allow us to apply the techniques developed in Peccati and Taqqu (2007) to study their stable convergence. Stable convergence entails convergence in law.

In what follows, we present a brief review of the main notions introduced in Peccati and Taqqu (2007), that will be needed in the subsequent sections. The reader is referred to Peccati and Taqqu (2007, Sections 2 and 3) for further details, proofs and examples. Throughout the following, $\mathfrak{H}$ denotes a real separable Hilbert space, with inner product $(\cdot,\cdot)_{\mathfrak{H}}$ and norm $\|\cdot\|_{\mathfrak{H}}$. All random objects are defined on an adequate probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**(i)** **(Resolutions of the identity; see e.g. Brodskii, 1971 and Yoshida, 1980):** A *continuous resolution of the identity* over $\mathfrak{H}$ is a collection $\pi = \{\pi_t : t \in [0,1]\}$ of orthogonal projections satisfying:
(a) $\pi_0 = 0$, and $\pi_1 = id.$;
(b) $\forall 0 \leq s < t \leq 1$, $\pi_s \mathfrak{H} \subseteq \pi_t \mathfrak{H}$;
(c) $\forall 0 \leq t_0 \leq 1$, $\forall h \in \mathfrak{H}$, $\lim_{t \to t_0} \|\pi_t h - \pi_{t_0} h\|_{\mathfrak{H}} = 0$.
A subset $F$ of $\mathfrak{H}$ is $\pi$-*reproducing* if the linear span of the class $\{\pi_t f : f \in F,\ t \in [0,1]\}$ is dense in $\mathfrak{H}$. The *rank* of $\pi$ is the smallest of the dimensions of all the closed subspaces generated by the $\pi$-reproducing subsets of $\mathfrak{H}$. The class of all continuous resolutions of the identity is denoted $\mathcal{R}(\mathfrak{H})$.

**(ii)** **(Isometric random fields):** In the following,

$$X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\} \tag{2.1}$$

denotes a collection of centered real-valued random variables, indexed by the elements of $\mathfrak{H}$ and satisfying the isometric relation

$$\mathbb{E}[X(h)X(h')] = (h, h')_{\mathfrak{H}},$$

for every $h, h' \in \mathfrak{H}$.

**(iii)** **($\pi$-Independent increments):** We define $\mathcal{R}_X(\mathfrak{H})$ to be the subset of $\mathcal{R}(\mathfrak{H})$ containing those $\pi$ such that the vector

$$\left(X((\pi_{t_1} - \pi_{t_0})h_1), X((\pi_{t_2} - \pi_{t_1})h_2) ..., X\left(\left(\pi_{t_m} - \pi_{t_{m-1}}\right)h_m\right)\right)$$

is composed of jointly independent random variables, for any choice of $m \geq 2$, $h_1, ..., h_m \in \mathfrak{H}$ and $0 \leq t_0 < t_1 < ... < t_m \leq 1$. The set $\mathcal{R}_X(\mathfrak{H})$ depends in general of $X$ and may of course be empty. If $X(\mathfrak{H})$ is Gaussian, then $\mathcal{R}_X(\mathfrak{H}) = \mathcal{R}(\mathfrak{H})$. When $\mathcal{R}_X(\mathfrak{H}) \neq \varnothing$ and $\pi \in \mathcal{R}_X(\mathfrak{H})$, we say that $X$ has *independent increments with respect to* $\pi$, or $\pi$-*independent* increments.

**(iv)** **(Filtrations):** To every $\pi \in \mathcal{R}_X(\mathfrak{H})$ is associated the filtration

$$\mathcal{F}_t^{\pi}(X) = \sigma\{X(\pi_t f) : f \in \mathfrak{H}\}, \quad t \in [0,1]. \tag{2.2}$$

**(v)** **(Infinite divisibility and Lévy-Khintchine exponents):** If $\mathcal{R}_X(\mathfrak{H})$ is not empty, for every $h \in \mathfrak{H}$, the law of $X(h)$ is infinitely divisible. As a consequence (see e.g. Sato, 1999), for every $h \in \mathfrak{H}$, there exists a unique pair $\left(c^2(h), \nu_h\right)$ such that $c^2(h) \in [0, +\infty)$ and $\nu_h$ is a measure on $\mathbb{R}$ satisfying

$$\nu_h(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2 \nu_h(dx) < +\infty; \tag{2.3}$$

moreover, for every $\lambda \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\exp(i\lambda X(h))] &= \exp\left[-\frac{c^2(h)\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x)\,\nu_h(dx)\right] && (2.4) \\ &\triangleq \exp[\psi_{\mathfrak{H}}(h;\lambda)]; && (2.5) \end{aligned}$$

$\psi_{\mathfrak{H}}(h;\lambda)$ is known as the *Lévy-Khintchine exponent* associated to $X(h)$. Relation (2.3) ensures that $\mathbb{E}X(h)^2 < +\infty$.

**(vi)** **(Adapted $\mathfrak{H}$-valued random elements):** By $L^2(\mathbb{P},\mathfrak{H},X) = L^2(\mathfrak{H},X)$ we denote the space of $\sigma(X)$-measurable and $\mathfrak{H}$-valued random variables $Y$ satisfying $\mathbb{E}\left[\|Y\|_{\mathfrak{H}}^2\right] < +\infty$ ($L^2(\mathfrak{H},X)$ is a Hilbert space, with inner product $(Y,Z)_{L^2(\mathfrak{H},X)} = \mathbb{E}\left[(Y,Z)_{\mathfrak{H}}\right]$). We associate to every $\pi \in \mathcal{R}_X(\mathfrak{H})$ the subspace $L^2_\pi(\mathfrak{H},X)$ of the $\pi$-*adapted* elements of $L^2(\mathfrak{H},X)$, that is: $Y \in L^2_\pi(\mathfrak{H},X)$ if, and only if, $Y \in L^2(\mathfrak{H},X)$ and, for every $t\in[0,1]$ and every $h \in \mathfrak{H}$,
$$(Y,\pi_t h)_{\mathfrak{H}} \in \mathcal{F}_t^\pi(X). \tag{2.6}$$
For any $\pi \in \mathcal{R}_X(\mathfrak{H})$, $L^2_\pi(\mathfrak{H},X)$ is a closed subspace of $L^2(\mathfrak{H},X)$.

**(vii)** **(Elementary adapted random elements):** For every $\pi \in \mathcal{R}_X(\mathfrak{H})$, $\mathcal{E}_\pi(\mathfrak{H},X)$ is the space of the *elementary* elements of $L^2_\pi(\mathfrak{H},X)$, i.e., $\mathcal{E}_\pi(\mathfrak{H},X)$ is the set of those elements of $L^2_\pi(\mathfrak{H},X)$ that are linear combinations of $\mathfrak{H}$-valued random variables of the type
$$h = \Phi(t_1)(\pi_{t_2}-\pi_{t_1})f, \tag{2.7}$$
where $t_2 > t_1$, $f\in\mathfrak{H}$ and $\Phi(t_1)$ is a random variable which is square-integrable and $\mathcal{F}_{t_1}^\pi(X)$ - measurable. We recall that, according to Peccati and Taqqu (2007, Lemma 3), $\forall\pi\in\mathcal{R}_X(\mathfrak{H})$, the span of the set $\mathcal{E}_\pi(\mathfrak{H},X)$ is dense in $L^2_\pi(\mathfrak{H},X)$.

**(viii)** **(Integrals of elementary random elements):** Fix $\pi\in\mathcal{R}_X(\mathfrak{H})$ and consider simple integrands of the form $h=\sum_{i=1}^n \lambda_i h_i \in \mathcal{E}_\pi(\mathfrak{H},X)$, where $\lambda_i\in\mathbb{R}$, $n\geq 1$, and $h_i$ is as in (2.7), i.e.
$$h_i = \Phi_i\left(t_1^{(i)}\right)\left(\pi_{t_2^{(i)}}-\pi_{t_1^{(i)}}\right)f_i,\quad f_i\in\mathfrak{H},\quad i=1,...,n, \tag{2.8}$$
with $t_2^{(i)} > t_1^{(i)}$, and $\Phi_i\left(t_1^{(i)}\right)\in\mathcal{F}^\pi_{t_1^{(i)}}(X)$ and square integrable. Then, the (generalized) stochastic integral of such a $h$ with respect to $X$ and $\pi$, is defined as
$$J_X^\pi(h) = \sum_{i=1}^n \lambda_i J_X^\pi(h_i) = \sum_{i=1}^n \lambda_i \Phi_i\left(t_1^{(i)}\right) X\left(\left(\pi_{t_2^{(i)}}-\pi_{t_1^{(i)}}\right)f_i\right). \tag{2.9}$$

**(ix)** **(Extension of $J_X^\pi$; see Peccati and Taqqu, 2007, Proposition 4):** Fix $\pi\in\mathcal{R}_X(\mathfrak{H})$. Then, there exists a unique linear extensions of $J_X^\pi$ to $L^2_\pi(\mathfrak{H},X)$ satisfying the following three conditions: (a) $J_X^\pi(h)$ equals the RHS of (2.9), for every $h=\sum_{i=1}^n\lambda_i h_i$ defined according to (2.7), (b) $J_X^\pi$ is a continuous operator, from $L^2_\pi(\mathfrak{H},X)$ to $L^2(\mathbb{P})$, and (c) for every $h,h'\in L^2_\pi(\mathfrak{H},X)$,
$$\mathbb{E}\left[J_X^\pi(h)\,J_X^\pi(h')\right] = (h,h')_{L^2_\pi(\mathfrak{H})}. \tag{2.10}$$
The random variable $J_X^\pi(h)$ is the (generalized) stochastic integral of $h$ with respect to $X$ and $\pi$.

