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Independence of Four Projective Criteria for the Weak Invariance Principle

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Abstract. Let $(X_i)_{i\in\mathbb{Z}}$ be a regular stationary process for a given filtration. The weak invariance principle holds under the condition $\sum_{i\in\mathbb{Z}}\|P_0(X_i)\|_2 < \infty$ (see Hannan, 1979; Dedecker and Merlevède, 2003; Dedecker et al., 2007). In this paper, we show that this criterion is independent of other known criteria: the martingale-coboundary decomposition of Gordin (see Gordin, 1969, 1973), the criterion of Dedecker and Rio (see Dedecker and Rio, 2000) and the condition of Maxwell and Woodroofe (see Maxwell and Woodroofe, 2000; Peligrad and Utev, 2005; Volný, 2006, 2007).

1. Introduction and Results

The aim of this paper is to study the relation between several criteria to get the weak invariance principle of Donsker in dependent case. In the paper by Durieu and Volný (2008), the independence between three of them is already shown. These criteria are the martingale-coboundary decomposition, the projective criterion of Dedecker and Rio and the Maxwell-Woodroofe condition. Here, we consider a fourth one ($\sum_{i\in\mathbb{Z}}\|P_0(X_i)\|_2 < \infty$) and we show that it is independent of the three others. Let us begin by the statements of the four criteria.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let T be a bijective bimeasurable transformation of Ω preserving μ. We assume $(\Omega, \mathcal{A}, \mu, T)$ is an ergodic dynamical system. Let $f : \Omega \longrightarrow \mathbb{R}$ be a measurable function with zero mean. We recall that the process $(f \circ T^i)_{i\in\mathbb{N}}$ satisfies the weak invariance principle if the process

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor tn\rfloor-1} f\circ T^k,\ t\in[0,1]$$

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converges in distribution to a Gaussian process in the space $D([0,1])$ provided with the Skorohod topology (see Billingsley, 1968). Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$ such that $T^{-1}\mathcal{F} \subset \mathcal{F}$. We denote by $\mathcal{F}_i$ the σ-algebra $T^{-i}\mathcal{F}$. The function f is called regular with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ if

$$\mathbb{E}(f|\mathcal{F}_{-\infty}) = 0 \text{ and } \mathbb{E}(f|\mathcal{F}_{+\infty}) = f.$$

In the sequel, we assume that f is a square integrable function and we write $\mathbb{L}^p$ for $\mathbb{L}^p(\mu)$, $p \geq 1$.

• The first criterion is the martingale-coboundary decomposition due to Gordin (1969). We will restrict our attention to the martingale-coboundary decomposition in $\mathbb{L}^1$ (see Gordin, 1973; Esseen and Janson, 1985 for a complete proof). We say that f admits such a decomposition if $f = m + g - g\circ T$ where $(m\circ T^i)_{i\in\mathbb{Z}} \subset \mathbb{L}^1$ is a martingale difference sequence and $g \in \mathbb{L}^1$. If $m \in \mathbb{L}^2$, then the central limit theorem holds. Further, if $\frac{1}{\sqrt{n}}\max_{i\leq n}|g\circ T^i|$ goes to 0 in probability, the weak invariance principle holds (see Hall and Heyde, 1980). If f is a regular function with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ then the martingale-coboundary decomposition in $\mathbb{L}^1$ is equivalent to

$$\sum_{i=0}^{\infty}\mathbb{E}(f\circ T^i|\mathcal{F}_0) \text{ and } \sum_{i=0}^{\infty} f\circ T^{-i} - \mathbb{E}(f\circ T^{-i}|\mathcal{F}_0) \text{ converge in } \mathbb{L}^1, \tag{1.1}$$

see Volný (1993). Remark that if the process $(f\circ T^i)_{i\in\mathbb{Z}}$ is adapted to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$, the second sum is equal to zero.

• The Dedecker and Rio criterion is satisfied if

$$\sum_{k=1}^{\infty} f\mathbb{E}(f\circ T^k|\mathcal{F}_0) \text{ converges in } \mathbb{L}^1. \tag{1.2}$$

According to Dedecker and Rio (2000), in the adapted case, this condition implies the weak invariance principle.