## 3. Stable convergence criteria

The convergence results for stochastic integrals obtained in Section 6 hold in the sense of the *stable convergence* towards *mixtures* of random probability measures. The notions of random probability, random characteristic functions, mixtures and stable convergence are recalled in the next definitions. The reader is referred e.g. to Jacod and Shiryaev (1987, Chapter 4), Peccati and Taqqu (2007) and the references therein for a more detailed characterization of stable convergence.

**Definition I** – Let $\mathcal{B}(\mathbb{R})$ denote the Borel $\sigma$-field on $\mathbb{R}$.

**(a)** A map $m(\cdot,\cdot)$, from $\mathcal{B}(\mathbb{R})\times\Omega$ to $\mathbb{R}$ is called a *random probability* (on $\mathbb{R}$) if, for every $C\in\mathcal{B}(\mathbb{R})$, $m(C,\cdot)$ is a random variable and, for $\mathbb{P}$-a.e. $\omega$, the map $C\mapsto m(C,\omega)$, $C\in\mathcal{B}(\mathbb{R})$, defines a probability measure on $\mathbb{R}$. The class of all random probabilities is noted $\mathbf{M}$, and, for $m\in\mathbf{M}$, we write $\mathbb{E}m(\cdot)$ to indicate the (deterministic) probability measure

$$\mathbb{E}m(C)\triangleq\mathbb{E}[m(C,\cdot)],\quad C\in\mathcal{B}(\mathbb{R});\tag{3.1}$$

$\mathbb{E}m$ is called a *mixture*.

**(b)** For a measurable map $\phi(\cdot,\cdot)$, from $\mathbb{R}\times\Omega$ to $\mathbb{C}$, we write $\phi\in\widehat{\mathbf{M}}$ whenever there exists $m\in\mathbf{M}$ such that

$$\phi(\lambda,\omega)=\widehat{m}(\lambda)(\omega),\quad\forall\lambda\in\mathbb{R},\text{ for }\mathbb{P}\text{-a.e. }\omega,\tag{3.2}$$

where $\widehat{m}(\cdot)$ is defined as

$$\widehat{m}(\lambda)(\omega)=\begin{cases}\int_{\mathbb{R}}\exp(i\lambda x)\,m(dx,\omega) & \text{if } m(\cdot,\omega)\text{ is a probability measure}\\ 1 & \text{otherwise.}\end{cases},\quad\lambda\in\mathbb{R};\tag{3.3}$$

$\widehat{m}$ is a *random characteristic function.*

**(c)** For a given $\phi\in\widehat{\mathbf{M}}$, we write $\phi\in\widehat{\mathbf{M}}_0$ whenever

$$\mathbb{P}\{\omega:\phi(\lambda,\omega)\neq0\ \ \forall\lambda\in\mathbb{R}\}=1$$

.

**Definition II** (see e.g. Jacod and Shiryaev, 1987, Chapter 4 or Xue, 1991) – Let $\mathcal{F}^*\subseteq\mathcal{F}$ be a $\sigma$-field, and let $m\in\mathbf{M}$. A sequence of real valued r.v.'s $\{X_n:n\geq1\}$ is said to *converge* $\mathcal{F}^*$*-stably* to $m(\cdot)$, written $X_n\to_{(s,\mathcal{F}^*)}m(\cdot)$, if, for every $\lambda\in\mathbb{R}$ and every bounded $\mathcal{F}^*$-measurable r.v. $Z$,

$$\lim_{n\to+\infty}\mathbb{E}[Z\times\exp(i\lambda X_n)]=\mathbb{E}[Z\times\widehat{m}(\lambda)],\tag{3.4}$$

where the notation is the same as in (3.3).

If $X_n$ converges $\mathcal{F}^*$-stably, then the conditional distributions $\mathcal{L}(X_n\mid A)$ converge for any $A\in\mathcal{F}^*$ such that $\mathbb{P}(A)>0$ (see again Jacod and Shiryaev (1987, Section 5, §5c)). By setting $Z=1$ in (3.4), we obtain that if $X_n\to_{(s,\mathcal{F}^*)}m(\cdot)$, then the law of the $X_n$'s converges weakly to $\mathbb{E}m(\cdot)$.

We shall now state the main result of Peccati and Taqqu (2007). The notation is the same as that introduced in Section 2. In particular, $\mathfrak{H}_n$, $n\geq1$, is a sequence of real separable Hilbert spaces, and, for each $n\geq1$,

$$X_n=X_n(\mathfrak{H}_n)=\{X_n(g):g\in\mathfrak{H}_n\},\tag{3.5}$$

is a centered, real-valued stochastic process, indexed by the elements of $\mathfrak{H}_n$ and such that $\mathbb{E}[X_n(f)X_n(g)]=(f,g)_{\mathfrak{H}_n}$.

**Theorem 1** (Peccati and Taqqu, 2007, Theorem 7)**.** *For every $n \geq 1$, let $\pi^{(n)} \in \mathcal{R}_{X_n}(\mathfrak{H}_n)$ (we implicitly assume that $\mathcal{R}_{X_n}(\mathfrak{H}_n)$ is not empty) and $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$. Suppose also there exists a sequence $\{t_n : n \geq 1\} \subset [0,1]$ and a collection of $\sigma$-fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\lim_{n \to +\infty} \mathbb{E}\left[\left\|\pi^{(n)}_{t_n} u_n\right\|^2_{\mathfrak{H}_n}\right] = 0 \tag{3.6}$$

*and*

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n). \tag{3.7}$$

*If*

$$\exp\left[\psi_{\mathfrak{H}_n}(u_n; \lambda)\right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R}, \tag{3.8}$$

*where $\psi_{\mathfrak{H}_n}(u_n; \lambda)$ is defined according to (2.5), $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$,*

$$\phi(\lambda) \in \vee_n \mathcal{U}_n \triangleq \mathcal{U}^*,$$

*then, as $n \to +\infty$,*

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n)\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \tag{3.9}$$

*and*

$$J^{\pi^{(n)}}_{X_n}(u_n) \to_{(s, \mathcal{U}^*)} m(\cdot), \tag{3.10}$$

*where $m \in \mathbf{M}$ verifies (3.2).*

Roughly speaking, Theorem 1 considers sequences of random integrands $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$ and states that, under the negligeability condition (3.6), the stable convergence of the sequence $\{J^{\pi^{(n)}}_{X_n}(u_n) : n \geq 1\}$ (which, in general, is composed of random variables whose law *is not* infinitely divisible) can be deduced from the convergence of the random Lévy-Khintchine exponents $\psi_{\mathfrak{H}_n}(u_n; \lambda)$, $n \geq 1$, which are indeed deterministic transforms of the random integrands $u_n$. It is shown in Peccati and Taqqu (2007) that the quantity $\exp\left[\psi_{\mathfrak{H}_n}(u_n; \lambda)\right]$ can be represented as the conditional characteristic function of a "decoupled" version of $J^{\pi^{(n)}}_{X_n}(u_n)$.

## 4. Independently scattered random measures and multiple integrals

We are now ready to focus on multiple integrals with respect to general independently scattered random measures (not necessarily Poisson, nor Gaussian) and corresponding limit theorems. We will show in Proposition 5 below that these multiple integrals are generalized stochastic integrals in the sense of Points (viii) and (ix) of Section 2. We will then use Theorem 1 to obtain new central and non-central limit theorems for these multiple integrals, extending some of the results proved in Nualart and Peccati (2005) and Peccati and Tudor (2004), which were established in the framework of multiple Wiener-Itô integrals with respect to Gaussian processes. The limits will be mixtures of infinitely divisible distributions.

Several applications are developed in the companion paper Peccati and Taqqu (2008a), where we study sequences of double integrals with respect to Poisson random measures, as well as quadratic functionals of generalized moving average processes. For a general discussion concerning multiple integrals with respect to random measures, see Engel (1982), Kwapień and Woyczyński (1992) and Rota and Wallstrom (1997). For limit theorems involving multiple stochastic integrals (and other related classes of random variables), see the two surveys by Surgailis Surgailis (2000a) and Surgailis (2000b), and the references therein.

From now on $(Z, \mathcal{Z}, \mu)$ stands for a standard Borel space, with $\mu$ a positive, non-atomic and $\sigma$-finite measure on $(Z, \mathcal{Z})$. We denote by $\mathcal{Z}_\mu$ the subset of $\mathcal{Z}$ composed of sets of finite $\mu$-measure. Observe that the $\sigma$-finiteness of $\mu$ implies that $\mathcal{Z} = \sigma(\mathcal{Z}_\mu)$.

**Definition III** – An *independently scattered random measure* $M$ on $(Z, \mathcal{Z})$, with *control measure* $\mu$, is a collection of random variables

$$M = \{M(B) : B \in \mathcal{Z}_\mu\},$$

indexed by the elements of $\mathcal{Z}_\mu$, and such that: (**i**) for every $B \in \mathcal{Z}_\mu$ $M(B) \in L^2(\mathbb{P})$, (**ii**) for every finite collection of disjoint sets $B_1, ..., B_m \in \mathcal{Z}_\mu$, the vector $(M(B_1), ..., M(B_d))$ is composed of mutually independent random variables; (**iii**) for every $B, C \in \mathcal{Z}_\mu$,

$$\mathbb{E}[M(B) M(C)] = \mu(C \cap B). \tag{4.1}$$

Let $\mathfrak{H}_\mu = L^2(Z, \mathcal{Z}, \mu)$ be the Hilbert space of real-valued and square-integrable functions on $(Z, \mathcal{Z})$ (with respect to $\mu$). Since relation (4.1) holds, it is easily seen that there exists a unique collection of centered and square-integrable random variables

$$X_M = X_M(\mathfrak{H}_\mu) = \{X_M(h) : h \in \mathfrak{H}_\mu\}, \tag{4.2}$$

such that the following two properties are verified: (a) for every elementary function $h \in \mathfrak{H}_\mu$ with the form $h(z) = \sum_{i=1,...,n} c_i \mathbf{1}_{B_i}(z)$, where $n = 1, 2, ...$, $c_i \in \mathbb{R}$ and $B_i \in \mathcal{Z}_\mu$ are disjoint, $X_M(h) = \sum_{i=1,...,n} c_i M(B_i)$, and (b) for every $h, h' \in \mathfrak{H}_\mu$

$$\mathbb{E}[X_M(h) X_M(h')] = (h, h')_{\mathfrak{H}_\mu} \triangleq \int_Z h(z) h'(z) \mu(dz). \tag{4.3}$$

Property (a) implies in particular that, $\forall B \in \mathcal{Z}_\mu$, $M(B) = X_M(\mathbf{1}_B)$. Note that $X_M(\mathfrak{H}_\mu)$ is a collection of random variables of the kind defined in formula (2.1) of Section 2. Since $M$ is independently scattered, for every $h \in \mathfrak{H}_\mu$, the random variable $X_M(h)$ has an infinitely divisible law. It follows that, for every $h \in \mathfrak{H}_\mu$, there exists a unique pair $(c^2(h), \nu_h)$ such that $c^2(h) \in [0, +\infty)$ and $\nu_h$ is a (Lévy) measure on $\mathbb{R}$ satisfying the three properties in (2.3), so that, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\lambda X_M(h))] = \exp[\psi_{\mathfrak{H}_\mu}(h; \lambda)], \tag{4.4}$$

where the Lévy-Khinchine exponent $\psi_{\mathfrak{H}_\mu}(h; \lambda)$ is defined by (2.5).

The random variable $X_M(h)$ is thus infinitely divisible with Lévy-Khintchine exponent $\psi_{\mathfrak{H}_\mu}(h; \lambda)$. While the random measure $M$ is defined on $Z$, the following characterization of $\psi_{\mathfrak{H}_\mu}(h; \lambda)$ (and hence $X_M(h)$) involves the space $Z \times \mathbb{R}$. Although in the same spirit as that of Rajput and Rosinski (1989), it involves the control measure $\mu$ directly (see e.g. (4.8)). The proof is based on techniques developed in Rajput and Rosinski (1989) (see also Kwapień and Woyczyński, 1991, Section 5) and is presented for the sake of completeness.

**Proposition 2.** *For every $B \in \mathcal{Z}_\mu$, let $\left(c^2(B), \nu_B\right)$ denote the pair such that $c^2(B) \in [0, +\infty)$, $\nu_B$ verifies (2.3) and*

$$\psi_{\mathfrak{H}_\mu}(\mathbf{1}_B; \lambda) = -\frac{\lambda^2}{2} c^2(B) + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x)\, \nu_B(dx). \tag{4.5}$$

*Then,*

(1) *The application $B \mapsto c^2(B)$, from $\mathcal{Z}_\mu$ to $[0, +\infty)$, extends to a unique $\sigma$-finite measure $c^2(dz)$ on $(Z, \mathcal{Z})$, such that $c^2(dz) \ll \mu(dz)$.*

(2) *There exists a unique measure $\nu$ on $(Z \times \mathbb{R}, \mathcal{Z} \times \mathcal{B}(\mathbb{R}))$ such that $\nu(B \times C) = \nu_B(C)$, for every $B \in \mathcal{Z}_\mu$ and $C \in \mathcal{B}(\mathbb{R})$.*

(3) *There exists a function $\rho_\mu : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$ such that (i) for every $z \in Z$, $\rho_\mu(z, \cdot)$ is a Lévy measure*[1] *on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $\int_Z x^2 \rho_\mu(z, dx) < +\infty$, (ii) for every $C \in \mathcal{B}(\mathbb{R})$, $\rho_\mu(\cdot, C)$ is a Borel measurable function, (iii) for every positive function $g(z, x) \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$,*

$$\int_Z \int_{\mathbb{R}} g(z, x)\, \rho_\mu(z, dx)\, \mu(dz) = \int_Z \int_{\mathbb{R}} g(z, x)\, \nu(dz, dx). \tag{4.6}$$

(4) *For every $(\lambda, z) \in \mathbb{R} \times Z$, define*

$$K_\mu(\lambda, z) = -\frac{\lambda^2}{2} \sigma_\mu^2(z) + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \rho_\mu(z, dx), \tag{4.7}$$

*where*

$$\sigma_\mu^2(z) = \frac{dc^2}{d\mu}(z)$$

*, then, for every $h \in \mathfrak{H}_\mu = L^2(Z, \mathcal{Z}, \mu)$, $\int_Z |K_\mu(\lambda h(z), z)|\, \mu(dz) < +\infty$ and the exponent $\psi_{\mathfrak{H}_\mu}$ in (4.4) is given by*

$$\begin{aligned} \psi_{\mathfrak{H}_\mu}(h; \lambda) &= \int_Z K_\mu(\lambda h(z), z)\, \mu(dz) & (4.8) \\ &= -\frac{\lambda^2}{2} \int_Z h^2(z)\, \sigma_\mu^2(z)\, \mu(dz) & (4.9) \\ &\quad + \int_Z \int_{\mathbb{R}} \left(e^{i\lambda h(z) x} - 1 - i\lambda h(z)\, x\right) \rho_\mu(z, dx)\, \mu(dz). \end{aligned}$$

**Proof.** The proof follows from results contained in Rajput and Rosinski (1989, Section II). Point 1 is indeed a direct consequence of Rajput and Rosinski (1989, Proposition 2.1 (a)). In particular, whenever $B \in \mathcal{Z}$ is such that $\mu(B) = 0$, then $M(B) = 0$, a.s.-$\mathbb{P}$ (by applying (4.1) with $B = C$), and therefore $c^2(B) = 0$, thus implying $c^2 \ll \mu$. Point 2 follows from the first part of the statement of Rajput and Rosinski (1989, Lemma 2.3). To establish Point 3 define, as in Rajput and Rosinski (1989, p. 456),

$$\gamma(A) = c^2(A) + \int_{\mathbb{R}} \min\left(1, x^2\right) \nu_A(dx),$$

whenever $A \in \mathcal{Z}_\mu$, and observe (see Rajput and Rosinski, 1989, Definition 2.2) that $\gamma(\cdot)$ can be canonically extended to a $\sigma$-finite and positive measure on $(Z, \mathcal{Z})$. Moreover, since $\mu(B) = 0$ implies $M(B) = 0$ a.s.-$\mathbb{P}$, the uniqueness of the Lévy-Khinchine characteristics implies as before $\gamma(A) = 0$, and therefore $\gamma(dz) \ll \mu(dz)$. Observe also that, by standard arguments, one can select a version of the density

[1] That is, $\rho_\mu(z, \{0\}) = 0$ and $\int_{\mathbb{R}} \min\left(1, x^2\right) \rho_\mu(z, dx) < +\infty$

$(d\gamma/d\mu)(z)$ such that $(d\gamma/d\mu)(z) < +\infty$ for every $z \in Z$. According to Rajput and Rosinski (1989, Lemma 2.3), there exists a function $\rho : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$, such that: (a) $\rho(z, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ for every $z \in Z$, (b) $\rho(\cdot, C)$ is a Borel measurable function for every $C \in \mathcal{B}(\mathbb{R})$, (c) for every positive function $g(z, x) \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$,

$$\int_Z \int_{\mathbb{R}} g(z, x)\, \rho(z, dx)\, \gamma(dz) = \int_Z \int_{\mathbb{R}} g(z, x)\, \nu(dz, dx). \tag{4.10}$$

In particular, by using (4.10) in the case $g(z, x) = \mathbf{1}_A(z)\, x^2$ for $A \in \mathcal{Z}_\mu$,

$$\int_A \int_{\mathbb{R}} x^2 \rho(z, dx)\, \gamma(dz) = \int_{\mathbb{R}} x^2 \nu_A(dx) < +\infty,$$

since $M(A) \in L^2(\mathbb{P})$, and we deduce that $\rho$ can be chosen in such a way that, for every $z \in Z$, $\int_{\mathbb{R}} x^2 \rho(z, dx) < +\infty$. Now define, for every $z \in Z$ and $C \in \mathcal{B}(\mathbb{R})$,

$$\rho_\mu(z, C) = \frac{d\gamma}{d\mu}(z)\, \rho(z, C),$$

and observe that, due to the previous discussion, the application $\rho_\mu : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$ trivially satisfies properties (i)-(iii) in the statement of Point 3, which is therefore proved.