• The Maxwell-Woodroofe condition (see Maxwell and Woodroofe, 2000) is satisfied if

$$\sum_{n=1}^{\infty}\frac{\|\mathbb{E}(S_n(f)|\mathcal{F}_0)\|_2}{n^{\frac{3}{2}}} < +\infty \tag{1.3}$$

where $S_n(f) = \sum_{i=0}^{n-1} f\circ T^i$. In the adapted case, Peligrad and Utev (2005) proved that this condition implies the weak invariance principle. In the general case, the weak invariance principle holds as soon as (1.3) and

$$\sum_{n=1}^{\infty}\frac{\|S_n(f) - \mathbb{E}(S_n(f)|\mathcal{F}_n)\|_2}{n^{\frac{3}{2}}} < +\infty,$$

(see Volný, 2006, 2007).

The independence between these three criteria is proved in Durieu and Volný (2008). Here we add a new criterion. Let us denote by $H_k = \mathbb{L}^2(\mathcal{F}_k)$ the space of $\mathcal{F}_k$-measurable functions which are square integrable and denote by P_k the orthogonal projection operator onto the space $H_k \ominus H_{k-1}$. For $f \in \mathbb{L}^2$,

$$P_k(f) = \mathbb{E}(f|\mathcal{F}_k) - \mathbb{E}(f|\mathcal{F}_{k-1}).$$

• Let f be a regular function for the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. As a consequence of a result given by Heyde (1974) (see Volný, 1993, Theorem 6) the central limit theorem holds as soon as

$$\sum_{i\in\mathbb{Z}} \|P_0(f\circ T^i)\|_2 < \infty. \tag{1.4}$$

In the adapted case, this result and the weak invariance principle were proved by Hannan (1973, 1979) under the assumption that T is weakly mixing. Hannan's weak invariance principle was proved without the extra assumption by Dedecker and Merlevède (2003, Corollary 3). Finally, in the general case, the weak invariance principle under (1.4) is due to Dedecker, Merlevède and Volný (2007, Corollary 2).

Our main result is the following theorem.

Theorem 1.1. *Conditions* (1.1)*,* (1.2)*,* (1.3) *and* (1.4) *are pairwise independent: in all ergodic dynamical system with positive entropy, for each couple of conditions among the four, there exists an* $\mathbb{L}^2$*-function satisfying the first condition but not the second one.*

With the results of Durieu and Volný (2008), it remains to prove the independence of (1.1), (1.2), (1.3) with (1.4).

2. Proof of Theorem 1.1

Let $(\Omega,\mathcal{A},\mu,T)$ be an ergodic dynamical system with entropy grea ter or equal than 1. Let $\mathcal{B}$ and $\mathcal{C}$ be two independent sub-σ-algebra of $\mathcal{A}$. Let $(e_i)_{i\in\mathbb{Z}}$ be a sequence of independent $\mathcal{B}$-measurable random variables in $\{-1,1\}$ such that $\mu(e_i=-1)=\mu(e_i=1)=\frac{1}{2}$ and $e_i=e_0\circ T^i$, $i\in\mathbb{Z}$. We denote by $\mathcal{F}_0$ the σ-algebra generated by $\mathcal{C}$ and e_i for $i\le 0$ and we set $\mathcal{F}_i=T^{-i}\mathcal{F}_0$. Note that the case of entropy in $(0,1)$ can be studied by using another Bernoulli shift.

We introduce three sequences with the following properties:
$(\theta_k)_{k\in\mathbb{N}}\subset(0,+\infty)$;
$(\rho_k)_{k\in\mathbb{N}}\subset(0,1)$ such that $\sum_{k\ge0}\rho_k<1$;
$(N_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ such that $N_{k+1}>N_k$.
We can always find a sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,1)$ such that $\sum_{k\ge0}\theta_k N_k\sqrt{\varepsilon_k}<\infty$. So, we fix such a sequence and denote by f the function defined by

$$f=\sum_{k=1}^{\infty}\theta_k e_{-N_k}\mathbb{1}_{A_k}$$

where the sequence of sets $(A_k)_{k\in\mathbb{N}}$ verifies:

- $\forall k\in\mathbb{N}$, $A_k\in\mathcal{C}$,
- the sets A_k are disjoint,
- $\exists a\in(0,1)$, $\forall k\in\mathbb{N}$, $a\rho_k\le\mu(A_k)\le\rho_k$,
- $\forall k\in\mathbb{N}$, $\forall i\in\{0,\dots,N_k\}$, $\mu(T^{-i}A_k\Delta A_k)\le\varepsilon_k$.

The construction of these sets is done in detail in Durieu and Volný (2008).

First, remark that the process $(f\circ T^i)_{i\in\mathbb{Z}}$ is adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ (then f is regular) and

$$f\in\mathbb{L}^2 \text{ if and only if } \sum_{k\ge1}\theta_k^2\rho_k<\infty.$$

The next proposition is proved in Durieu and Volný (2008).

Proposition 2.1. *For the function f previously defined,*

i. (1.1) $\Leftrightarrow$ $\sum_{k\geq 1}\theta_k\rho_k\sqrt{N_k}<\infty$;
ii. (1.2) $\Leftrightarrow$ $\sum_{k\geq 1}\theta_k^2\rho_k\sqrt{N_k}<\infty$;
iii. (1.3) $\Leftrightarrow$ $\sum_{n\geq 1}n^{-\frac{3}{2}}\left(\sum_{k\geq 1}\theta_k^2\min(n,N_k)\rho_k\right)^{\frac{1}{2}}<\infty$.

We can state an analogous result for condition (1.4).

Proposition 2.2. *For the function f previously defined,*

$$\sum_{i\in\mathbb{Z}}\|P_0(f\circ T^i)\|_2<\infty \textit{ if and only if } \sum_{k\geq 1}\theta_k\sqrt{\rho_k}<\infty.$$

This proposition is proved in Section 3.

Counterexamples. Now we give two counterexamples proving Theorem 1.1.

(1) We consider the function f defined by the sequences $\theta_k=\frac{2^k}{k}$, $\rho_k=\frac{1}{4^k}$, $N_k=k$. Then $f\in\mathbb{L}^2$ and using Proposition 2.1, we get:
   a) $\sum_{k\geq 1}\theta_k\rho_k\sqrt{N_k}=\sum_{k\geq 1}\frac{1}{2^k\sqrt{k}}<\infty$ and then (1.1) is verified.
   b) $\sum_{k\geq 1}\theta_k^2\rho_k\sqrt{N_k}=\sum_{k\geq 1}k^{-\frac{3}{2}}<\infty$ and then (1.2) is verified.
   c) $\sum_{k\geq 1}\theta_k^2\min(n,N_k)\rho_k=\sum_{k\geq 1}\frac{\min(n,k)}{k^2}$.
   But $\sum_{k=1}^{n}\frac{1}{k}\leq 1+\ln(n)$ and $\sum_{k=n+1}^{\infty}\frac{n}{k^2}\leq 1$. Then
   $\sum_{n\geq 1}n^{-\frac{3}{2}}\left(\sum_{k\geq 1}\theta_k^2\min(n,N_k)\rho_k\right)^{\frac{1}{2}}\leq\sum_{n\geq 1}n^{-\frac{3}{2}}\sqrt{\ln(n)+2}<\infty$ and (1.3) is verified.
   d) $\sum_{k\geq 1}\theta_k\sqrt{\rho_k}=\sum_{k\geq 1}\frac{1}{k}$ diverges and then by Proposition 2.2, (1.4) is not satisfied.

   This counterexample shows that none of the conditions (1.1), (1.2) and (1.3) implies (1.4).

(2) Now, if we consider the function f defined by the sequences: $\theta_k=\frac{2^k}{k^{\frac{3}{2}}}$, $\rho_k=\frac{1}{4^k}$, $N_k=2^{4k}$, then $f\in\mathbb{L}^2$ and:
   a) $\sum_{k\geq 1}\theta_k\rho_k\sqrt{N_k}=\sum_{k\geq 1}\frac{2^k}{k^{\frac{3}{2}}}$ diverges and then (1.1) does not hold.
   b) $\sum_{k\geq 1}\theta_k^2\rho_k\sqrt{N_k}=\sum_{k\geq 1}\frac{2^{2k}}{k^3}$ diverges and then (1.2) does not hold.
   c) $\sum_{k\geq 1}\theta_k^2\min(n,N_k)\rho_k=\sum_{k\geq 1}\frac{\min(n,2^{4k})}{k^3}\geq\sum_{k\geq\lfloor\frac{\ln n}{4\ln 2}\rfloor}\frac{n}{k^3}\geq 8\ln^2 2\frac{n}{\ln^2 n}$.
   Then $\sum_{n\geq 1}n^{-\frac{3}{2}}\left(\sum_{k\geq 1}\theta_k^2\min(n,N_k)\rho_k\right)^{\frac{1}{2}}\geq 8\ln^2 2\sum_{n\geq 1}\frac{1}{n\ln n}$ diverges and (1.3) does not hold.
   d) On the other hand, $\sum_{k\geq 1}\theta_k\sqrt{\rho_k}=\sum_{k\geq 1}\frac{1}{k^{\frac{3}{2}}}<\infty$ and then (1.4) holds.

   This shows that (1.4) does not imply any conditions (1.1), (1.2) or (1.3).