To Prove point 4, first define a function $h \in \mathfrak{H}_\mu$ to be *simple* if $h(z) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(z)$, where $a_i \in \mathbb{R}$, and $(A_1, ..., A_n)$ is a finite collection of disjoints elements of $\mathcal{Z}_\mu$. Of course, the class of simple functions (which is a linear space) is dense in $\mathfrak{H}_\mu$, and therefore for every $h \in \mathfrak{H}_\mu$ there exists a sequence $h_n$, $n \geq 1$, of simple functions such that $\int_Z (h_n(z) - h(z))^2 \mu(dz) \to 0$. As a consequence, since $\mu$ is $\sigma$-finite there exists a subsequence $n_k$ such that $h_{n_k}(z) \to h(z)$ for $\mu$-a.e. $z \in Z$ (and therefore for $\gamma$-a.e. $z \in Z$) and moreover, for every $A \in \mathcal{Z}$, the random sequence $X_M(\mathbf{1}_A h_n)$ (where we use the notation (4.2)) is a Cauchy sequence in $L^2(\mathbb{P})$, and hence it converges in probability. In the terminology of Rajput and Rosinski (1989, p. 460), this implies that every $h \in \mathfrak{H}_\mu$ is $M$-integrable, and that, for every $A \in \mathcal{Z}$, the random variable $X_M(h\mathbf{1}_A)$, defined according to (4.2), coincides with $\int_A h(z)\, M(dz)$, i.e. the integral of $h$ with respect to the restriction of $M(\cdot)$ to $A$, as defined in Rajput and Rosinski (1989, p. 460). As a consequence, by using a slight modification of Rajput and Rosinski (1989, Proposition 2.6)[2], the function $K_0$ on $\mathbb{R} \times Z$ given by

$$K_0(\lambda, z) = -\frac{\lambda^2}{2}\sigma_0^2(z) + \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) \rho(z, dx),$$

where $\sigma_0^2(z) = \left(dc^2/d\gamma\right)(z)$, is such that $\int_Z |K_0(\lambda h(z), z)|\, \gamma(dz) < +\infty$ for every $h \in \mathfrak{H}_\mu$, and also

$$\mathbb{E}\left[\exp(i\lambda X_M(h))\right] = \int_Z K_0(\lambda h(z), z)\, \gamma(dz).$$

Relation (4.4) and the fact that, by definition,

$$K_\mu(\lambda h(z), z) = K_0(\lambda h(z), z)\, \frac{d\gamma}{d\mu}(z), \quad \forall z \in Z, \forall h \in \mathfrak{H}_\mu, \forall \lambda \in \mathbb{R},$$

yield (4.8). ■

[2]The difference lies in the choice of the truncation.

**Examples** – (a) If $M$ is a centered Gaussian measure with control $\mu$, then $\nu = 0$ and, for $h \in \mathfrak{H}_\mu$,

$$\psi_{\mathfrak{H}_\mu}(h;\lambda) = -\frac{\lambda^2}{2}\int_Z h^2(z)\,\mu(dz).$$

(b) If $M$ is a centered Poisson measure with control $\mu$, then $c^2(\cdot) = 0$ and $\rho_\mu(z, dx) = \delta_1(dx)$ for all $z \in Z$, where $\delta_1$ is the Dirac mass at $x = 1$, and therefore, for $h \in \mathfrak{H}_\mu$,

$$\psi_{\mathfrak{H}_\mu}(h;\lambda) = \int_Z \left(e^{i\lambda h(z)} - 1 - i\lambda h(z)\right)\mu(dz);$$

$\exp\left(\psi_{\mathfrak{H}_\mu}(h;\lambda)\right)$ is then the characteristic function of $\int_Z h(z)\,dM(z)$. One can take e.g. $Z = \mathbb{R}\times\mathbb{R}$, and $\mu(dx, du) = dx\nu(du)$, where $\nu(du)$ is a measure on $\mathbb{R}$ satisfying $\int u^2\nu(du) < +\infty$. In this case, by considering a function $h : \mathbb{R} \to \mathbb{R}$ with support in $[0, +\infty)$ and such that $\int h(x)^2\,dx < +\infty$, we can associate to $M$ the following (centered and square integrable) *stationary moving average process*:

$$Y_t^h = \int_{-\infty}^{t}\int_{\mathbb{R}} uh(t-x)\,M(du, dx), \quad t \geq 0. \tag{4.11}$$

For instance, if $h(x) = \sqrt{2\lambda}\exp(-\lambda x)\,\mathbf{1}_{x>0}$ for some $\lambda > 0$, $Y^h$ is called a *Ornstein-Uhlenbeck Lévy process* of parameter $\lambda$; when $h(x) = \lambda^{\kappa-1}\frac{\sqrt{2\lambda}}{\Gamma(\kappa)}x^{\kappa-1}\exp(-\lambda x)\,\mathbf{1}_{x>0}$ for $\lambda > 0$ and $\kappa > 1/2$, $Y^h$ is called *fractional Ornstein-Uhlenbeck Lévy process* of parameters $\lambda$ and $\kappa$. The reader is referred to Peccati and Taqqu (2008a) for asymptotic results involving moving average processes, and to Wolpert and Taqqu (2005) for several applications of such processes to network modeling.

We now want to define multiple integrals, of functions vanishing on diagonals, with respect to the random measure $M$. To this end, fix $d \geq 2$ and set $\mu^d$ to be the canonical product measure on $\left(Z^d, \mathcal{Z}^d\right)$ induced by $\mu$. We introduce the following standard notation: (i) $L^2\left(\mu^d\right) \triangleq L^2\left(Z^d, \mathcal{Z}^d, \mu^d\right)$ is the class of real-valued and square-integrable functions on $\left(Z^d, \mathcal{Z}^d\right)$; (ii) $L_s^2\left(\mu^d\right)$ is the subset of $L^2\left(\mu^d\right)$ composed of square integrable and symmetric functions; (iii) $L_{s,0}^2\left(\mu^d\right)$ is the subset of $L_s^2\left(\mu^d\right)$ composed of square integrable and symmetric functions vanishing on diagonals.

Now define $\mathcal{S}_{s,0}\left(\mu^d\right)$ to be subset of $L_{s,0}^2\left(\mu^d\right)$ composed of functions with the form

$$f(z_1, ..., z_d) = \sum_{\sigma\in\mathfrak{S}_d} \mathbf{1}_{B_1}\left(z_{\sigma(1)}\right)\cdots\mathbf{1}_{B_d}\left(z_{\sigma(d)}\right), \tag{4.12}$$

where $B_1, ..., B_d \in \mathcal{Z}_\mu$ are pairwise disjoint sets, and $\mathfrak{S}_d$ is the group of all permutations of $\{1, ..., d\}$. Recall (see e.g. Rota and Wallstrom, 1997, Proposition 3) that $\mathcal{S}_{s,0}\left(\mu^d\right)$ is total in $L_{s,0}^2\left(\mu^d\right)$. For $f \in L_{s,0}^2\left(\mu^d\right)$ as in (4.12), we set

$$I_d^M(f) = d!M(B_1)\times M(B_2)\times\cdots\times M(B_d) \tag{4.13}$$

to be the *multiple integral*, of order $d$, of $f$ with respect to $M$. It is well known (see for instance Rota and Wallstrom, 1997, Theorem 5) that there exists a unique linear extension of $I_d^M$, from $\mathcal{S}_{s,0}\left(\mu^d\right)$ to $L_{s,0}^2\left(\mu^d\right)$, satisfying the following: (a) for every $f \in L^2_{s,0}\left(\mu^d\right)$, $I_d^M(f)$ is a centered and square-integrable random variable, and (b) for every $f, g \in L^2_{s,0}\left(\mu^d\right)$

$$\mathbb{E}\left[I_d^M(f)\, I_d^M(g)\right] = d!\,(f,g)_{L^2(\mu^d)} \triangleq d! \int_{Z^d} f(\mathbf{z}_d)\, g(\mathbf{z}_d)\, \mu^d(d\mathbf{z}_d), \qquad (4.14)$$

where $\mathbf{z}_d = (z_1, ..., z_d)$ stands for a generic element of $Z^d$. Note that, by construction, if $d \neq d'$, $\mathbb{E}\left[I_d^M(f)\, I_{d'}^M(g)\right] = 0$ for every $f \in L^2_{s,0}\left(\mu^d\right)$ and every $g \in L^2_{s,0}\left(\mu^{d'}\right)$. Again, for $f \in L^2_{s,0}\left(\mu^d\right)$, $I_d^M(f)$ is called the multiple integral, of order $d$, of $f$ with respect to $M$. When $f \in L^2_s\left(\mu^d\right)$ (hence, $f$ does not necessarily vanish on diagonals) we define

$$I_d^M(f) \triangleq I_d^M\left(f \mathbf{1}_{Z_0^d}\right), \qquad (4.15)$$

where

$$Z_0^d \triangleq \left\{(z_1, ..., z_d) \in Z^d : \text{ the } z_j\text{'s are all different }\right\}, \qquad (4.16)$$

so that (since $\mu$ is non atomic, and therefore the product measures do not charge diagonals), for every $f, g \in L^2\left(\mu^d\right)$, $\mathbb{E}\left[I_d^M(f)\, I_d^M(g)\right] = d! \int_{Z_0^d} f(\mathbf{z}_d)\, g(\mathbf{z}_d)\, \mu^d(d\mathbf{z}_d)$ $= d!\,(f,g)_{L^2(\mu^d)}$. Note that, for $d = 1$, one usually sets $L^2_{s,0}\left(\mu^1\right) = L^2_s\left(\mu^1\right) = L^2\left(\mu^1\right) = \mathfrak{H}_\mu$, and $I_1^M(f) = X_M(f)$, $f \in \mathfrak{H}_\mu$.