Remarks.

i. In fact, we showed a little more than Theorem 1.1. We got that the three conditions (1.1), (1.2) and (1.3) together are independent of (1.4).
ii. To show that (1.4) does not imply (1.1), it is enough to consider a linear process $f=\sum_{i\in\mathbb{Z}}a_i\xi_i$ where $(\xi_i)_{i\in\mathbb{Z}}$ is an iid sequence with $\mu(\xi_0=1)=\mu(\xi_0=-1)=\frac{1}{2}$ and $a_i=\frac{1}{i^{\frac{3}{2}}}$.

3. Proof of Proposition 2.2

First of all, $(f\circ T^i)_{i\in\mathbb{Z}}$ is adapted to the filtration and then for all $i<0$,

$$P_0(f\circ T^i)=0.$$

For $i\geq 0$, we have

$$\begin{aligned}P_0(f\circ T^i) &= \mathbb{E}(f\circ T^i|\mathcal{F}_0)-\mathbb{E}(f\circ T^i|\mathcal{F}_{-1})\\ &= \sum_{k\geq 1}\theta_k\left[\mathbb{E}(e_{-N_k+i}|\mathcal{F}_0)-\mathbb{E}(e_{-N_k+i}|\mathcal{F}_{-1})\right]\mathbb{1}_{A_k}\circ T^i.\end{aligned}$$

Since e_j is $\mathcal{F}_0$-measurable for $j\leq 0$ and independent of $\mathcal{F}_0$ for $j>0$,

$$\begin{aligned}\mathbb{E}(e_{-N_k+i}|\mathcal{F}_0)-\mathbb{E}(e_{-N_k+i}|\mathcal{F}_{-1}) &= \begin{cases} e_0 & \text{if} \quad i=N_k\\ 0 & \text{otherwise}\end{cases}\\ &= e_0\mathbb{1}_{\{i=N_k\}}.\end{aligned}$$

Thus

$$\begin{aligned}P_0(f\circ T^i) &= \sum_{k\geq 1}\theta_k e_0\mathbb{1}_{\{i=N_k\}}\mathbb{1}_{A_k}+\sum_{k\geq 1}\theta_k e_0\mathbb{1}_{\{i=N_k\}}(\mathbb{1}_{T^{-i}A_k}-\mathbb{1}_{A_k})\\ &= \qquad I_1(i) \qquad + \qquad I_2(i) \qquad .\end{aligned}$$

For I_2, we use the fact that $\mu(A_k\Delta T^{-i}A_k)\leq\varepsilon_k$ for $0\leq i\leq N_k$ to get

$$\begin{aligned}\|I_2(i)\|_2 &\leq \sum_{k\geq 1}\theta_k\mathbb{1}_{\{i=N_k\}}\|e_0\mathbb{1}_{A_k\Delta T^{-i}A_k}\|_2\\ &\leq \sum_{k\geq 1}\theta_k\mathbb{1}_{\{i=N_k\}}\sqrt{\varepsilon_k}.\end{aligned}$$

Remark for each $i\geq 0$, there is at most one integer k such that $N_k=i$ and for each $k\geq 1$, there exists an integer i such that $i=N_k$. We deduce

$$\begin{aligned}\sum_{i\geq 0}\|I_2(i)\|_2 &\leq \sum_{i\geq 0}\sum_{k\geq 1}\theta_k\mathbb{1}_{\{i=N_k\}}\sqrt{\varepsilon_k}\\ &= \sum_{k\geq 1}\theta_k\sqrt{\varepsilon_k}\end{aligned}$$

which is finite by the assumptions.

Thus, $\sum_{i\geq 0}\|P_i(f)\|_2$ is converging if and only if $\sum_{i\geq 0}\|I_1(i)\|_2$ is converging. Now for a fixed i, since the sets A_k are disjoint and since there is at most one k such that $N_k=i$, we have

$$\begin{aligned}\|I_1(i)\|_2 &= \sqrt{\sum_{k\geq 1}\theta_k^2\mathbb{1}_{\{i=N_k\}}\mu(A_k)}\\ &= \sum_{k\geq 1}\theta_k\mathbb{1}_{\{i=N_k\}}\sqrt{\mu(A_k)}.\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{i\geq 0}\|I_1(i)\|_2 &= \sum_{i\geq 0}\sum_{k\geq 1}\theta_k\mathbb{1}_{\{i=N_k\}}\sqrt{\mu(A_k)}\\ &= \sum_{k\geq 1}\theta_k\sqrt{\mu(A_k)}.\end{aligned}$$

We can conclude the proof using $a\rho_k \leq \mu(A_k) \leq \rho_k$.

□

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