In what follows, we shall show that, for some well chosen resolutions $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$, every multiple integral of the type $I_d^M(f)$, $f \in L^2_{s,0}\left(\mu^d\right)$, can be represented in the form of a generalized adapted integral of the kind introduced in Section 3. As a consequence, the asymptotic behavior of $I_d^M(f)$ can be studied by means of Theorem 1.

## 5. Representation of multiple integrals

Under the notation and assumptions of the previous section, consider a "continuous" increasing family $\{Z_t : t \in [0,1]\}$ of elements of $\mathcal{Z}$, such that $Z_0 = \varnothing$, $Z_1 = Z$, $Z_s \subseteq Z_t$ for $s < t$, and, for every $g \in L^1(\mu)$ and every $t \in [0,1]$,

$$\lim_{s \to t} \int_{Z_s} g(x)\, \mu(dx) = \int_{Z_t} g(x)\, \mu(dx). \qquad (5.1)$$

For example, for $Z = [0,1]^2$, one can take $Z_0 = \{\varnothing\}$ and $Z_t = [0,t]^2$ or $Z_t = [(1-t)/2, (1+t)/2]^2$ for $t \in (0,1]$. For $Z = \mathbb{R}^2$, one can take $Z_0 = \{\varnothing\}$ and $Z_t = [\log(1-t), -\log(1-t)]^2$ for $t \in (0,1]$. The dominated convergence theorem ensures that (5.1) is satisfied for all these choices.

To each $t \in [0,1]$, we associate the following projection operator $\pi_t : \mathfrak{H}_\mu \mapsto \mathfrak{H}_\mu$: $\forall f \in \mathfrak{H}_\mu$,

$$\pi_t f(z) = \mathbf{1}_{Z_t}(z)\, f(z), \quad z \in Z, \qquad (5.2)$$

so that, since $M$ is independently scattered, the continuous resolution of the identity $\pi = \{\pi_t : t \in [0,1]\}$ is such that, $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$ (note that specifying $Z_t$, $t \in [0,1]$, is equivalent to specifying $\pi_t$, $t \in [0,1]$). Note also that, thanks to (5.1) and by uniform continuity, for every $f \in \mathfrak{H}_\mu$, every $t \in (0,1]$ and every sequence of partitions of $[0,t]$,

$$t^{(n)} = \left\{0 = t_0^{(n)} < t_1^{(n)} < ... < t_{r_n}^{(n)} = t\right\}, \quad n \geq 1, \qquad (5.3)$$

such that $mesh\left(t^{(n)}\right) \triangleq \max_{i=0,\dots,r_n-1}\left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$,

$$\max_{i=0,\dots,r_n-1}\left\|\left(\pi_{t_{i+1}^{(n)}} - \pi_{t_i^{(n)}}\right) f\right\|_{\mathfrak{H}_\mu}^2 \to 0, \tag{5.4}$$

and in particular, for every $B \in \mathcal{Z}_\mu$,

$$\max_{i=0,\dots,r_n-1} \mu\left(B \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) \to 0. \tag{5.5}$$

The following result contains the key of the subsequent discussion.

**Proposition 3.** *For every $d \geq 2$, every random variable of the form $I_d^M\left(f\mathbf{1}_{Z_t^d}\right) = I_d^M\left(f\mathbf{1}_{Z_t}^{\otimes d}\right)$, for some $f \in L_{s,0}^2\left(\mu^d\right)$ and $t \in (0,1]$, can be approximated in $L^2(\mathbb{P})$ by linear combinations of random variables of the type*

$$M\left(B_1 \cap Z_{t_1}\right) \times M\left(B_2 \cap \left(Z_{t_2} \backslash Z_{t_1}\right)\right) \times \cdots \times M\left(B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)\right), \tag{5.6}$$

*where the $t_1, \dots, t_d$ are rational, $0 \leq t_1 < t_2 < \cdots < t_d \leq t$ and $B_1, \dots, B_d \in \mathcal{Z}_\mu$ are disjoint. In particular, $I_d^M\left(f\mathbf{1}_{Z_t^d}\right) \in \mathcal{F}_t^\pi$, where the filtration $\mathcal{F}_t^\pi$, $t \in [0,1]$, is defined as in (2.2).*

**Remark** – Observe that, if $f \in \mathcal{S}_{s,0}\left(\mu^d\right)$ is such that

$$f\left(z_1, \dots, z_d\right) = \sum_{\sigma \in \mathfrak{S}_d} \mathbf{1}_{B_1 \cap Z_{t_1}}\left(z_{\sigma(1)}\right) \cdots \mathbf{1}_{B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)}\left(z_{\sigma(d)}\right), \tag{5.7}$$

then, by (4.13),

$$d!M\left(B_1 \cap Z_{t_1}\right) \times M\left(B_2 \cap \left(Z_{t_2} \backslash Z_{t_1}\right)\right) \times \cdots \times M\left(B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)\right) = I_d^M(f). \tag{5.8}$$

**Proof.** Observe first that, for every $f \in L_{s,0}^2\left(\mu^d\right)$, every $t \in (0,1]$ and every sequence of rational numbers $t_n \to t$, $I_d^M\left(f\mathbf{1}_{Z_{t_n}^d}\right) \to I_d^M\left(f\mathbf{1}_{Z_t^d}\right)$ in $L^2(\mathbb{P})$. By density, it is therefore sufficient to prove the statement for multiple integrals of the type $I_d^M\left(f\mathbf{1}_{Z_t^d}\right)$, where $t \in \mathbb{Q} \cap (0,1]$ and $f \in \mathcal{S}_{s,0}\left(\mu^d\right)$ is as in (5.7). Start with $d = 2$. In this case,

$$\frac{1}{2} I_2^M\left(f\mathbf{1}_{Z_t^2}\right) = M\left(B_1 \cap Z_t\right) M\left(B_2 \cap Z_t\right)$$

with $B_1, B_2$ disjoints, and also, for every partition $\{0 = t_0 < t_1 < \dots < t_r = t\}$ (with $r \geq 1$) of $[0,t]$,

$$\begin{aligned}
\frac{1}{2} I_2^M(f) &= \sum_{i=1}^r M\left(B_1 \cap \left(Z_{t_i} \backslash Z_{t_{i-1}}\right)\right) \sum_{j=1}^r M\left(B_2 \cap \left(Z_{t_j} \backslash Z_{t_{j-1}}\right)\right) \\
&= \sum_{1 \leq i \neq j \leq r} M\left(B_1 \cap \left(Z_{t_i} \backslash Z_{t_{i-1}}\right)\right) M\left(B_2 \cap \left(Z_{t_j} \backslash Z_{t_{j-1}}\right)\right) + \\
&\quad + \sum_{i=1}^r M\left(B_1 \cap \left(Z_{t_i} \backslash Z_{t_{i-1}}\right)\right) M\left(B_2 \cap \left(Z_{t_i} \backslash Z_{t_{i-1}}\right)\right) \triangleq \Sigma_1 + \Sigma_2.
\end{aligned}$$

The summands in the first sum $\Sigma_1$ have the desired form (5.6). It is therefore sufficient to prove that for every sequence of partitions $t^{(n)}$, $n \geq 1$, as in (5.3) and such that $mesh\left(t^{(n)}\right) \to 0$ and the $t_1^{(n)}, ..., t_{r_n}^{(n)}$ are rational,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{r_n} M\left(B_1 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) M\left(B_2 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right)\right)^2\right] = 0. \quad (5.9)$$

Since $B_1$ and $B_2$ are disjoint, and thanks to the isometric properties of $M$,

$$\begin{aligned} & \mathbb{E}\left[\left(\sum_{i=1}^{r_n} M\left(B_1 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) M\left(B_2 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right)\right)^2\right] \\ = & \sum_{i=1}^{r_n} \mu\left(B_1 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) \mu\left(B_2 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) \\ \leq & \mu\left(B_1\right) \max_{i=1,\dots,r_n} \mu\left(B_2 \cap \left(Z_{t_i^{(n)}} \backslash Z_{t_{i-1}^{(n)}}\right)\right) \to 0, \end{aligned}$$

thanks to (5.5). Now fix $d \geq 3$, and consider a random variable of the type

$$F = M\left(B_1 \cap Z_t\right) \times \cdots \times M\left(B_{d-1} \cap Z_t\right) \times M\left(B_d \cap Z_t\right), \quad (5.10)$$

where $B_1, ..., B_d \in \mathcal{Z}_\mu$ are disjoint. The above discussion yields that $F$ can be approximated by linear combinations of random variables of the type

$$\begin{aligned} & M\left(B_1 \cap Z_t\right) \times \cdots \times M\left(B_{d-3} \cap Z_t\right) \quad (5.11) \\ & \times \left[M\left(B_{d-2} \cap Z_t\right) \times M\left(B_{d-1} \cap \left(Z_s \backslash Z_r\right)\right) \times M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right)\right], \end{aligned}$$

where $r < s < u < v \leq t$ are rational. We will proceed by induction focusing first on the terms in the brackets in (5.11). Express $Z_t$ as the union of five disjoint sets $Z_t = (Z_t \backslash Z_v) \cup (Z_v \backslash Z_u) \cup (Z_u \backslash Z_s) \cup (Z_s \backslash Z_r) \cup Z_r$, and decompose $M\left(B_{d-2} \cap Z_t\right)$ accordingly. One gets

$$\begin{aligned} & M\left(B_{d-2} \cap Z_t\right) M\left(B_{d-1} \cap \left(Z_s \backslash Z_r\right)\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right) \quad (5.12) \\ = & M\left(B_{d-2} \cap \left(Z_s \backslash Z_r\right)\right) M\left(B_{d-1} \cap Z_s \backslash Z_r\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right) \\ & + M\left(B_{d-2} \cap \left(Z_v \backslash Z_u\right)\right) M\left(B_{d-1} \cap Z_s \backslash Z_r\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right) \\ & + M\left(B_{d-2} \cap \left(Z_u \backslash Z_s\right)\right) M\left(B_{d-1} \cap Z_s \backslash Z_r\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right) \\ & + M\left(B_{d-2} \cap \left(Z_t \backslash Z_v\right)\right) M\left(B_{d-1} \cap Z_s \backslash Z_r\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right) \\ & + M\left(B_{d-2} \cap \left(Z_r \backslash Z_0\right)\right) M\left(B_{d-1} \cap Z_s \backslash Z_r\right) M\left(B_d \cap \left(Z_v \backslash Z_u\right)\right). \end{aligned}$$

Observe that the last three summands involve disjoint subsets of $Z$ and hence are of the form (5.6). Since each of the first two summands involve two identical subsets of $Z$ (e.g. $(Z_s \backslash Z_r)$) and a disjoint subset (e.g. $(Z_v \backslash Z_u)$), they can be dealt with in the same way as (5.9) above. Thus, linear combinations of the five summands on the RHS of (5.12) can be approximated by linear combinations of random variables of the type

$$M\left(C_1 \cap \left(Z_{t_2} \backslash Z_{t_1}\right)\right) M\left(C_2 \cap \left(Z_{t_3} \backslash Z_{t_2}\right)\right) M\left(C_3 \cap \left(Z_{t_4} \backslash Z_{t_3}\right)\right),$$

where $C_1, C_2, C_3 \in \mathcal{Z}_\mu$ are disjoints, and $t_1 < t_2 < t_3 < t_4 \leq t$ are rational. The general result is obtained by induction on $d$. ■

Proposition 3 will be used to prove that, whenever there exists $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$ defined as in formula (5.2), multiple integrals can be represented as generalized adapted integrals of the kind described in Section 2. To do this, we introduce a partial ordering on $Z$ as follows: for every $z', z \in Z$,

$$z' \prec_\pi z \tag{5.13}$$

if, and only if, there exists $t \in \mathbb{Q} \cap (0,1)$ such that $z' \in Z_t$ and $z \in Z_t^c$, where $Z_t^c$ stands for the complement of $Z_t$ (we take $t \in \mathbb{Q}$ for measurability purposes). For instance, suppose that $Z = [0,1]^2$ and $Z_t = [0,t]^2$, $t \in [0,1]$; then, for every $z = \left(z^{(1)}, z^{(2)}\right) \in [0,1]^2$, $z' \prec_\pi z$ if, and only if, $z' \in [0,\overline{z})^2$, where $\overline{z} \triangleq \max\left(z^{(1)}, z^{(2)}\right)$ (please draw a picture).

For a fixed $d \geq 2$, we define the $\pi$-*purely non-diagonal* subset of $Z^d$ as

$$Z^d_{\pi,0} = \left\{(z_1, ..., z_d) \in Z^d : z_{\sigma(1)} \prec_\pi z_{\sigma(2)} \prec_\pi \cdots \prec_\pi z_{\sigma(d)}, \text{ for some } \sigma \in \mathfrak{S}_d\right\}.$$

Note that $Z^d_{\pi,0} \in \mathcal{Z}^d$, and also that not every pair of distinct points of $Z$ can be ordered, that is, in general, $Z^d_{\pi,0} \neq Z^d_0$, where $d \geq 2$ and $Z^d_0$ is defined in (4.16) (for illustration, think again of $Z = [0,1]^2$, $Z_t = [0,t]^2$, $t \in [0,1]$; indeed $((1/8, 1/4), (1/4, 1/4)) \in Z^2_0$, but $(1/4, 1/4)$ and $(1/8, 1/4)$ cannot be ordered). However, because of the continuity condition (5.1) and for every $d \geq 2$, the class of the elements of $Z^d_0$ whose components cannot be ordered has measure $\mu^d$ equal to zero, as indicated by the following corollary.

**Corollary 4.** *For every $d \geq 2$ and every $f \in L^2_{s,0}\left(\mu^d\right)$,*

$$I^M_d(f) = I^M_d\left(f \mathbf{1}_{Z^d_{\pi,0}}\right).$$

*As a consequence, $\mu^d\left(Z^d_0 \backslash Z^d_{\pi,0}\right) = 0$, where $Z_0$ is defined in (4.16).*

**Proof.** First observe that the class of r.v.'s of the type $I^M_d\left(f \mathbf{1}_{Z^d_{\pi,0}}\right)$, $f \in L^2_{s,0}\left(\mu^d\right)$ is a closed vector space. Plainly, every $f \in \mathcal{S}_{s,0}\left(\mu^d\right)$ with the form (5.7) is such that $f(\mathbf{z}_d) = f(\mathbf{z}_d)\mathbf{1}_{Z^d_{\pi,0}}(\mathbf{z}_d)$ for every $\mathbf{z}_d \in Z^d$. Since, by Proposition 3 and relation (5.8), the class of functions of the type (5.7) are total in $L^2_{s,0}\left(\mu^d\right)$, the result is obtained by a density argument. The last assertion follows from the facts that $\forall f \in L^2_{s,0}\left(\mu^d\right)$ one has $f = f\mathbf{1}_{Z^d_{\pi,0}}$, a.e.-$\mu^d$, and $I^M_d(f) = I^M_d(g)$ if and only if $f = g$, a.e.-$\mu^d$. ■

For $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$ as in formula (5.2), the vector spaces $L^2_\pi(\mathfrak{H}_\mu, X_M)$ and $\mathcal{E}_\pi(\mathfrak{H}_\mu, X_M)$, composed respectively of adapted and elementary adapted elements of $L^2(\mathfrak{H}_\mu, X_M)$, are defined as in Section 2 (in particular, *via* formulae (2.6) and (2.7)). Recall that, according to Point (vii) in Section 2, the closure of $\mathcal{E}_\pi(\mathfrak{H}_\mu, X_M)$ coincides with $L^2_\pi(\mathfrak{H}_\mu, X_M)$. For every $h \in L^2_\pi(\mathfrak{H}_\mu, X_M)$, the random variable $J^\pi_{X_M}(h)$ is defined as in Point (ix) in Section 2 and formula (2.9). The following result states that every multiple integral with respect to $M$ is indeed a generalized adapted integral of the form $J^\pi_{X_M}(h)$, for some $h \in L^2_\pi(\mathfrak{H}_\mu, X_M)$.

In what follows, for every $d \geq 1$, every $f \in L^2_{s,0}\left(\mu^d\right)$ and every fixed $z \in Z$, the symbol $f(z, \cdot)\mathbf{1}(\cdot \prec_\pi z)$ stands for the element of $L^2_{s,0}\left(\mu^{d-1}\right)$, given by

$$(z_1, ..., z_{d-1}) \mapsto f(z, z_1, ..., z_{d-1}) \prod_{j=1}^{d-1} \mathbf{1}_{(z_j \prec_\pi z)}. \tag{5.14}$$

For instance, when $Z = [0,1]^2$ and $Z_t = [0,t]^2$, $t \in [0,1]$, for every fixed $z = \left(z^{(1)}, z^{(2)}\right) \in Z$ the kernel $f(z,\cdot)\,\mathbf{1}(\cdot \prec_\pi z)$ is given by the application

$$(z_1,...,z_{d-1}) \mapsto f(z,z_1,...,z_{d-1})\,\mathbf{1}_{\cap_{j=1}^{d-1}\{z_j<\max\left(z^{(1)},z^{(2)}\right)\}}.$$

**Theorem 5.** *Fix $d \geq 2$, and let $f \in L^2_{s,0}\left(\mu^d\right)$. Then,*

(1) *The random function*

$$z \mapsto h_\pi(f)(z) = d \times I^M_{d-1}\left(f(z,\cdot)\,\mathbf{1}(\cdot \prec_\pi z)\right), \quad z \in Z, \tag{5.15}$$

*is an element of $L^2_\pi(\mathfrak{H}_\mu, X_M)$;*

(2) *One has*

$$I^d_M(f) = J^\pi_{X_M}(h_\pi(f)),$$

*where $h_\pi(f)$ is defined as in (5.15).*

*Moreover, if a random variable $F \in L^2(\mathbb{P})$ has the form $F = \sum_{d=1}^\infty I^d_M\left(f^{(d)}\right)$, where $f^{(d)} \in L^2_{s,0}\left(\mu^d\right)$ for $d \geq 1$ and the series is convergent in $L^2(\mathbb{P})$, then*

$$F = J^\pi_{X_M}(h_\pi(F)), \tag{5.16}$$

*where*

$$h_\pi(F)(z) = \sum_{d=1}^\infty h_\pi\left(f^{(d)}\right)(z), \quad z \in Z, \tag{5.17}$$

*and the series in (5.17) is convergent in $L^2_\pi(\mathfrak{H}_\mu, X_M)$.*

**Proof.** It is clear that $h_\pi(f) \in L^2(\mathfrak{H}_\mu, X_M)$ (the class of square integrable, but not necessarily adapted processes). To prove that $h_\pi(f) \in L^2_\pi(\mathfrak{H}_\mu, X_M)$, and hence adaptedness, observe that, thanks to Proposition 3, if $g \in L^2_{s,0}\left(\mu^d\right)$ has support in $Z^d_t$ for some $t \in (0,1]$, then $I^d_M(g) \in \mathcal{F}^\pi_t$. Now, for any fixed $z \in Z_t$, $t \in (0,1]$, the symmetric function (on $Z^{d-1}$) $f(z,\cdot)\,\mathbf{1}(\cdot \prec_\pi z)$ has support in $Z^d_t$ by definition of the order relation $\prec_\pi$. Then, for every $b \in \mathfrak{H}_\mu$ and $t \in (0,1]$,

$$(h_\pi(f), \pi_t b)_{\mathfrak{H}_\mu} = \int_{Z_t} h_\pi(f)(z)\, b(z)\, \mu(dz) = d\int_{Z_t} b(z)\, I^M_{d-1}\left(f(z,\cdot)\,\mathbf{1}(\cdot \prec_\pi z)\right)\mu(dz) \in \mathcal{F}^\pi_t,$$

and therefore $h_\pi(f) \in L^2_\pi(\mathfrak{H}_\mu, X_M)$. This proves Point 1. By density, it is sufficient to prove Point 2 for random variables of the type $I^d_M(f)$, where $f \in \mathcal{S}_{s,0}\left(\mu^d\right)$ is as in (5.7). Indeed, for such an $f$ and for every $(z,z_1,...,z_{d-1}) \in Z^d$

$$\begin{aligned} & f(z,z_1,...,z_{d-1}) \prod_{j=1}^{d-1} \mathbf{1}_{(z_j \prec_\pi z)} \\ = & \sum_{\sigma \in \mathfrak{S}_{d-1}} \mathbf{1}_{B_1 \cap Z_{t_1}}\left(z_{\sigma(1)}\right) \cdots \mathbf{1}_{B_{d-1} \cap \left(Z_{t_{d-1}} \backslash Z_{t_{d-2}}\right)}\left(z_{\sigma(d-1)}\right) \mathbf{1}_{B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)}(z), \end{aligned}$$

so that

$$d \times h_\pi(f)(z) = d\,(d-1)!\,M(B_1 \cap Z_{t_1}) \times \cdots \times M\left(B_{d-1} \cap \left(Z_{t_{d-1}} \backslash Z_{t_{d-2}}\right)\right) \mathbf{1}_{B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)}(z),$$

and finally, thanks to (2.9) and (5.8),

$$J^\pi_{X_M}(h_\pi(f)) = d!\,M(B_1 \cap Z_{t_1}) \times \cdots \times M\left(B_d \cap \left(Z_{t_d} \backslash Z_{t_{d-1}}\right)\right) = I^d_M(f).$$

The last assertion in the statement is an immediate consequence of the orthogonality relations between multiple integrals of different orders. ■

**Remarks** – (1) The function $h_\pi(f)$ in (5.15) is random for $d \geq 2$ and equal to $f$ when $d = 1$.

(2) Formula (5.15) implies that, for $t \in [0,1]$ and $f \in L^2_{s,0}\left(\mu^d\right)$,

$$I^d_M\left(f\mathbf{1}_{Z^d_t}\right) = J^\pi_{X_M}\left(\pi_t h_\pi(f)\right), \tag{5.18}$$

and therefore, since $t \mapsto J^\pi_{X_M}\left(\pi_t h_\pi(f)\right)$ is a $\mathcal{F}^\pi_t$-martingale (see Peccati and Taqqu, 2007),

$$\mathbb{E}\left[I^d_M(f) \mid \mathcal{F}^\pi_t\right] = I^d_M\left(f\mathbf{1}_{Z^d_t}\right), \quad t \in [0,1]. \tag{5.19}$$

(3) The random process

$$z \mapsto dI^M_{d-1}\left(f(z,\cdot)\right) \triangleq D_z I^M_d(f)$$

is a "formal" Malliavin-Shikegawa derivative of the random variable $I^M_d(f)$, whereas $z \mapsto dI^M_{d-1}\left(f(z,\cdot)\mathbf{1}(\cdot \prec_\pi z)\right)$ is the projection of $D_z I^M_d(f)$ on the space of adapted integrands $L^2_\pi(\mathfrak{H}_\mu, X_M)$. In this sense, formula (5.16) can be interpreted as a "generalized Clark-Ocone formula", in the same spirit of the results proved by L. Wu in Wu (1990) in the framework of abstract Wiener spaces.

## 6. Stable and weak convergence of multiple integrals

We now state the announced convergence results, which are consequences of Theorem 1 and Theorem 5. In what follows, $(Z_n, \mathcal{Z}_n, \mu_n)$, $n \geq 1$, is a sequence of measurable spaces and, for each $n$, $M_n$ is an independently scattered random measures on $(Z_n, \mathcal{Z}_n)$ with control $\mu_n$ (the $M_n$'s are defined on the same probability space); also $\mathfrak{H}_{\mu_n} = L^2(Z_n, \mathcal{Z}_n, \mu_n)$. The collection of random variables $X_{M_n} = X_{M_n}(\mathfrak{H}_{\mu_n})$ is defined through formula (4.2), with Lévy-Khinchine exponent $\psi_{\mathfrak{H}_{\mu_n}}(h,\lambda)$, $h \in \mathfrak{H}_{\mu_n}$, $\lambda \in \mathbb{R}$, given by (4.4). Moreover, for every $n \geq 1$, we choose the continuous resolution of the identity $\pi^{(n)} = \left\{\pi^{(n)}_t : t \in [0,1]\right\} \in \mathcal{R}_{X_{M_n}}(\mathfrak{H}_{\mu_n})$ as

$$\pi^{(n)}_t h(z) = \mathbf{1}_{Z_{n,t}}(z)\, h(z), \quad z \in Z, \quad h \in \mathfrak{H}_{\mu_n}, \tag{6.1}$$

where $Z_{n,t}$, $t \in [0,1]$ is an increasing collection of measurable sets such that $Z_{n,0} = \varnothing$, $Z_{n,1} = Z_n$ and verifying the continuity condition (5.1).

**Theorem 6.** *Under the previous notation and assumptions, let $d_n$, $n \geq 1$, be a sequence of natural numbers such that $d_n \geq 1$, and let $\pi^{(n)} \in \mathcal{R}_{X_{M_n}}(\mathfrak{H}_{\mu_n})$ be as in (6.1). Let also $f^{(n)}_{d_n} \in L^2_{s,0}\left(\mu^{d_n}_n\right)$, $n \geq 1$, and suppose there exists a sequence $\{t_n : n \geq 1\} \subset [0,1]$ and $\sigma$-fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\lim_{n\to+\infty} d_n! \left\| f^{(n)}_{d_n} \mathbf{1}_{Z^{d_n}_{n,t_n}} \right\|^2_{L^2\left(\mu^{d_n}_n\right)} = 0 \tag{6.2}$$

*and*

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}^{\pi^{(n)}}_{t_n}(X_{M_n}). \tag{6.3}$$

*Define also $h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right) \in L^2_{\pi^{(n)}}(\mathfrak{H}_{\mu_n}, X_{M_n})$ via formula (5.15) when $d_n \geq 2$, and set $h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right) = f^{(n)}_{d_n}$ when $d_n = 1$. If*

$$\exp\left[\int_{Z_n} K_{\mu_n}\left(\lambda h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right)(z), z\right)\mu_n(dz)\right] \xrightarrow{\mathbb{P}} \phi(\lambda,\omega), \quad \forall \lambda \in \mathbb{R}, \tag{6.4}$$

*where $K_{\mu_n}(t,z)$, $(t,z) \in \mathbb{R} \times Z$, is given by (4.7), $\phi \in \widehat{\mathbf{M}}_0$ and $\phi(\lambda) \in \vee_n \mathcal{U}_n \triangleq \mathcal{U}^*$, then, as $n \to +\infty$,*

$$\mathbb{E}\left[\exp\left(i\lambda I_{d_n}^{M_n}\left(f_{d_n}^{(n)}\right)\right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}\left(X_{M_n}\right)\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \tag{6.5}$$

*and*

$$I_{d_n}^{M_n}\left(f_{d_n}^{(n)}\right) \to_{(s,\mathcal{U}^*)} m(\cdot), \tag{6.6}$$

*where $m \in \mathbf{M}$ is as in (3.2).*

**Proof.** It is sufficient to observe that, thanks to (5.18) and the isometries properties of $I_{d_n}^{M_n}$ (see (4.14)) and $J_{X_{M_n}}^{\pi^{(n)}}$ (see (2.10)),

$$\begin{aligned} d_n! \left\| f_{d_n}^{(n)} \mathbf{1}_{Z_{n,t_n}^{d_n}} \right\|_{L^2\left(\mu_n^{d_n}\right)}^2 &= \mathbb{E}\left[ I_{d_n}^{M_n}\left(f_{d_n}^{(n)} \mathbf{1}_{Z_{n,t_n}^{d_n}}\right)^2\right] \\ &= \mathbb{E}\left[ J_{X_{M_n}}^{\pi^{(n)}}\left(\pi_{t_n}^{(n)} h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)\right)^2\right] \\ &= \left\| \pi_{t_n}^{(n)} h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)\right\|_{L^2_{\pi^{(n)}}(\mathfrak{H}_{\mu_n}, X_{M_n})}^2 . \end{aligned}$$

Moreover, according to Proposition 2,

$$\int_{Z_n} K_{\mu_n}\left(\lambda h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)(z), z\right)\mu_n(dz) = \psi_{\mathfrak{H}_{\mu_n}}\left(h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right), \lambda\right).$$

The conclusion is now a direct consequence of Theorem 1. ■

**Remark** – Theorem 6 can be immediately extended to sequences of random variables of the type $F_n = \sum_d I_{M_n}^d\left(f_n^{(d)}\right)$, $n \geq 1$, by using the last part of Theorem 5 (just replace $h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)$ with $h_{\pi^{(n)}}(F_n)$).

When $\phi(\lambda)$ is deterministic the statement of Theorem 6 can be simplified. Indeed, in this case, one can take $t_n = 0$ and $\mathcal{U}_n = \{\Omega, \varnothing\}$ for every $n$, so that conditions (6.2) and (6.3) become immaterial. Note also that, when $\phi(\lambda)$ is deterministic, formulae (6.5) and (6.6) are both equivalent to the weak convergence of the law of $I_{d_n}^{M_n}\left(f_{d_n}^{(n)}\right)$ towards the deterministic probability measure $m$. In the following Corollary, $\phi$ is deterministic but $h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)$ (defined in (5.15)) is random if $d_n \geq 2$.

**Corollary 7.** *Let the notation and assumptions of Theorem 6 prevail, and suppose moreover that $\phi(\lambda)$ is the Fourier transform of a deterministic probability measure $m$. If*

$$\exp\left[\int_{Z_n} K_{\mu_n}\left(\lambda h_{\pi^{(n)}}\left(f_{d_n}^{(n)}\right)(z), z\right)\mu_n(dz)\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \tag{6.7}$$

*then, as $n \to +\infty$, the law of $I_{d_n}^{M_n}\left(f_{d_n}^{(n)}\right)$ converges weakly to $m$.*

**Remark** – The exponential appearing in the RHS of (6.7) is the random characteristic function of a decoupled version of the generalized adapted integral $J^{\pi^{(n)}}_{X_{M_n}}\left(h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right)\right)$. Indeed, for every $n \geq 1$ and every $\lambda \in \mathbb{R}$, one has that

$$\exp\left[\int_{Z_n} K_{\mu_n}\left(\lambda h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right)(z), z\right)\mu_n(dz)\right] = \mathbb{E}\left[\exp\left(i\lambda\int_{Z_n} h_{\pi^{(n)}}\left(f^{(n)}_{d_n}\right)(z)\widetilde{M}_n(dz)\right) \mid M_n\right],$$

where $\widetilde{M}_n$ is an independent copy of $M_n$.

To conclude the section, we specialize Corollary 7 to the case where $\pi^{(n)}$, $\mu_n$, $Z_{n,t}$ and $Z_n$ do depend on $n$, the stochastic integration is performed with respect to a Poisson random measure, and the limit in (6.7) is Gaussian. This means that: (i) $\phi(\lambda) = \exp(-\lambda^2/2)$, (ii) $d_n = d \geq 2$, (iii) $(Z_n, \mathcal{Z}_n, \mu_n) = (Z, \mathcal{Z}, \mu)$ is a $\sigma$-finite measure space, (iv) $\pi^{(n)} = \pi$ is the resolution of the identity given by (6.1) for some increasing sequence $\{Z_{n,t}\} = \{Z_t\}$, and (v) $M_n = \hat{N}$ is a random centered Poisson measure with control $\mu$. Under (i)-(v), the expression on the RHS of (6.7) becomes

$$\int_Z K_\mu\left(\lambda h_\pi\left(f^{(n)}_d\right)(z), z\right)\mu(dz) = \int_Z\left(e^{i\lambda h_\pi\left(f^{(n)}_d\right)(z)} - 1 - i\lambda h_\pi\left(f^{(n)}_d\right)(z)\right)\mu(dz),$$

where $h_\pi(f^{(n)}_d)$ is once again given by $h_\pi(f^{(n)}_d) = d \times I^{\hat{N}}_{d-1}(f(z,\cdot)\mathbf{1}(\cdot \prec_\pi z))$ (see (5.15)).

The next results conclude the section, and are the starting point of Peccati and Taqqu (2008a). The first provides sufficient conditions for the multiple integral $I^{\hat{N}}_d\left(f^{(n)}_d\right)$ to converge in law to a $N(0,1)$ random variables.

**Corollary 8.** *Under the above notation, let assumptions (i)-(v) be satisfied. If*

$$\int_Z\left(e^{i\lambda h_\pi\left(f^{(n)}_d\right)(z)} - 1 - i\lambda h_\pi\left(f^{(n)}_d\right)(z)\right)\mu(dz) \xrightarrow{\mathbb{P}} -\lambda^2/2, \quad \forall\lambda \in \mathbb{R},$$

*then, the sequence $I^{\hat{N}}_d\left(f^{(n)}\right)$ converges in law to a standard Gaussian random variable.*

Observe that when $d = 2$, the random function $h_\pi(f)(z)$, $z \in Z$, becomes

$$\begin{aligned} h_\pi(f)(z) &= 2I^{\hat{N}}_1(f(z,\cdot)\mathbf{1}(\cdot \prec_\pi z)) \\ &= 2\hat{N}(f(z,\cdot)\mathbf{1}(\cdot \prec_\pi z)) = \int_Z f(z,x)\mathbf{1}(x \prec_\pi z)\hat{N}(dx). \quad (6.8) \end{aligned}$$

The following result, involving the case $d = 2$, is obtained by combining relation (6.8) with Corollary 8.

**Corollary 9.** *Consider a sequence of the type*

$$F_n = I^{\hat{N}}_2(f_n), \quad n \geq 1,$$

*where $f_n \in L^2_{s,0}(\mu^2)$ and, for every fixed $z \in Z$, define (as above) $f_n\mathbf{1}(\cdot \prec_\pi z) \in L^2(\mu)$ to be the application $y \mapsto f_n(y,z)\mathbf{1}_{(y \prec_\pi z)}$. Suppose that, for every $\lambda \in \mathbb{R}$,*

$$\int_Z \exp\left(i\lambda 2\hat{N}(f_n\mathbf{1}(\cdot \prec_\pi z)) - 1 - i\lambda 2\hat{N}(f_n\mathbf{1}(\cdot \prec_\pi z))\right)\mu(dz) \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} -\frac{\lambda^2}{2}. \quad (6.9)$$

*Then, $F_n \overset{law}{\to} N(0,1)$, where $N(0,1)$ is a standard Gaussian random variable.*

## 7. Applications

To conclude the paper, we shall briefly describe some of the contents of the companion paper Peccati and Taqqu (2008a), where the results of Section 6 (especially Corollary 9) are applied to obtain CLTs for vectors of single and double integrals with respect to a centered Poisson measure. These results involve moving average Lévy processes of the type

$$Y_t^h = \int_{-\infty}^{t} \int_{\mathbb{R}} uh\,(t-x)\,M\,(du,dx)\,, \quad t \geq 0,$$

where:

(i) $M\,(\cdot)$ is a centered Poisson measure on $\mathbb{R} \times \mathbb{R}$, with control $\nu\,(du)\,dx$ ($dx$ is Lebesgue measure),

(ii) $\nu$ is a positive measure satisfying such that $\int_{\mathbb{R}} u^j \nu\,(du) < +\infty$, $j = 2, 4, 6$,

(iii) $h$ has support in $[0, +\infty)$ and is such that $\int_{\mathbb{R}} [h\,(x)^2 + h\,(x)^4] dx < +\infty$.

We obtain CLTs involving the following classes of random variables:

– **Quadratic forms**. We will establish conditions on $M$ and $h$ to have that quadratic forms of the type

$$\beta\,(n) \sum_{1 \leq t,s \leq n} a\,(t,s)\,Y_t^h Y_s^h \quad \text{and} \quad \gamma\,(T) \int_0^T \int_0^T c\,(s,t)\,Y_t^h Y_s^h ds dt, \tag{7.1}$$

where $a\,(\cdot,\cdot)$ and $c\,(\cdot,\cdot)$ are real kernels and $\beta\,(\cdot)$ and $\gamma\,(\cdot)$ are appropriate normalizations, converge in law to a Gaussian random variable (see Bhansali et al., 2007 for general results about quadratic forms);

– **Quadratic functionals and path variances.** We will obtain CLTs (as $T \to +\infty$) for the quadratic functional

$$\int_0^T \left(Y_t^h\right)^2 dt, \tag{7.2}$$

and for the path-variance of $Y_t^h$, that is defined as

$$\frac{1}{T} \int_0^T \left( Y_t^h - \frac{1}{T} \int_0^T Y_u^h du \right)^2 dt. \tag{7.3}$$

The asymptotic behavior of the functionals in (7.1), (7.2) and (7.3) will be characterized by means of limit theorems for single and double stochastic integrals with respect to $M$. In particular, the following multiplication formula (see Peccati and Taqqu, 2008a and the references therein for general statements) will be used: for every $s, t \geq 0$

$$Y_t^h Y_s^h = \mathbb{E} Y_t^h Y_s^h + \int_{-\infty}^{t \wedge s} \int_{\mathbb{R}} u^2 h\,(t-x)^2\,M\,(du,dx) + I_2^M\,(H)\,,$$

where the symmetric kernel $H$, on $(\mathbb{R} \times \mathbb{R})^2$, is given by

$$H\,(u,t;v,s) = uvh\,(t-x)\,h\,(s-y)\,.$$

Finally, we stress that the papers de Blasi et al. (2008) and Peccati and Prünster contain several applications of the asymptotic properties of objects such as (7.2) and (7.3) to prior specification in Bayesian non parametric statistics. In particular, we use the fact that non centered versions of random processes such $Y_t^h$ are used to model the evolution of random hazard rates (see e.g. Ibrahim et al., 2001).

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